# LUCAS PSEUDOPRIMES AND THE PELL CONIC 

ANTONIO J. DI SCALA*, NADIR MURRU, AND CARLO SANNA ${ }^{\dagger}$


#### Abstract

We show a connection between the Lucas pseudoprimes and the Pell conic equipped with the Brahmagupta product introducing the Pell pseudoprimes.


## 1. Introduction

Primality testing is a very important topic, especially in cryptography (see, e.g., [22] for an overview). The most classical primality tests are: Fermat and Euler's test [17], Lucas test [3], Solovay-Strassen primality test [19], Rabin-Miller primality test [13, 15], Baillie-PSW primality test [3], and AKS primality test [2].

The Lucas test is based on some properties of Lucas sequences. Given two integers $P>0$ and $Q$ such that $D:=P^{2}-4 Q \neq 0$, the Lucas sequences $\left(U_{k}\right)_{k \geq 0}$ and $\left(V_{k}\right)_{k \geq 0}$ associated to $(P, Q)$ are defined by

$$
\begin{array}{ll}
U_{0}=0, & V_{0}=2, \\
U_{1}=1, & V_{1}=P, \\
U_{k}=P U_{k-1}-Q U_{k-2}, & V_{k}=P V_{k-1}-Q V_{k-2},
\end{array}
$$

for every integer $k \geq 2$. We will write $U_{k}(P, Q)$ and $V_{k}(P, Q)$ when it is necessary to show the dependency on $P$ and $Q$. The Lucas test is based on the fact that when $n$ is an odd prime with $\operatorname{gcd}(n, Q)=1$, we have

$$
\begin{equation*}
U_{n-\left(\frac{D}{n}\right)} \equiv 0 \quad(\bmod n), \tag{1}
\end{equation*}
$$

where $\left(\frac{D}{n}\right)$ denotes the Jacobi symbol. If $n$ is composite but (1) still holds, then $n$ is called a Lucas pseudoprime with parameters $P$ and $Q$ [3]. Lucas pseudoprimes have been widely studied [6, 7, 20, 21]. Some authors also studied primality tests using more general linear recurrence sequences $[1,9,10]$.

In this paper, we highlight how the Lucas test can be introduced in an equivalent way by means of the Brahmagupta product and the Pell's equation.

The Pell's equation is the Diophantine equation

$$
x^{2}-D y^{2}=1,
$$

where $D$ is a fixed squarefree integer. It is well known that given two solutions $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ the Brahmagupta product

$$
\left(x_{1}, y_{1}\right) \otimes_{D}\left(x_{2}, y_{2}\right):=\left(x_{1} x_{2}+D y_{1} y_{2}, x_{1} y_{2}+x_{2} y_{1}\right)
$$

yields another solution of the Pell's equation (see, e.g., [4]). Given a ring $\mathcal{R}$, we can consider the Pell conic

$$
\mathcal{C}=\mathcal{C}_{D}(\mathcal{R}):=\left\{(x, y) \in \mathcal{R} \times \mathcal{R}: x^{2}-D y^{2}=1\right\} .
$$

[^0]In particular, if $\mathcal{R}$ is a field then $\left(\mathcal{C}, \otimes_{D}\right)$ is a group with identity ( 1,0 ). Moreover, when $\mathcal{R}=\mathbb{Z}_{p}$ (the field of residue classes modulo a prime number $p$ ), we have $|\mathcal{C}|=p-\left(\frac{D}{p}\right)$ (see, e.g., [14]). Consequently, when $n$ is an odd prime, we have

$$
\begin{equation*}
(\widetilde{x}, \widetilde{y})^{\otimes n-\left(\frac{D}{n}\right)} \equiv(1,0) \quad(\bmod n) \tag{2}
\end{equation*}
$$

for all $(\widetilde{x}, \widetilde{y}) \in \mathcal{C}_{D}\left(\mathbb{Z}_{p}\right)$, where the power is evaluated with respect to $\otimes:=\otimes_{D}$. We say that an odd composite integer $n$ such that $\operatorname{gcd}(n, \widetilde{y})=1$ and

$$
y_{n} \equiv 0 \quad(\bmod n)
$$

where $\left(x_{n}, y_{n}\right)=(\tilde{x}, \tilde{y})^{\otimes n-(D / n)}$, is a Pell pseudoprime with parameters $D$ and $(\widetilde{x}, \widetilde{y}) \in \mathcal{C}_{D}\left(\mathbb{Z}_{n}\right)$. The possibility of using the properties of the Pell conic for constructing a primality test has been highlighted in [12], but the author does not provide an extensive study about it and only focuses on a conic of the kind $x^{2}-D y^{2}=4$. Moreover, in [11] the author used the conic $x^{2}+3 y^{2}=4$ for testing numbers of the form $3^{n} h \pm 1$.

Remark 1.1. The term Pell pseudoprime is already used for the Lucas pseudoprimes with parameters $P=2$ and $Q=-1$. Indeed, in this case, the sequence $U_{n}$ is known as the Pell sequence (A000129 in OEIS [18]). Furthermore, the term Pell pseudoprime is also used to indicate a composite integer $n$ such that

$$
U_{n} \equiv\left(\frac{2}{n}\right) \quad(\bmod n)
$$

for $P=2$ and $Q=-1$ (A099011 in OEIS).
The relation between Lucas pseudoprimes and Pell pseudoprimes is given by the following result.

Theorem 1.2. On the one hand, if $n$ is a Lucas pseudoprime with parameters $P>0$ and $Q=1$, then $n$ is a Pell pseudoprime with parameters $D=P^{2}-4$ and $(\widetilde{x}, \widetilde{y}) \equiv\left(2^{-1} P, 2^{-1}\right)$ $(\bmod n)$. On the other hand, if $n$ is a Pell pseudoprime with parameter $D$ and $(\widetilde{x}, \widetilde{y})$, then $n$ is a Lucas pseudoprime with parameters $P=2 \widetilde{x}$, and $Q=1$.

## 2. Proof of Theorem 1.2

Lemma 2.1. Let $\widetilde{x}, \widetilde{y} \in \mathbb{Z}$ and let $D$ be a nonzero integer. We have

$$
(\widetilde{x}, \widetilde{y})^{\otimes k}=\left(\frac{1}{2} V_{k}(P, Q), \widetilde{y} U_{k}(P, Q)\right),
$$

for every integer $k \geq 0$, where $P:=2 \widetilde{x}, Q:=\widetilde{x}^{2}-D \widetilde{y}^{2}$, and the Brahmagupta product is computed respect to $D$.

Proof. It is clear from the definition of Brahmagupta product that $(\widetilde{x}, \widetilde{y})^{\otimes k}=\left(x_{k}, y_{k}\right)$, where $x_{k}, y_{k} \in \mathbb{Z}$ are defined by $(\widetilde{x}+\sqrt{D} \widetilde{y})^{k}=x_{k}+\sqrt{D} y_{k}$. Conjugating this last equality we get $(\widetilde{x}-\sqrt{D} \widetilde{y})^{k}=x_{k}-\sqrt{D} y_{k}$, from which in turn we obtain

$$
x_{k}=\frac{(\widetilde{x}+\sqrt{D} \widetilde{y})^{k}+(\widetilde{x}-\sqrt{D} \widetilde{y})^{k}}{2} \quad \text { and } \quad y_{k}=\frac{(\widetilde{x}+\sqrt{D} \widetilde{y})^{k}-(\widetilde{x}-\sqrt{D} \widetilde{y})^{k}}{2 \sqrt{D}} .
$$

It is well known [16, Ch. 1] that

$$
V_{k}(P, Q)=\alpha^{k}+\beta^{k} \quad \text { and } \quad U_{k}=\frac{\alpha^{k}-\beta^{k}}{\alpha-\beta}
$$

where $\alpha, \beta$ are the roots of $X^{2}-P X+Q$. Since $P:=2 \widetilde{x}$ and $Q:=\widetilde{x}^{2}-D \widetilde{y}^{2}$, we have $\alpha=\widetilde{x}+\sqrt{D} \widetilde{y}$ and $\beta=\widetilde{x}-\sqrt{D} \widetilde{y}$, so that $x_{k}=\frac{1}{2} V_{k}(P, Q)$ and $y_{k}=\widetilde{y} U_{k}(P, Q)$, as desired.

Suppose that $n$ is a Lucas pseudoprime with parameters $P>0$ and $Q=1$. Let $\widetilde{x}, \widetilde{y} \in \mathbb{Z}$ be such that $(\widetilde{x}, \widetilde{y}) \equiv\left(2^{-1} P, 2^{-1}\right)(\bmod n)$ and put $D:=P^{2}-4$. We have

$$
\widetilde{x}^{2}-D \widetilde{y}^{2} \equiv\left(2^{-1} P\right)^{2}-\left(P^{2}-4\right) 2^{-2} \equiv 1 \quad(\bmod n),
$$

so that $(\widetilde{x}, \widetilde{y}) \in \mathcal{C}_{D}\left(\mathbb{Z}_{n}\right)$. Moreover, by Lemma 2.1 with $k=n-(D \mid n)$ and since $n$ is a Lucas pseudoprime, we have

$$
(\widetilde{x}, \widetilde{y})^{\otimes k}=\left(\frac{1}{2} V_{k}(P, Q), \widetilde{y} U_{k}(P, Q)\right) \equiv\left(\frac{1}{2} V_{k}(P, Q), 0\right) \quad(\bmod n) .
$$

Hence, $n$ is a Pell pseudoprime with parameters $D=P^{2}-4$ and $(\widetilde{x}, \widetilde{y}) \equiv\left(2^{-1} P, 2^{-1}\right)(\bmod n)$.
Now suppose that $n$ is Pell pseudoprime with parameters $D$ and $(\widetilde{x}, \widetilde{y}) \in \mathcal{C}$. Let $P=2 \widetilde{x}$ and $Q=1$. Note that since $n$ is Pell pseudoprime, by definition we have $\operatorname{gcd}(\widetilde{y}, n)=1$.

By Lemma 2.1 with $k=n-(D \mid n)$ and since $n$ is a Pell pseudoprime, we have

$$
\left(\frac{1}{2} V_{k}(P, Q), \widetilde{y} U_{k}(P, Q)\right) \equiv(\widetilde{x}, \widetilde{y})^{\otimes k},
$$

so that $U_{k}(P, Q) \equiv 0(\bmod n)$. Hence, $n$ is a Lucas pseudoprime with paraments $P=2 \widetilde{x}$ and $Q=1$.

## 3. Further remarks

Let us note that, fixed the parameters $P$ and $Q=1$ for the Lucas test (for checking, e.g., the primality of all the integers in a certain range), there is not a corresponding Pell test with fixed parameters $D, \widetilde{x}$ and $\widetilde{y}$ as integer numbers. Indeed, given any $P$ and $Q=1$, we have seen that $D=P^{2}-4$ and $(\widetilde{x}, \widetilde{y}) \equiv\left(2^{-1} P, 2^{-1}\right)(\bmod n)$ are the corresponding parameters of the Pell test, but these values depend on the integer $n$ we are testing.

Moreover, in general, we are not able to fix the integer parameters $D, \widetilde{x}, \widetilde{y}$ in the Pell test for checking the primality of all the integers in a given range, because it is necessary that $\widetilde{x}^{2}-D \widetilde{y}^{2} \equiv 1(\bmod n)$ and this can not be true for any integer $n$. For overcoming these issues, the use of a parametrization of the conic $\mathcal{C}$ can be helpful. In [5], the authors provided the following map

$$
\Phi:\left\{\begin{array}{l}
\mathcal{R} \cup\{\alpha\} \rightarrow \mathcal{C} \\
a \mapsto\left(\frac{a^{2}+D}{a^{2}-D}, \frac{2 a}{a^{2}-D}\right)
\end{array}\right.
$$

where $\alpha \notin \mathcal{R}$ is the point at the infinity of such a parametrization of $\mathcal{C}$. When $\mathcal{R}$ is a field and $t^{2}-D$ is irreducible in $\mathcal{R}$, the map is always defined, otherwise there are values of $a$ such that $\Phi(a)$ can not be evaluated. In this way, we can consider the Pell test with fixed parameters $D$ and $a$, in the sense that $\widetilde{x}=\left(a^{2}+D\right) /\left(a^{2}-D\right)$ and $\widetilde{y}=2 a /\left(a^{2}-D\right)$. However, given a Pell test with parameters $D$ and $a$, there is not always a corresponding Lucas test with fixed parameters $P$ and $Q=1$ as integer numbers. Indeed, the correspondence is given by considering $P=2 \widetilde{x}$. We see some examples for clarifying these considerations.

Example 3.1. Fixed $P=3$ and $Q=1$, the first Lucas pseudoprime is 21, indeed we have $\left(\frac{5}{21}\right)=1$ and

$$
U_{20}=102334155 \equiv 0 \quad(\bmod 21) .
$$

It is also a Pell pseudoprime for $D=P^{2}-4=5, \widetilde{x}=P / 2(\bmod 21)=12, \widetilde{y}=1 / 2(\bmod 21)=$ 11 , indeed

$$
(12,11)^{\otimes 20} \equiv(13,0) \quad(\bmod 21) .
$$

The second Lucas pseudoprime, in this case, is 323 and it is a Pell pseudoprime for $D=$ $P^{2}-4=5, \widetilde{x}=P / 2(\bmod 3) 23=163, \widetilde{y}=1 / 2(\bmod 3) 23=162$, which are different from the previous parameters (the point $(163,162)$ does not belong to $\mathcal{C}$ for $D=5$ and $R=\mathbb{Z}_{21}$ ).

Example 3.2. If we consider the parameters $D=3, \widetilde{x}=8, \widetilde{y}=66$, in the interval $[1,100]$, we can only test the integers $3,5,9,15,17,45,51,85$, since for the other integers $m \in[1,100]$ the oint $(8,65)$ does not belong to $\mathcal{C}$ for $D=5$ and $R=\mathbb{Z}_{m}$. For instance, we can test the integer $n=85$ and observing that it is a Pell pseudoprime in this case, consequently it
is a Lucas pseudoprime for $P=16$ and $Q=1$. Let us note that $\left(\frac{D}{n}\right)=\left(\frac{3}{85}\right)=1$ and $\left(\frac{P^{2}-4 Q}{n}\right)=\left(\frac{252}{85}\right)=1$.
On the other hand, we can test the integer 85 with the Pell using different parameters, e.g., $D=3, x_{1}=7, y_{1}=4(\operatorname{being}(7,4) \in \mathcal{C}$ in this case) and we have

$$
(7,4)^{\otimes 84} \equiv(76,15) \quad(\bmod 85),
$$

i.e., $n$ is not a Pell pseudoprime. The corresponding Lucas test is given by $P=14$ and $Q=1$ and we get that 85 in not a Lucas pseudoprime, since

$$
U_{84} \equiv 25 \quad(\bmod 8) 5
$$

Example 3.3. Given $P=4$ and $Q=1$, the Lucas pseudoprimes up to 5000 are

$$
65,209,629,679,901,989,1241,1769,1961,1991,2509,2701,2911,3007,3439,3869 .
$$

When $P$ is even, we are always able to find an equivalent Pell test, providing all the same pseudoprimes of the Lucas test. Indeed, it is sufficient to choice $D$ and $a$ such that ( $a^{2}+$ $D) /\left(a^{2}-D\right)$ is the integer number $P / 2$. For instance in this case, taking $D=3$ and $a=3$, we have $\widetilde{x}=2$ and $\widetilde{y}=1$.

Example 3.4. Given $P=3$ and $Q=1$ the Lucas pseudoprimes up to 5000 are

$$
21,323,329,377,451,861,1081,1819,2033,2211,3653,3827,4089,4181 .
$$

Also for $P$ odd, we are always able to find an equivalent Pell test exploiting the above parametrization. In this case, we search for $a$ and $D$ integers such that $\left(a^{2}+D\right) /\left(a^{2}-D\right)$ is equal to the fraction $P / 2$. For instance, considering $D=5$ and $a=5$, we have $\widetilde{x}=3 / 2$ and $\widetilde{y}=1 / 2$. Let us note that in this case the values of $\widetilde{x}$ and $\widetilde{y}$ will be different as integer numbers, depending on the integer we are testing.

Example 3.5. Let us see some Pell tests that do not have an equivalent Lucas test with fixed integer parameters. Given $D=6$ and $a=4$, the Pell pseudoprimes up to 3000 are

$$
77,187,217,323,341,377,1763,2387,
$$

for this test we have $\widetilde{x}=11 / 5$ and $\widetilde{y}=4 / 5$.
Given $D=23$ and $a=32$, the Pell pseudoprimes up to 3000 are

$$
323,1047,1247,1745,2813,
$$

for this test we have $\widetilde{x}=1047 / 1001$ and $\widetilde{y}=64 / 1001$.
Given $D=21$ and $a=49$, the Pell pseudoprimes up to 3000 are

$$
253,473,779,2627,2641
$$

for this test we have $\widetilde{x}=173 / 170$ and $\widetilde{y}=7 / 170$.
Given $D=29$ and $a=48$, the Pell pseudoprimes up to 3000 are

$$
989,1101,1457,1991,2449,2679
$$

for this test we have $\widetilde{x}=2333 / 2275$ and $\widetilde{y}=96 / 2275$.
Remark 3.6. The Lucas test with parameters $P$ and $Q=1$ is equivalent to the Pell test with parameters $D=P^{2}-4$ and $a=P+2$. Indeed, in this case, exploiting the parametrization $\Phi$, we get $x_{1}=P / 2$ and $y_{1}=1 / 2$. Note that using this method, the Pell test equivalent to the Lucas test considered in Example 3.3 has parameters $D=12$ and $a=6$. This means that there are Pell tests with different parameters which are equivalent to each others.

Remark 3.7. Considering the identity (2), it is possible to define a stronger test. Indeed some Pell pseudoprimes may not satisfy (2) as in example (3.1) for the Pell pseduoprime 21. The test determined by (2) has an equivalent formulation by means of the Lucas sequences. In this case, we can define a pseudoprime as an odd integer $n$ such that

$$
U_{n-\left(\frac{D}{n}\right)} \equiv 0 \quad(\bmod n) \quad \text { and } \quad U_{n-\left(\frac{D}{n}\right)+1} \equiv 1 \quad(\bmod n)
$$

where as usual $D=P^{2}-4 Q$ and $P, Q$ parameters that define the Lucas sequence. This test does not appear in literature with a specific name, but when $D$ is chosen with the Selfridge method [3], the sequence of pseudoprimes coincides with the Frobenius pseudoprimes with respect to the Fibonacci polynomial [8].

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Politecnico di Torino, Dipartimento di Scienze Matematiche, Torino, Italy
E-mail address: antonio.discala@polito.it
Università degli Studi di Torino, Department of Mathematics, Torino, Italy
E-mail address: nadir.murru@unito.it
Università degli Studi di Genova, Department of Mathematics, Genova, Italy
E-mail address: carlo.sanna.dev@gmail.com


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