# A complement to a recent paper on some infinite sums with the zeta values 

Iaroslav V. Blagouchine


#### Abstract

Recently, several new results related to the evaluation of the series $\sum(-1)^{n} \zeta(n) /(n+k)$ were published. In this short note we show that this series also possesses an interesting connection to the values of the $\zeta$-function on the critical line and to the Euler constant.

Keywords: Zeta-function, zeta values, critical line, closed-form evaluation, integral representation, Euler constant, complex integration, Cauchy residue theorem.


## I. Introduction

In a recent paper Coppo [4] investigated the series

$$
\begin{equation*}
v_{k} \equiv \sum_{j=2}^{\infty}(-1)^{j} \frac{\zeta(j)}{j+k}, \quad k \in\{-1,0\} \cup \mathbb{N}, \tag{1}
\end{equation*}
$$

and obtained various interesting properties by comparing different closed-form expressions for it. The same series was earlier studied in [1, p. 413, Eq. (38)], where we also obtained a closed-form expression for it 1 In this short note, we devise yet another expression for the same series, showing that there exist an intimate connection between the values of the $\zeta$-function on the critical line $1 / 2+i t$, $t \in \mathbb{R}$, Euler's constant $\gamma$ and the fundamental values of the zeta function at positive integers $\zeta(n)$, $n=2,3,4, \ldots$ Furthermore, the obtained expression may also be useful in that sense that it also holds for non-integer and even complex values of $k$.

## II. The results

We present our results in the form of two theorems with a corollary. Since the proofs of both theorems are quite similar, for the purpose of brevity we provide the proof only for the first theorem.

Theorem 1. The series $v_{k}$ is closely connected to the values of the $\zeta$-function on the critical line. In particular, $v_{k}$ may be evaluated via the following integral with an exponentially decreasing kernel:

$$
\begin{equation*}
v_{\omega}=-\frac{1}{(\omega+1)^{2}}+\frac{\gamma}{\omega+1}-\frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1 / 2 \pm i x)}{(1 / 2 \pm i x+\omega) \operatorname{ch} \pi x} d x, \quad \operatorname{Re} \omega>-\frac{1}{2} \tag{2}
\end{equation*}
$$

Theorem 2. For any $\omega \in \mathbb{C}$ such that $\operatorname{Re} \omega>-3 / 2$, we also have

$$
\begin{equation*}
v_{\omega}=\frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3 / 2 \pm i x)}{(3 / 2 \pm i x+\omega) \operatorname{ch} \pi x} d x \tag{3}
\end{equation*}
$$

[^0]Corollary. Putting $\omega=k, k \in \mathbb{N}_{0}$, in previous theorems and comparing (1) to the expressions obtained in (1), p. 413, Eq. (38)], [4] and [2], enable us to evaluate the integrals in (2) and (3) in a closed-form. For instance,

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} \frac{\zeta(1 / 2 \pm i x)}{(1 / 2 \pm i x) \operatorname{ch} \pi x} d x=-2, \\
& \int_{-\infty}^{+\infty} \frac{\zeta(1 / 2+i x)}{(3 / 2+i x) \operatorname{ch} \pi x} d x=\ln 2 \pi-\frac{5}{2}, \\
& \int_{-\infty}^{+\infty} \frac{\zeta(1 / 2+i x)}{(5 / 2+i x) \operatorname{ch} \pi x} d x=\ln 2 \pi-4 \ln A-\frac{11}{9}, \\
& \int_{-\infty}^{+\infty} \frac{\zeta(3 / 2 \pm i x)}{(3 / 2 \pm i x) \operatorname{ch} \pi x} d x=2 \gamma, \\
& \int_{-\infty}^{+\infty} \frac{\zeta(3 / 2 \pm i x)}{(1 / 2 \pm i x) \operatorname{ch} \pi x} d x=2 \int_{0}^{1} \frac{\Psi(x+1)+\gamma}{x} d x=2 \kappa_{1}+\frac{\pi^{2}}{6}-\gamma^{2}-2 \gamma_{1},
\end{aligned}
$$

where $A \equiv e^{\frac{1}{12}-\zeta^{\prime}(-1)}=1.282427129 \ldots$ is the Glaisher-Kinkelin constant, $\Psi$ is the digamma function (the logarithmic derivative of the $\Gamma$-function), $\gamma_{1}=-0.07281584548 \ldots$ is the first Stieltjes constant and $\kappa_{1}=0.5290529699 \ldots$ is a constant related to Gregory's coefficients $G_{n}$, see [2], Appendix], as well as the sequence A270859 from the OEIS for more digits of $\kappa_{1}$. We do not know if such a "simple" result for these integrals could be found for algebraic or just rational values of $\omega$.

Proof. Consider the following line integral taken along a contour $C$ consisting of the interval $[-R,+R]$, $R \in \mathbb{N}$, on the real axis, and a semicircle of the radius $R$ in the upper half-plane, denoted $C_{R}$,

$$
\begin{equation*}
\oint_{C} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z} d z=\int_{-R}^{+R} \frac{\zeta(1 / 2-i x)}{(1 / 2-i x+\omega) \operatorname{ch} \pi x} d x+\int_{C_{R}} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z} d z \tag{4}
\end{equation*}
$$

with $\omega \in \mathbb{C}, \operatorname{Re} \omega>-\frac{1}{2}$. On the contour $C_{R}$ the last integral may be bounded as follows:

$$
\begin{align*}
& \left|\int_{C_{R}} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z} d z\right|=R\left|\int_{0}^{\pi} \frac{\zeta\left(1 / 2-i R e^{i \varphi}\right) e^{i \varphi}}{\left(1 / 2-i R e^{i \varphi}+\omega\right) \operatorname{ch}\left(\pi R e^{i \varphi}\right)} d \varphi\right| \\
& \quad \leqslant R \max _{\varphi \in[0, \pi]}\left|\frac{\zeta\left(1 / 2-i R e^{i \varphi}\right)}{1 / 2-i \operatorname{Re} e^{i \varphi}+\omega}\right| \cdot I_{R} \leqslant \max _{\varphi \in[0, \pi]}\left|\zeta\left(1 / 2-i R e^{i \varphi}\right)\right| \cdot I_{R} \tag{5}
\end{align*}
$$

where we denoted

$$
I_{R} \equiv \int_{0}^{\pi} \frac{d \varphi}{\left|\operatorname{ch}\left(\pi R e^{i \varphi}\right)\right|}, \quad R>0
$$

for brevity. Now, in the half-plane $\sigma>1$, the absolute value of $\zeta(\sigma+i t)$ may be always bounded by a constant $C=\zeta(\sigma)$, which decreases and tends to 1 as $\sigma \rightarrow \infty$. In contrast, in the strip $0 \leqslant \sigma \leqslant 1$ the function $|\zeta(\sigma+i t)|$ is unbounded; presently, the rate of grow is still not known, but it follows from the general theory of Dirichlet series that it cannot be faster than $O\left(|t|^{1-\sigma}\right),|t| \geqslant \frac{1}{2}$, in the strip $\frac{1}{2} \leqslant \sigma<1$,
see [3, Theorem 35, pp. 99-102] ${ }^{2}$ Hence, $\operatorname{since} \sin \varphi \geqslant 0$ and if $R$ is large enough, this rough estimate gives us

$$
\left|\zeta\left(1 / 2-i R e^{i \varphi}\right)\right|=|\zeta(1 / 2+R \sin \varphi-i R \cos \varphi)|=O(\sqrt{R})
$$

in the interval $\varphi \in[0, \pi]$. On the other hand, as $R$ tends to infinity and remains integer the integral $I_{R}$ tends to zero as $O(1 / R)$. To show this, we first note that

$$
\frac{1}{\left|\operatorname{ch}\left(\pi R e^{i \varphi}\right)\right|}=\frac{\sqrt{2}}{\sqrt{\operatorname{ch}(2 \pi R \cos \varphi)+\cos (2 \pi R \sin \varphi)}}=O\left(e^{-\pi R|\cos \varphi|}\right), \quad R \rightarrow \infty,
$$

because $0 \leqslant \varphi \leqslant \pi$ and $R \in \mathbb{N}$. Since $\left|\operatorname{ch}\left(\pi R e^{i \varphi}\right)\right|^{-1}$ is symmetric about $\varphi=\frac{1}{2} \pi$, we may write

$$
\begin{align*}
I_{R} & =\int_{0}^{\frac{\pi}{2}} \frac{2 \sqrt{2}}{\sqrt{\operatorname{ch}(2 \pi R \cos \varphi)+\cos (2 \pi R \sin \varphi)}} d \varphi \\
& =O\left(\int_{0}^{\frac{\pi}{2}} e^{-\pi R \cos \varphi} d \varphi\right)=O\left(\int_{0}^{\frac{\pi}{2}} e^{-\pi R \sin \vartheta} d \vartheta\right), \quad R \rightarrow \infty . \tag{6}
\end{align*}
$$

Now, from the well-known inequality

$$
\frac{2 \vartheta}{\pi} \leqslant \sin \vartheta \leqslant \vartheta, \quad \vartheta \in\left[0, \frac{1}{2} \pi\right]
$$

we deduce that

$$
\begin{equation*}
\frac{1-e^{-\frac{1}{2} \pi^{2} R}}{\pi R} \leqslant \int_{0}^{\frac{\pi}{2}} e^{-\pi R \sin \vartheta} d \vartheta \leqslant \frac{1-e^{-\pi R}}{2 R} \tag{7}
\end{equation*}
$$

whence $I_{R}=O(1 / R)$ at $R \rightarrow \infty$. Inserting both latter results into (5), we obtain

$$
\left|\int_{C_{R}} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+k) \operatorname{ch} \pi z} d z\right|=O\left(R^{-1 / 2}\right) \rightarrow 0 \quad \text { as } \quad R \rightarrow \infty, R \in \mathbb{N} .
$$

Hence, making $R \rightarrow \infty$, equality (4) becomes

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \frac{\zeta(1 / 2-i x)}{(1 / 2-i x+\omega) \operatorname{ch} \pi x} d x=\oint_{C} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z} d z, \quad \operatorname{Re} \omega>-\frac{1}{2} \tag{8}
\end{equation*}
$$

where the latter integral is taken around an infinitely large semicircle in the upper half-plane, denoted C. The integrand is not a holomorphic function: in $C$ it has the simple poles at $z=z_{n} \equiv i\left(n-\frac{1}{2}\right)$, $n=2,3,4, \ldots$, due to the hyperbolic secant, and a double pole at $z=\frac{i}{2}$, due to both the hyperbolic

[^1]secant and the $\zeta$-function 3 Therefore, by the Cauchy residue theorem
\[

$$
\begin{gathered}
\oint_{C} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z} d z=2 \pi i\left\{\operatorname{res}_{z=\frac{i}{2}} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z}+\sum_{n=2}^{\infty} \operatorname{res}_{z=z_{n}} \frac{\zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z}\right\} \\
=2 \pi i\left\{\lim _{z \rightarrow \frac{i}{2}} \frac{\partial}{\partial z} \frac{\left(z-\frac{1}{2} i\right)^{2} \zeta(1 / 2-i z)}{(1 / 2-i z+\omega) \operatorname{ch} \pi z}+\left.\sum_{n=2}^{\infty} \frac{\zeta(1 / 2-i z)}{\pi(1 / 2-i z+\omega) \operatorname{sh} \pi z}\right|_{z=i\left(n-\frac{1}{2}\right)}\right\} \\
=2 \pi i\left\{\frac{1}{\pi i} \cdot \frac{\gamma(1+\omega)-1}{(1+\omega)^{2}}+\frac{1}{\pi i} \sum_{n=2}^{\infty}(-1)^{n+1} \frac{\zeta(n)}{n+\omega}\right\}
\end{gathered}
$$
\]

Equating the left part of (8) with the last result yields (2).

## References

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[^2]
[^0]:    Email address: iaroslav.blagouchine@univ-tln.fr (Iaroslav V. Blagouchine)
    ${ }^{1}$ Except for the case $k=-1$, which was not studied there.

[^1]:    ${ }^{2}$ There exist, of course, more sharp estimations, such as Huxley's estimations or Lindelöf hypothesis, but we do not need them for our proof (see, for more details, e.g. [10, Chapt. XIII], [5], [8], [7], 6], [9]).

[^2]:    ${ }^{3}$ Note that the pole $z=-i\left(\frac{1}{2}+\omega\right)$, due to the denominator $(1 / 2-i z+\omega)$, is located in the lower half-plane.

