THE SCHUR FUNCTOR OF UNIVERSAL ENVELOPING PRE-LIE ALGEBRA

VLADIMIR DOTSENKO AND OISÍN FLYNN–CONNOLLY

ABSTRACT. Motivated by the classification problem of left invariant locally flat affine structures on Lie groups, Segal proved in 1994 a version of the Poincaré–Birkhoff–Witt for universal enveloping pre-Lie algebras of Lie algebras; these algebras were studied in more detail by Bolgar in 1996. Recent work of the first author and Tamaroff implies that a stronger version of such theorem holds, meaning that the PBW isomorphisms have a strong functoriality property. By contrast with the classical PBW theorem, neither of the abovementioned results leads to a description of the Schur functor one has to use to compute the underlying space of the universal enveloping algebra. In this paper, we compute the corresponding Schur functor in terms of combinatorics of rooted trees.

1. INTRODUCTION

The algebraic structure of a pre-Lie algebra, or a right-symmetric algebra, is formally defined as a vector space V equipped with a binary operation $\lhd$ satisfying the identity

$$(a_1 \lhd a_2) \lhd a_3 - a_1 \lhd (a_2 \lhd a_3) = (a_1 \lhd a_3) \lhd a_2 - a_1 \lhd (a_3 \lhd a_2).$$

Any pre-Lie algebra is a Lie algebra with respect to the commutator $[a_1, a_2] := a_1 \lhd a_2 - a_2 \lhd a_1$, so one may consider universal enveloping pre-Lie algebras of Lie algebras. They have been first studied by Segal in [Seg94] who was motivated by geometric questions such as the classification problem of left invariant locally flat affine structures on Lie groups; the relevance of pre-Lie algebras for such purposes was discovered by Vinberg [Vin63]. Segal found a certain basis of nonassociative words for the universal enveloping algebra; his answer does not depend on the Lie algebra structure on $\mathfrak{g}$, and therefore one can say that an analogue of the classical Poincaré–Birkhoff–Witt (PBW) theorem holds for universal enveloping pre-Lie algebras of Lie algebras, along the lines of the general approach to PBW theorems proposed by Mikhalev and Shestakov in [MS14]. Moreover, Bolgar proved in [Bol96] that for the natural filtration on $\mathsf{U}_{PreLie}(\mathfrak{g})$, the associated graded pre-Lie algebra is isomorphic to the universal enveloping pre-Lie algebra of the abelian Lie algebra with the same underlying vector space as $\mathfrak{g}$, which brings universal enveloping pre-Lie algebras closer to the context of the category theoretical approach to PBW theorems proposed by the first author and Tamaroff in [DT18] who proved that a functorial PBW theorem holds in this case using a previous result [Dot19] of the first author. Neither of the abovementioned results, however, leads to an explicit description of the Schur functor of $\mathsf{U}_{PreLie}(\mathfrak{g})$. Our goal is to furnish such a description.

The classical Poincaré–Birkhoff–Witt theorem in the associative case is centered around the space of symmetric tensors $S(\mathfrak{g})$. It is well known that monomials in pre-Lie are rooted trees whose vertices are labelled by generators: implicit in the works of Cayley [Cay57], it has been proved rigorously in [CL01]. The main result of this paper gives an explicit description of the Schur functor of the universal enveloping pre-Lie algebra of a Lie algebra in a way that is reminiscent of the classical Poincaré–Birkhoff–Witt theorem and at the same time highlights the appealing combinatorics of rooted trees.

Theorem (Th. 1)**.** *Let $\mathfrak{g}$ be a Lie algebra, and $\mathsf{U}_{PreLie}(\mathfrak{g})$ its universal enveloping pre-Lie algebra. There is a vector space isomorphism*

$$\mathsf{U}_{PreLie}(\mathfrak{g}) \cong RT_{\neq 1}(S(\mathfrak{g})),$$

where $RT_{\neq 1}$ is the species of rooted trees for which no vertex has exactly one child; moreover, these isomorphisms can be chosen in a way that is natural with respect to Lie algebra morphisms.

Our proof uses a modification of the homological criterion of freeness [DT18, Prop. 4.1] that allows us to utilise the underlying rooted tree structure, bringing in standard techniques for working with graph complexes [Wil15]. The species we found has been previously studied by graph theorists who established that it can also be viewed as the species of “labelled connected P_4-free chordal graphs”, see [CW03] and the entry A058863 in [Slo] for details.

Our result has some immediate applications, of which we give three: a proof of a similar result for the operad of F-manifold algebras [Dot19], a formula for the permutative bar homology of an associative commutative algebra, and a hint that can hopefully be used to construct a conjectural good triple of operads [Lod08] $(\mathcal{X}^c, \mathit{PreLie}, \mathit{Lie})$ that would allow one to prove a Milnor–Moore theorem for universal enveloping pre-Lie algebras.

We conclude the paper by proving, using a similar method, a simpler result concerning associative universal enveloping algebras of pre-Lie algebras. For an operad $\mathcal{P}$ and a $\mathcal{P}$-algebra V, the associative universal enveloping algebra, also known as universal multiplicative enveloping algebra, is the associative algebra $\mathsf{UA}(V)$ whose category of left modules is equivalent to the category of operadic V-modules. For a pre-Lie algebra V, this object was studied in [KU04, Th. 1] by means of noncommutative Gröbner bases, and its monomial basis was constructed. In this paper, we use the approach to associative universal enveloping algebras of [Kho18] to prove the following functorial version of the description of associative universal enveloping algebras of pre-Lie algebras:

Theorem (Th. 2)**.** *Let V be a pre-Lie algebra, and $\mathsf{UA}(V)$ its associative universal enveloping algebra. There is a vector space isomorphism*

$$\mathsf{UA}(V) \cong T(V) \otimes S(V).$$

Moreover, these isomorphisms can be chosen in a way that is natural with respect to pre-Lie algebra morphisms.

This is a short note, and we do not intend to overload it with excessive recollections. We refer the reader to [LV12] for relevant information on symmetric operads, Koszul duality, and operadic twisting cochains, and to [BLL98] for information on combinatorics of species. All operads in this paper are defined over a field $\Bbbk$ of characteristic zero, and are assumed weight graded and connected. All chain complexes are homological (with the differential of degree -1). When writing down elements of operads, we use Latin letters as placeholders; when working with algebras over operads that carry nontrivial homological degrees, there are extra signs which arise from applying operations to arguments via the usual Koszul sign rule.

2. Functorial PBW theorems

We begin with a brief recollection of the recent results on functorial Poincaré–Birkhoff–Witt theorems. In fact, there exist two kinds of universal enveloping algebras, and two kinds of functorial Poincaré–Birkhoff–Witt theorems.

2.1. **Universal enveloping $\mathcal{Q}$-algebras of $\mathcal{P}$-algebras.** The universal enveloping algebras of the first kind are defined whenever one is given a morphism of operads

$$\phi\colon \mathcal{P}\to\mathcal{Q}.$$

Such a morphism leads to a natural functor ϕ^* from the category of $\mathcal{Q}$-algebras to the category of $\mathcal{P}$-algebras (pullback of the structure). This functor admits a left adjoint $\phi_!$ computed via the relative composite product formula [Rez96]

$$\phi_!(V)=\mathcal{Q}\circ_{\mathcal{P}} V,$$

where V in the latter formula is regarded as a "constant analytic endofunctor" (a symmetric sequence supported at arity zero); this left adjoint is called the universal enveloping $\mathcal{Q}$-algebra of the $\mathcal{P}$-algebra V.

In joint work with Tamaroff [DT18], the first author gave a categorical definition of what it means for the datum $(\mathcal{P},\mathcal{Q},\phi)$ to have the PBW property: by definition, one requires that there exists an endofunctor $\mathcal{X}$ such that the underlying object of the universal enveloping $\mathcal{Q}$-algebra of any $\mathcal{P}$-algebra V is isomorphic to $\mathcal{X}(V)$ naturally with respect to $\mathcal{P}$-algebra morphisms. According to [DT18, Th. 3.1], the datum $(\mathcal{P},\mathcal{Q},\phi)$ has the PBW property if and only if the right $\mathcal{P}$-module action on $\mathcal{Q}$ via ϕ is free; in this case, the endofunctor $\mathcal{X}$ that generates that right module satisfies the above condition:

$$\phi_!(V)\cong\mathcal{X}(V)$$

naturally with respect to $\mathcal{P}$-algebra morphisms.

2.2. **Associative universal enveloping algebras of $\mathcal{P}$-algebras.** The universal enveloping algebra of the second kind is defined for any algebra V over an operad $\mathcal{P}$: it is an associative algebra $\mathsf{UA}(V)$ whose category of left modules is equivalent to the category of V-modules defined by means of operad theory; in the "pre-operad" literature, this object is often referred to as the universal multiplicative enveloping algebra. Let us briefly summarise the relevant background information here, following [Fre09, Kho18]. First, one considers a particular type of $\{1,2\}$-coloured operads, namely those whose structure operations can either have all inputs and the output of colour 1 or all inputs but one of colour 1 and the remaining input as well as the output of colour 2. Such an operad is a pair $(\mathcal{Q},\mathcal{R})$, where $\mathcal{Q}$ is a usual operad, and $\mathcal{R}$ is a right $\mathcal{Q}$-module in the category of twisted associative algebras, or in other words, a *Ass*-$\mathcal{Q}$-bimodule.

For a usual operad $\mathcal{P}$, one can consider the derivative $\partial(\mathcal{P})$ defined by

$$\partial(\mathcal{P})(I) := \mathcal{P}(I \sqcup \{\star\}),$$

and define a $\{1,2\}$-coloured analytic endofunctor $(\mathcal{P}, \partial(\mathcal{P}))$, where by definition the input $\star$ and the output of $\partial(\mathcal{P})$ are of colour 2. This endofunctor has a $\{1,2\}$-coloured operad structure arising from the operad structure on $\mathcal{P}$, and algebras (V, M) over this coloured operad are precisely a $\mathcal{P}$-algebra V and a V-module M. As we assume all operads connected, the augmentation $\mathcal{P} \to \mathbb{1}$ of the operad $\mathcal{P}$ may be used to make the pair $(\mathcal{P}, \partial(\mathbb{1}))$ a $\{1,2\}$-coloured operad. The unit $\eta\colon \mathbb{1} \to \mathcal{P}$ of the operad $\mathcal{P}$ gives rise to a morphism of two-coloured operads

$$\psi\colon (\mathcal{P}, \partial(\mathbb{1})) \to (\mathcal{P}, \partial(\mathcal{P})),$$

and if one denotes by $\Bbbk$ the trivial V-module (that is, the module on which all the operations of the augmentation ideal of $\mathcal{P}$ vanish), we have

$$(V, \mathsf{UA}(V)) \cong (\mathcal{P}, \partial(\mathcal{P}))_{(\mathcal{P}, \partial(\mathbb{1}))}(V, \Bbbk).$$

From this observation and [DT18, Th. 3.1], it immediately follows that a functorial PBW type theorem for associative universal enveloping algebras holds if and only if the endofunctor $\partial(\mathcal{P})$ is a free right $\mathcal{P}$-module; in this case, the endofunctor $\mathcal{Y}$ that generates that right module satisfies $\mathcal{Y}(A) \cong \mathsf{UA}(A)$ naturally with respect to $\mathcal{P}$-algebra morphisms.

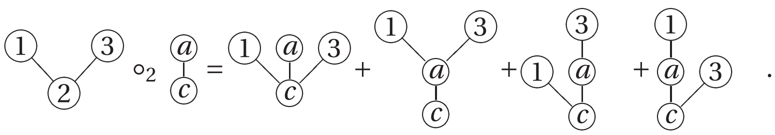
3. The universal enveloping pre-Lie algebra of a Lie algebra

In this section, we prove the main result of this paper, the functorial version of the Poincaré–Birkhoff–Witt theorem for pre-Lie algebras [Seg94, Th. 2] which gives a precise description of the underlying vector space of the universal enveloping algebra via a combinatorially defined analytic endofunctor.

To achieve that goal, we recall that according to [CL01], the underlying endofunctor of the operad *PreLie* is the linearisation of the species of rooted trees RT, and the operad structure is defined in a simple combinatorial way: if $T_1 \in RT(I)$, and $T_2 \in RT(J)$, then for $i \in I$, the element $T_1 \circ_i T_2$ is given by

$$T_1 \circ_i T_2 = \sum_{f\colon \mathrm{in}(T_1,i) \to \mathrm{vert}(T_2)} T_1 \circ_i^f T_2.$$

Here $\mathrm{in}(T_1, i)$ is the set of incoming edges of the vertex i in T_1 and $\mathrm{vert}(T_2)$ is the set of all vertices of T_2; the tree $T_1 \circ_i^f T_2$ is obtained by replacing the vertex i of the tree T_1 by the tree T_2, and grafting the subtree corresponding to the input v of i at the vertex $f(v)$ of T_2. For example, we have



Theorem 1. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathsf{U}_{PreLie}(\mathfrak{g})$ its universal enveloping pre-Lie algebra. There is a vector space isomorphism*

$$\mathsf{U}_{PreLie}(\mathfrak{g}) \cong RT_{\neq 1}(S(\mathfrak{g})),$$

where $RT_{\neq 1}$ is the species of rooted trees for which no vertex has exactly one child; moreover, these isomorphisms can be chosen in a way that is natural with respect to Lie algebra morphisms.

Proof. One of the tools for proving freeness of modules over weight graded operads is the homological criterion of freeness [DT18, Prop. 4.1] that states that a weight graded right $\mathcal{P}$-module $\mathcal{M}$ over a connected weight graded operad $\mathcal{P}$ is free if and only if the homology of the complex

$$(\mathcal{M}\circ_\kappa \mathsf{B}_\bullet(\overline{\mathcal{P}}), d)$$

is concentrated in degree zero. Here the differential of the complex combines the differential of the bar complex $\mathsf{B}_\bullet(\overline{\mathcal{P}})$ of the augmentation ideal of $\mathcal{P}$ with the twisting cochain $\kappa\colon \mathsf{B}_\bullet(\overline{\mathcal{P}})\to\mathcal{P}$ arising from the identity map on $\mathcal{P}$. If the operad $\mathcal{P}$ is Koszul, one may replace the bar complex by its homology given by the Koszul dual cooperad; this results in a much smaller complex

$$\mathsf{K}_\bullet(\mathcal{M},\mathcal{P}) := (\mathcal{M}\circ_\tau \mathcal{P}^{¡}, d),$$

where the whole differential comes from the twisting cochain $\tau\colon \mathcal{P}^{¡}\to\mathcal{P}$ arising from the identity map on the generators. As in [DT18, Prop. 4.1], if the homology of this complex is concentrated in degree zero, it represents the Schur functor that freely generates $\mathcal{M}$ as a right $\mathcal{P}$-module.

Since the operad *Lie* is known to be Koszul, as we just explained, to establish that this operad is free as a right *Lie*-module, one has to consider the chain complex

$$\mathsf{K}_\bullet(\mathit{PreLie},\mathit{Lie}) := (\mathit{PreLie}\circ_\tau \mathit{Lie}^{¡}, d).$$

Since the Koszul dual cooperad $\mathit{Lie}^{¡}$ is one-dimensional in each arity, the n-th component $\mathsf{K}_\bullet(\mathit{PreLie},\mathit{Lie})(n)$ of this chain complex has a basis of rooted trees whose vertices are decorated by disjoint subsets of $\{1,\dots,n\}$. The combinatorial description of composition in the operad *PreLie* leads to a description of the differential which is reminiscent of the usual graph complex differential [Wil15]: it is equal to the sum of all possible ways to split a vertex of a tree into two vertices connected with an edge and to distribute the subset labelling that vertex between the two new vertices; such an element appears with the sign arising from the decomposition maps in the cooperad $\mathit{Lie}^{¡}$.

To compute the homology of this complex, we shall utilise a fairly standard filtration argument. Let us define the *frame* of a rooted tree as the longest path starting from the root and consisting of vertices that have exactly one child (the last point of the frame is the first vertex with at least two children or a leaf). Clearly, the differential either increases the cardinality of the frame by one, or leaves it unchanged. Let us define for each d the space $F^d\mathsf{K}_\bullet(\mathit{PreLie},\mathit{Lie})(n)$ as the span of all decorated trees for which the sum of the size of the frame and the homological degree does not exceed d. These spaces form an increasing filtration of the complex $\mathsf{K}_\bullet(\mathit{PreLie},\mathit{Lie})(n)$. This complex is finite-dimensional, therefore we may use the spectral sequence associated to the filtration. In the associated graded complex, the differential must increase the size of the frame at each step. The frame of a rooted tree can be identified with an element of $\mathit{Ass}(I)$, where I is the set of vertices of the frame, therefore, the frame of a decorated rooted tree from $\mathsf{K}_\bullet(\mathit{PreLie},\mathit{Lie})(n)$ can be identified with an element of $\mathit{Ass}\circ\mathit{Lie}^{¡}(J)$, where J is the union of all subsets used to decorate the frame. Since the associated graded differential only modifies the frame, the associated graded chain complex is isomorphic to the direct sum of complexes with the fixed set J decorating the frame. For each such complex, the homology of the $\mathit{Ass}\circ\mathit{Lie}^{¡}(J)$

is simply $\mathit{Com}(J)$ concentrated in degree zero, since the operad Ass is free as a right Lie-module, and the generators may be identified with the underlying endofunctor of Com. The remaining part of the differential deals separately with the trees grafted at the last vertex of the frame, and an inductive argument applies, showing that the homology is concentrated in degree zero and is isomorphic to the endofunctor $RT_{\neq 1}\circ \mathit{Com}$. As a consequence, the underlying object of $\mathsf{U}_{\mathit{PreLie}}(\mathfrak{g})$ is naturally isomorphic to

$$\mathit{PreLie}\circ_{\mathit{Lie}}(\mathfrak{g})\cong (RT_{\neq 1}\circ \mathit{Com}\circ \mathit{Lie})\circ_{\mathit{Lie}}(\mathfrak{g})\cong RT_{\neq 1}\circ \mathit{Com}(\mathfrak{g})\cong RT_{\neq 1}(S(\mathfrak{g})),$$

as required. □

4. Applications

In this section, we record several consequences of our main result, and new directions prompted by it.

4.1. ***F*-manifold algebras.** Recall [Dot19] that the operad FMan of F-manifold algebras is generated by a symmetric binary operation $-\circ-$ and a skew-symmetric binary operation $[-,-]$ satisfying the associativity relation and the Jacobi identity

$$(a_1\circ a_2)\circ a_3=a_1\circ(a_2\circ a_3),$$
$$[[a_1,a_2],a_3]+[[a_2,a_3],a_1]+[[a_3,a_1],a_2]=0,$$

and related to each other by the Hertling–Manin relation [HM99]

$$[a_1\circ a_2,a_3\circ a_4]=[a_1\circ a_2,a_3]\circ a_4+[a_1\circ a_2,a_4]\circ a_3+a_1\circ[a_2,a_3\circ a_4]+a_2\circ[a_1,a_3\circ a_4]-\\(a_1\circ a_3)\circ[a_2,a_4]-(a_2\circ a_3)\circ[a_1,a_4]-(a_2\circ a_4)\circ[a_1,a_3]-(a_1\circ a_4)\circ[a_2,a_3].$$

Corollary 1. *The operad FMan is a free right Lie-module; the Schur functor of generators is isomorphic to $RT_{\neq 1}\circ \mathit{Com}$.*

Proof. As explained in [DT18], the strategy of the proof of the main theorem of [Dot19] can be used to show that the operad FMan is free as a right Lie-module (essentially, that proof exhibits an explicit basis of tree monomials in the shuffle operad associated to FMan, and the shape of those monomials allows to apply an argument identical to that of [Dot13, Th. 4(2)]. Moreover, that same theorem establishes that the operad FMan is an associated graded of the operad PreLie for a certain filtration, so the underlying Schur functors of FMan and PreLie are the same, and the result follows. □

We note that our calculation implies that the underlying Schur functor of the operad FMan is $RT_{\neq 1}\circ \mathit{Poisson}$. Given that the Hertling-Manin relation may be viewed [HM99] as a weakened version of the derivation rule in a Poisson algebra, it would be interesting to give a more direct proof of this result.

4.2. **Permutative homology of commutative algebras.** Suppose that $f\colon \mathcal{P}\to\mathcal{Q}$ is a map of Koszul operads. Using the Koszul dual map $f^{\text{¡}}\colon \mathcal{P}^{\text{¡}}\to\mathcal{Q}^{\text{¡}}$, one makes $\mathcal{P}^{\text{¡}}$ a right $\mathcal{Q}^{\text{¡}}$-comodule. In [Gri14, Th. 3.7(II)], Griffin establishes that if that right comodule is cofree with $\mathcal{X}$ as the Schur functor of cogenerators, then for each $\mathcal{Q}$-algebra A, one has an isomorphism of complexes

$$\mathsf{B}_{\mathcal{P}}(f^*A)\cong\mathcal{X}\circ\mathsf{B}_{\mathcal{Q}}(A).$$

Here $f^* A$ is the $\mathcal{P}$-algebra obtained from the algebra A by the pullback of structure via f. We may apply this result in the case where $\mathcal{P} = \mathit{Perm}$ is the operad of permutative algebras [Cha01] (which is the Koszul dual of the operad *PreLie*), $\mathcal{Q} = \mathit{Com}$, and $f\colon \mathit{Perm} \to \mathit{Com}$ is the obvious projection (this map is discussed in [Gri14, Example 3.8], but without the knowledge of the Schur functor of cogenerators it is not possible to write down a specific formula).

Corollary 2. *Let A be a commutative associative algebra, and let $\mathsf{B}_{\mathit{Perm}}(A)$ be bar complex of that algebra considered as a permutative algebra. There is an isomorphism of complexes*

$$\mathsf{B}_{\mathit{Perm}}(A) \cong RT_{\neq 1}(\mathit{Com}(\mathsf{B}_{\mathit{Com}}(A))).$$

The Künneth formula for the composite product of Schur functors implies the same result on the level of homology. This ultimately leads to a question of a definition of a homotopy invariant notion of the universal enveloping pre-Lie algebra (of a Lie algebra up to homotopy). We are informed that a general framework for addressing questions like that is developed in an upcoming paper of Khoroshkin and Tamaroff [KT20]; it would be interesting to apply their methods in this particular case.

One may also look at the permutative (co)homology with coefficients of a commutative associative algebra A; an interesting example is given by the deformation complex of the algebra A in the category of permutative algebras. According to [Gri14, Th. 4.6], for questions like that it is reasonable to use information on the structure of the operad *PreLie* as a *Lie*-bimodule. That seems a much harder question; the corresponding bimodule is certainly not free, and not much is known about it. It would be interesting to construct its minimal resolution by free resolution of *PreLie* by free *Lie*-bimodules. We hope to address this question elsewhere.

4.3. A conjectural good triple $(\mathcal{X}^c, \mathit{PreLie}, \mathit{Lie})$.

In the case of universal enveloping associative algebras of Lie algebras, the celebrated Milnor–Moore theorem [MM65] states that a connected graded cocommutative bialgebra is the universal enveloping associative algebra of its Lie algebra of primitive elements. Conceptually, that result is a manifestation of the general theory of good triples of operads [Lod08]: in modern terms, it follows from the fact that there exists a good triple

$$(\mathit{Com}^c, \mathit{Ass}, \mathit{Lie}).$$

Thus, a direct generalisation of that criterion to the case of universal enveloping pre-Lie algebras would require to find a good triples of operads

$$(\mathcal{X}^c, \mathit{PreLie}, \mathit{Lie}),$$

so that each universal enveloping algebra $\mathsf{U}_{\mathit{PreLie}}(\mathfrak{g})$ has a coalgebra structure over the cooperad $\mathcal{X}^c$ for which $\mathfrak{g}$ is a space of primitive elements. To the best of our knowledge, the question of existence of such good triple remains open; it was asked by Loday about fifteen years ago and is recorded as [Mar07, Problem 7.3]. Theorem 1 offers a useful hint for this question, implying that that the underlying species of the cooperad $\mathcal{X}$ must be $RT_{\neq 1} \circ \mathit{Com}$.

5. The associative universal enveloping algebra of a pre-Lie algebra

We conclude our paper with another, slightly simpler result on universal enveloping algebras, a functorial version of the result of [KU04, Th. 1]. A possibility of such a result is indicated in [Kho18, Th. 5.4]; however, the proof of that paper utilises the shuffle operad criterion of freeness [Dot13, Th. 4], and therefore no filtrations of algebras can be obtained from filtrations of operads, and no direct conclusion about Schur functors can be made. The following argument rectifies this problem, using combinatorics related to that of the vertebrate species of Joyal [Joy81] (albeit a bit different).

Theorem 2. *Let V be a pre-Lie algebra, and $\mathsf{UA}(V)$ its associative universal enveloping algebra. There is a vector space isomorphism*

$$\mathsf{UA}(V) \cong T(V) \otimes S(V).$$

Moreover, these isomorphisms can be chosen in a way that is natural with respect to pre-Lie algebra morphisms.

Proof. To analyse the associative universal enveloping algebra $\mathsf{UA}(V)$, one has to consider the right *PreLie*-module $\partial(\mathit{PreLie})$. We note that the corresponding Schur functor is the linearisation of the species $\partial(RT)$ of rooted trees where all of the vertices but one are labelled.

Each element of $\partial(RT)$ has a *barebone* which consists of all vertices on the path from the root to the unlabelled vertex and all the children of the unlabelled vertex. We consider the filtration $F^\bullet\partial(\mathit{PreLie})$ by the length of the path from the root to the unlabelled vertex, so that $F^k\partial(\mathit{PreLie})$ is the span of all trees for which that path has length at least k. From the combinatorial definition of the operad insertion for the operad *PreLie*, it follows that this filtration is compatible with the right pre-Lie module structure: insertion can either keep the length of the path the same or increase it. The associated graded module $\mathrm{gr}_F\,\partial(\mathit{PreLie})$ has a simpler structure, for which insertion at each vertex i along the path to the unlabelled vertex is a simple combinatorial insertion (utilised in the combinatorial definition of the nonassociative permutative operad [Liv06]). This immediately implies that as a right *PreLie*-module, $\mathrm{gr}_F\,\partial(\mathit{PreLie})$ is a free module generated by the species of barebones. By a standard spectral sequence argument, the right *PreLie*-module $\partial(\mathit{PreLie})$ is also free with the same species of generators.

Let us look at the species of barebones closer. Since the vertices on the path from the root to the unlabelled vertex are ordered, and the children of the unlabelled vertex are unordered, the species of possible barebones are described by the Cauchy product $\mathit{Ord}\times\mathit{Set}$, where *Ord* is the species of linear orders (whose linearisation is the operad *uAss* of unital associative algebras), and *Set* is the species of sets (whose linearisation is the operad *uCom* of unital commutative algebras). Thus, our argument above implies that we have the right module isomorphism

$$\partial(\mathit{PreLie}) \cong (\mathit{uAss}\otimes\mathit{uCom})\circ\mathit{PreLie}.$$

From that, one deduces that

$$\begin{aligned}\mathsf{UA}(V) &\cong \partial(\mathit{PreLie})\circ_{\mathit{PreLie}}(V)\cong\\ &((\mathit{uAss}\otimes\mathit{uCom})\circ\mathit{PreLie})\circ_{\mathit{PreLie}}(V)\cong(\mathit{uAss}\otimes\mathit{uCom})(V)\cong T(V)\otimes S(V),\end{aligned}$$

the isomorphism being natural in V, as required. □

Let us connect this result with the existing work defining cohomology of pre-Lie algebras, the concrete definition of Dzhumadildaev [Dzh99] and the general approach of Balavoine [Bal98]. According to [Kho18, Cor. 4.4], the Poincaré–Birkhoff–Witt property for the algebra $\mathsf{UA}(V)$ implies that this algebra is a nonhomogeneous (quadratic-linear) Koszul algebra [Pos93]. Since the associated graded homogeneous quadratic algebra is $T(V)\otimes S(V)$, the Koszul dual of the algebra $\mathsf{UA}(V)$ is the coalgebra

$$(\Bbbk\oplus V)\otimes\Lambda(V)\cong\Lambda(V)\oplus(V\otimes\Lambda(V))$$

equipped with a differential arising from the linear terms in the nonhomogeneous relations of $\mathsf{UA}(V)$. Thus, one may construct the Koszul resolution of the trivial left $\mathsf{UA}(V)$-module $\Bbbk$, and use it to define cohomology of V with coefficients in a left module M (equivalently, a left $\mathsf{UA}(V)$-module M). The shape of the complex computing that cohomology is consistent with the general approach of Balavoine [Bal98] whose results imply that the operadic cohomology of a pre-Lie algebra V with coefficients in a left module M (equivalently, a left $\mathsf{UA}(V)$-module M) is computed by a complex

$$C^\bullet(V,M):=(\mathrm{Hom}_\Bbbk(V\otimes\Lambda(V),M),d),$$

where the differential d is defined by including that complex in a bigger complex, the deformation complex of the square-free extension algebra $V\oplus M$ (this also mostly agrees with the formulas of [Dzh99], up to mismatch in low degrees where the latter paper forces some changes by hand). The latter complex includes also the terms $\mathrm{Hom}_\Bbbk(\Lambda(V),M)$ arising from the above formula.

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UMR 7501, UNIVERSITÉ DE STRASBOURG ET CNRS, 7 RUE RENÉ-DESCARTES, 67000 STRASBOURG CEDEX, FRANCE

E-mail address: `vdotsenko@unistra.fr`

LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UNIVERSITÉ PARIS-SUD, 91405 ORSAY CEDEX, FRANCE

E-mail address: `flynncoo@tcd.ie`