UNITARILY INVARIANT VALUATIONS AND TUTTE'S SEQUENCE

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Abstract. We prove Fu's power series conjecture which relates the algebra of isometry invariant valuations on complex space forms to a formal power series from combinatorics which was introduced by Tutte. The n-th coefficient of this series is the number of triangulations of a triangle with $3n$ internal edges; or the number of intervals in Tamari's lattice Y_n.

1. Statement of the result and background

Thanks to the groundbreaking work of Alesker [2, 4, 5], the space of valuations on manifolds (i.e. finitely additive functionals on some class of sufficiently regular sets) was endowed with a product structure which satisfies a version of Poincaré duality.

The product of invariant valuations on isotropic manifolds (i.e. Riemannian manifolds such that the isometry group acts transitively on the sphere bundle) encodes the kinematic formulas on such spaces [8, 10]. Using this powerful new branch of integral geometry, which is called *algebraic integral geometry*, it was possible to write down in explicit form the kinematic formulas on all complex space forms [9, 10]. We refer to [7, 15] for a survey on these developments.

The starting point is a theorem by J. Fu which describes the algebra of invariant valuations on hermitian space $(\mathbb{C}^n, \mathbb{U}(n))$.

Theorem 1.1. *The algebra* $\mathrm{Val}^{\mathrm{U}(n)}$ *of translation-invariant, continuous and* $\mathrm{U}(n)$*-invariant valuations on* $\mathbb{C}^n$ *is isomorphic to the polynomial algebra*

$$\mathbb{R}[t,s]/(f_{n+1}, f_{n+2}),$$

where $\deg t = 1, \deg s = 2$*, and*

$$\log(1+t+s) = \sum_{i \geq 1} f_i(t,s)$$

is the decomposition into homogeneous components.

The valuation $t \in \mathrm{Val}^{\mathrm{U}(n)}$ equals, up to normalization, the first intrinsic volume. The valuation s equals on a compact convex subset $K \subset \mathbb{C}^n$ the

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measure of complex hyperplanes intersecting $K \subset \mathbb{C}^n$ (again up to normalization).

This theorem was the main entry into a deep study of the integral geometry of $(\mathbb{C}^n, \mathrm{U}(n))$. In [9], several geometrically interesting bases of $\mathrm{Val}^{\mathrm{U}(n)}$ were introduced, their mutual relations were described and the kinematic formulas were explicitly written down.

After these results in the flat case, the next challenge was to find a similar approach in the curved case. Given a real number λ, we denote by $\mathbb{CP}^n_\lambda$ the complex space form of constant holomorphic curvature 4λ, endowed with the group G_λ of holomorphic isometries. If $\lambda > 0$, then $\mathbb{CP}^n_\lambda$ is a rescaling of complex projective space. If $\lambda < 0$, then $\mathbb{CP}^n_\lambda$ is a rescaling of complex hyperbolic space, while $\mathbb{CP}^n_0$ is the flat space $\mathbb{C}^n$.

A natural guess is that the algebra structure of $\mathcal{V}(\mathbb{CP}^n_\lambda)^{G_\lambda}$ can be described in a way analogous to Theorem 1.1, with some modification of the f_i depending on λ. Based on numerical evidence, J. Fu stated in several talks around 2008 and in written form in [15] the following conjecture.

Conjecture 1.2 (Fu's power series conjecture)**.** *Define Tutte's series as*

$$\tau(\lambda) := \sum_{i=1}^\infty \frac{2(4i+1)!}{(i+1)!(3i+2)!}\lambda^i = \lambda + 3\lambda^2 + 13\lambda^3 + 68\lambda^4 + 399\lambda^5 + \dots$$

Then the algebra of invariant valuations on $\mathbb{CP}^n_\lambda$ is isomorphic to

$$\mathbb{R}[[t,s]]/(\bar{f}^\lambda_{n+1}, \bar{f}^\lambda_{n+2}, \bar{f}^\lambda_{n+3}, \dots),$$

where the formal power series $\bar{f}^\lambda_k(t,s)$ is defined as the degree k-part in the expansion of

$$\log\left(1 + t + s + \tau(\lambda)\right).$$

Here t is of degree 1, s is of degree 2, and λ is of degree (-2).

In this conjecture, t denotes a certain multiple of the first intrinsic volume (which may be defined canonically on any Riemannian manifold), and s is the average Euler characteristic of the intersection with a totally geodesic complex hyperplane in $\mathbb{CP}^n_\lambda$.

The first few terms of the conjecture can be confirmed using the template method and some formulas involving binomial coefficients. These formulas can be shown using Zeilberger's algorithm, but get more and more involved for higher orders of λ, compare the discussion in [15].

Interestingly, the sequence of coefficients $1, 3, 13, 68, 399, 2530, \dots$ has various combinatorial interpretations. Tutte [18] has shown that the coefficient of $\lambda^i, i > 0$ in $\tau(\lambda)$ is the number of non-isomorphic planar triangulations of a triangle with $3i$ internal edges. In [13, Section 4], the same coefficient appears as the number of description trees of type $(1,1)$ and size i. In [12] it was shown that this coefficient also equals the number of intervals in Tamari's lattice Y_i. In [6] some explicit bijections explainig these numerical coincidences are constructed. See [17] for many other appearances of this sequence.

Some years after the statement of the conjecture, and without using it, the integral geometry of complex space forms was worked out in [10]. A surprising result, which is based on some computations and not yet fully understood, is the following.

Theorem 1.3. *The map given by* $t \mapsto t, s \mapsto \frac{s}{1-\lambda s}$ *induces an algebra isomorphism between* $\mathrm{Val}^{\mathrm{U}(n)}$ *and* $\mathcal{V}(\mathbb{CP}^n_\lambda)^{G_\lambda}$.

Despite the progress in integral geometry of complex space forms (see also [1, 3, 11, 19, 20]), the original conjecture remained previously open. In this paper we are going to prove it.

Theorem 1. *Fu's power series conjecture is true.*

By Theorem 1.3, the map $t \mapsto t, s \mapsto \frac{s}{1+\lambda s}$ is an isomorphism from $\mathcal{V}(\mathbb{CP}^n_\lambda)^{G_\lambda}$ to $\mathrm{Val}^{\mathrm{U}(n)}$. Therefore, the theorem is equivalent to saying that $\mathrm{Val}^{\mathrm{U}(n)}$ is isomorphic to $\mathbb{R}[[t,s]]/(f^\lambda_{n+1}, f^\lambda_{n+2}, \ldots)$, where $f^\lambda_k(t,s)$ is the degree k-component in

$$\log\left(1+t+\frac{s}{1+\lambda s}+\tau(\lambda)\right).$$

We will prove it in this form.

Our proof is a mixture of the template method and some algebraic manipulations of generating functions. One of the main ingredients is the fact that $\tau(\lambda)$ satisfies an algebraic equation over $\mathbb{R}(\lambda)$, which implies that some auxiliary power series in s and λ, which comes out of the template method, is of non-positive degree (where $\deg s = 2, \deg \lambda = -2$). This part of the proof uses the theory of holonomic functions. We refer to the very recent lecture notes [16] for more information on this subject.

We do not know if there is a more direct link between integral geometry and the combinatorics of triangulations, Tamari's lattice or description trees. It is a priori not even clear that the coefficients in the power series in Fu's conjecture have to be integers and not just any real numbers. Also the way they appear in the conjecture (via the relations satisfied in some algebra) does not seem to be related to any counting of objects.

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2. The template method

We first need some preparations. All computations will be done with formal power series, so no convergence is required.

Lemma 2.1. *For all $m \geq 0$ we have*

$$\sum_{\substack{k \equiv m \mod 2 \\ k>0}} \binom{k+m}{\frac{k+m}{2}} \frac{x^k}{k} = Q_1^m\left(\frac{1}{x}\right) + Q_2^m\left(\frac{1}{x}\right)\sqrt{1-4x^2}$$
$$\underbrace{-\binom{m}{\frac{m}{2}}\log(1+\sqrt{1-4x^2})}_{\text{if } m \text{ is even}}, \tag{1}$$

where Q_1^m, Q_2^m are polynomials of degree m. If m is even, then Q_1^m, Q_2^m are even polynomials, and Q_2^m does not contain an absolute term. If m is odd, then Q_1^m, Q_2^m are odd polynomials, and the logarithmic term does not appear.

Proof. Define

$$Q_1^m(y) := \sum_{\substack{1 \leq i \leq m \\ i \equiv m \mod 2}} \binom{m-i}{\frac{m-i}{2}} \frac{y^i}{i},$$
$$Q_2^m(y) := -\sum_{\substack{1 \leq i \leq m \\ i \equiv m \mod 2}} \frac{2^{m-i} i!!(m-1)!!}{i(i-1)!!m!!} y^i,$$

where we use the standard notation $i!! = i \cdot (i-2) \cdot (i-4) \cdot \ldots$. We take the derivative of (1) and use the binomial series

$$\sum_{l=0}^\infty \binom{2l}{l} x^l = \frac{1}{\sqrt{1-4x^2}}$$

to find that, with this choice of Q_1^m, Q_2^m, the formula is correct up to a constant. If m is odd, then both sides of the equation are odd, hence the constant must be 0. If m is even, we may add the constant to Q_1^m. □

Definition 2.2. *A formal power series $b = b(s, \lambda)$ is called* of degree $\leq 2m$ *if each monomial $s^i \lambda^j$ appearing in b has $2i - 2j \leq 2m$. Equivalently,*

$$b^{(k)}(0) := \left(\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0}\right)^k b(s, \lambda)$$

is a polynomial in s of degree $\leq 2(m+k)$ for each $k \geq 0$.

It is clear that the sum of two series of degrees $\leq 2m$ is again of degree $\leq 2m$ and that the product of a series of degree $\leq 2m_1$ and a series of degree $\leq 2m_2$ is a series of degree $\leq 2(m_1 + m_2)$.

If $c > 0$ and b has no constant term, then

$$\log(c+b) = \log(c) + \log\left(1 + \frac{b}{c}\right) = \log(c) + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \frac{b^n}{c^n}$$

is a well-defined formal power series. If moreover b is of degree ≤ 0, then $\log(c+b)$ is also of degree ≤ 0.

Proposition 2.3. *Let* $r := \frac{s}{1+\lambda s} + \tau(\lambda)$*, where* $\tau(\lambda)$ *is Tutte's sequence. Then the formal power series*

$$b(s,\lambda) := s + (1+\lambda s)\sqrt{(1+r)^2 - 4s}$$

is of degree ≤ 0*.*

We postpone the technical proof to the next section.

Given a formal power series p in t, s, let us write

$$\int_{\mathbb{C}^n} p := p_{2n}(D_n^{\mathbb{C}}),$$

where p_{2n} denotes the $(2n)$-homogeneous component of p and $D_n^{\mathbb{C}} \subset \mathbb{C}^n$ is the unit ball.

Let us write $p \equiv q$ if p, q are formal power series in t, s, λ such that $\int_{\mathbb{C}^n} p = \int_{\mathbb{C}^n} q$ for all $n \geq 0$. If $p \equiv q$, then

$$\int_{\mathbb{C}^n} sp = \int_{\mathbb{C}^{n-1}} p = \int_{\mathbb{C}^{n-1}} q = \int_{\mathbb{C}^n} sq,$$

hence $sp \equiv sq$. By [15, Equation (134)] we have

$$t^k \equiv \begin{cases} \binom{k}{\frac{k}{2}} s^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 2.4. *Let* $r = \frac{s}{1+\lambda s} + \tau(\lambda)$ *as above. There exists a formal power series* $h_m(s,\lambda) \in \mathbb{R}(s,\lambda)$ *of degree* $\leq 2m$ *such that*

$$t^m \log(1+t+r) \equiv h_m(s,\lambda). \tag{2}$$

Proof. We compute

$$\begin{aligned}
t^m \log(1+t+r) &= \sum_{n=1}^{\infty} t^m \frac{(-1)^{n+1}}{n} (t+r)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=0}^{n} \binom{n}{k} t^{k+m} r^{n-k} \\
&\equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k \equiv m \mod 2} \binom{n}{k} \binom{k+m}{\frac{k+m}{2}} s^{\frac{k+m}{2}} r^{n-k} \\
&= \underbrace{\binom{m}{\frac{m}{2}} s^{\frac{m}{2}} \log(1+r)}_{\text{if } m \text{ even}} + \sum_{k \equiv m, k>0} \binom{k+m}{\frac{k+m}{2}} s^{\frac{k+m}{2}} \sum_{n=k}^{\infty} \frac{(-1)^{n+1}}{n} \binom{n}{k} r^{n-k} \\
&= \underbrace{\binom{m}{\frac{m}{2}} s^{\frac{m}{2}} \log(1+r)}_{\text{if } m \text{ even}} + (-1)^{m+1} \sum_{k \equiv m, k>0} \binom{k+m}{\frac{k+m}{2}} \frac{s^{\frac{k+m}{2}}}{k(1+r)^k} \\
&= h_m(s,\lambda),
\end{aligned}$$

where

$$h_m(s,\lambda) := \binom{m}{\frac{m}{2}} s^{\frac{m}{2}} \left(\log(1+r) + \log\left(1+\sqrt{1-\frac{4s}{(1+r)^2}}\right)\right)$$
$$+ (-1)^{m+1} s^{\frac{m}{2}} \left(Q_1^m\left(\frac{1+r}{\sqrt{s}}\right) + Q_2^m\left(\frac{1+r}{\sqrt{s}}\right)\sqrt{1-\frac{4s}{(1+r)^2}}\right),$$

and the logarithmic terms only appear if m is even.

It remains to show that h_m is of degree $\leq 2m$. Consider a monomial in the Q_1-term. Let $0 \leq i \leq m, i \equiv m \mod 2$. Then

$$\begin{aligned} s^{\frac{m}{2}}\left(\frac{1+r}{\sqrt{s}}\right)^i &= s^{\frac{m-i}{2}}\left(1+\frac{s}{1+\lambda s}+\tau(\lambda)\right)^i \\ &= s^{\frac{m-i}{2}}\sum_{a=0}^{i}\binom{i}{a}\left(\frac{s}{1+\lambda s}\right)^a(1+\tau(\lambda))^{i-a} \\ &= \sum_{a=0}^{i}\binom{i}{a}s^{\frac{m-i}{2}+a}\frac{(1+\tau(\lambda))^{i-a}}{(1+\lambda s)^a}. \end{aligned}$$

Since $a \leq i$, each summand is of degree $\leq 2m$.

Next, we consider a monomial in the Q_2-term. Let $1 \leq i \leq m, i \equiv m \mod 2$. Then

$$\begin{aligned} s^{\frac{m}{2}}\left(\frac{1+r}{\sqrt{s}}\right)^i\sqrt{1-\frac{4s}{(1+r)^2}} &= s^{\frac{m-1}{2}}\left(\frac{1+r}{\sqrt{s}}\right)^{i-1}\sqrt{(1+r)^2-4s} \\ &= s^{\frac{m-1}{2}}\left(\frac{1+r}{\sqrt{s}}\right)^{i-1}\frac{1}{1+\lambda s}(b(s,\lambda)-s). \end{aligned}$$

By Proposition 2.3, $(b(s,\lambda)-s)$ is of degree ≤ 2. As we have seen above, $s^{\frac{m-1}{2}}\left(\frac{1+r}{\sqrt{s}}\right)^{i-1}$ is of degree $\leq 2(m-1)$. Hence the whole term is of degree $\leq 2m$.

Let us finally consider the logarithmic term, which only appears if m is even. We have

$$\begin{aligned} \log(1+r)+\log&\left(1+\sqrt{1-\frac{4s}{(1+r)^2}}\right) \\ &= \log\left(1+r+\sqrt{(1+r)^2-4s}\right) \\ &= \log\left(1+\lambda s+(1+\lambda s)\tau(\lambda)+b(s,\lambda)\right)-\log(1+\lambda s). \end{aligned}$$

By Proposition 2.3, this term is of degree ≤ 0. Since we multiply it by $\binom{m}{\frac{m}{2}}s^{\frac{m}{2}}$, we get a term of degree $\leq m \leq 2m$. $\square$

Proof of Theorem 1. Let us write

$$f_n^\lambda = \sum_{i=0}^\infty f_{n,i}(s,t)\lambda^i, \quad \deg f_{n,i}(s,t) = n+2i,$$

and

$$h_m(s,\lambda) = \sum_{i \geq 0} h_{m,i}(s)\lambda^i.$$

Let us show that $\int_{\mathbb{C}^n} t^m s^l f_{n+1}^\lambda = 0$ for all n, m, l or equivalently that $\int_{\mathbb{C}^n} t^m s^l f_{n+1,i} = 0$ for all n, i, m, l. Alesker duality [4, Theorem 0.8] then implies that $f_{n+1,i} = 0$ for all i and hence $f_{n+1}^\lambda = 0$ on $\mathbb{C}^n$.

By degree reasons, we have to show this only if $n = m + 2l + 2i + 1$. Now

$$\begin{aligned} \int_{\mathbb{C}^{m+2l+2i+1}} t^m s^l f_{m+2i+2,i}(t,s) &= \int_{\mathbb{C}^{m+2l+2i+1}} t^m s^l \sum_q f_{q,i}(t,s) \\ &= \int_{\mathbb{C}^{m+2l+2i+1}} s^l h_{m,i}(s), \end{aligned}$$

where the last equation follows from comparing the coefficients of λ^i in (2).

By Lemma 2.4, we have $\deg s^l h_{m,i} \leq 2l + 2(m+i) < 2(m+2l+2i+1)$, hence the integral on the right hand side vanishes. This finishes the proof that $f_{n+1}^\lambda = 0$ on $\mathbb{C}^n$ for all n. If $i > 1$, then $f_{n+i}^\lambda = f_{(n+i-1)+1}^\lambda = 0$ on $\mathbb{C}^{n+i-1}$ by what we have shown and hence $f_{n+i}^\lambda = 0$ on $\mathbb{C}^n$ by restriction from $\mathbb{C}^{n+i-1}$ to $\mathbb{C}^n$. □

3. Proof of Proposition 2.3

The proof in this section is based on some terminology and results from D-modules and holonomic functions. We refer to [14, 16, 21] for more information on this subject. For the reader's convenience, we have spelled out the details so that it should be possible to understand the proof without any prior knowledge of holonomic functions.

Let us first sketch the idea. Tutte has shown that $\tau(\lambda)$ is an algebraic function [18, Equations (4.8), (4.9)] (see also [13, Theorem 4], [12, Equation (10)]). The formal power series $b(s,\lambda)$ is therefore the composition of a holonomic and an algebraic function and is itself holonomic [16, Proposition 2.3]. The holonomic rank turns out to be 4. In particular, if we consider b as a formal power series in λ with coefficients in the space of formal power series in s, it satisfies some differential equation of degree 4 whose coefficients are polynomials in s and λ. The differential equation gives a recursive relation for the higher derivatives $\left.\frac{d^i}{d\lambda^i}\right|_{\lambda=0} b(s,\lambda)$ which will be enough to prove the proposition.

Proof of Proposition 2.3.

Step 1: Define an algebraic power series $\phi(\lambda)$ by the equation $P(\lambda,\phi(\lambda))=0$, where $P(\lambda,\phi):=\phi-\lambda(1+\phi)^4\in\mathbb{R}[\lambda,\phi]$ is irreducible. Then, by [12, Equation (10)], $\tau(\lambda)=\phi(\lambda)(1-\phi(\lambda)-\phi(\lambda)^2)$.

Taking derivatives yields

$$\phi'(\lambda)=-\frac{\frac{\partial P}{\partial\lambda}(\lambda,\phi(\lambda))}{\frac{\partial P}{\partial\phi}(\lambda,\phi(\lambda))}=\frac{(1+\phi(\lambda))^4}{1-4\lambda(1+\phi(\lambda))^3}.$$

Applying the extended Euclidean algorithm to P and $\frac{\partial P}{\partial\phi}$ gives polynomials $U,V\in\mathbb{R}(\lambda)[\phi]$ such that $UP+V\frac{\partial P}{\partial\phi}=1$, and we deduce that

$$\phi'(\lambda)=-\frac{V\frac{\partial P}{\partial\lambda}(\lambda,\phi(\lambda))}{V\frac{\partial P}{\partial\phi}(\lambda,\phi(\lambda))}=-V\frac{\partial P}{\partial\phi}(\lambda,\phi(\lambda)).$$

Let $P_0\in\mathbb{R}(\lambda)[\phi]$ be the remainder of $-V\frac{\partial P}{\partial\phi}$ by division by P. Then $\phi'(\lambda)=P_0(\lambda,\phi(\lambda))$. Doing the computations explicitly, we find that

$$P_0(\lambda,\phi)=\frac{12\lambda\phi^3+52\lambda\phi^2+4\lambda\phi-36\lambda+9\phi}{(256\lambda-27)\lambda}.$$

Step 2: Set

$$F(s,\lambda,\phi):=\sqrt{\left[1+\lambda s+s+(1+\lambda s)\phi(1-\phi-\phi^2)\right]^2-4s(1+\lambda s)^2}.$$

Then $b(s,\lambda)=s+F(s,\lambda,\phi(\lambda))$. Noting that F^2 is a polynomial in s,λ,ϕ, we see that

$$\begin{aligned}
\frac{\partial F}{\partial\lambda}&=\frac{1}{2F^2}\frac{\partial F^2}{\partial\lambda}F=\tilde P_1F,\\
\frac{\partial F}{\partial\phi}&=\frac{1}{2F^2}\frac{\partial F^2}{\partial\phi}F=\tilde P_2F,\\
\frac{\partial F}{\partial s}&=\frac{1}{2F^2}\frac{\partial F^2}{\partial s}F=\tilde P_3F,
\end{aligned}$$

where $\tilde P_1,\tilde P_2,\tilde P_3$ are rational functions of s,λ,ϕ. This means that the formal power series F is holonomic of rank 1. Arguing as in Step 1, we find $P_1,P_2\in\mathbb{R}(s,\lambda)[\phi]$ of degree ≤ 3 such that

$$\begin{aligned}
\frac{\partial F}{\partial\lambda}(s,\lambda,\phi(\lambda))&=P_1(s,\lambda,\phi(\lambda))F(s,\lambda,\phi(\lambda))\\
\frac{\partial F}{\partial\phi}(s,\lambda,\phi(\lambda))&=P_2(s,\lambda,\phi(\lambda))F(s,\lambda,\phi(\lambda)).
\end{aligned}$$

By the chain rule we find that

$$\begin{aligned}
\frac{d}{d\lambda}F(s,\lambda,\phi(\lambda))&=\frac{\partial F}{\partial\lambda}(s,\lambda,\phi(\lambda))+\frac{\partial F}{\partial\phi}(s,\lambda,\phi(\lambda))\phi'(\lambda)\\
&=\left[P_1(s,\lambda,\phi(\lambda))+P_0(\lambda,\phi(\lambda))P_2(s,\lambda,\phi(\lambda))\right]F(s,\lambda,\phi(\lambda)).
\end{aligned}$$

Reducing $P_1 + P_0P_2$ modulo P, we find $Q_1 \in \mathbb{R}(s,\lambda)[\phi]$ of degree ≤ 3 with

$$\frac{d}{d\lambda}F(s,\lambda,\phi(\lambda)) = Q_1(s,\lambda,\phi(\lambda))F(s,\lambda,\phi(\lambda)).$$

Step 3 Since $\phi(\lambda)$ is algebraic of rank 4 and F is holonomic of degree 1, it follows that $F(s,\lambda,\phi(\lambda))$ is holonomic of degree 4. Let us write down a differential equation.

By induction, we construct $Q_i \in \mathbb{R}(s,\lambda)[\phi]$ of degree ≤ 3 in ϕ such that

$$\frac{d^iF}{d\lambda^i}(s,\lambda,\phi(\lambda)) = Q_i(s,\lambda,\phi(\lambda))F(s,\lambda,\phi(\lambda)). \tag{3}$$

With $Q_0 := 1$ and with Q_1 as in the previous step, the cases $i = 0, 1$ are already done.

Once Q_i is defined, we take the derivative to find that

$$\frac{d^{i+1}F}{d\lambda^{i+1}}(s,\lambda,\phi(\lambda)) = \tilde{Q}_{i+1}(s,\lambda,\phi(\lambda))F(s,\lambda,\phi(\lambda)),$$

where

$$\begin{aligned}\tilde{Q}_{i+1}(s,\lambda,\phi) := {} & \frac{\partial Q_i}{\partial\lambda}(s,\lambda,\phi) + \frac{\partial Q_i}{\partial\phi}(s,\lambda,\phi)P_0(\lambda,\phi)\\ & + Q_i(s,\lambda,\phi)Q_1(s,\lambda,\phi) \in \mathbb{R}(s,\lambda)[\phi].\end{aligned}$$

We let $Q_{i+1} \in \mathbb{R}(s,\lambda)[\phi]$ be the remainder of $\tilde{Q}_{i+1}$ by division by P. Then (3) is satisfied with i replaced by $(i+1)$.

Step 4 The space of polynomials in ϕ of degree ≤ 3 with coefficients in $\mathbb{R}(s,\lambda)$ is a vector space of dimension 4 over the field $\mathbb{R}(s,\lambda)$. Hence there exist rational functions $\tilde{R}_i \in \mathbb{R}(s,\lambda)$ which are not all zero with

$$\sum_{i=0}^4 \tilde{R}_i(s,\lambda)\frac{1}{i!}Q_i(s,\lambda,\phi) = 0.$$

Clearing denominators and multiplying by $F(s,\lambda,\phi(\lambda))$ we find polynomials $R_i \in \mathbb{R}[s,\lambda]$ with

$$\sum_{i=0}^4 R_i(s,\lambda)\frac{1}{i!}\frac{d^iF}{d\lambda^i}(s,\lambda,\phi(\lambda)) = 0.$$

Write $R_i = \sum_k^\infty \frac{1}{k!}R_{ik}(s)\lambda^k$ with $R_{ik} \in \mathbb{R}[s]$. The explicit computation of these polynomials is tedious, but a computer algebra software can handle this very quickly. We only need the following properties.

$$R_{ik} = 0 \quad \text{if } k < i-1, \tag{4}$$

$$R_{i,i-1} = c_i(s-1), c_i > 0, i = 1,\ldots,4. \tag{5}$$

Step 5 Write

$$b(s,\lambda) = \sum_{j=0}^\infty \frac{b_j}{j!}\lambda^j, \quad b_j \in \mathbb{R}(s).$$

We have

$$0 = R_0(b(\lambda)-s) + \sum_{i=1}^4 R_i b^{(i)}(\lambda) = R_0(b(\lambda)-s) + \sum_{i=1}^4 \sum_{j,k=0}^\infty \frac{R_{ik}}{k!}\frac{b_{j+i}}{j!}\lambda^{j+k}.$$

Comparing the coefficient of $\lambda^l, l \geq 0$ and using (4) yields

$$\sum_{i=1}^4 \binom{l}{l+1-i} R_{i,i-1} b_{l+1} + \sum_{m=0}^l \sum_{i=0}^4 \binom{l}{m-i} R_{i,l+i-m} b_m = R_{0,l} s.$$

By (5), $\sum_{i=1}^4 \binom{l}{l+1-i} R_{i,i-1}$ is a non-zero scalar multiple of $(s-1)$. Under the assumption that b_m is a polynomial in s for all $m \leq l$, we can deduce that $(s-1)b_{l+1}$ is a polynomial in s.

Step 6 We argue as in Step 4, but this time using that the polynomials $\frac{1}{i!}Q_i(s,\lambda,\phi), i=1,\ldots,5$ are linearly dependent. We then find polynomials $\hat{R}_i \in \mathbb{R}[s,\lambda], i=1,\ldots,5$ such that

$$\sum_{i=1}^5 \hat{R}_i(s,\lambda)\frac{1}{i!}\frac{d^iF}{d\lambda^i}(s,\lambda,\phi(\lambda)) = 0.$$

We decompose

$$\hat{R}_i = \sum_k^\infty \frac{1}{k!}\hat{R}_{ik}(s)\lambda^k.$$

We only need that

$$\deg \hat{R}_i \leq 2(3-i), \quad \hat{R}_{ik} = 0 \text{ if } k < i-2, \tag{6}$$

and

$$\hat{R}_{i,i-2} = \hat{c}_i(3s+1), \hat{c}_i > 0, i = 2,\ldots,5. \tag{7}$$

Comparing coefficients as above gives us for every $l \geq 0$

$$\sum_{i=2}^5 \binom{l}{l+2-i}\hat{R}_{i,i-2} b_{l+2} + \sum_{m=0}^{l+1}\sum_{i=1}^5 \binom{l}{m-i}\hat{R}_{i,l+i-m} b_m = 0. \tag{8}$$

By (7), $\sum_{i=2}^5 \binom{l}{l+2-i}\hat{R}_{i,i-2}$ is a non-zero multiple of $(3s+1)$. Hence $(3s+1)b_{l+2}$ is a polynomial in s provided that all $b_m, m \leq l+1$ are polynomials. In this case, $(s-1)b_{l+2}$ is also a polynomial in s by the previous step. Since $(s-1)$ and $(3s+1)$ are relatively prime, this implies that b_{l+2} is a polynomial in s. Since $b_0 = 1, b_1 = 3s+1$ are polynomials, induction on l thus shows that b_l is a polynomial for all l.

We prove by induction that the polynomial $b_j(s)$ is of degree $\leq 2j$ in s for each j, which is equivalent to the statement of the proposition. Since $b_0 = 1, b_1 = 3s+1$, the cases $j = 0,1$ are our induction start.

The induction hypothesis and (6) imply that for each $1 \leq i \leq 5, 0 \leq m \leq l+1$ we have

$$\deg \hat{R}_{i,l+i-m} b_m = \deg \hat{R}_{i,l+i-m} + \deg b_m \leq 2(l-m+3) + 2m = 2(l+3).$$

From (8) and $\deg(3s+1) = 2$ we conclude that $\deg b_{l+2} \leq 2(l+2)$. $\square$

References

[1] Judit Abardia, Eduardo Gallego, and Gil Solanes. The Gauss-Bonnet theorem and Crofton-type formulas in complex space forms. *Israel J. Math.*, 187:287–315, 2012.

[2] Semyon Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.*, 11(2):244–272, 2001.

[3] Semyon Alesker. Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.*, 63(1):63–95, 2003.

[4] Semyon Alesker. The multiplicative structure on continuous polynomial valuations. *Geom. Funct. Anal.*, 14(1):1–26, 2004.

[5] Semyon Alesker. Theory of valuations on manifolds: a survey. *Geom. Funct. Anal.*, 17(4):1321–1341, 2007.

[6] Olivier Bernardi and Nicolas Bonichon. Intervals in Catalan lattices and realizers of triangulations. *J. Combin. Theory Ser. A*, 116(1):55–75, 2009.

[7] Andreas Bernig. Algebraic integral geometry. In *Global Differential Geometry*, volume 17 of *Springer Proceedings in Mathematics*, pages 107–145. Springer, Berlin Heidelberg, 2012.

[8] Andreas Bernig and Joseph H. G. Fu. Convolution of convex valuations. *Geom. Dedicata*, 123:153–169, 2006.

[9] Andreas Bernig and Joseph H. G. Fu. Hermitian integral geometry. *Ann. of Math.*, 173:907–945, 2011.

[10] Andreas Bernig, Joseph H. G. Fu, and Gil Solanes. Integral geometry of complex space forms. *Geom. Funct. Anal.*, 24(2):403–492, 2014.

[11] Andreas Bernig, Joseph H. G. Fu, and Gil Solanes. Dual curvature measures in Hermitian integral geometry. In *Analytic aspects of convexity*, volume 25 of *Springer INdAM Ser.*, pages 1–17. Springer, Cham, 2018.

[12] F. Chapoton. Sur le nombre d'intervalles dans les treillis de Tamari. *Sém. Lothar. Combin.*, 55:Art. B55f, 18 pp. (electronic), 2005/07.

[13] Robert Cori and Gilles Schaeffer. Description trees and Tutte formulas. *Theoret. Comput. Sci.*, 292(1):165–183, 2003. Selected papers in honor of Jean Berstel.

[14] S. C. Coutinho. *A primer of algebraic D-modules*, volume 33 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[15] Joseph H.G. Fu. Algebraic integral geometry. In Eduardo Gallego and Gil Solanes, editors, *Integral Geometry and Valuations*, Advanced Courses in Mathematics - CRM Barcelona, pages 47–112. Springer Basel, 2014.

[16] Anna-Laura Sattelberger and Bernd Sturmfels. D-Modules and Holonomic Functions. Preprint arXiv:1910.01395.

[17] N.J.A. Sloane. The on-line encyclopedia of integer sequences. https://oeis.org/, 2006.

[18] W. T. Tutte. A census of planar maps. *Canadian J. Math.*, 15:249–271, 1963.

[19] Thomas Wannerer. Integral geometry of unitary area measures. *Adv. Math.*, 263:1–44, 2014.

[20] Thomas Wannerer. The module of unitarily invariant area measures. *J. Differential Geom.*, 96(1):141–182, 2014.

[21] Doron Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32(3):321–368, 1990.

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