A complement to a recent paper on some infinite sums with the zeta values

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Abstract

Recently, several new results related to the evaluation of the series $\sum (-1)^n \zeta(n)/(n+k)$ were published. In this short note we show that this series also possesses an interesting connection to the values of the ζ -function on the critical line and to the Euler constant.

Keywords: Zeta–function, zeta values, critical line, closed–form evaluation, integral representation, Euler constant, complex integration, Cauchy residue theorem.

I. Introduction

In a recent paper Coppo [4] investigated the series

$$\nu_k \equiv \sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{j+k}, \qquad k \in \{-1,0\} \cup \mathbb{N},$$
(1)

and obtained various interesting properties by comparing different closed–form expressions for it. The same series was earlier studied in [1, p. 413, Eq. (38)], where we also obtained a closed–form expression for it.¹ In this short note, we devise yet another expression for the same series, showing that there exist an intimate connection between the values of the ζ –function on the critical line 1/2 + it, $t \in \mathbb{R}$, Euler's constant γ and the fundamental values of the zeta function at positive integers $\zeta(n)$, $n = 2, 3, 4, \ldots$ Furthermore, the obtained expression may also be useful in that sense that it also holds for non–integer and even complex values of k.

II. The results

We present our results in the form of two theorems with a corollary. Since the proofs of both theorems are quite similar, for the purpose of brevity we provide the proof only for the first theorem.

Theorem 1. The series v_k is closely connected to the values of the ζ -function on the critical line. In particular, v_k may be evaluated via the following integral with an exponentially decreasing kernel:

$$\nu_{\omega} = -\frac{1}{(\omega+1)^2} + \frac{\gamma}{\omega+1} - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 \pm ix)}{(1/2 \pm ix + \omega) \operatorname{ch} \pi x} \, dx \,, \qquad \operatorname{Re} \omega > -\frac{1}{2} \tag{2}$$

Theorem 2. For any $\omega \in \mathbb{C}$ such that $\operatorname{Re} \omega > -3/2$, we also have

$$\nu_{\omega} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(3/2 \pm ix + \omega) \operatorname{ch} \pi x} \, dx \,. \tag{3}$$

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¹Except for the case k = -1, which was not studied there.

Corollary. Putting $\omega = k, k \in \mathbb{N}_0$, in previous theorems and comparing (1) to the expressions obtained in [1, p. 413, Eq. (38)], [4] and [2], enable us to evaluate the integrals in (2) and (3) in a closed–form. For instance,

$$\int_{-\infty}^{+\infty} \frac{\zeta(1/2 \pm ix)}{(1/2 \pm ix) \operatorname{ch} \pi x} dx = -2,$$

$$\int_{-\infty}^{+\infty} \frac{\zeta(1/2 + ix)}{(3/2 + ix) \operatorname{ch} \pi x} dx = \ln 2\pi - \frac{5}{2},$$

$$\int_{-\infty}^{+\infty} \frac{\zeta(1/2 + ix)}{(5/2 + ix) \operatorname{ch} \pi x} dx = \ln 2\pi - 4 \ln A - \frac{11}{9},$$

$$\int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(3/2 \pm ix) \operatorname{ch} \pi x} dx = 2\gamma,$$

$$\int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(1/2 \pm ix) \operatorname{ch} \pi x} dx = 2\int_{0}^{1} \frac{\Psi(x+1) + \gamma}{x} dx = 2\kappa_{1} + \frac{\pi^{2}}{6} - \gamma^{2} - 2\gamma_{1},$$

where $A \equiv e^{\frac{1}{12}-\zeta'(-1)} = 1.282427129...$ is the Glaisher-Kinkelin constant, Ψ is the digamma function (the logarithmic derivative of the Γ -function), $\gamma_1 = -0.07281584548...$ is the first Stieltjes constant and $\kappa_1 = 0.5290529699...$ is a constant related to Gregory's coefficients G_n , see [2, Appendix], as well as the sequence A270859 from the OEIS for more digits of κ_1 . We do not know if such a "simple" result for these integrals could be found for algebraic or just rational values of ω .

Proof. Consider the following line integral taken along a contour *C* consisting of the interval [-R, +R], $R \in \mathbb{N}$, on the real axis, and a semicircle of the radius *R* in the upper half-plane, denoted *C*_{*R*},

$$\oint_{C} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} \, dz = \int_{-R}^{+R} \frac{\zeta(1/2 - ix)}{(1/2 - ix + \omega) \operatorname{ch} \pi x} \, dx + \int_{C_{R}} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} \, dz \,, \qquad (4)$$

with $\omega \in \mathbb{C}$, Re $\omega > -\frac{1}{2}$. On the contour C_R the last integral may be bounded as follows:

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$$\left| \int_{C_R} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} dz \right| = R \left| \int_{0}^{\pi} \frac{\zeta(1/2 - iRe^{i\varphi}) e^{i\varphi}}{(1/2 - iRe^{i\varphi} + \omega) \operatorname{ch} (\pi Re^{i\varphi})} d\varphi \right|$$
$$\leq R \max_{\varphi \in [0,\pi]} \left| \frac{\zeta(1/2 - iRe^{i\varphi})}{1/2 - iRe^{i\varphi} + \omega} \right| \cdot I_R \leq \max_{\varphi \in [0,\pi]} \left| \zeta(1/2 - iRe^{i\varphi}) \right| \cdot I_R$$
(5)

where we denoted

$$I_R \equiv \int_0^{\pi} \frac{d\varphi}{|\operatorname{ch}(\pi R e^{i\varphi})|}, \qquad R > 0,$$

for brevity. Now, in the half–plane $\sigma > 1$, the absolute value of $\zeta(\sigma + it)$ may be always bounded by a constant $C = \zeta(\sigma)$, which decreases and tends to 1 as $\sigma \to \infty$. In contrast, in the strip $0 \le \sigma \le 1$ the function $|\zeta(\sigma + it)|$ is unbounded; presently, the rate of grow is still not known, but it follows from the general theory of Dirichlet series that it cannot be faster than $O(|t|^{1-\sigma})$, $|t| \ge \frac{1}{2}$, in the strip $\frac{1}{2} \le \sigma < 1$,

see [3, Theorem 35, pp. 99–102].² Hence, since $\sin \varphi \ge 0$ and if *R* is large enough, this rough estimate gives us

$$\left|\zeta\left(1/2 - iRe^{i\varphi}\right)\right| = \left|\zeta\left(1/2 + R\sin\varphi - iR\cos\varphi\right)\right| = O\left(\sqrt{R}\right)$$

in the interval $\varphi \in [0, \pi]$. On the other hand, as *R* tends to infinity and remains integer the integral *I*_{*R*} tends to zero as O(1/R). To show this, we first note that

$$\frac{1}{|\operatorname{ch}(\pi R e^{i\varphi})|} = \frac{\sqrt{2}}{\sqrt{\operatorname{ch}(2\pi R \cos\varphi) + \cos(2\pi R \sin\varphi)}} = O(e^{-\pi R |\cos\varphi|}), \qquad R \to \infty,$$

because $0 \leq \varphi \leq \pi$ and $R \in \mathbb{N}$. Since $|\operatorname{ch}(\pi Re^{i\varphi})|^{-1}$ is symmetric about $\varphi = \frac{1}{2}\pi$, we may write

$$I_{R} = \int_{0}^{\frac{\pi}{2}} \frac{2\sqrt{2}}{\sqrt{\operatorname{ch}(2\pi R \cos \varphi) + \cos(2\pi R \sin \varphi)}} d\varphi$$
$$= O\left(\int_{0}^{\frac{\pi}{2}} e^{-\pi R \cos \varphi} d\varphi\right) = O\left(\int_{0}^{\frac{\pi}{2}} e^{-\pi R \sin \vartheta} d\vartheta\right), \qquad R \to \infty.$$
(6)

Now, from the well-known inequality

$$\frac{2\vartheta}{\pi} \leqslant \sin \vartheta \leqslant \vartheta, \qquad \vartheta \in \left[0, \frac{1}{2}\pi\right]$$

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we deduce that

$$\frac{1 - e^{-\frac{1}{2}\pi^2 R}}{\pi R} \leqslant \int_{0}^{\frac{1}{2}} e^{-\pi R \sin \vartheta} d\vartheta \leqslant \frac{1 - e^{-\pi R}}{2R},$$
(7)

whence $I_R = O(1/R)$ at $R \to \infty$. Inserting both latter results into (5), we obtain

$$\left| \int_{C_R} \frac{\zeta(1/2 - iz)}{(1/2 - iz + k) \operatorname{ch} \pi z} \, dz \right| = O\left(R^{-1/2}\right) \to 0 \quad \text{as} \quad R \to \infty, \, R \in \mathbb{N}$$

Hence, making $R \rightarrow \infty$, equality (4) becomes

$$\int_{-\infty}^{+\infty} \frac{\zeta(1/2 - ix)}{(1/2 - ix + \omega) \operatorname{ch} \pi x} dx = \oint_{C} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} dz, \qquad \operatorname{Re} \omega > -\frac{1}{2}, \tag{8}$$

where the latter integral is taken around an infinitely large semicircle in the upper half-plane, denoted *C*. The integrand is not a holomorphic function: in *C* it has the simple poles at $z = z_n \equiv i \left(n - \frac{1}{2}\right)$, $n = 2, 3, 4, \ldots$, due to the hyperbolic secant, and a double pole at $z = \frac{i}{2}$, due to both the hyperbolic

²There exist, of course, more sharp estimations, such as Huxley's estimations or *Lindelöf hypothesis*, but we do not need them for our proof (see, for more details, e.g. [10, Chapt. XIII], [5], [8], [7], [6], [9]).

secant and the ζ -function.³ Therefore, by the Cauchy residue theorem

$$\oint_{C} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} \, dz = 2\pi i \left\{ \operatorname{res}_{z = \frac{i}{2}} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} + \sum_{n=2}^{\infty} \operatorname{res}_{z = z_n} \frac{\zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} \right\}$$
$$= 2\pi i \left\{ \lim_{z \to \frac{i}{2}} \frac{\partial}{\partial z} \frac{(z - \frac{1}{2}i)^2 \zeta(1/2 - iz)}{(1/2 - iz + \omega) \operatorname{ch} \pi z} + \sum_{n=2}^{\infty} \frac{\zeta(1/2 - iz)}{\pi(1/2 - iz + \omega) \operatorname{sh} \pi z} \right|_{z = i(n - \frac{1}{2})} \right\}$$
$$= 2\pi i \left\{ \frac{1}{\pi i} \cdot \frac{\gamma(1 + \omega) - 1}{(1 + \omega)^2} + \frac{1}{\pi i} \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\zeta(n)}{n + \omega} \right\}.$$

Equating the left part of (8) with the last result yields (2).

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³Note that the pole $z = -i(\frac{1}{2} + \omega)$, due to the denominator $(1/2 - iz + \omega)$, is located in the lower half-plane.