

An explicit upper bound for Siegel zeros of imaginary quadratic fields

D. Ralaivaosaona F. B. Razakarinoro

Abstract

For any integer $d \geq 3$ such that $-d$ is a fundamental discriminant, we show that the Dirichlet L -function associated with the real primitive character $\chi(\cdot) = \left(\frac{-d}{\cdot}\right)$ does not vanish on the positive part of the interval $[1 - 6.5/\sqrt{d}, 1]$.

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1 Introduction

For a fundamental discriminant D , the arithmetic function defined by the Kronecker symbol $\chi(n) = \left(\frac{D}{n}\right)$ is a real primitive Dirichlet character and its associated L -function is defined by the series

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

The series on the right-hand side only makes sense when $\operatorname{Re}(s) > 1$, but it is well known that the function $L(s, \chi)$ has an analytic continuation defined over the whole complex plane. The locations of the zeros of $L(s, \chi)$ are particularly important in number theory. One of the most important open problem in mathematics – the Generalized Riemann Hypothesis (GRH) – asserts that all zeros with positive real parts lie precisely on the vertical line $\operatorname{Re}(s) = \frac{1}{2}$.

Siegel zeros or sometimes called Landau-Siegel zeros are hypothetical real zeros of the L -functions that lie very close to 1. The existence of these zeros has not yet been ruled out, but it is known that $L(s, \chi)$ has at most one simple zero in an interval of the form $(1 - c/\log |D|, 1)$, see Page [12]. Morrill and Trudgian, in [11], recently gave an explicit version of the latter statement with $c = 1.011$ using Pintz's refinement of Page's theorem. The largest positive zero of $L(s, \chi)$, if it exists, will be denoted by β throughout this paper.

We are interested in the upper bound on β , or equivalently the lower bound on the distance from β to 1, for the case $D = -d$ where $d \geq 3$. It is well known that there exists an absolute constant $c > 0$ such that $1 - \beta > c/\sqrt{d}$, see Haneke [10], Goldfeld and Schinzel [9], and Pintz [13]. In particular, it is shown in the Goldfeld-Schinzel's paper that

$$1 - \beta > \left(\frac{6}{\pi} + o(1)\right) \frac{1}{\sqrt{d}} \quad \text{as } d \rightarrow \infty. \quad (1)$$

Pintz achieved a similar result, but with a different method. He improved the constant $\frac{6}{\pi}$ to $\frac{12}{\pi}$, and then improved it further to $\frac{16}{\pi}$ following Schinzel's remark, see the footnote on page 277 of [13]. We are unaware of any result of the form $1 - \beta > c/\sqrt{d}$ with an explicit constant $c > 0$ prior to this work. Known explicit results have an additional $(\log d)^2$ term in the denominator, see [7, Lemma 3], [8], and [2]. Most of

these papers made use of explicit upper bounds for $L'(1, \chi)$. We do not follow this route, instead, we use the method of Goldfeld and Schinzel in [9].

It is worth noting that $L(s, \chi)$ does not have positive real zeros for at least a positive proportion of fundamental discriminants $-d$, see [4]. Moreover, Watkins' computational results in [17] show that the same holds for all $L(s, \chi)$ with fundamental discriminants $-d$ such that $d \leq 300000000$. The following theorem is our main result.

Theorem 1. *Let $d > 300000000$ such that $-d$ is a fundamental discriminant. Let $L(s, \chi)$ be the Dirichlet L -function associated with the primitive character $\chi(n) = \left(\frac{-d}{n}\right)$. If there exists $\beta > 0$ such that $L(\beta, \chi) = 0$, then*

$$1 - \beta > \frac{6.5}{\sqrt{d}}. \quad (2)$$

Another Watkins' paper [16] provides a classification of all imaginary quadratic fields with class number less or equal to 100. The combination of the results from [17] and [16] guarantees that we may only consider the case where the class number $h(-d)$ of the corresponding imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ is at least 101. We will see that a higher class number gives a better constant in (2). In fact, we have the following asymptotic result in terms of the class number.

Theorem 2. *Let d and β be as in Theorem 1 and let $h(-d)$ be the class number of the quadratic field $\mathbb{Q}(\sqrt{-d})$. Then, we have*

$$1 - \beta > \left(2\pi + o(1)\right) \frac{h(-d)}{(\log h(-d))^2 \sqrt{d}} \text{ as } h(-d) \rightarrow \infty. \quad (3)$$

It is also possible to obtain an explicit bound for the $o(1)$ term in (3) in terms of $h(-d)$. But this will not be very useful unless we have an explicit lower bound for the class number $h(-d)$, which is a much harder problem.

This paper is organized as follows: In Section 2 we prove two preliminary results, one on the sum of reciprocal prime powers and the other on the sum of reciprocal ideal norms. The proof of Theorem 1 and Theorem 2 are done in Section 3 and Section 4 respectively. We conclude with a short discussion on possible improvements of Theorem 1 in Section 5.

2 Preliminary results

2.1 Sum of reciprocal prime powers

We are going to need explicit estimates for the sum $\sum_{p^\alpha \leq x} p^{-\alpha}$, where the sum is taken over the prime powers p^α not exceeding x . It is clear that $\sum_{p^\alpha \leq x} p^{-\alpha}$ is greater than the sum of reciprocal primes $\sum_{p \leq x} p^{-1}$ but they are asymptotically equal as $x \rightarrow \infty$. It is well known that for $x \geq 3$, we have

$$\sum_{p \leq x} p^{-1} = \log \log x + B_1 + o(1),$$

where B_1 is known as the Mertens constant, ref. Sequence A077761 in the OEIS. Dusart [6] recently provided an explicit bound for the error term in the above estimate. It is shown, see [6, Theorem 5.6], that for every $x \geq 2278383$, we have

$$\left| \sum_{p \leq x} p^{-1} - \log \log x - B_1 \right| \leq \frac{0.2}{(\log x)^3}. \quad (4)$$

We use this result to obtain an explicit estimate for the sum of reciprocal prime-powers.

Proposition 1. For every $x \geq 2$, we have

$$-\frac{1.75}{(\log x)^2} \leq \sum_{p^\alpha \leq x} p^{-\alpha} - \log \log x - B_2 \leq \min \left\{ \frac{0.2}{(\log x)^3}, 10^{-4} \right\}, \quad (5)$$

where

$$B_2 = B_1 + \sum_{\alpha \geq 2} \sum_p p^{-\alpha} = 1.03465\dots$$

The constant B_1 is sometimes referred to as the prime reciprocal constant, so we could analogously call B_2 as the prime power reciprocal constant. B_2 also appears in the OEIS as Sequence A083342.

Remark 1. The lower bound in (5) could be made of the form $c/(\log x)^3$ as in Dusart's result above, but we chose to use the asymptotically weaker bound in the proposition as it gives a slightly better approximation for small values of x , see the comparison with the exact error in Figure 1.

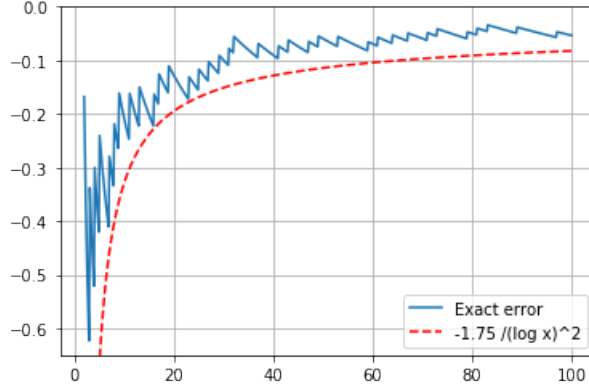


Figure 1: Lower bound in (5) and exact error.

Proof. For $x \geq 2$, we have

$$0 \leq \sum_{p^\alpha \leq x} p^{-\alpha} - \sum_{p \leq x} p^{-1} \leq \sum_{\alpha \geq 2} \sum_p p^{-\alpha}.$$

Let us denote the double summation on the right-hand side by C . It is easy to check that it is convergent. In fact, one has

$$C = \lim_{N \rightarrow \infty} \sum_{\alpha \geq 2} \sum_{p \leq N} p^{-\alpha} = \lim_{N \rightarrow \infty} \sum_{p \leq N} \sum_{\alpha \geq 2} p^{-\alpha} = \sum_p \frac{1}{p^2 - p}.$$

This implies that

$$\sum_{p^\alpha \leq x} p^{-\alpha} \leq \sum_{p \leq x} p^{-1} + C. \quad (6)$$

Now, for the lower bound, we have

$$\sum_{p^\alpha \leq x} p^{-\alpha} - \sum_{p \leq x} p^{-1} = C - \sum_{\alpha \geq 2} \sum_{p^\alpha > x} p^{-\alpha},$$

and

$$\sum_{\alpha \geq 2} \sum_{p^\alpha > x} p^{-\alpha} \leq \sum_{p > \sqrt{x}} \frac{1}{p^2 - p} \leq \sum_{n > \sqrt{x}} \frac{1}{n^2 - n} = \frac{1}{\lceil \sqrt{x} \rceil - 1}.$$

Thus

$$\sum_{p^\alpha \leq x} p^{-\alpha} \geq \sum_{p \leq x} p^{-1} + C - \frac{1}{\lceil \sqrt{x} \rceil - 1}. \quad (7)$$

Combining (6) and (7), with Dusart's bound (4), we obtain the following: for $x \geq 2278383$,

$$-\frac{0.2}{(\log x)^3} - \frac{1}{\lceil \sqrt{x} \rceil - 1} \leq \sum_{p^\alpha \leq x} p^{-\alpha} - \log \log x - B_1 - C \leq \frac{0.2}{(\log x)^3}. \quad (8)$$

It is easy to check that for $x \geq 2278383$, the latter bounds imply the estimates (5) in the statement of the proposition (if $x \geq 2278383$, then $0.2/(\log x)^3 < 10^{-4}$). It now remains to check that (5) also holds for all $x < 2278383$. We can use a computer check for this, but we need to be cautious because x can take any real value. First, for $x \geq 2$, we let

$$\varepsilon(x) := \sum_{p^\alpha \leq x} p^{-\alpha} - \log \log x - B_2.$$

Then, we can easily show from this definition that if p^α is the greatest prime power $\leq x$ then

$$\varepsilon(x) \leq \varepsilon(p^\alpha).$$

We verified numerically with a computer program that $\varepsilon(p^\alpha) < 0$ for all prime powers p^α in the interval $[2, 2278383]$, which proves the upper bound in (5). Similarly, if p^α is the least prime power $\geq x$, then we have

$$\varepsilon(x) + \frac{1.75}{(\log x)^2} \geq \varepsilon(p^\alpha) + \frac{1.75}{(\log p^\alpha)^2} - \begin{cases} p^{-\alpha} & \text{if } x < p^\alpha \\ 0 & \text{if } x = p^\alpha. \end{cases}$$

Again, we checked with a computer program that $\varepsilon(p^\alpha) + \frac{1.75}{(\log p^\alpha)^2} - p^{-\alpha} > 0$ for all prime powers p^α in the interval $[2, 2278421]$ (the number 2278421 is the smallest prime power greater than 2278383). Therefore, we deduce that $\varepsilon(x) + \frac{1.75}{(\log x)^2} > 0$ for all $x \in [2, 2278383]$, which completes the proof of the proposition. \square

2.2 Exploiting the class number

The approach of Goldfeld and Schinzel involves reciprocal sums of norms of ideals of the form

$$\sum_{N(\mathfrak{a}) \leq x} \frac{1}{N(\mathfrak{a})}, \quad (9)$$

where $x \geq 1$ and \mathfrak{a} runs over all nonzero ideals of the ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$. In order to understand such sums, let us recall some useful results from the classical theory of imaginary quadratic fields. For each positive integer a let $\nu(a)$ denote the number of representations of a as a norm of an ideal of $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ that is not divisible by any rational integer > 1 . Such an ideal can be written uniquely in the form

$$\left[a, \frac{b + \sqrt{-d}}{2} \right] := \left\{ an + \frac{b + \sqrt{-d}}{2} m : n, m \in \mathbb{Z} \right\},$$

where $a \geq 1$, $-a < b \leq a$, and $b^2 \equiv -d \pmod{4a}$. Moreover, every other ideal can be written in the form $u[a, \frac{b + \sqrt{-d}}{2}]$, where u is a positive integer, and the norm of such an ideal is $u^2 a$. So we can rewrite the sum in (9) as follows:

$$\sum_{N(\mathfrak{a}) \leq x} \frac{1}{N(\mathfrak{a})} = \sum_{u^2 a \leq x} \frac{\nu(a)}{u^2 a}.$$

We have the following important lemma concerning the arithmetic function $\nu(\cdot)$. It was given without proof in [9], so we will provide a quick proof here.

Lemma 1. *The function $\nu(\cdot)$ is multiplicative with*

$$\nu(p^\alpha) = \begin{cases} 1 + \chi(p) & \text{if } p \nmid d \text{ or } \alpha = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The multiplicativity of $\nu(\cdot)$ follows easily from the unique factorization property of the ideals of $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$. As for the formula for $\nu(p^\alpha)$, we use the characterization of prime ideals in $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$:

- If $\chi(p) = 0$ and $\alpha = 1$, then the only ideal with norm p is the ideal \mathfrak{p} with $(p) = \mathfrak{p}^2$.
- If $\chi(p) = 1$, then we have the factorization $(p) = \mathfrak{p}_1 \mathfrak{p}_2$ with $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$. Hence, we have $(p^\alpha) = \mathfrak{p}_1^\alpha \mathfrak{p}_2^\alpha$. Thus, the only ideals with norm p^α that are not divisible by rational integers are \mathfrak{p}_1^α and \mathfrak{p}_2^α , since any other choice will have to be divisible by both \mathfrak{p}_1 and \mathfrak{p}_2 , i.e., divisible by (p) .
- If $\chi(p) = -1$, then (p) is a prime ideal with norm p^2 . Hence, there are no ideals with norm p^α if α is odd. But if α is even, then any ideal with norm p^α will be divisible by (p) .

The only remaining case is when $\chi(p) = 0$ and $\alpha \geq 2$. However, since $-d$ is a fundamental discriminant, the only possibility for this to happen is for d to be divisible by 4, $p = 2$, and $\alpha = 2$ or 3. But again, in this case, the only ideal with norm 2^α is the ideal \mathfrak{p}^α , where $\mathfrak{p}^2 = (2)$. Since $\alpha \geq 2$, such an ideal is divisible by (2) . \square

When studying sums over norms of ideals like (9), it is often useful to consider the Dedekind zeta function for $\mathbb{Q}(\sqrt{-d})$. Let

$$\zeta_{-d}(s) := \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where \mathfrak{a} runs over all nonzero ideals of $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ and $\text{Re}(s) > 1$. Lemma 1 implies that

$$\zeta_{-d}(s) = \zeta(s)L(s, \chi), \quad (10)$$

which immediately provides an analytic continuation for $\zeta_{-d}(s)$. Equation (10) is well known, but also follows easily from Lemma 1. Indeed, for $\text{Re}(s) > 1$, we have

$$\begin{aligned} \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} &= \sum_{u^2 \mathfrak{a}} \frac{\nu(\mathfrak{a})}{u^{2s} a^s} \\ &= \sum_{u^2} \frac{1}{u^{2s}} \sum_{\mathfrak{a}} \frac{\nu(\mathfrak{a})}{a^s} \\ &= \prod_p (1 - p^{-2s})^{-1} \prod_{p, \chi(p)=0} (1 + p^{-s}) \prod_{p, \chi(p)=1} \left(1 + 2 \frac{p^{-s}}{1 - p^{-s}}\right) \\ &= \prod_p (1 - p^{-s})^{-1} \prod_{p, \chi(p)=-1} (1 + p^{-s})^{-1} \prod_{p, \chi(p)=1} (1 - p^{-s})^{-1} \\ &= \zeta(s)L(s, \chi). \end{aligned}$$

Every ideal $[a, \frac{b+\sqrt{-d}}{2}]$ corresponds to a binary quadratic form $ax^2 + bxy + cy^2$, where $d = 4ac - b^2$. Such a form is called reduced if $-a < b \leq a < c$ or $0 \leq b \leq a = c$. The number of reduced forms is known as the class number of $\mathbb{Q}(\sqrt{-d})$, and we denote it by $h(-d)$. Watkins in [16] gives all negative fundamental discriminant with class

number less or equal to 100. The largest absolute value of such discriminants is 2383747 (whose class number is 98). Moreover, it is shown in another Watkins' paper [17] that for $d \leq 300000000$, the function $L(s, \chi)$ does not have positive real zeros. Hence, we can assume from now on that $d > 300000000$, and so

$$h(-d) \geq 101. \quad (11)$$

Lemma 2. *Let $h(-d)$ be the class number of a quadratic field of discriminant $-d$ with $d > 300000000$. Then, we have*

$$\sum_{a \leq \frac{1}{2}\sqrt{d}} \frac{\nu(a)}{a} \leq \frac{h(-d)}{11}. \quad (12)$$

Proof. Notice first that for an ideal $[a, \frac{b+\sqrt{-d}}{2}]$ with norm $a \leq \frac{1}{2}\sqrt{d}$, the corresponding quadratic form $ax^2 + bxy + cy^2$ is reduced. To see this, note that for $d > 4$, equality cannot hold for $a \leq \frac{1}{2}\sqrt{d}$ since, otherwise, $d/4$ would not be squarefree. So

$$4a^2 < d = 4ac - b^2 \leq 4ac,$$

which yields $a < c$. The above observation implies that each ideal class of $\mathbb{Q}(\sqrt{-d})$ contains at most one ideal of the form $[a, \frac{b+\sqrt{-d}}{2}]$ with norm $a \leq \frac{1}{2}\sqrt{d}$, and in particular, we have

$$\sum_{a \leq \frac{1}{2}\sqrt{d}} \nu(a) \leq h(-d).$$

On the other hand, using Lemma 1, we can show that $\nu(a) \leq 2^{w(a)}$, where $w(n)$ denotes the number of distinct prime divisors of n , with $w(1) = 0$. Hence, we have

$$\sum_{a \leq \frac{1}{2}\sqrt{d}} \frac{\nu(a)}{a} \leq \sum_{a \leq \frac{1}{2}\sqrt{d}} \frac{2^{w(a)}}{a}.$$

One can simply verify with a calculator that $\sum_{n=1}^{34} 2^{w(n)} = 101$. This implies that there can only be at most 101 ideals of the form $[a, \frac{b+\sqrt{-d}}{2}]$ with norm a less or equal to 34. But since in our case, the class number $h(-d)$ is at least 101, we may write

$$\sum_{a \leq \frac{1}{2}\sqrt{d}} \frac{\nu(a)}{a} \leq \sum_{n=1}^{34} \frac{2^{w(n)}}{n} + \frac{h(-d) - 101}{35}.$$

This is because the sum is larger if more small numbers a are represented as norms of ideals. So in the above, 101 ideals have norms from 1 to 34 and the norms of the rest must be at least 35. Hence, by evaluating the sum on the right in the above, we obtain

$$\sum_{a \leq \frac{1}{2}\sqrt{d}} \frac{\nu(a)}{a} \leq 9.161 + \frac{h(-d) - 101}{35} \leq \frac{h(-d)}{11},$$

for $h(-d) \geq 101$. □

3 Proof of Theorem 1

The proof relies on estimates of sums of the form (9) when x is slightly larger than $\frac{1}{2}\sqrt{d}$. To make this precise, we consider an auxiliary function $f(d) \geq 1$, to be specified later. We set

$$x = \frac{1}{2}\sqrt{d}f(d).$$

From now on, we may assume that there exists $\beta > 0$ such that $L(\beta, \chi) = 0$ and that

$$1 - \beta \leq \frac{6.5}{\sqrt{d}}, \quad (13)$$

for otherwise, there will be nothing to prove. Then, we define the integral

$$I := \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta_{-d}(s+\beta) \frac{x^s}{s(s+2)(s+3)} ds. \quad (14)$$

As we can see in the next lemma, this integral allows us to estimate the sum of reciprocal norms of ideals that we mentioned in the previous section.

Lemma 3. *We have*

$$I \leq \frac{1}{6} x^{1-\beta} \sum_{N(\mathfrak{a}) \leq x} \frac{1}{N(\mathfrak{a})}.$$

Before proving this lemma, let us first recall Perron's Formula, see [1, p.243] for example: if y is any positive real number and $c > 0$, then we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds = \begin{cases} 1 & \text{if } y > 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1, \end{cases} \quad (15)$$

where by $\int_{c-i\infty}^{c+i\infty}$, we mean $\lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT}$.

Proof of Lemma 3. We begin by the following partial fraction decomposition:

$$\frac{1}{s(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}. \quad (16)$$

Hence, by (15) we obtain

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{y^s}{s(s+2)(s+3)} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{6} - \frac{y^{-2}}{2} + \frac{y^{-3}}{3} & \text{if } y \geq 1. \end{cases} \quad (17)$$

Since

$$\zeta_{-d}(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \sum_{u^2 a} \frac{\nu(a)}{u^{2s} a^s}$$

converges absolutely for $\text{Re}(s) > 1$, we have

$$\begin{aligned} I &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^{s+\beta}} \frac{x^s}{s(s+2)(s+3)} ds \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^\beta} \left(\frac{x}{N(\mathfrak{a})} \right)^s \frac{1}{s(s+2)(s+3)} ds. \end{aligned}$$

Swapping summation and integration and using (17) (setting $y = \frac{x}{N(\mathfrak{a})}$) yield

$$\begin{aligned} I &= \sum_{N(\mathfrak{a}) \leq x} \frac{1}{N(\mathfrak{a})^\beta} \left[\frac{1}{6} - \frac{N(\mathfrak{a})^2}{2x^2} + \frac{N(\mathfrak{a})^3}{3x^3} \right] \\ &\leq \frac{1}{6} \sum_{N(\mathfrak{a}) \leq x} \frac{1}{N(\mathfrak{a})^\beta} \quad \left(\text{since } \frac{1}{6} - \frac{1}{2y^2} + \frac{1}{3y^3} \leq \frac{1}{6} \text{ for any } y \geq 1 \right) \\ &\leq \frac{x^{1-\beta}}{6} \sum_{N(\mathfrak{a}) \leq x} \frac{1}{N(\mathfrak{a})}, \end{aligned}$$

which complete the proof of the lemma. \square

3.1 Lower bound on I

By shifting the path of integration of the integral I to $\operatorname{Re}(s) = -\beta$, Equation (14) can now be written as

$$I = \frac{L(1, \chi)x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)} + \frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} \zeta(s+\beta)L(s+\beta, \chi) \frac{x^s}{s(s+2)(s+3)} ds, \quad (18)$$

where the first term on right-hand side comes from the simple pole of the integrand at $s = 1 - \beta$. Note that $s = 0$ is also a singularity but it is removable since we assumed that $L(\beta, \chi) = 0$. Let us denote the integral on the right-hand side of (18) by J , i.e.,

$$J := \frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} \zeta(s+\beta)L(s+\beta, \chi) \frac{x^s}{s(s+2)(s+3)} ds.$$

Then, one has

$$|J| \leq \frac{x^{-\beta}}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta(it)||L(it, \chi)|}{\sqrt{(\beta^2 + t^2)((2-\beta)^2 + t^2)((3-\beta)^2 + t^2)}} dt.$$

On the other hand, using our assumptions (13) and $d > 300000000$, we deduce that

$$\beta^2 > \left(1 - 6.5/\sqrt{300000000}\right)^2 > 0.9996^2 > 0.999.$$

Therefore,

$$|J| < \frac{x^{-\beta}}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta(it)||L(it, \chi)|}{\sqrt{(0.999 + t^2)(1 + t^2)(4 + t^2)}} dt. \quad (19)$$

In order to find an upper bound for the above integral we need to obtain explicit bounds for $|\zeta(it)|$ and $|L(it, \chi)|$. The following explicit result can be found in [14]: for $|t| \geq 3$,

$$|\zeta(1 + it)| \leq \frac{3}{4} \log |t|. \quad (20)$$

Similarly, for $|L(it, \chi)|$, Dudek in [5] obtained

$$|L(1 + it, \chi)| \leq \log d + \log(e(|t| + 14/5)). \quad (21)$$

Using (20), (21), and the functional equations for the respective functions, we obtain the following lemma.

Lemma 4. *For any real number t such that $|t| \geq 3$, we have*

$$|\zeta(it)| \leq \frac{3}{\sqrt{32\pi}} \sqrt{|t|} \log |t|, \quad \text{and}$$

$$|L(it, \chi)| \leq 0.4 \sqrt{d|t|} (\log d + \log(e(|t| + 14/5))).$$

Proof. The functional equation of the Riemann zeta function gives

$$|\zeta(it)| = \pi^{-1} \sinh(\pi|t|/2) |\Gamma(1 - it)| |\zeta(1 - it)|.$$

Since

$$\begin{aligned} \sinh(\pi|t|/2) |\Gamma(1 - it)| &= \sqrt{\frac{\pi}{2} |t| \tanh(\pi|t|/2)} \\ &= \sqrt{\frac{\pi}{2} |t| \left(1 - \frac{2}{e^{\pi|t|} + 1}\right)} \\ &\leq \sqrt{\frac{\pi}{2} |t|}, \end{aligned}$$

we deduce from (20) that for $|t| \geq 3$ we have

$$|\zeta(it)| \leq \frac{3}{\sqrt{32\pi}} \sqrt{|t|} \log |t|.$$

Similarly the functional equation for $L(s, \chi)$ is as follows: if

$$\Lambda(s, \chi) = \left(\frac{\pi}{d}\right)^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi),$$

then

$$\Lambda(1-s, \chi) = \frac{ik^{1/2}}{\tau(\chi)} \Lambda(s, \chi),$$

where $\tau(\chi) = \sum_{k=1}^d \chi(k) \exp(2\pi ik/d)$ (here χ is real and $\chi(-1) = -1$). Using the fact that $|\tau(\chi)| = d^{1/2}$ and replacing s by it yield

$$\left(\frac{\pi}{d}\right)^{-1} \left| \Gamma\left(\frac{2-it}{2}\right) \right| |L(1-it, \chi)| = \left(\frac{\pi}{d}\right)^{-1/2} \left| \Gamma\left(\frac{1+it}{2}\right) \right| |L(it, \chi)|.$$

Hence

$$|L(it, \chi)| = \left(\frac{d}{\pi}\right)^{1/2} \left| \Gamma\left(\frac{2-it}{2}\right) \right| \left| \Gamma\left(\frac{1+it}{2}\right) \right|^{-1} |L(1-it, \chi)|.$$

Moreover, we have

$$\left| \Gamma\left(\frac{2-it}{2}\right) \right| = \sqrt{\frac{\pi|t|}{2 \sinh(\pi|t|/2)}} \quad \text{and} \quad \left| \Gamma\left(\frac{1+it}{2}\right) \right| = \sqrt{\frac{\pi}{\cosh(\pi|t|/2)}}.$$

Thus,

$$\left| \Gamma\left(\frac{2-it}{2}\right) \right| \left| \Gamma\left(\frac{1+it}{2}\right) \right|^{-1} = \sqrt{\frac{|t|}{2} \coth(\pi|t|/2)}.$$

Therefore, we deduce that

$$|L(it, \chi)| = \left(\frac{d}{\pi}\right)^{1/2} \sqrt{\frac{1}{2}|t| \coth(\pi|t|/2)} |L(1-it, \chi)|. \quad (22)$$

If $|t| \geq 3$, then $e^{\pi|t|} \geq e^{3\pi} > 12391$, so

$$\sqrt{\frac{1}{2}|t| \coth(\pi|t|/2)} = \sqrt{\frac{|t|}{2} \left(1 + \frac{2}{e^{\pi|t|}-1}\right)} < \sqrt{\frac{1}{2}|t| \left(1 + \frac{2}{12390}\right)} < 0.708\sqrt{|t|}.$$

Thus for $|t| \geq 3$, we have

$$|L(it, \chi)| < \frac{0.708}{\sqrt{\pi}} \sqrt{d|t|} |L(1-it, \chi)| < 0.4\sqrt{d|t|} |L(1-it, \chi)|.$$

The proof of the lemma is complete by using (21) to estimate the right-hand side. \square

Another consequence of the calculations in the proof above is that we also have

$$|\zeta(it)L(it, \chi)| = \frac{\sqrt{d}}{2\pi} |t\zeta(1-it)L(1-it, \chi)| \quad \text{for } t \in \mathbb{R} \setminus \{0\}. \quad (23)$$

In view of (19), we consider the following integrals:

$$\begin{aligned}
J_1 &:= \frac{1}{2\pi} \int_{-3}^3 \frac{|t\zeta(1-it)|}{\sqrt{(0.999+t^2)((1+t^2)(4+t^2))}} dt, \\
J_2 &:= \frac{1}{2\pi} \int_{-3}^3 \frac{|t\zeta(1-it)| \log(e(|t|+14/5))}{\sqrt{(0.999+t^2)((1+t^2)(4+t^2))}} dt, \\
J_3 &:= \frac{0.6}{\sqrt{2\pi}} \int_3^\infty \frac{t \log t}{\sqrt{(0.999+t^2)((1+t^2)(4+t^2))}} dt, \\
J_4 &:= \frac{0.6}{\sqrt{2\pi}} \int_3^\infty \frac{t \log t \log(e(t+14/5))}{\sqrt{(0.999+t^2)((1+t^2)(4+t^2))}} dt.
\end{aligned}$$

By (19), (23) and Lemma 4, we have

$$|J| \leq \frac{x^{-\beta}}{2\pi} \sqrt{d} \left((J_1 + J_3) \log d + J_2 + J_4 \right). \quad (24)$$

On the other hand, we can use a computer algebra system such as SageMath or Mathematica to calculate the J_i 's numerically. We obtained the following numerical values (with high accuracy)

$$\begin{aligned}
J_1 &= 0.19692\dots, \\
J_2 &= 0.45203\dots, \\
J_3 &= 0.15661\dots, \\
J_4 &= 0.61360\dots
\end{aligned}$$

Rounding this values up at the 3rd digit, and using (24), we have

$$|J| \leq \frac{x^{-\beta}}{2\pi} \sqrt{d} \left(0.354 \log d + 1.067 \right) < \frac{x^{-\beta}}{2\pi} \left(0.354 + \frac{1.067}{\log d} \right) \sqrt{d} \log d.$$

Thus using $d \geq 300000000$ to estimate the term in brackets, we deduce that

$$|J| < 0.066 x^{-\beta} \sqrt{d} \log d. \quad (25)$$

Returning to the integral I . Recall from Equation (18) that we have

$$I = \frac{L(1, \chi) x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)} + J.$$

Hence, using the estimate (25) for J that we just achieved, we get

$$I \geq \frac{x^{1-\beta}}{(1-\beta)} \left(\frac{L(1, \chi)}{(3-\beta)(4-\beta)} - 0.066 (1-\beta) \frac{\sqrt{d} \log d}{x} \right).$$

Since $x = \frac{1}{2} \sqrt{d} f(d)$, we deduce that

$$I \geq \frac{x^{1-\beta}}{(1-\beta)} \left(\frac{L(1, \chi)}{(3-\beta)(4-\beta)} - 0.132 (1-\beta) \frac{\log d}{f(d)} \right).$$

In addition, by the class number formula for $d > 4$, we have $L(1, \chi) = \frac{\pi h(-d)}{\sqrt{d}}$. So we finally get a lower estimate of I

$$I \geq \frac{x^{1-\beta}}{(1-\beta)\sqrt{d}} \left(\frac{\pi h(-d)}{(3-\beta)(4-\beta)} - 0.132 \frac{(1-\beta)\sqrt{d} \log d}{f(d)} \right). \quad (26)$$

3.2 Upper bound on I

We recall the bound from Lemma 3,

$$I \leq \frac{1}{6} x^{1-\beta} \sum_{N(\mathbf{a}) \leq x} N(\mathbf{a})^{-1}.$$

where $x = \frac{1}{2}\sqrt{d}f(d)$ and $f(d) \geq 1$ a function of d to be chosen later. Here we aim to estimate the sum on the right-hand side. For this, we choose another auxiliary function $\ell(d)$ that satisfies $f(d) \leq \ell(d)$. We have

$$\sum_{N(\mathbf{a}) \leq x} N(\mathbf{a})^{-1} = \sum_{u^2 a \leq x} \frac{\nu(a)}{u^2 a} \leq \frac{\pi^2}{6} \sum_{a \leq x} \frac{\nu(a)}{a}. \quad (27)$$

We split the last sum into two parts

$$\sum_{a \leq x} \frac{\nu(a)}{a} = \sum_{a \leq \frac{1}{2}\sqrt{d}} \frac{\nu(a)}{a} + \sum_{\frac{1}{2}\sqrt{d} < a \leq x} \frac{\nu(a)}{a} =: S_0 + S_1.$$

Furthermore, we also split S_1 ,

$$S_1 = \sum' \frac{\nu(a)}{a} + \sum'' \frac{\nu(a)}{a},$$

where $\sum' \frac{\nu(a)}{a}$ denotes the sum over all a with $\frac{1}{2}\sqrt{d} < a \leq x$ such that a has a prime divisor $p^\alpha > \ell(d)$. Hence, Lemma 1 yields

$$\sum' \frac{\nu(a)}{a} = \sum' \frac{\nu(p^\alpha b)}{p^\alpha b} \leq \sum_{b < x/\ell(d)} \frac{\nu(b)}{b} \sum_{\Delta(b) < p^\alpha \leq x/b} (2p^{-\alpha}), \quad (28)$$

where

$$\Delta(b) := \max \left\{ \ell(d), \frac{1}{2b}\sqrt{d} \right\}.$$

Recalling our notation from Section 2 that

$$\sum_{p^\alpha \leq y} p^{-\alpha} = \log \log y + B_2 + \varepsilon(y).$$

So

$$\begin{aligned} \sum_{\Delta(b) < p^\alpha \leq x/b} (2p^{-\alpha}) &= 2 \log \left(\frac{\log(x/b)}{\log \Delta(b)} \right) + 2 \left(\varepsilon(x/b) - \varepsilon(\Delta(b)) \right) \\ &= 2 \log \left(\frac{\log(f(d)) + \log(\frac{1}{2b}\sqrt{d})}{\log \Delta(b)} \right) + 2 \left(\varepsilon(x/b) - \varepsilon(\Delta(b)) \right) \\ &\leq 2 \log \left(1 + \frac{\log f(d)}{\log \Delta(b)} \right) + 2 \left(\max_{y \geq \ell(d)} \varepsilon(y) - \min_{y \geq \ell(d)} \varepsilon(y) \right) \\ &\leq 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)} \right) + 2 \left(\max_{y \geq \ell(d)} \varepsilon(y) - \min_{y \geq \ell(d)} \varepsilon(y) \right). \end{aligned}$$

Moreover, since $f(d) \leq \ell(d)$, we have $x/\ell(d) \leq \frac{1}{2}\sqrt{d}$. Thus

$$\sum_{b < x/\ell(d)} \frac{\nu(b)}{b} \leq S_0.$$

Therefore, we deduce from (28), that

$$\sum' \frac{\nu(a)}{a} \leq S_0 \left(2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)} \right) + 2 \left(\max_{y \geq \ell(d)} \varepsilon(y) - \min_{y \geq \ell(d)} \varepsilon(y) \right) \right). \quad (29)$$

As for the sum $\sum'' \frac{\nu(a)}{a}$, each positive integer a contributing to this sum has no prime power divisor $> \ell(d)$. So the number of distinct prime divisors of such an a is at least

$$k_0 := \left\lceil \frac{\log(\frac{1}{2}\sqrt{d})}{\log \ell(d)} \right\rceil.$$

The latter and the multiplicative property of the function $\nu(a)$ imply that

$$\begin{aligned} \sum'' \frac{\nu(a)}{a} &\leq \sum_{k \geq k_0} \frac{1}{k!} \left(\sum_{p^\alpha \leq \ell(d)} \frac{\nu(p^\alpha)}{p^\alpha} \right)^k \\ &\leq \frac{\sigma^{k_0}}{k_0!} \left(1 + \frac{\sigma}{(k_0+1)} + \frac{\sigma^2}{(k_0+1)(k_0+2)} + \cdots \right), \end{aligned}$$

where

$$\sigma := \sum_{p^\alpha \leq \ell(d)} \frac{2}{p^\alpha}.$$

If we choose $\ell(d)$ in such a way $k_0 + 1 > \sigma$, then

$$1 + \frac{\sigma}{(k_0+1)} + \frac{\sigma^2}{(k_0+1)(k_0+2)} + \cdots \leq 1 + \frac{\sigma}{(k_0+1)} + \frac{\sigma^2}{(k_0+1)^2} + \cdots = \frac{1+k_0}{1+k_0-\sigma}.$$

Hence, we obtain

$$\sum'' \frac{\nu(a)}{a} \leq \frac{(1+k_0) \sigma^{k_0}}{(1+k_0-\sigma) k_0!}. \quad (30)$$

Putting everything together, and using the result in Lemma 2 that $S_0 \leq \frac{h(-d)}{11}$, we arrive at the following estimate

$$\sum_{N(\mathfrak{a}) \leq x} N(\mathfrak{a})^{-1} \leq \frac{\pi^2}{66} h(-d) \left(1 + 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)} \right) + \text{Er}(d, \ell(d)) \right), \quad (31)$$

where

$$\text{Er}(d, \ell(d)) := 2 \left(\max_{y \geq \ell(d)} \varepsilon(y) - \min_{y \geq \ell(d)} \varepsilon(y) \right) + \frac{11(1+k_0)\sigma^{k_0}}{(1+k_0-\sigma)h(-d)k_0!}.$$

This expression is not easy work with so let us derive a simpler bound for it. From Proposition 1, we know that

$$\max_{y \geq \ell(d)} \varepsilon(y) - \min_{y \geq \ell(d)} \varepsilon(y) \leq \frac{1.75}{(\log \ell(d))^2} + \min \left\{ \frac{0.2}{(\log \ell(d))^3}, 10^{-4} \right\}.$$

If $\frac{0.2}{(\log \ell(d))^3} < 10^{-4}$, then $\log \ell(d) > 12$, so

$$\frac{1.75}{(\log \ell(d))^2} + \frac{0.2}{(\log \ell(d))^3} \leq \frac{1}{(\log \ell(d))^2} \left(1.75 + \frac{0.2}{\log \ell(d)} \right) < \frac{1.8}{(\log \ell(d))^2}.$$

On the other hand if $\frac{0.2}{(\log \ell(d))^3} \geq 10^{-4}$, then $\log \ell(d) < 13$, so

$$\frac{1.75}{(\log \ell(d))^2} + 10^{-4} \leq \frac{1}{(\log \ell(d))^2} \left(1.75 + \frac{(\log \ell(d))^2}{10000} \right) < \frac{1.8}{(\log \ell(d))^2}.$$

Therefore, we get

$$2 \left(\max_{y \geq \ell(d)} \varepsilon(y) - \min_{y \geq \ell(d)} \varepsilon(y) \right) < \frac{3.6}{(\log \ell(d))^2}. \quad (32)$$

On the other hand, by Stirling's formula, we have

$$k_0! \geq \sqrt{2\pi k_0} \left(\frac{k_0}{e}\right)^{k_0}.$$

Which implies that

$$\frac{\sigma^{k_0}}{k_0!} \leq \frac{1}{\sqrt{2\pi k_0}} \left(\frac{e\sigma}{k_0}\right)^{k_0}.$$

All these together with $h(-d) \geq 101$, we obtain

$$E(d, \ell(d)) \leq \frac{3.6}{(\log \ell(d))^2} + \frac{11(1+k_0)}{101(1+k_0-\sigma)} \frac{1}{\sqrt{2\pi k_0}} \left(\frac{e\sigma}{k_0}\right)^{k_0}. \quad (33)$$

3.3 Final steps

The purpose here is to choose suitable values for $f(d)$ and $\ell(d)$, but before we do that, let us first list all the constraints (on $f(d)$ and $\ell(d)$) that we assumed earlier. For $d > 300000000$, we require

- $f(d) \geq 1$,
- $f(d) \leq \ell(d)$, and
- $k_0 + 1 > \sigma$ (both sides of the inequality depend on $\ell(d)$).

Case $\log(d) \leq 42$

We choose $f(d) = 1$ ($\ell(d)$ will not be needed here), then (27), Lemma 2 and Lemma 3 yield

$$I \leq \frac{1}{6} x^{1-\beta} \sum_{N(\mathfrak{a}) \leq \frac{1}{2}\sqrt{d}} N(\mathfrak{a})^{-1} \leq \frac{\pi^2}{396} x^{1-\beta} h(-d).$$

This and the lower bound of I in (26) imply

$$\frac{1}{(1-\beta)\sqrt{d}} \left(\frac{\pi}{(3-\beta)(4-\beta)} - 0.132 \frac{(1-\beta)\sqrt{d} \log d}{h(-d)} \right) \leq \frac{\pi^2}{396}.$$

Recall our assumption in (13) that $1-\beta \leq \frac{6.5}{\sqrt{d}}$. Since $d > 300000000$, the latter implies that $\beta > 0.999$, so

$$\frac{\pi}{(3-\beta)(4-\beta)} > 0.523.$$

Using $h(-d) \geq 101$, and the assumption (13) again (to estimate the term $(1-\beta)\sqrt{d}$ inside the brackets above), we obtain

$$\begin{aligned} (1-\beta)\sqrt{d} &> \frac{396}{\pi^2} \left(0.523 - 0.132 \frac{6.5 \log d}{101} \right) \\ &> 20.984 - 0.341 \log d. \end{aligned}$$

The latter is greater than 6.6 (if $\log(d) \leq 42$), contradicting (13).

Case $42 < \log(d) \leq 100$

Here, we choose $f(d) = \ell(d) = 16$. The combination (26), (31) and Lemma 3 gives

$$\begin{aligned} &\frac{1}{(1-\beta)\sqrt{d}} \left(\frac{\pi}{(3-\beta)(4-\beta)} - 0.132 \frac{(1-\beta)\sqrt{d} \log d}{101f(d)} \right) \\ &\leq \frac{\pi^2}{396} \left(1 + 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)} \right) + \text{Er}(d, \ell(d)) \right). \end{aligned}$$

This implies that

$$(1 - \beta)\sqrt{d} > \frac{20.984 - 0.341 \frac{\log d}{f(d)}}{1 + 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)}\right) + \text{Er}(d, \ell(d))}.$$

The numerator is obtained in the same way as in the previous case. Since $\ell(d) = 16$, we have

$$\sigma = 2 \sum_{p^\alpha \leq 16} p^{-\alpha} < 3.786.$$

Moreover, from (33), we get

$$\begin{aligned} E(d, \ell(d)) &\leq \frac{3.6}{(\log 16)^2} + \frac{11(1+k_0)}{101(1+k_0-3.786)} \frac{1}{\sqrt{2\pi k_0}} \left(\frac{10.3}{k_0}\right)^{k_0} \\ &< 0.469 + 0.044 \frac{(1+k_0)}{(1+k_0-3.786)} \frac{1}{\sqrt{k_0}} \left(\frac{10.3}{k_0}\right)^{k_0}. \end{aligned}$$

Similarly, since $f(d) = \ell(d) = 16$,

$$20.984 - 0.341 \frac{\log d}{f(d)} \geq 20.984 - 0.022 \log d,$$

and

$$1 + 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)}\right) = 1 + 2 \log(2) < 2.387.$$

Thus, we finally obtain

$$(1 - \beta)\sqrt{d} > \frac{20.984 - 0.022 \log d}{2.856 + 0.044 \frac{(1+k_0)}{(1+k_0-3.786)} \frac{1}{\sqrt{k_0}} \left(\frac{10.3}{k_0}\right)^{k_0}} \quad (34)$$

where, here

$$k_0 = \left\lceil \frac{\log d - \log 4}{2 \log 16} \right\rceil.$$

The right-hand side of (34) is still difficult to estimate manually, so we did this numerically and the result is shown in Figure 2. The corners in the graph correspond to the points where the value of k_0 changes from an integer to the next. The minimum occurs in the first corner where k_0 changes from 8 to 9 i.e. when $\log d$ is close to $16 \log 16 - \log 4 \approx 45.747$. At this point we still have $(1 - \beta)\sqrt{d} > 6.53$ (when $k_0 \geq 11$ the corners become less apparent because the second term in the denominator contributes very little). So, it is clear that we also obtain $(1 - \beta)\sqrt{d} > 6.5$ for all d such that $42 < \log(d) \leq 100$, which contradicts (13).

Case $\log d > 100$

Just as in the previous case, we also have the bound

$$(1 - \beta)\sqrt{d} > \frac{20.984 - 0.341 \frac{\log d}{f(d)}}{1 + 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)}\right) + \text{Er}(d, \ell(d))}, \quad (35)$$

but we choose $f(d) = \ell(d) = 0.5 \log(\frac{1}{2}\sqrt{d})$, which we simply abbreviate as t to make the reading easier. So from here, we will write everything in terms of t . The condition $\log d > 100$ implies that $t > 24.65$. Let us begin by estimating the terms in the denominator of (35). We have

$$k_0 = \left\lceil \frac{2t}{\log t} \right\rceil \geq \frac{2t}{\log t},$$

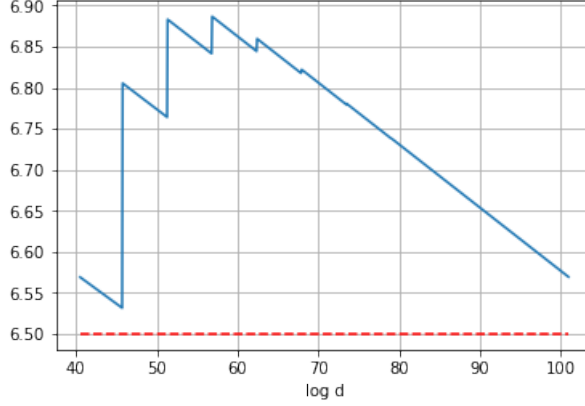


Figure 2: Numerical plot for the case $42 < \log(d) \leq 100$

and since the right-hand side is at least 15.3, we may assume that $k_0 \geq 16$. By Proposition 1, we have

$$\sigma = 2 \sum_{p^\alpha \leq t} p^{-\alpha} \leq 2 \log \log t + 2.07.$$

Thus, we deduce that

$$\frac{e\sigma}{k_0} \leq \frac{e \log t (2 \log \log t + 2.07)}{2t}$$

It is easy to verify that the term on the right-hand side is a decreasing function of t when $t > 24.65$. Hence, we get

$$\frac{e\sigma}{k_0} < \frac{e \log(24.65) (2 \log \log(24.65) + 2.07)}{49.3} < 0.778.$$

In particular, we verified that $1 + k_0 > \sigma$. We also have

$$\frac{1 + k_0}{1 + k_0 - \sigma} = \frac{1}{1 - \frac{\sigma}{1+k_0}} < \frac{1}{1 - \frac{0.778}{e}} < 1.401.$$

Using these numerical estimates and the fact that $k_0 \geq 16$, we obtain

$$\frac{11(1+k_0)}{101(1+k_0-\sigma)} \frac{1}{\sqrt{2\pi k_0}} \left(\frac{e\sigma}{k_0}\right)^{k_0} < 0.016 (0.778)^{16} < 0.0003. \quad (36)$$

We can see that the contribution from this term is very small. Let us now look the remaining terms in the denominator of the right-hand side of (35). Since we have chosen $f(d) = \ell(d) = t$, we have

$$1 + 2 \log \left(1 + \frac{\log f(d)}{\log \ell(d)}\right) + \frac{3.6}{(\log \ell(d))^2} = 1 + 2 \log 2 + \frac{3.6}{(\log t)^2} < 2.737. \quad (37)$$

The numerical value is obtain by rounding up the value at $t = 24.65$. Combining (36) and (37), we finally get

$$(1 - \beta)\sqrt{d} > \frac{20.984 - 0.341 \frac{\log d}{t}}{2.738} > 7.663 - 0.125 \frac{4t + \log 4}{t} > 7$$

for $t > 24.65$. Once again, we obtain a constant strictly greater than 6.5 which bounds $(1 - \beta)\sqrt{d}$ from below. Since we have shown that this is the case for all possible values of $\log d \geq \log(300000000)$ the proof of Theorem 1 is complete.

4 Proof of Theorem 2

We begin by the following consequence Theorem 1 in Goldfeld-Schinzel [9] : if β exists, then

$$1 - \beta \geq \left(\frac{6}{\pi^2} + o(1) \right) \frac{L(1, \chi)}{\sum_{a \leq \frac{1}{4}\sqrt{d}} \frac{\nu(a)}{a}} \quad \text{as } d \rightarrow \infty. \quad (38)$$

For now, we need know how to estimate sums of the form $\sum_{a \leq x} \frac{\nu(a)}{a}$. We start with the following observation which we already used in the proof Lemma 2: for any $1 \leq x \leq \frac{1}{2}\sqrt{d}$, we have

$$\sum_{a \leq x} \frac{\nu(a)}{a} \leq \sum_{a \leq y} \frac{2^{w(a)}}{a} \quad \text{whenever } \sum_{a \leq y} 2^{w(a)} \geq h(-d). \quad (39)$$

The next lemma gives asymptotic estimates of the sums involving $2^{w(a)}$.

Lemma 5. *As $y \rightarrow \infty$, we have*

$$\sum_{n \leq y} 2^{w(n)} = \frac{6}{\pi^2} y \log y + O(y) \quad \text{and} \quad \sum_{n \leq y} \frac{2^{w(n)}}{n} = \frac{3}{\pi^2} (\log y)^2 + O(\log y).$$

Proof. For each $n \geq 1$, the number $2^{w(n)}$ is equal to the number of squarefree divisors of n , i.e.,

$$2^{w(n)} = \sum_{d|n} |\mu(d)|.$$

Therefore,

$$\begin{aligned} \sum_{n \leq y} 2^{w(n)} &= \sum_{n \leq y} \sum_{d|n} |\mu(d)| \\ &= \sum_{d \leq y} |\mu(d)| \sum_{q \leq \frac{y}{d}} 1 \\ &= y \sum_{d \leq y} \frac{|\mu(d)|}{d} + O(y). \end{aligned} \quad (40)$$

To estimate the sum in the last line, we use a well known estimate for the counting function of squarefree integers

$$\sum_{n \leq y} |\mu(n)| = \frac{6}{\pi^2} y + O(\sqrt{y}). \quad (41)$$

This is not too difficult to prove, we can even find a proof with an explicit error term in [3]. Hence, applying Abel's identity, we have

$$\sum_{n \leq y} \frac{|\mu(n)|}{n} = \frac{1}{y} \sum_{n \leq y} |\mu(n)| + \int_1^y \frac{1}{t^2} \left(\sum_{n \leq t} |\mu(n)| \right) dt. \quad (42)$$

The first term is obviously bounded, and the second can be estimated using (41). Thus

$$\sum_{n \leq y} \frac{|\mu(n)|}{n} = \frac{6}{\pi^2} \log y + O(1). \quad (43)$$

The estimate of $\sum_{n \leq y} 2^{w(n)}$ in the lemma now follows from (40). For the second estimate in the lemma, we use Abel's identity again

$$\sum_{n \leq y} \frac{2^{w(n)}}{n} = \frac{1}{y} \sum_{n \leq y} 2^{w(n)} + \int_1^y \frac{1}{t^2} \left(\sum_{n \leq t} 2^{w(n)} \right) dt.$$

Then, we use the first estimate in the lemma, that we just proved, to estimate both terms on the right-hand side, and we obtain

$$\sum_{n \leq y} \frac{2^{w(n)}}{n} = \frac{3}{\pi^2} (\log y)^2 + O(\log y),$$

which completes the proof of the lemma. \square

One can find explicit upper bounds of the sums in Lemma 5 in [15, Lemma 12]. Lower bounds can also be achieved using the same proof provided in that paper. We are now ready to prove the asymptotic formula in Theorem 2.

Proof of Theorem 2. We choose a positive number $y = y(d)$ in such a way that

$$\sum_{a \leq y-1} 2^{w(a)} < h(-d) \leq \sum_{a \leq y} 2^{w(a)},$$

then, by the first estimate in Lemma 5, we have $h(-d) = \frac{6}{\pi^2} y(\log y) + O(y)$. Thus, writing y in terms of $h(-d)$, we obtain

$$y = \left(\frac{\pi^2}{6} + O\left(\frac{1}{\log h(-d)} \right) \right) \frac{h(-d)}{\log h(-d)}. \quad (44)$$

Similarly, using the second estimate in Lemma 5, we have

$$\sum_{a \leq \frac{1}{4}\sqrt{d}} \frac{\nu(a)}{a} \leq \sum_{a \leq y} \frac{2^{w(a)}}{a} = \frac{3}{\pi^2} (\log y)^2 + O(\log y).$$

Then, once again, expressing the right-hand side in terms of $h(-d)$ using (44) yields

$$\sum_{a \leq \frac{1}{4}\sqrt{d}} \frac{\nu(a)}{a} \leq \left(\frac{3}{\pi^2} + O\left(\frac{\log \log h(-d)}{\log h(-d)} \right) \right) (\log h(-d))^2.$$

Plugging this into (38) completes the proof. \square

5 Concluding remarks

About further improvements of Theorem 1, one might be able to push the constant 6.5 to about 7 by carefully choosing the values of $f(d)$ and $\ell(d)$. Another idea is to replace the term $s(s+2)(s+3)$ in the definition of the integral I in (14) with $s(s+a)(s+b)$, then choose a and b that give the best result. We have tried this and found out that $s(s+2)(s+3)$ is already very close to optimal. Replacing it will either make an insignificant improvement on the final result or worsen it. What could really make a difference is any improvement of the bound in Lemma 2, with $h(-d) \geq 101$ we could only get the factor 11. We do not know if one could do significantly better than that.

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D. Ralaivaosaona, DEPARTMENT OF MATHEMATICAL SCIENCES, STELLENBOSCH UNIVERSITY, SOUTH AFRICA

E-mail address, D. Ralaivaosaona: naina@sun.ac.za

F. B. Razakarinoro, DEPARTMENT OF MATHEMATICAL SCIENCES, STELLENBOSCH UNIVERSITY, SOUTH AFRICA

E-mail address, F. B. Razakarinoro: brice@aims.ac.za