Nonstandard Cayley Automatic Representations for Fundamental Groups of Torus Bundles Over the Circle

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Abstract. We construct a new family of Cayley automatic representations of semidirect products $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ for which none of the projections of the normal subgroup $\mathbb{Z}^n$ onto each of its cyclic components is finite automaton recognizable. For $n = 2$ we describe a family of matrices from $\mathrm{GL}(2, \mathbb{Z})$ corresponding to these representations. We are motivated by a problem of characterization of all possible Cayley automatic representations of these groups.

Keywords: FA–presentable structure, Cayley automatic representation, semidirect product, Pell's equation

1 Introduction and Preliminaries

Thurston and Epstein showed that a fundamental group of a closed 3–manifold is automatic if and only if none of its prime factors is a closed manifold modelled on nilgeometry or solvgeometry [10, Chapter 12]. A fundamental group of a closed manifold modelled on nilgeometry or solvgeometry has a finite index subgroup isomorphic to $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where A is unipotent or Anosov, respectively. These groups are not automatic due to [10, Theorems 8.2.8 and 8.1.3]. To include all fundamental groups of closed 3–manifolds, the class of automatic groups had been extended by Bridson and Gilman [6], Baumslag, Shapiro and Short [1]; see also autostackable groups proposed by Brittenham, Hermiller and Holt [8]. In this paper we use the concept of Cayley automatic groups, extending the class of automatic groups, proposed by Kharlampovich, Khoussainov and Miasnikov [13].

All semidirect products of the form $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ are Cayley automatic [13, Proposition 13.5]. These groups are the fundamental groups of torus bundles over the circle and they play important role in group theory. Bridson and Gersten studied the Dehn function for this family groups [7]. In this paper we construct a new family of Cayley automatic representations for semidirect products $\mathbb{Z}^n \rtimes_A \mathbb{Z}$. These representations demonstrate unforeseen behaviour violating a basic property, to be explained below in this section, known for representations described

in [13, Proposition 10.5]. They also reveal an unexpected connection with Pell's equation. The results of this paper are based on the original construction of FA–presentation for $(\mathbb{Z}^2, +)$ found by Nies and Semukhin [18].

In general, we are interested in the following question: Given a Cayley automatic group, is there any way to characterize all of its Cayley automatic representations in terms of some numerical characteristics or by any other means? Despite the generality of the notion of Cayley automatic groups which retains only computational mechanism of automatic groups, it is possible to partly answer this question for some Cayley automatic groups in terms of a certain numerical characteristic which is intimately related to the Dehn function. We discuss it in more details in the end of this section. In the following few paragraphs we briefly recall the notion of Cayley automatic groups and representations, and a standard way to construct such representations for semidirect products $\mathbb{Z}^n \rtimes_A \mathbb{Z}$.

Let Σ be a finite alphabet. We denote by $\Sigma_\diamond$ the alpahbet $\Sigma \cup \{\diamond\}$, where $\diamond \notin \Sigma$ is called a padding symbol. The convolution $w_1 \otimes \cdots \otimes w_m \in \Sigma_\diamond^m$ of strings $w_1, \ldots, w_m \in \Sigma^*$ is the string of length $\max\{|w_1|, \ldots, |w_m|\}$ obtained by placing $w_1, \ldots, w_m$ one under another and adding the padding symbol $\diamond$ at the end of each string to make their lengths equal. More formally, the kth symbol of $w_1 \otimes \cdots \otimes w_m$ is $(\sigma_1, \ldots, \sigma_m)^\top$, where σ_i, $i = 1, \ldots, m$ is the kth symbol of w_i if $k \leqslant |w_i|$ and $\diamond$ otherwise. The convolution $\otimes R$ of a m–ary relation $R \subseteq \Sigma^{*m}$ is defined as $\otimes R = \{w_1 \otimes \cdots \otimes w_m \mid (w_1, \ldots, w_m) \in R\}$. The relation R is called FA–recognizable if $\otimes R$ is recognized by a finite automaton.

Let $\mathcal{A} = (A; R_1^{m_1}, \ldots, R_\ell^{m_\ell}, f_1^{k_1}, \ldots, f_r^{k_r})$ be a structure, where A is the domain, $R_i^{m_i} \subseteq A^{m_i}, i = 1, \ldots, \ell$ is a m_i–ary relation over A and $f_j^{k_j} : A^{k_j} \to A$, $j = 1, \ldots, r$ is a k_j–ary operation on A. Assume that there exist a regular language $L \subseteq \Sigma^*$ and a bijection $\psi : L \to A$ such that all relations $\psi^{-1}(R_i^{m_i}) = \{(w_1, \ldots, w_{m_i}) \in \Sigma^{*m_i} \mid (\psi(w_1), \ldots, \psi(w_{m_i})) \in R_i^{m_i}\}$, $i = 1, \ldots, \ell$ and $\psi^{-1}(\mathrm{Graph}(f_j)) = \{(w_1, \ldots, w_{k_j}, w_{k_j+1}) \in \Sigma^{*(k_j+1)} \mid f_j(\psi(w_1), \ldots, \psi(w_{k_j})) = \psi(w_{k_j+1})\}$, $j = 1, \ldots, r$ are FA–recognizable. In this case the structure $\mathcal{A}$ is called FA–presentable and the bijection $\psi : L \to A$ is called FA–presentation of $\mathcal{A}$ [15,5,16]. For a recent survey of the theory of FA–presentable structures we refer the reader to [21]. A finitely generated group G is called Cayley automatic if the labelled directed Cayley graph $\Gamma(G, S)$ is a FA–presentable structure for some generating set $S \subseteq G$ [13]. Cayley automatic groups form a special class of FA–presentable structures and they naturally generalize automatic groups retaining its basic algorithmic properties. We call a FA–presentation $\psi : L \to G$ of $\Gamma(G, S)$ a Cayley automatic representation of the group G.

We recall that every element of a group $\mathbb{Z}^n \rtimes_A \mathbb{Z}$, where $A \in \mathrm{GL}(n, \mathbb{Z})$, is given as a pair (b, h), where $b \in \mathbb{Z}$ and $h \in \mathbb{Z}^n$. The group multiplication is given by $(b_1, h_1) \cdot (b_2, h_2) = (b_1 + b_2, A^{b_2} h_1 + h_2)$. The maps $b \mapsto (b, \mathbf{0})$ and $h \mapsto (0, h)$ give the natural embeddings of $\mathbb{Z}$ and $\mathbb{Z}^n$ into $\mathbb{Z}^n \rtimes_A \mathbb{Z}$, respectively, where 0 and $\mathbf{0}$ denote the identities of the groups $\mathbb{Z}$ and $\mathbb{Z}^n$, respectively. Let $g_0 = (1, \mathbf{0})$ and $g_i = (0, e_i)$, where $e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)^t \in \mathbb{Z}^n$. The elements $g_0, g_1, \ldots, g_n$ generate the group $\mathbb{Z}^n \rtimes_A \mathbb{Z}$. The right multiplication by $g_i, i = 0, 1, \ldots, n$ is as follows: for a given $g = (b, h) \in \mathbb{Z}^n \rtimes_A \mathbb{Z}$, $gg_0 = (b+1, Ah)$ and $gg_i = (b, h + e_i)$.

Let $\psi_1 : L_1 \to \mathbb{Z}$ be a Cayley automatic representation of $\mathbb{Z}$ and $\psi_2 : L_2 \to \mathbb{Z}^n$ be a Cayley automatic representations of $\mathbb{Z}^n$ such that the automorphism of $\mathbb{Z}^n$ given by the matrix A is FA–recognizable. Then, due to [13, Theorem 10.3], one gets a Cayley automatic representation $\psi : L \to \mathbb{Z}^n \rtimes_A \mathbb{Z}$ as follows: $L = L_1 L_2$ (we may assume that $L_1 \subset \Sigma_1$, $L_2 \subset \Sigma_2$ and $\Sigma_1 \cap \Sigma_2 = \varnothing$) and for given $u \in L_1$ and $v \in L_2$, $\psi(uv) = (\psi_1(u), \psi_2(v))$. A standard way to construct $\psi_2 : L_2 \to \mathbb{Z}^n$ is to take a FA–presentation $\varphi : L_0 \to \mathbb{Z}$ of the structure $(\mathbb{Z}, +)$, for example a binary representation, and define L_2 as $L_2 = \{w_1 \otimes \cdots \otimes w_n \mid w_i \in L_0, i = 1, \ldots, n\}$ and ψ_2 as $\psi_2(w_1 \otimes \cdots \otimes w_n) = (\varphi(w_1), \ldots, \varphi(w_n))$ for every $w_1, \ldots, w_n \in L_0$. Clearly, for such a representation ψ_2 every automorphism of $\mathbb{Z}^n$ is FA–recognizable. Therefore, ψ_1 and φ as above give a Cayley automatic representation of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$. We call such a representation standard. Every standard Cayley automatic representation $\psi : L \to \mathbb{Z}^n \rtimes_A \mathbb{Z}$ satisfies the following basic properties:

a) The language $L_{\mathbb{Z}^n} = \psi^{-1}(\mathbb{Z}^n)$ of the strings representing elements in the subgroup $\mathbb{Z}^n \trianglelefteq \mathbb{Z}^n \rtimes_A \mathbb{Z}$ is regular and the relation $R_A = \{(u, v) \in L_{\mathbb{Z}^n} \times L_{\mathbb{Z}^n} \mid A\psi(u) = \psi(v)\}$ is FA–recognizable.

b) For each projection $p_i : \mathbb{Z}^n \to \mathbb{Z}^n$, $i = 1, \ldots, n$, on the ith component given by $p_i((z_1, \ldots, z_n)) = (0, \ldots, 0, z_i, 0, \ldots, 0)$ the relation $P_i = \{(u, v) \in L_{\mathbb{Z}^n} \times L_{\mathbb{Z}^n} \mid p_i \psi(u) = \psi(v)\}$ is FA–recognizable.

In this paper we construct Cayley automatic representations of groups $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ for which the property a) holds but the property b) does not hold – in other words, these representations are nonstandard. Namely, in Section 2 we construct Cayley automatic representations of $\mathbb{Z}^n$ for which every projection $p_i : \mathbb{Z}^n \to \mathbb{Z}^n$, $i = 1, \ldots, n$ is not FA–recognizable while some nontrivial automorphisms $A \in \mathrm{GL}(n, \mathbb{Z})$ are FA–recognizable. A family of these automorphisms for the case $n = 2$ is described in Section 3. Taking such a representation as ψ_2 and an arbitrary Cayley automatic representation $\psi_1 : L_1 \to \mathbb{Z}$ one obtains a Cayley automatic representation of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ as described above. Clearly, for this representation the property a) holds while the property b) does not hold. In this paper we primarily focus on the case $n = 2$ briefly discussing the case $n > 2$. Section 4 concludes the paper.

Apart from the importance of semidirect products $\mathbb{Z}^n \rtimes_A \mathbb{Z}$, let us explain another reason motivated us to study Cayley automatic representations of this family of groups violating at least one of the properties a) or b). We first briefly recall some notation and results. For a given f.g. group G with some finite set of generators $A \subseteq G$, we denote by A^{-1} the set of inverses of the elements of A in G and by d_A the word metric in G with respect to A. We denote by $\pi : (A \cup A^{-1})^* \to G$ the canonical map sending a word $w \in (A \cup A^{-1})^*$ to the corresponding group element $\pi(w)$. For the rest of the section we assume that $L \subseteq (A \cup A^{-1})^*$ [1]. We denote by $L^{\leqslant n}$ the language $L^{\leqslant n} = \{w \in L \mid |w| \leqslant n\}$.

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[1] We recall that every FA–presentable structure has a FA–presentation over a binary alphabet [5]. The alphabet $A \cup A^{-1}$ always has at least two symbols. The case of FA–presentable structures over a unary alphabet is special, see [5,17,14].

For a Cayley automatic representation $\psi : L \to G$ we denote by h the function: $h(n) = \max\{d_A(\psi(w), \pi(w)) | w \in L^{\leqslant n}\}$. The function h had been introduced in [3] as a measure of deviation of Cayley automatic representation ψ from π, i.e., from being automatic in the classical sense of Thurston. For two nondecreasing functions $h : [Q_1, +\infty) \to \mathbb{R}^+$ and $f : [Q_2, +\infty) \to \mathbb{R}^+$, where $[Q_1, +\infty), [Q_2, +\infty) \subseteq \mathbb{N}$, we say that $h \preceq f$ if there exist positive integer constants K, M and N such that for all $n \geqslant N$: $h(n) \leqslant Kf(Mn)$. A f.g. group is said to be in $\mathcal{B}_f$ if there exists a Cayley automatic representation ψ for which the function $h \preceq f$. It was shown that the identity function $\mathfrak{i}(n) = n$ is the sharp lower bound of the function h (in the sense of $\preceq$) for all Cayley automatic representations of the Baumslag–Solitar groups $BS(p, q), 1 \leqslant p < q$ [3, Theorem 11] and the wreath products $G \wr H$, if H is virtually cyclic and G is in the class $\mathcal{B}_\mathfrak{i}$ [2].

We recall that the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2 \rtimes_T \mathbb{Z}$ for some lower triangular matrix T, see Remark 15. The result of [4, Theorem 5.1] shows that if a Cayley automatic representation of the Heisenberg group $\psi : L \to \mathcal{H}_3(\mathbb{Z})$ satisfies certain conditions, then the function h is bounded from below by the exponential function $\mathfrak{e}(n) = \exp(n)$. In particular, for every Cayley automatic representation $\psi : L \to \mathcal{H}_3(\mathbb{Z})$ satisfying the properties a) and b) the function h has the exponential lower bound: $\mathfrak{e} \preceq h$. The lower bounds for all possible Cayley automatic representations of the Heisenberg group and the groups $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, if $A \in \mathrm{GL}(2, \mathbb{Z})$ is a matrix with two real eigenvalues not equal to ± 1, known to us are given by the functions $\sqrt[3]{n}$ and $\mathfrak{i}$, respectively, see [4, Corollary 2.4]. However, it is not known whether or not these lower bounds are sharp. These observations motivated us to seek nonstandard Cayley automatic representations for a whole family of groups $\mathbb{Z}^n \rtimes_A \mathbb{Z}$, $A \in \mathrm{GL}(n, \mathbb{Z})$. While we construct nonstandard representations for a large family of groups $\mathbb{Z}^n \rtimes_A \mathbb{Z}$, see Theorem 10 for the case $n = 2$, it does not contain nilpotent groups including the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$. This leads us to think that the case of nilpotent groups is special.

2   Nies–Semukhin FA–presentations of $(\mathbb{Z}^n, +)$

Nies and Semukhin constructed a FA–presentation of $(\mathbb{Z}^2, +)$ for which no nontrivial cyclic subgroup is FA–recognizable [18, § 6]. Let us briefly recall their construction. The group $\mathbb{Z}^2$ is identified with the additive group of the quotient ring $\mathbb{Z}[x]/\langle p_3 \rangle$, where $p_3(x) = x^2 + x - 3$ [2]. A polynomial $a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ is called reduced if $|a_i| \leqslant 2$ for all $i = 0, \ldots, n$. For given $f, g \in \mathbb{Z}[x]$, it is said that $f \sim g$ if p_3 divides $f - g$. In [18, Proposition 6.2] it is then shown that every $f(x) \in \mathbb{Z}[x]$ is equivalent to a reduced polynomial $\widetilde{f}(x)$. Let $\Sigma = \{-2, -1, 0, 1, 2\}$. Each reduced polynomial $a_n x^n + \cdots + a_0$ is represented by a string $a_0 \ldots a_n$ over the alphabet Σ. Two strings $u = a_0 \ldots a_n$ and $v = b_0 \ldots b_m$ from Σ^* are said to be equivalent

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[2] In [18, Remark 6.1] it is said that one can use a polynomial $x^2 + x - q$ for a prime $q \geqslant 3$.

$(u \sim v)$ if $a_n x^n + \cdots + a_0 \sim b_m x^m + \cdots + b_0$. It is then shown that this equivalence relation defined on Σ^* is FA–recognizable. Let $llex$ be the length–lexicographical order on Σ^* with respect to the ordering $-2 < -1 < 0 < 1 < 2$. A regular domain for a presentation of $\mathbb{Z}^2$ is defined as $\mathrm{Dom} = \{w \in \Sigma^* : (\forall u <_{llex} w)\, u \not\sim w\}$. Then a FA–recognizable relation $R(x_1, x_2, x_3) \subset \Sigma^{*3}$ is defined such that for every pair $x_1, x_2 \in \Sigma^*$ there exists a unique $x_3 \in \Sigma^*$ for which $(x_1, x_2, x_3) \in R$ and if $(x_1, x_2, x_3) \in R$, then for the corresponding polynomials f_1, f_2 and f_3: $f_1 + f_2 \sim f_3$. It enables to define a FA–recognizable relation $\mathrm{Add}(x, y, z)$ on Dom as follows: $\mathrm{Add} = \{(x, y, z) : x, y, z \in \mathrm{Dom} \wedge \exists w(R(x, y, w) \wedge (w \sim z))\}$. Clearly, the structure $(\mathrm{Dom}, \mathrm{Add})$ is isomorphic to $(\mathbb{Z}^2, +)$.

Now we notice that the Nies–Semukhin construction can be generalized for a given polynomial $t(x) = x^2 + px - q \in \mathbb{Z}[x]$ for which $1 + |p| < |q|$. Again, we identify $\mathbb{Z}^2$ with the additive group of the quotient ring $\mathbb{Z}[x]/\langle t \rangle$. The inequality $1 + |p| < |q|$ implies that $|q| \geqslant 2$. We say that a polynomial $a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ is reduced if $|a_i| < |q|$ for all $i = 0, \ldots, n$ and two polynomials $f, g \in \mathbb{Z}[x]$ are equivalent $f \sim g$ if t divides $f - g$. For a given real r we denote by $[r]$ the integral part of r: $[r] = \max\{m \in \mathbb{Z} \,|\, m \leqslant r\}$ if $r \geqslant 0$ and $[r] = \min\{m \in \mathbb{Z} \,|\, m \geqslant r\}$ if $r < 0$.

Proposition 1. *Every polynomial $f(x) \in \mathbb{Z}[x]$ is equivalent to a reduced polynomial $\widetilde{f}(x)$.*

Proof. Let $f(x) = a_n x^n + \cdots + a_0$ and $k_0 = \left[\frac{a_0}{q}\right]$. Since $x^2 + px \sim q$, $f(x) \sim f_1(x) = b_n x^n + \cdots + b_0$, where $b_0 = a_0 - k_0 q$, $b_1 = a_1 + k_0 p$, $b_2 = a_2 + k_0$ and $b_i = a_i$ for $i > 2$. If $|a_0| < |q|$, then $f_1(x) = f_0(x)$. Otherwise, we get that $\sum_{i=0}^n |a_i| > \sum_{i=0}^n |b_i|$. Let $k_1 = \left[\frac{b_1}{q}\right]$. Since $x^3 + px^2 \sim qx$, $f_1(x) \sim f_2(x) = c_n x^n + \cdots + c_0$, where $c_0 = b_0$, $c_1 = b_1 - k_1 q$, $c_2 = b_2 + k_1 p$, $c_3 = b_3 + k_1$ and $c_i = b_i$ for $i > 3$. If $|b_1| < |q|$, then $f_2(x) = f_1(x)$. Otherwise, we get that $\sum_{i=0}^n |b_i| > \sum_{i=0}^n |c_i|$. We have: $|c_0| = |b_0| < |q|$ and $|c_1| < |q|$. If we continue in this way, the process will terminate after a finite number of iterations producing a reduced polynomial $\widetilde{f}(x)$ at the last iteration. $\square$

Remark 2. It can be seen that if the inequality $1 + |p| < |q|$ is not satisfied, then the procedure described in Proposition 1 fails to produce a reduced polynomial for some input polynomials $f(x)$. For example, let $t(x) = x^2 + 2x - 3$ and $f(x) = 2x + 6$. Applying the procedure from Proposition 1 one gets an infinite sequence of polynomials $f_i(x) = 2x^{i+1} + 6x^i$ which never terminates.

Let $\Sigma_q = \{-(|q| - 1), \ldots, |q| - 1\}$. We represent a reduced polynomial $a_n x^n + \cdots + a_0$ by a string $a_0 \ldots a_n$ over the alphabet Σ_q. Similarly, we say that two strings $a_0 \ldots a_n$ and $b_0 \ldots b_m$ over Σ_q are equivalent if the polynomials $a_n x_n + \cdots + a_0$ and $b_m x^m + \cdots + b_0$ are equivalent. An algorithm checking whether two given reduced polynomials $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ are equivalent is the same, up to minor changes, as it is described by Nies and Semukhin for the case $t(x) = x^2 + x - 3$, see [18, § 6]. We first check if q divides $a_0 - b_0$; if not, $f \not\sim g$. We remember two carries $r_0 = p\frac{a_0 - b_0}{q}$ and $r_1 = \frac{a_0 - b_0}{q}$, and

then verify whether q divides $r_0 + a_1 - b_1$; if not, $f \nsim g$. Otherwise, we update the carries: $r_0 \to r_1 + p\frac{r_0+a_1-b_1}{q}$ and $r_1 \to \frac{r_0+a_1-b_1}{q}$, and then verify whether q divides $r_0 + a_2 - b_2$. Proceeding in this way we check if $f \sim g$ or not. Initially, $|r_1| \leqslant 1 \leqslant |q| - 1$ and $|r_0| \leqslant |p| < (|q|-1)^2$. Since q divides $r_0 + a_i - b_i$ at every step of our process unless $f \nsim g$, we can change the formulas for updating carries as follows: $r_0 \to r_1 + p\left[\frac{r_0+a_i-b_i}{q}\right]$ and $r_1 \to \left[\frac{r_0+a_i-b_i}{q}\right]$. Now, if $|r_1| \leqslant |q| - 1$ and $|r_0| \leqslant (|q|-1)^2$, then $\left|\left[\frac{r_0+a_i-b_i}{q}\right]\right| \leqslant \left[\frac{(|q|-1)^2+2(|q|-1)}{|q|}\right] = |q| - 1$ and $\left|r_1 + p\left[\frac{r_0+a_i-b_i}{q}\right]\right| \leqslant (|q|-1) + |p|\left|\left[\frac{r_0+a_i-b_i}{q}\right]\right| \leqslant (|q|-1) + (|q|-2)(|q|-1) = (|q|-1)^2$. This shows that $|r_1|$ and $|r_0|$ are always bounded by $|q|-1$ and $(|q|-1)^2$. This algorithm requires only a finite amount of memory, so the equivalence relation $\sim$ is FA–recognizable.

Similarly, one can construct a FA–recognizable relation $R(u,v,w) \subset \Sigma_q^*$ such that for every pair $(u,v) \in \Sigma_q^*$ there exists a unique $w \in \Sigma_q^*$ for which $(u,v,w) \in R$ and if $(u,v,w) \in R$ then for the corresponding polynomials f_u, f_v and f_w: $f_u + f_v \sim f_w$. Again, the construction of such a relation R is the same, up to minor changes, as it is described by Nies and Semukhin for the case $t(x) = x^2 + x - 3$. Let $u = a_0 \ldots a_n$ and $v = b_0 \ldots b_m$. Then a string $w = c_0 \ldots c_k$ for which $(u,v,w) \in R$ is obtained as follows. Let c_0 be an integer such that $|c_0| < |q| - 1$, c_0 has the same sign as $a_0 + b_0$ and $c_0 \equiv a_0 + b_0 \pmod q$. We remember two carries $r_0 = p\left[\frac{a_0+b_0}{q}\right]$ and $r_1 = \left[\frac{a_0+b_0}{q}\right]$. We put c_1 to be an integer such that $|c_1| \leqslant |q| - 1$, c_1 has the same sign as $r_0 + a_1 + b_1$ and $c_1 \equiv r_0 + a_1 + b_1 \pmod q$, and update the carries as $r_0 \to r_1 + p\left[\frac{r_0+a_1+b_1}{q}\right]$ and $r_1 \to \left[\frac{r_0+a_1+b_1}{q}\right]$. This process is continued until the string w is generated. The formulas for updating carries are $r_0 \to r_1 + p\left[\frac{r_0+a_i+b_i}{q}\right]$ and $r_1 \to \left[\frac{r_0+a_i+b_i}{q}\right]$. The proof that $|r_1|$ and $|r_0|$ are bounded by $(|q|-1)$ and $(|q|-1)^2$, respectively, is the same as in the paragraph above, so the relation R is FA–recognizable.

Fixing the ordering $-(|q|-1) < \cdots < (|q|-1)$ on Σ_q, the domain Dom and the relation Add are then defined in exactly the same way as by Nies and Semuhkhin, see the first paragraph of this section. So, for every pair of integers p and q, for which $1 + |p| < |q|$, we obtain a regular domain $\text{Dom}_{p,q}$ and a FA–recognizable relation $\text{Add}_{p,q}$ for which $(\text{Dom}_{p,q}, \text{Add}_{p,q})$ is isomorphic to $(\mathbb{Z}^2, +)$. For given p and q satisfying the inequality $1 + |p| < |q|$, we denote by $\psi_{p,q} : \text{Dom}_{p,q} \to \mathbb{Z}^2$ the representation of $(\mathbb{Z}^2, +)$ described above. Let $g \in \mathbb{Z}[x]$ be some fixed polynomial. Clearly, if $f_1 \sim f_2$, then $f_1 g \sim f_2 g$. Therefore, multiplication by g induces a map from $\mathbb{Z}[x]/\langle t \rangle$ to $\mathbb{Z}[x]/\langle t \rangle$ which sends an equivalence class $[f]_\sim$ to the equivalence class $[fg]_\sim$. So, by Proposition 1, multiplication by g induces a map $\varphi_g : \text{Dom}_{p,q} \to \text{Dom}_{p,q}$.

Proposition 3. *For every representation $\psi_{p,q}$ the function $\varphi_g : \text{Dom}_{p,q} \to \text{Dom}_{p,q}$ is FA–recognizable.*

Proof. Since the equivalence relation $\sim$ and Add are FA–recognizable, it is enough only to show that multiplication by a monomial x is FA–recognizable. It

is true because for a string $u = a_0 \ldots a_n \in \mathrm{Dom}_{p,q}$ the string $\varphi_x(u)$ is equivalent to the shifted string $0a_0 \ldots a_n$. Clearly, such shifting of strings is FA–recognizable.   □

Nies and Semukhin showed that every nontrivial cyclic subgroup $\langle z \rangle$ of $\mathbb{Z}^2$ is not FA–recognizable for the representation $\psi_{1,3}$ [18, § 6]. We will show that each of the two cyclic components of $\mathbb{Z}^2$ is not FA–recognizable for every representation $\psi_{p,q}$, if $\gcd(p,q) = 1$. Let $\xi = [1]_\sim$, where 1 is the polynomial $f(x) = 1$; also, ξ corresponds to the single–letter string $1 \in \mathrm{Dom}_{p,q}$: $\psi_{p,q}(1) = \xi$. Let us show that the cyclic subgroup generated by ξ is not FA–recognizable with respect to $\psi_{p,q}$, if $\gcd(p,q) = 1$. We will use arguments analogous to the ones in [18, § 6] with relevant modifications. It is straightforward that [18, Lemma 6.3] claiming that for given two equivalent reduced polynomials $f(x)$ and $g(x)$, $x^k | f$ implies $x^k | g$, holds valid. It is said that $f(x) \in \mathbb{Z}[x]$ starts with k zeros in reduced form if there exists a reduced polynomial $g(x)$ for which $f \sim g$ and $x^k | g(x)$: in this case the string representing $g(x)$ starts with k zeros. For a given $k > 0$, the polynomial q^k starts with at least k zeros in reduced form because $q^k \sim x^k(x + p)^k$.

Assume now that $L_\xi = \psi_{p,q}^{-1}(\langle \xi \rangle)$ is regular and recognized by a finite automaton with k_0 states. The string $\psi_{p,q}^{-1}([q^{k_0}]_\sim) \in L_\xi$ starts with at least k_0 zeros, i.e., $\psi_{p,q}^{-1}([q^{k_0}]_\sim) = 0^k u$ for $k \geqslant k_0$ and some $u \in \Sigma_q^*$, which does not have 0 as the first symbol. By pumping lemma, there exist k_1, k_2 and $0 < d \leqslant k_0$, for which $k_1 + d + k_2 = k$, such that $s_i = 0^{k_1 + di + k_2} u \in L_\xi$ for all $i \geqslant 0$. Since $s_i \in L_\xi$, we have a sequence of integers n_i, $i \geqslant 0$ for which $\psi_{p,q}(s_i) = [n_i]_\sim$, so n_i starts with $k_1 + di + k_2$ zeros in reduced form. For a given integer n, if it starts with at least one zero in reduced form, then $q \,|\, n$: it is because $n = q\ell + r$ for some ℓ and $r \in \{0, \ldots, |q| - 1\}$, so if $r \neq 0$ then $n \sim x(x + p)\ell + r$ starts with no zeros in reduced form.

Proposition 4. *Assume that $\gcd(p,q) = 1$. If $n = q\ell$ starts with $m > 0$ zeros in reduced form, then ℓ starts with $m - 1$ zeros in reduced form.*

Proof. Let $f(x) = x^i(b_j x^{j-i} + \cdots + b_i)$ be a reduced polynomial equivalent to ℓ, where $b_i \neq 0$. We have $n = q\ell \sim x^{i+1}(x+p)(b_j x^{j-i} + \cdots + b_i)$. Since $\gcd(p,q) = 1$ and $|b_i| < |q|$, $q \nmid pb_i$. Therefore, n starts with $i + 1$ zeros in reduced form, so $i = m - 1$. Therefore, ℓ starts with $m - 1$ zeros in reduced form.   □

Thus, if $\gcd(p,q) = 1$, by Proposition 4, we obtain that $q^{k_1 + di + k_2} | n_i$, so $n_i = q^{k_1 + di + k_2} m_i$ for some nonzero integer m_i. Let α and β be the roots of the polynomial $t(x) = x^2 + px - q$. We have $\alpha\beta = -q$, so $|\alpha\beta| = |q|$. Therefore, either $|\alpha|$ or $|\beta|$ must be less or equal than $\sqrt{|q|}$. So, let us assume that $|\alpha| \leqslant \sqrt{|q|}$. For every two equivalent polynomials $f \sim g$: $f(\alpha) = g(\alpha)$. Let f_i be the reduced polynomials corresponding to the strings s_i. If $|\alpha| > 1$, then $|f_i(\alpha)|$ is bounded from above by $(|q| - 1)|u||\alpha|^{|s_i| - 1}$, where $|s_i| = k_1 + di + k_2 + |u|$ is the length of the string s_i; it is because there are only at most $|u|$ nonzero coefficients of the polynomial f_i and the absolute value of each of which is less than or equal to $|q| - 1$. Therefore, $|f_i(\alpha)| \leqslant C_1 |\alpha|^{di}$, where $C_1 = (|q| - 1)|u||\alpha|^{k_1 + k_2 + |u| - 1}$. If $|\alpha| \leqslant 1$, then $|f_i(\alpha)| \leqslant C_2$, where $C_2 = (|q| - 1)|u|$. In both cases we obtain that

$|f_i(\alpha)| \leqslant C\sqrt{|q|}^{di}$ for some constant C. On the other hand, since $f_i \sim n_i$, $f_i(\alpha) = n_i = q^{k_1+di+k_2}m_i$. Therefore, $|f_i(\alpha)| = |q|^{k_1+di+k_2}|m_i| \geqslant |q|^{di}$. Thus, we obtain that $|q|^{di} \leqslant C\sqrt{|q|}^{di}$ for all $i \geqslant 0$, which apparently leads to a contradiction since $|q| > 1$. Thus, L_ξ is not regular.

Let $\eta = [x]_\sim$, where x is the polynomial $f(x) = x$; also, η corresponds to the string $01 \in \mathrm{Dom}_{p,q}$: $\psi_{p,q}(01) = \eta$. Clearly, $\mathbb{Z}^2$ is the direct sum of its cyclic subgroups $\langle\xi\rangle$ and $\langle\eta\rangle$. Let $L_\eta = \psi_{p,q}^{-1}(\langle\eta\rangle)$. We notice that $L_\xi = \{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_x(w) \in L_\eta\}$. The inclusion $L_\xi \subseteq \{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_x(w) \in L_\eta\}$ is straightforward. For the inclusion $\{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_x(w) \in L_\eta\} \subseteq L_\xi$ it is enough to notice that if $\psi_{p,q}(w) = [sx+r]_\sim$, then $\varphi_x(w) = [x(sx+r)]_\sim = [s(-px+q)+rx]_\sim = [(r-sp)x+sq]_\sim$ which is equal to $[kx]_\sim$ for some $k \in \mathbb{Z}$ only if $sq = 0$. The map $\varphi_x : \mathrm{Dom}_{p,q} \to \mathrm{Dom}_{p,q}$ is FA–recognizable, by Proposition 3. So, the regularity of L_η implies the regularity of L_ξ. Therefore, L_η is not regular. Clearly, the fact that L_ξ and L_η are not regular implies that the projections of $\mathbb{Z}^2$ onto its cyclic components $\langle\xi\rangle$ and $\langle\eta\rangle$ are not FA–recognizable. Let us summarize the results we obtained in the following theorem.

Theorem 5. *For every pair of integers p and q for which $1 + |p| < |q|$ the map $\psi_{p,q} : \mathrm{Dom}_{p,q} \to \mathbb{Z}^2$ gives a FA–presentation of $(\mathbb{Z}^2, +)$. Moreover, if $\gcd(p,q) = 1$, then none of the two cyclic components of $\mathbb{Z}^2$ and the projections onto theses components is FA–recognizable with respect to $\psi_{p,q}$.*

Remark 6. Let $z(x) = ax + b$ be a polynomial in $\mathbb{Z}[x]$ and $\zeta = [z]_\sim$ be the corresponding element in $\mathbb{Z}^2$. For a given integer $m > 1$ let $\delta = m\zeta \in \mathbb{Z}^2$. We denote by φ_m the map φ_g for the constant polynomial $g(x) = m$. Let $L_\zeta = \psi_{p,q}^{-1}(\langle\zeta\rangle)$ and $L_\delta = \psi_{p,q}^{-1}(\langle\delta\rangle)$. Then we have: $L_\zeta = \{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_m(w) \in L_\delta\}$. The inclusion $L_\zeta \subseteq \{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_m(w) \in L_\delta\}$ is straightforward. In order to prove the inclusion $\{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_m(w) \in L_\delta\} \subseteq L_\zeta$ we notice that if $\psi_{p,q}(w) = [sx+r]_\sim$, then $\varphi_m(w) = [m(sx+r)]_\sim$ which is equal to $k\delta = [kmz]_\sim = [kmax + kmb]_\sim$ for some $k \in \mathbb{Z}$ iff $kma = ms$ and $kmb = mr$. Clearly, this holds iff $s = ka$ and $r = kb$, so $\psi_{p,q}(w) = [kax+kb]_\sim = k\zeta$ which implies that $w \in L_\zeta$. Therefore, if L_ζ is not regular, then L_δ is not regular. In particular, if $\gcd(p,q) = 1$, then none of the cyclic subgroups $\langle m\xi\rangle$ and $\langle m\eta\rangle$ is FA–recognizable with respect to $\psi_{p,q}$ for every nonzero integer m.

Remark 7. In order to guarantee that all nontrivial cyclic subgroups of $\mathbb{Z}^2$ are not FA–recognizable with respect to $\psi_{p,q}$, one should additionally require that the polynomial $t(x) = x^2 + px - q$ is irreducible in $\mathbb{Z}[x]$. Let $\gamma = [g]_\sim$ for some $g \in \mathbb{Z}[x]$, $g \not\sim 0$, and $L_\gamma = \psi_{p,q}^{-1}(\langle\gamma\rangle)$. We have: $L_\xi = \{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_g(w) \in L_\gamma\}$. The inclusion $L_\xi \subseteq \{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_g(w) \in L_\gamma\}$ is again straightforward. In order to prove the inclusion $\{w \in \mathrm{Dom}_{p,q} \,|\, \varphi_g(w) \in L_\gamma\} \subseteq L_\xi$ we notice that if $\psi_{p,q}(w) = [sx+r]_\sim$, then $\varphi_g(w) = [g(sx+r)]_\sim$ which is equal to $[gk]_\sim$ for some $k \in \mathbb{Z}$ iff the polynomial t divides $g(sx+r-k)$. Since t is irreducible and t does not divide g, then $s = 0$ and $r = k$. Therefore, by Proposition 3, if L_γ is regular, then L_ξ is regular. So, L_γ is not regular.

Remark 8. Moreover, if t is irreducible in $\mathbb{Z}[x]$, every nonzero endomorphism of $\mathbb{Z}^2$ with nontrivial kernel is not FA–recognizable. This immediately follows from the observation that the image of a such endomorphism is a cyclic subgroup of $\mathbb{Z}^2$ which is not FA–recognizable with respect to $\psi_{p,q}$ by Remark 7.

Now, let $n > 2$ and $t(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1 x - q$ be a polynomial with integer coefficients for which $1 + |p_{n-1}| + \cdots + |p_1| < |q|$. We identify the group $\mathbb{Z}^n$ with the additive group of the ring $\mathbb{Z}[x]/\langle t \rangle$. We denote by $\overline{p}$ a tuple $\overline{p} = \langle p_1, \ldots, p_{n-1} \rangle$. Clearly, one gets a FA–presentation $\psi_{\overline{p},q} : \mathrm{Dom}_{\overline{p},q} \to \mathbb{Z}^n$ of $(\mathbb{Z}^n, +)$ in exactly the same way as it is described for the case $n = 2$. It can be seen that all arguments presented in this section hold valid up to the following minor modifications. For an algorithm recognizing the equivalence $\sim$, one should use n carries $r_0, r_1, \ldots, r_{n-1}$ updated as follows: $r_0 \to r_1 + p_1 \left[\frac{r_0 + a_i - b_i}{q} \right], r_1 \to r_2 + p_2 \left[\frac{r_0 + a_i - b_i}{q} \right], \ldots, r_{n-2} \to r_{n-1} + p_{n-1}\left[\frac{r_0 + a_i - b_i}{q}\right], r_{n-1} \to \left[\frac{r_0 + a_i - b_i}{q}\right]$. Let us verify the inequalities: $r_0 \leqslant (|q| - 1)^2, r_1 \leqslant (|q| - 1)(1 + |p_{n-1}| + |p_{n-2}| + \cdots + |p_2|), \ldots, |r_{n-2}| \leqslant (|q| - 1)(1 + |p_{n-1}|)$ and $|r_{n-1}| \leqslant |q| - 1$. Initially these inequalities are satisfied.

Suppose now that they hold for a current iteration of the algorithm. Since $|r_0| \leqslant (|q| - 1)^2$ for the current iteration, then $\left| \left[\frac{r_0 + a_i - b_i}{q} \right] \right| \leqslant |q| - 1$ which implies that $|r_{n-1}| \leqslant |q| - 1$ for the next iteration of the algorithm. Since we assumed that $|r_{n-1}| \leqslant |q| - 1$ for the current iteration, then $r_{n-1} + p_{n-1}\left[\frac{r_0 + a_i - b_i}{q} \right] \leqslant (|q| - 1)(1 + |p_{n-1}|)$ which implies that $|r_{n-2}| \leqslant (|q| - 1)(1 + |p_{n-1}|)$ for the next iteration. In the same way we prove the inequalities for $r_{n-3}, \ldots, r_1$ for the next iteration. Finally, since we assumed that $r_1 \leqslant (|q| - 1)(1 + |p_{n-1}| + \cdots + |p_2|)$ for the current iteration, for the next iteration we have: $|r_0| \leqslant (|q| - 1)(1 + |p_{n-1}| + |p_{n-2}| + \cdots + |p_1|) \leqslant (|q| - 1)^2$. So, the algorithm requires only a finite amount of memory. The same remains true for an algorithm recognizing the addition. In order to get the analogue of Proposition 4 for $n > 2$, one should simply change p to p_1. Also, clearly, there is a root α of polynomial $t(x)$ for which $|\alpha| \leqslant \sqrt[n]{|q|}$.

We call all presentations $\psi_{\overline{p},q}$ satisfying the conditions $1 + |p_{n-1}| + \cdots + |p_1| < |q|$ and $\gcd(p_1, q) = 1$ Nies–Semukhin FA–presentations. So, in exactly the same way as for $n = 2$, we obtain that for every Nies–Semukhin FA–presentation the language L_ξ is not regular. Let $\eta_i = [x^i]_\sim$ for $i = 1, \ldots, n - 1$. It is clear that $\mathbb{Z}^n$ is the direct sum of its cyclic subgroups $\langle \xi \rangle$ and $\langle \eta_1 \rangle, \ldots, \langle \eta_{n-1} \rangle$. Let $L_{\eta_i} = \psi_{\overline{p},q}^{-1}(\langle \eta_i \rangle)$ for $i = 1, \ldots, n - 1$. Similarly to the case $n = 2$ we obtain that $L_\xi = \{w \in \mathrm{Dom}_{\overline{p},q} \,|\, \varphi_x(w) \in L_{\eta_1}\}$. The inclusion $L_\xi \subseteq \{w \in \mathrm{Dom}_{\overline{p},q} \,|\, \varphi_x(w) \in L_{\eta_1}\}$ is straightforward. In order to prove the inclusion $\{w \in \mathrm{Dom}_{\overline{p},q} \,|\, \varphi_x(w) \in L_{\eta_1}\} \subseteq L_\xi$ we notice that if $\psi_{\overline{p},q}(w) = [sx^{n-1} + r_{n-2}x^{n-2} + \cdots + r_0]_\sim$, then $\varphi_x(w) = [x(sx^{n-1} + r_{n-2}x^{n-2} + \cdots + r_0)]_\sim = [(r_{n-2} - sp_{n-1})x^{n-1} + \cdots + (r_0 - sp_1)x + sq]_\sim$ which is equal to $[kx]_\sim$ for some $k \in \mathbb{Z}$ only if $s, r_{n-2}, \ldots, r_2$ and r_1 are equal to zero. Thus, the regularity of L_{η_1} implies the regularity of L_ξ. Therefore, L_{η_1} is not regular.

Then we consecutively prove that each of the languages $L_{\eta_2}, \ldots, L_{\eta_{n-1}}$ is not regular using the observation that $L_{\eta_i} = \{w \in \mathrm{Dom}_{\overline{p},q} \,|\, \varphi_x(w) \in L_{\eta_{i+1}}\}$ for $i = 1, \ldots, n - 2$. The inclusion $L_{\eta_i} \subseteq \{w \in \mathrm{Dom}_{\overline{p},q} \,|\, \varphi_x(w) \in L_{\eta_{i+1}}\}$ is

straightforward. In order to prove the inclusion $L_{\eta_i} \subseteq \{w \in \mathrm{Dom}_{\overline{p},q} \,|\, \varphi_x(w) \in L_{\eta_{i+1}}\}$ for $i = 1, \ldots, n-2$ we notice that if $\psi_{\overline{p},q}(w) = [sx^{n-1} + r_{n-2}x^{n-2} + \cdots + r_0]_\sim$, then $\varphi_x(w) = [kx^{i+1}]_\sim$ for some $k \in \mathbb{Z}$ only if $s = 0$ and $r_j = 0$ for $j \neq i$. The following theorem generalizes Theorem 5 for the case $n > 2$.

Theorem 9. *For every tuple $\overline{p} = \langle p_1, \ldots, p_{n-1} \rangle$ and an integer q for which $1 + |p_{n-1}| + \cdots + |p_1| < |q|$ the map $\psi_{\overline{p},q} : \mathrm{Dom}_{\overline{p},q} \to \mathbb{Z}^n$ gives a FA–presentation of $(\mathbb{Z}^n, +)$. If $\gcd(p_1, q) = 1$, then none of the cyclic components of $\mathbb{Z}^n$ and the projections onto these components is FA–recognizable with respect to $\psi_{\overline{p},q}$.*

Clearly, Remarks 6 and 7 hold valid also for the case $n > 2$. In particular, for every nonzero integer m each cyclic subgroup $\langle m\eta_1 \rangle, \ldots, \langle m\eta_{n-1} \rangle$ and $\langle m\xi \rangle$ is not FA–recognizable with respect to a Nies–Semukhin FA–presentation. Furthermore, if a polynomial $t(x)$ is irreducible, then none of the cyclic subgroups of $\mathbb{Z}^n$ is FA–recognizable.

3   FA–recognizable automorphisms of $\mathbb{Z}^n$

In this section until the last paragraph we discuss the case $n = 2$. By Proposition 3, for a polynomial $g \in \mathbb{Z}[x]$, multiplication by g induces a FA–recognizable map $\varphi_g : \mathrm{Dom}_{p,q} \to \mathrm{Dom}_{p,q}$. Clearly, if $f \sim g$, then $\varphi_g = \varphi_f$. Therefore, since every polynomial from $\mathbb{Z}[x]$ is equivalent to a polynomial of degree at most one, we may assume that $g(x) = ax + b$ for $a, b \in \mathbb{Z}$. Let $h(x) = h_1 x + h_2$, for $h_1, h_2 \in \mathbb{Z}$. The equivalence class $[h]_\sim$ is identified with $(h_1, h_2) \in \mathbb{Z}^2$. We have: $g(x)h(x) = (ax + b)(h_1 x + h_2) = ah_1 x^2 + (ah_2 + bh_1)x + bh_2 \sim ah_1(-px + q) + (ah_2 + bh_1)x + bh_2 = ((b - ap)h_1 + ah_2)x + aqh_1 + bh_2$. Clearly, $\xi = [1]_\sim$ and $\eta = [x]_\sim$, already defined in Section 2, generate the group $\mathbb{Z}^2$. We denote by H_1 and H_2 the cyclic subgroups of $\mathbb{Z}^2$ generated by η and ξ, respectively. Thus, multiplication by g induces an endomorphism of $\mathbb{Z}^2 = H_1 \oplus H_2$ given by a matrix $A = \begin{pmatrix} b - ap & a \\ aq & b \end{pmatrix}$. The condition that $A \in \mathrm{GL}(2, \mathbb{Z})$ yields the equations $b^2 - abp - a^2 q = \pm 1$. The latter is equivalent to $(2b - ap)^2 - (p^2 + 4q)a^2 = \pm 4$. Let $c = 2b - ap$. Then we have:

$$A = \begin{pmatrix} \frac{c - ap}{2} & a \\ aq & \frac{c + ap}{2} \end{pmatrix}, \tag{1}$$

where p, q, a and c satisfy one of the following two equations:

$$c^2 - (p^2 + 4q)a^2 = \pm 4. \tag{2}$$

For given p and q, the trivial solutions of (1), $a = 0$ and $c = \pm 2$, correspond to the matrices $A = \pm I$. We will assume that $a \neq 0$. Let $n = p^2 + 4q$. Clearly, nontrivial solutions of (2) exist only if $n \geqslant -4$. The following theorem can be verified by direct calculations.

Theorem 10. *For a given $n \geqslant -4$, the matrices A defined by (1) together with the coefficients p and q for which p, q, a and c satisfy: $1 + |p| < |q|$, $\gcd(p, q) = 1$, $n = p^2 + 4q$, $a \neq 0$ and the equation $c^2 - na^2 = \pm 4$ are as follows:*

- For $n = -4$, $A = \pm \begin{pmatrix} -r & 1 \\ -(r^2+1) & r \end{pmatrix}$, $p = 2r$ and $q = -(r^2+1)$, where $r \in (-\infty, -4] \cup [4, +\infty)$ and $r \equiv 0 \pmod 2$.

- For $n = -3$, $A = \pm \begin{pmatrix} -r & 1 \\ -(r^2+r+1) & (r+1) \end{pmatrix}$ or $A = \pm \begin{pmatrix} -(r+1) & 1 \\ -(r^2+r+1) & r \end{pmatrix}$, $p = 2r+1$ and $q = -(r^2+r+1)$, where $r \in (-\infty, -3] \cup [2, +\infty)$ and either $r \equiv 0 \pmod 3$ or $r \equiv 2 \pmod 3$.

- For $n = 0$, $n = -1$ and $n = -2$, there exist no nontrivial solutions.

- For $n = m^2 > 0$, nontrivial solutions exist only if $n = 1$ or $n = 4$. For $n = 1$, $A = \pm \begin{pmatrix} -(2r+1) & 2 \\ -2(r^2+r) & (2r+1) \end{pmatrix}$, $p = 2r+1$ and $q = -(r^2+r)$, where $r \in (-\infty, -4] \cup [3, +\infty)$. For $n = 4$, $A = \pm \begin{pmatrix} -r & 1 \\ 1-r^2 & r \end{pmatrix}$, $p = 2r$ and $q = 1 - r^2$, where $r \in (-\infty, -4] \cup [4, +\infty)$ and $r \equiv 0 \pmod 2$.

- For a positive nonsquare integer n, the equality $n = p^2 + 4q$ implies that either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. For these two cases we have:

  • For $n = 4s$, $A = \pm \begin{pmatrix} x-ra & a \\ a(s-r^2) & x+ra \end{pmatrix}$ or $A = \pm \begin{pmatrix} -x-ra & a \\ a(s-r^2) & -x+ra \end{pmatrix}$, $p = 2r$ and $q = s - r^2$, where $x > 0$ and $a > 0$ give a solution of Pell's equation or negative Pell's equation:

  $$x^2 - sa^2 = \pm 1,$$

  and r either satisfies the inequality $|r| < \sqrt{s} - 1$ or the inequality $|r| > \sqrt{s+2} + 1$. Also, it is required that $\gcd(r, s) = 1$ and $r \not\equiv s \pmod 2$.

  • For $n \equiv 1 \pmod 4$, $A = \pm \begin{pmatrix} \frac{c-pa}{2} & a \\ a\frac{n-p^2}{4} & \frac{c+pa}{2} \end{pmatrix}$ or $A = \pm \begin{pmatrix} \frac{-c-pa}{2} & a \\ a\frac{n-p^2}{4} & \frac{-c+pa}{2} \end{pmatrix}$, $p \equiv 1 \pmod 2$ and $q = \frac{n-p^2}{4}$, where $c > 0$ and $a > 0$ give a solution of one of the following Pell–type equations:

  $$c^2 - na^2 = \pm 4,$$

  and p either satisfies the inequality $|p| < \sqrt{n} - 2$ or the inequality $|p| > \sqrt{n+8} + 2$. Also, it is required that $\gcd(p, n) = 1$.

Remark 11. We recall that for a nonsquare integer $n > 0$ Pell's equation $x^2 - ny^2 = 1$ has infinitely many solutions which are recursively generated, using Brahmagupta's identity: $(x_1^2 - ny_1^2)(x_2^2 - ny_2^2) = (x_1 x_2 + n y_1 y_2)^2 - n(x_1 y_2 + y_1 x_2)^2$, from the fundamental solution – the one for which positive x and y are minimal. The fundamental solution can be found, for example, using continued fraction of $\sqrt{n}$. All solutions of negative Pell's equation $x^2 - ny^2 = -1$ are also generated from its fundamental solution. However, solutions of negative Pell's equation do not always exist. The first 54 numbers for which solutions exist are given by the sequence A031396 in OEIS [20]. Similarly, for the Pell–type equations $c^2 - na^2 = 4$ and $c^2 - na^2 = -4$, all solutions are recursively generated from the fundamental solutions. For the latter equation solutions exist if and only if they exist for the

equation $x^2 - ny^2 = -1$. Furthermore, by Cayley's theorem, if the fundamental solution (u, v) of the equation $c^2 - na^2 = 4$ is odd (i.e., both u and v are odd), then $((u^2 - 3)u/2, (u^2 - 1)v/2)$ gives the fundamental solution of the equation $x^2 - ny^2 = 1$. Similarly, the odd fundamental solution (u, v) of the equation $c^2 - na^2 = -4$ leads to the fundamental solution $((u^2 + 3)u)/2, ((u^2 + 1)v)/2)$ of the equation $x^2 - ny^2 = -1$ [19]. If the fundamental solution is even then it is obtained from the fundamental solution of the corresponding Pell's equation by multiplication by 2.

Remark 12. We note that Pell's equation already appeared in the proof that the ring $(\mathbb{Z}(\sqrt{n}), \mathbb{Z}, +, <, =; \cdot)$ has for every positive natural number n a semi-automatic presentation [11]. This is not surprising since the technique used in the construction of a such semiautomatic presentation is similar to the Nies–Semukhin construction [18, § 6].

Remark 13. For a fixed pair p and q, the matrices (1) with coefficients satisfying (2) form a submonoid $\mathcal{S}_{p,q}$ in $\mathrm{GL}(2, \mathbb{Z})$. Let $\mathcal{P}$ be the set of all pairs (p, q) for which $1 + |p| < |q|$, $\gcd(p, q) = 1$ and $n = p^2 + 4q$ is equal to either $-4, -3, 1, 4$ or a nonsquare positive integer. Then a set of all matrices given by Theorem 10 together with the matrices $\pm I$ is the union $\mathcal{S} = \bigcup_{(p,q) \in \mathcal{P}} \mathcal{S}_{p,q}$. For different pairs $(p, q), (p', q') \in \mathcal{P}$ we clearly have $\mathcal{S}_{p,q} \cap \mathcal{S}_{p',q'} = \{\pm I\}$. Moreover, it can be verified that each of these submonoids $\mathcal{S}_{p,q}$ is isomorphic to one of the groups: $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z} \times \mathbb{Z}_2$. Namely, from Theorem 10 we obtain the following. For $n = -4$, $n = -3$ and $n = 1, 4$, $\mathcal{S}_{p,q}$ is a finite group isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. For a positive nonsquare integer n, $\mathcal{S}_{p,q} \cong \mathbb{Z} \times \mathbb{Z}_2$.

Remark 14. Let $(p, q) \in \mathcal{P}$ such that the polynomial $t(x) = x^2 + px - q$ is irreducible in $\mathbb{Z}[x]$. One can easily construct an infinite family of not FA–recognizable automorphisms of $\mathbb{Z}^2$ with respect to the representation $\psi_{p,q}$. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{S}_{p,q}$. For a matrix $A' = A + D$, where $D = \begin{pmatrix} k\ell & kn \\ m\ell & mn \end{pmatrix}$ is a nonzero singular matrix, $\det A' = \det A$ iff $m(a_{11}n + a_{12}\ell) + k(a_{21}n + a_{22}\ell) = 0$. The latter equation admits infinitely many solutions for k, l, m and n. Since A is FA–recognizable with respect to $\psi_{p,q}$, assuming that A' is FA–recognizable with respect to $\psi_{p,q}$, we get that $D = A' - A$ must be FA–recognizable with respect to $\psi_{p,q}$. But D is not FA–recognizable (see Remark 8), so A' is not FA–recognizable.

Remark 15. There exist automorphisms of $\mathbb{Z}^2$ which are not FA–recognizable with respect to every representation $\psi_{p,q}$, $(p, q) \in \mathcal{P}$. For example, all automorphisms of $\mathbb{Z}^2$ given by the matrices $T_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for nonzero integer n are not FA–recognizable. This follows from the fact that I is FA–recognizable but, by Remark 6, the endomorphisms $T_n - I$ for $n \neq 0$ are not FA–recognizable. In particular, none of the representations $\psi_{p,q}$, $(p, q) \in \mathcal{P}$ can be used to construct a Cayley automatic representation for the Heisenberg group $\mathcal{H}_3(\mathbb{Z}) \cong \mathbb{Z}^2 \rtimes_{T_1} \mathbb{Z}$.

Remark 16. We note that for two conjugate matrices A and $B = TAT^{-1}$ in $\mathrm{GL}(2,\mathbb{Z})$ the groups $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_B \mathbb{Z}$ are isomorphic. An algorithm for solving conjugacy problem in $\mathrm{GL}(2,\mathbb{Z})$ is described in [9]; see also an algorithm for solving conjugacy problem in $\mathrm{SL}(2,\mathbb{Z})$ using continued fractions [12, § 7.2]. It can be verified that for the cases $n = -4, -3, 1, 4$ each of the matrices from Theorem 10 is conjugate to one of the following matrices in $\mathrm{GL}(2,\mathbb{Z})$:
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If n is a positive nonsquare integer, every matrix from Theorem 10, which is in $\mathrm{SL}(2,\mathbb{Z})$, is Anosov. Moreover, in this case, for a pair $(p, q) \in \mathcal{P}$ satisfying $n = p^2 + 4q$ the matrices from $\mathcal{S}_{p,q}$ generate infinitely many conjugacy classes in $\mathrm{GL}(2,\mathbb{Z})$. The latter immediately follows from the observation that for different values of c, which is the trace of the matrix (1), we have different conjugacy classes.

Similarly to the case $n = 2$, one can get a family of FA–recognizable automorphisms $A \in \mathrm{GL}(n,\mathbb{Z})$ with respect to the Nies–Semukhin FA–presentations $\psi_{\overline{p},q}$ of $\mathbb{Z}^n$ for $n > 2$. By Proposition 3 (its analogue clearly holds also for the case $n > 2$), multiplication by a polynomial $g \in \mathbb{Z}[x]$ induces a FA–recognizable map $\varphi_g : \mathrm{Dom}_{\overline{p},q} \to \mathrm{Dom}_{\overline{p},q}$; also, equivalent polynomials $f \sim g$ induce the same map: $\varphi_f = \varphi_g$. So, we may assume that $g(x) = a_{n-1}x^{n-1} + \cdots + a_0$ for $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. The matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ corresponding to the linear map φ_g depends on the tuples $\langle p_1, \ldots, p_{n-1} \rangle$, $\langle a_0, \ldots, a_{n-1} \rangle$ and the integer q. In this paper we do not give full classification of all such matrices for the case $n > 2$.

4   Conclusion and Open Questions

In this paper we generalize the Nies–Semukhin FA–presentation of $(\mathbb{Z}^2, +)$, originally constructed for the polynomial $x^2 + x - 3$, to a polynomial $x^2 + px - q$ such that $1 + |p| < |q|$ and $\gcd(p, q) = 1$. We also show how this construction is generalized for $(\mathbb{Z}^n, +), n > 2$. Based on this, we construct a new family of Cayley automatic representations of groups $\mathbb{Z}^n \rtimes_A \mathbb{Z}, A \in \mathrm{GL}(n,\mathbb{Z})$ that violate the basic property known for standard representations – projections $p_i : \mathbb{Z}^n \to \mathbb{Z}^n, i = 1, \ldots, n$ are FA–recognizable, i.e., the property b) in Section 1. For $n = 2$ we describe the set of matrices $\mathcal{S} \subseteq \mathrm{GL}(2,\mathbb{Z})$ corresponding to this family of nonstandard representations and show its connection with Pell's equation. Let us pose the following questions that are apparent from the results of this paper.

- Is there a nonstandard representation, e.g., preserving the property a) and violating the property b), for the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$?
- What is the set of conjugacy classes of the set of matrices $\mathcal{S}$ in $\mathrm{GL}(2,\mathbb{Z})$?
- Is there any Anosov $A \in \mathrm{SL}(2,\mathbb{Z})$ which is not FA–recognizable with respect to every Nies–Semukhin FA–presentation $\psi_{p,q}, (p, q) \in \mathcal{P}$?

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