

Accepted by Amer. Math. Monthly as a Note

**EACH POSITIVE RATIONAL NUMBER HAS THE FORM**  
 $\varphi(m^2)/\varphi(n^2)$

DMITRY KRACHUN AND ZHI-WEI SUN

ABSTRACT. In this note we show that each positive rational number can be written as  $\varphi(m^2)/\varphi(n^2)$ , where  $\varphi$  is Euler's totient function and  $m$  and  $n$  are positive integers.

Let  $\varphi$  be Euler's totient function. For distinct primes  $p_1, \dots, p_k$  and positive integers  $a_1, \dots, a_k$ , it is well known that

$$\varphi(p_1^{a_1} \cdots p_k^{a_k}) = \prod_{i=1}^k p_i^{a_i-1} (p_i - 1).$$

(See, e.g., [1, p. 20].) Thus, if  $n$  has the prime factorization  $\prod_{i=1}^k p_i^{a_i}$  (where  $p_1, \dots, p_k$  are distinct primes and  $a_1, \dots, a_k$  are positive integers), then

$$\varphi(n^2) = \prod_{i=1}^k p_i^{2a_i-1} (p_i - 1) = n\varphi(n).$$

For positive integers  $m$  and  $n$  with  $\varphi(m^2) = \varphi(n^2)$ , we have  $m = n$  by comparing the prime factorizations of  $\varphi(m^2)$  and  $\varphi(n^2)$ . The sequence  $\varphi(n^2)$  ( $n = 1, 2, 3, \dots$ ) is available from [2].

Section 4 of [3] contains many challenging conjectures on representations of positive rational numbers. For example, Sun [3, Conjecture 4.4] conjectured that any positive rational number can be written as  $m/n$ , where  $m$  and  $n$  are positive integers such that the sum of the  $m$ th prime and the  $n$ th prime is a square. Motivated by this, in this note we establish the following new result.

**Theorem 1.** *Any positive rational number can be written as  $\varphi(m^2)/\varphi(n^2)$ , where  $m$  and  $n$  are positive integers.*

*Proof.* We claim a stronger result: If  $p_1 < p_2 < \cdots < p_k$  are distinct primes and  $a_1, \dots, a_k$  are integers, then there are positive integers  $m$  and  $n$  with  $mn$  not divisible by any prime greater than  $p_k$  such that  $p_1^{a_1} \cdots p_k^{a_k} = \varphi(m^2)/\varphi(n^2)$ .

We prove the claim by induction on  $p_k$ .

---

*Key words and phrases.* Euler's totient function, representation of rational numbers.  
2010 *Mathematics Subject Classification.* Primary 11A25; Secondary 11D85.

The base of the induction is  $p_k = 2$ . For any  $a \in \mathbb{Z}$ , clearly

$$2^{2a} = \frac{2^{2(a+b)-1}}{2^{2b-1}} = \frac{\varphi(2^{2(a+b)})}{\varphi(2^{2b})}$$

for each integer  $b > |a|$ , and

$$2^{2a+1} = \begin{cases} \varphi(2^{2(a+1)})/\varphi(1^2) & \text{if } a \geq 0, \\ \varphi(1^2)/\varphi(2^{-2a}) & \text{if } a < 0. \end{cases}$$

Now let  $q$  be an odd prime and assume that the claim holds whenever  $p_k < q$ . Let  $q_1 < \dots < q_k = q$  be distinct primes and let  $r = \prod_{i=1}^k q_i^{a_i}$  with  $a_1, \dots, a_k \in \mathbb{Z}$ . Set  $r_0 = r/q_k^{a_k}$  if  $2 \mid a_k$ , and  $r_0 = r/((q_k - 1)q_k^{a_k})$  if  $2 \nmid a_k$ . Clearly, all the primes in the factorization of  $r_0$  are smaller than  $q_k = q$ . By the induction hypothesis, there are positive integers  $m_0$  and  $n_0$  with  $m_0 n_0$  not divisible by any prime  $p \geq q_k$  such that

$$\frac{\varphi(m_0^2)}{\varphi(n_0^2)} = r_0.$$

Obviously, we may take  $m_0 = n_0 = 1$  if  $r_0 = 1$ .

*Case 1.*  $2 \mid a_k$ .

In this case, we take positive integers  $b$  and  $c$  with  $b - c = a_k/2$ , and set  $m = m_0 q^b$  and  $n = n_0 q^c$ . Then

$$\frac{\varphi(m^2)}{\varphi(n^2)} = \frac{\varphi(m_0^2)}{\varphi(n_0^2)} \times \frac{q^{2b-1}(q-1)}{q^{2c-1}(q-1)} = r_0 q^{2(b-c)} = \prod_{i=1}^k q_i^{a_i} = r.$$

*Case 2.*  $2 \nmid a_k$ .

When  $a_k > 0$ , for  $m = m_0 q^{(a_k+1)/2}$  and  $n = n_0$ , we have

$$\frac{\varphi(m^2)}{\varphi(n^2)} = \frac{\varphi(m_0^2)}{\varphi(n_0^2)} \times q^{a_k}(q-1) = r_0 q^{a_k}(q-1) = \prod_{i=1}^k q_i^{a_i} = r.$$

If  $a_k < 0$ , then there are positive integers  $m$  and  $n$  with  $mn$  not divisible by any prime greater than  $q_k$  such that  $\prod_{i=1}^k q_i^{-a_i} = \varphi(n^2)/\varphi(m^2)$  and hence  $\prod_{i=1}^k q_i^{a_i} = \varphi(m^2)/\varphi(n^2)$ .

In view of the above, the claim holds and hence so does the theorem.  $\square$

**Examples.** We have

$$\frac{19}{47} = \frac{19 \times 19673280}{47 \times 19673280} = \frac{\varphi(39330^2)}{\varphi(55836^2)}$$

with

$$39330 = 2 \times 3^2 \times 5 \times 19 \times 23 \quad \text{and} \quad 55836 = 2^2 \times 3^3 \times 11 \times 47.$$

Also,

$$\frac{47}{58} = \frac{47 \times 1700160}{58 \times 1700160} = \frac{\varphi(14476^2)}{\varphi(20010^2)}$$

with

$$14476 = 2^2 \times 7 \times 11 \times 47 \quad \text{and} \quad 20010 = 2 \times 3 \times 5 \times 23 \times 29.$$

**Acknowledgment.** The authors would like to thank the two referees for their helpful comments. The work is supported by the NSFC-RFBR Cooperation and Exchange Program (grants NSFC 11811530072 and RFBR 18-51-53020-GFEN-a). The second author is also supported by the National Natural Science Foundation of China (grant no. 11971222).

#### REFERENCES

- [1] Ireland, K., Rosen, M. (1990). *A Classical Introduction to Modern Number Theory*, 2nd ed. Graduate Texts in Mathematics, Vol. 84. New York: Springer.
- [2] OEIS Foundation Inc. (1991). The On-Line Encyclopedia of Integer Sequences. [oeis.org/A002618](http://oeis.org/A002618)
- [3] Sun, Z.-W. (2017). Conjectures on representations involving primes. In: Nathanson, M. ed. *Combinatorial and Additive Number Theory II*. Springer Proceedings in Mathematics & Statistics, Vol. 220. Cham: Springer, pp. 279–310.

(DMITRY KRACHUN) ST. PETERSBURG DEPARTMENT OF STEKLOV MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES, FONTANKA 27, 191023, ST. PETERSBURG, RUSSIA

*E-mail address:* [dmitrykrachun@gmail.com](mailto:dmitrykrachun@gmail.com)

(ZHI-WEI SUN, CORRESPONDING AUTHOR) DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* [zwsun@nju.edu.cn](mailto:zwsun@nju.edu.cn)