Numerator polynomials of the Riordan matrices

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Abstract

Riordan matrices are infinite lower triangular matrices corresponding to the certain operators in the space of formal power series. Generalized Euler polynomials $g_n(x) = (1-x)^{n+1} \sum_{m=0}^{\infty} p_n(m) x^m$, where $p_n(m)$ is the polynomial of degree $\leq n$, are the numerator polynomials of the generating functions of diagonals of the ordinary Riordan matrices. Generalized Narayana polynomials $h_n(x) = (1-x)^{2n+1} \sum_{m=0}^{\infty} (m+1)\dots(m+n) p_n(m) x^m$ are the numerator polynomials of the generating functions of diagonals of the exponential Riordan matrices. In paper, the properties of these two types of numerator polynomials and the constructive relationships between them are considered. Separate attention is paid to the numerator polynomials of Riordan matrices associated with the family of series $_{(\beta)}a(x) = a\left(x_{(\beta)}a^{\beta}(x)\right)$.

1 Introduction

Transformations in the space of formal power series over the field of real or complex numbers and the corresponding matrices are the subject of our study. We associate the rows and columns of matrices with the generating functions of their elements, i.e., formal power series. Thus, the expression $Aa(x) = b(x)$ means that the column vector multiplied by the matrix A has the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$, resultant column vector has the generating function $b(x) = \sum_{n=0}^{\infty} b_n x^n$. nth coefficient of the series $a(x)$, nth row, nth descending diagonal and nth column of the matrix A will be denoted respectively by

$$[x^n]\, a(x), \qquad [n, \rightarrow]\, A, \qquad [n, \searrow]A, \qquad Ax^n.$$

Matrix $(f(x), g(x))$, nth column of which has the generating function $f(x) g^n(x)$, $g_0 = 0$, is called Riordan matrix (Riordan array) [1] – [3]. It is the product of two matrices that correspond to the operators of multiplication and composition of series:

$$(f(x), g(x)) = (f(x), x)(1, g(x)),$$

$$(f(x), x)\, a(x) = f(x)\, a(x), \qquad (1, g(x))\, a(x) = a(g(x)),$$

$$(f(x), g(x))(b(x), a(x)) = (f(x)\, b(g(x)), a(g(x))).$$

If $f_0 \neq 0$, $g_1 \neq 0$, matrix $(f(x), g(x))$ is called proper. Proper Riordan matrices form a group, called the Riordan group.

Matrices

$$|e^x|^{-1}(f(x), g(x))|e^x| = (f(x), g(x))_E,$$

where $|e^x|$ is the diagonal matrix: $|e^x|\, x^n = x^n/n!$, are called exponential Riordan matrices [4], [5]. Denote $[n, \rightarrow](f(x), g(x))_E = s_n(x)$. Then

$$(f(x), g(x))_E(1 - \varphi x)^{-1} = |e^x|^{-1}(f(x), g(x))\, e^{\varphi x} = |e^x|^{-1} f(x) \exp(\varphi g(x)),$$

or

$$\sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n = f(x) \exp(\varphi g(x)).$$

1

If $f_0 \neq 0$, $g_1 \neq 0$, sequence of polynomials $s_n(x)$ is called Sheffer sequence, and in the case $f(x) = 1$ binomial sequence. Properties of the Sheffer sequences are subject of study of the umbral calculus [6]. Matrix

$$P = \left(\frac{1}{1-x}, \frac{x}{1-x} \right) = (e^x, x)_E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called Pascal matrix. Power of the Pascal matrix is defined by

$$P^\varphi = \left(\frac{1}{1-\varphi x}, \frac{x}{1-\varphi x} \right) = (e^{\varphi x}, x)_E.$$

Along with the lower triangular Riordan matrices, we will consider the "square" matrices $(b(x), a(x))$, $b_0 \neq 0$, $a_0 = 1$, whose nth column has the generating function $b(x) a^n(x)$. Upper triangular matrix $(1, 1+x)$, transposed to the Pascal matrix and coinciding with the matrix of shift operator, also belongs to this type of matrices:

$$(1, 1+x) = P^T = E = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Matrix $(b(x), a(x))$ can be multiplied from the right by the matrix with the finite columns and from the left by the matrix with the finite rows. Since $[n, \rightarrow](b(x), a(x)) = [n, \searrow](b(x), xa(x))$, then the matrix$(b(x), a(x))$ is a useful tool for studying the matrix $(b(x), xa(x))$. Denote

$$[n, \rightarrow](b(x), a(x) - 1) = w_n(x) = \sum_{m=0}^{n} w_m x^m.$$

Since

$$(b(x), a(x)) = (b(x), a(x) - 1)(1, 1+x), \qquad [n, \rightarrow](1, 1+x) = \frac{x^n}{(1-x)^{n+1}},$$

then

$$[n, \rightarrow](b(x), a(x)) = \sum_{m=0}^{n} \frac{w_m x^m}{(1-x)^{m+1}} = \frac{\sum_{m=0}^{n} w_m x^m (1-x)^{n-m}}{(1-x)^{n+1}} = \frac{g_n(x)}{(1-x)^{n+1}}.$$

If $b(x) = 1$, $a(x) = e^x$, then $g_n(x) = A_n(x)/n!$, where $A_n(x)$ are the Euler polynomials:

$$\frac{A_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} m^n x^m, \qquad A_n(1) = n!.$$

For example,

$$A_0(x) = 1, \qquad A_1(x) = x, \qquad A_2(x) = x + x^2, \qquad A_3(x) = x + 4x^2 + x^3,$$

$$A_4(x) = x + 11x^2 + 11x^3 + x^4.$$

In this connection, we will called polynomials $g_n(x)$ the generalized Euler polynomials.

"Square" Riordan matrices (called convolution arrays) and numerator polynomials of the generating functions of their rows were considered in the series of papers [7] – [11]. In [12] such matrices are called generalized Riordan arrays. Concept of generalized Euler polynomials (called p_n-associated Eulerian polynomials) in general form is represented in [13].

Mentioned series of papers was written before the advent of the Riordan matrices theory. In modern literature, the topic of numerator polynomials of the Riordan matrices has not been raised. Exception consists of the three works [14] – [16], in which the generating functions of the diagonals of exponential Riordan matrices are considered. These works provided an incentive for writing the present paper.

In paper, we study the transformations in which the numerator polynomials of Riordan matrices participate, and build matrices of these transformations according to certain rules. These matrices serve as tools for further research. In Section 2, we consider the numerator polynomials of ordinary Riordan matrices (generalized Euler polynomials) and the transformations associated with them. Using matrices of these transformations, we obtain a general formula for the numerator polynomials of the matrices $\left(1, x_{(\beta)}a(x)\right)$, where $_{(\beta)}a(x)$ is the generalized binomial series:

$$_{(\beta)}a^\varphi(x) = \sum_{n=0}^\infty \frac{\varphi}{\varphi + \beta n}\binom{\varphi + \beta n}{n}x^n.$$

In Section 3, we consider the numerator polynomials of exponential Riordan matrices (generalized Narayana polynomials) and the transformations associated with them. In Section 4, we introduce matrices of the operators mapping generalized Euler polynomials to generalized Narayana polynomials. Using these matrices, we obtain a general formula for the numerator polynomials of the matrices $\left(1, x_{(\beta)}a(x)\right)_E$, where $_{(\beta)}a(x)$ is the generalized binomial series. In Section 5, we give an idea of the generalized Lagrange series $_{(\beta)}a(x)$ defined by the identity $_{(\beta)}a(x) = a\left(x_{(\beta)}a^\beta(x)\right)$, or $\left(1, xa^{-\beta}(x)\right)^{-1} = \left(1, x_{(\beta)}a^\beta(x)\right)$. In other words, we extend the generalization that underlies the generalized binomial series to any formal power series $a(x)$, $a_0 = 1$. In the remainder of paper, we will consider the transformations associated with these series.

2 Generalized Euler polynomials

In [13], the transformations associated with the generalized Euler polynomials are presented from a general point of view. In this section, we consider them from point of view of the Riordan matrix theory. We denote

$$[n, \rightarrow](b(x), \log a(x)) = c_n(x) = \sum_{m=0}^n c_m x^m, \qquad b_0 \neq 0, \qquad a_0 = 1,$$

$$[n, \rightarrow](b(x), \log a(x))_E = s_n(x) = \sum_{m=0}^n s_m x^m, \qquad [n, \rightarrow](b(x), a(x)) = \frac{g_n(x)}{(1-x)^{n+1}}.$$

Then

$$b(x)a^m(x) = \sum_{n=0}^\infty \frac{s_n(m)}{n!}x^n, \qquad \frac{g_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^\infty \frac{s_n(m)}{n!}x^m.$$

Since

$$(b(x), a(x)) = (b(x), \log a(x))(1, e^x),$$

3

then

$$\frac{g_n(x)}{(1-x)^{n+1}} = \sum_{p=0}^{n} \frac{c_p A_p(x)/p!}{(1-x)^{p+1}} = \frac{1}{n!}\sum_{p=0}^{n} \frac{s_p A_p(x)}{(1-x)^{p+1}} = \frac{\frac{1}{n!}\sum_{p=0}^{n} s_p(1-x)^{n-p}A_p(x)}{(1-x)^{n+1}}.$$

We introduce the matrices U_n:

$$U_n x^p = (1-x)^{n+1}\frac{1}{n!}\sum_{m=0}^{\infty} m^p x^m = \frac{1}{n!}(1-x)^{n-p}A_p(x), \qquad 0 \le p \le n.$$

For example,

$$U_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad U_2 = \frac{1}{2!}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad U_3 = \frac{1}{3!}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 1 & 1 \\ 3 & -2 & 0 & 4 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Then $U_n s_n(x) = g_n(x)$. Let the symbols $(\varphi)_n$, $[\varphi]_n$ denote respectively the descending and ascending factorial:

$$(\varphi)_n = \varphi(\varphi-1)\dots(\varphi-n+1), \qquad [\varphi]_n = \varphi(\varphi+1)\dots(\varphi+n-1).$$

Since

$$\frac{x^p}{(1-x)^{n+1}} = \sum_{m=0}^{\infty}\binom{m+n-p}{n}x^m = \sum_{m=0}^{\infty}\frac{(m+n-p)_n}{n!}x^m, \qquad 0 \le p \le n,$$

then

$$U_n^{-1}x^p = (x+n-p)_n = (x)_p[x+1]_{n-p}.$$

For example,

$$U_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad U_2^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad U_3^{-1} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 11 & 2 & -1 & 2 \\ 6 & 3 & 0 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We will consider the products of matrices of various orders with infinite matrices. Thus, the matrix of order n corresponds to infinite matrix, the columns elements of which, starting from the nth column, $n = 0, 1, 2, \dots$, and the rows elements of which, starting from the nth row, are equal to zero. We introduce the matrices J_n corresponding to the operator rearranging a coefficients of polynomial of degree n in the reverse order: $J_n c(x) = x^n c(1/x)$, where $c(x)$ is a polynomial of degree $\le n$. For example,

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.1.
$$U_n E(1,-x) U_n^{-1} = (-1)^n J_n.$$

Proof.

$$E(1,-x)(x)_p[x+1]_{n-p} = (-x-1)_p[-x]_{n-p} = (-1)^n(x)_{n-p}[x+1]_p,$$

4

or
$$E\left(1,-x\right)U_n^{-1} = (-1)^n U_n^{-1} J_n. \quad \square$$

Theorem 2.2. *Polynomials* $(-1)^n J_n g_n\left(x\right)$ *are the numerator polynomials of the matrix* $\left(b\left(x\right)a^{-1}\left(x\right), xa^{-1}\left(x\right)\right)$.
Proof. By the Theorem 2.1.

$$(-1)^n J_n g_n\left(x\right) = U_n E\left(1,-x\right)s_n\left(x\right).$$

Since

$$\left(b\left(x\right),\log a\left(x\right)\right)\left(1,-x\right)\left(e^x,x\right) = \left(b\left(x\right)a^{-1}\left(x\right),\log a^{-1}\left(x\right)\right),$$

then

$$E\left(1,-x\right)s\left(x\right) = s_n\left(-x-1\right) = [n,\rightarrow]\left(b\left(x\right)a^{-1}\left(x\right),\log a^{-1}\left(x\right)\right)_E. \quad \square$$

Theorem 2.3.

$$g_n\left(1\right) = b_0(a_1)^n.$$

Proof. Denote $U_n x^p = U_{n,p}\left(x\right)$. Since

$$a_1 = [x]\log a\left(x\right), \qquad b_0(a_1)^n = [x^n]\,s_n\left(x\right); \qquad U_{n,p}\left(1\right) = 0, \qquad p < n; \qquad U_{n,n}\left(1\right) = 1,$$

then

$$g_n\left(1\right) = \sum_{p=0}^{n} s_p U_{n,p}\left(1\right) = s_n = b_0(a_1)^n. \quad \square$$

Case, when $a_1 = 0$ and the degree of polynomial $s_n\left(x\right)$ is less than n, is possible. This possibility is reflected in the following theorem.
Theorem 2.4. *If* $c_{n-m}\left(x\right)$ *is a polynomial of degree* $n - m$, *then*

$$U_n c_{n-m}\left(x\right) = (1-x)^m \frac{(n-m)!}{n!}U_{n-m}c_{n-m}\left(x\right).$$

Respectively, if $d_{n-m}\left(x\right)$ *is a polynomial of degree* $\le n - m$, *then*

$$U_n^{-1}(1-x)^m d_{n-m}\left(x\right) = \frac{n!}{(n-m)!}U_{n-m}^{-1}d_{n-m}\left(x\right).$$

Proof. Let I_n is the identity matrix of order $n + 1$. It is obvious that

$$\left((1-x)^{-m},x\right)U_n I_{n-m} = \frac{(n-m)!}{n!}U_{n-m},$$

or

$$U_n I_{n-m} = \left((1-x)^m,x\right)\frac{(n-m)!}{n!}U_{n-m}.$$

Respectively,

$$U_n^{-1}\left((1-x)^m,x\right)I_{n-m} = \frac{n!}{(n-m)!}U_{n-m}^{-1}. \quad \square$$

We introduce the matrices $V_n = J_n E J_n = \left((1+x)^{n+1},x\right)P^{-1}I_n$. For example,

$$V_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad V_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

If $c(x)$ is a polynomial of degree $\leq n$, then

$$V_n c(x) = (1+x)^n c\left(\frac{x}{1+x}\right), \qquad V_n^{-1} c(x) = (1-x)^n c\left(\frac{x}{1-x}\right).$$

We found in the introduction that $V_n^{-1} w_n(x) = g_n(x)$, where $w_n(x) = [n, \rightarrow](b(x), a(x) - 1)$, and hence $U_n^{-1} V_n^{-1} w_n(x) = s_n(x)$. By the Theorem 2.4. we find:

$$U_n^{-1} V_n^{-1} x^p = U_n^{-1}(1-x)^{n-p} x^p = \frac{n!}{p!}(x)_p = \frac{n!}{p!}\sum_{m=0}^{p} s(p, m) x^m,$$

where $s(p, m)$ are the Stirling numbers of the first kind. Hence

$$V_n U_n x^p = \frac{1}{n!}\sum_{m=0}^{p} m! S(p, m) x^m,$$

where $S(p, m)$ are the Stirling numbers of the second kind. For example, $U_3^{-1} V_3^{-1}$, $V_3 U_3$:

$$3!\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2!} & 0 \\ 0 & 0 & 0 & \frac{1}{3!} \end{pmatrix}, \qquad \frac{1}{3!}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2! & 0 \\ 0 & 0 & 0 & 3! \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Numerator polynomials of the matrix $(1, xa(x))$ are of independent interest. We introduce special notation for them and the associated polynomials:

$$[n, \searrow](1, xa(x)) = \frac{\alpha_n(x)}{(1-x)^{n+1}}, \qquad [n, \searrow](1, xa^{-1}(x)) = \frac{\alpha_n^{(-1)}(x)}{(1-x)^{n+1}},$$

$$[n, \rightarrow](1, \log a(x))_E = u_n(x), \qquad [n, \rightarrow](1, a(x) - 1) = v_n(x).$$

Since $[n, \searrow](a^{-1}(x), xa^{-1}(x)) = (1/x)[n, \searrow](1, xa^{-1}(x))$, $n > 0$, then by the Theorem 2. 2.

$$\alpha_n^{(-1)}(x) = (-1)^n x J_n \alpha_n(x).$$

Example 2.1.

$$a(x) = \frac{1+x}{1-x}, \qquad a(x) - 1 = \frac{2x}{1-x}, \qquad v_n(x) = 2^n x\left(\frac{1}{2} + x\right)^{n-1}, \qquad n > 0,$$

$$\alpha_n(x) = V_n^{-1} v_n(x) = 2x(1+x)^{n-1},$$

$$u_n(x) = U_n^{-1} \alpha_n(x) = 2\sum_{p=0}^{n}\binom{n-1}{p-1}(x)_p[x+1]_{n-p} =$$

$$= U_n^{-1} V_n^{-1} v_n(x) = n!\sum_{p=0}^{n}\binom{n-1}{p-1}\frac{2^p}{p!}(x)_p.$$

Example 2.2. Let $(1, xg(x))^{-1} = (1, xb^{-1}(x))$, $g_0 = 1$, $b_0 = 1$. Then by the Lagrange inversion theorem

$$[x^n] g^m(x) = \frac{m}{m+n}[x^n] b^{m+n}(x) = [x^n]\left(1 - x(\log b(x))'\right) b^{m+n}(x).$$

Denote $[x^n] g^m(x) = g_n^{(m)}$, $[x^n]\left(1 - x(\log b(x))'\right) b^m(x) = c_n^{(m)}$. Then

$$[n, \rightarrow](1, xg(x)) = \sum_{m=0}^{n} g_{n-m}^{(m)} x^m = \sum_{m=0}^{n} c_{n-m}^{(n)} x^m = [n, \rightarrow]\left(\left(1 - x(\log b(x))'\right) b^n(x), x\right).$$

6

If

$$g(x) = \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}},$$

then

$$b(x) = \sqrt{1 + x}, \qquad 1 - x(\log b(x))' = \left(1 + \frac{x}{2}\right) \frac{1}{1 + x},$$

$$[2n, \rightarrow](1, xg(x)) = \left(\frac{1}{2} + x\right) x^n (1 + x)^{n-1}, \qquad n > 0.$$

Let

$$a(x) = \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right)^2.$$

Then

$$a(x) - 1 = x\sqrt{a(x)} = x\left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right),$$

$$v_{2n}(x) = \left(\frac{1}{2} + x\right) x^n (1 + x)^{n-1}, \qquad \alpha_{2n}(x) = V_{2n}^{-1} v_{2n}(x) = \frac{1}{2}(1 + x) x^n,$$

$$u_{2n}(x) = U_{2n}^{-1} \alpha_{2n}(x) = \frac{1}{2}(x)_n [x + 1]_n + \frac{1}{2}(x)_{n+1}[x + 1]_{n-1} = \prod_{m=0}^{n-1} (x^2 - m^2).$$

We find the generating function of the sequence of polynomials $\alpha_n(t)$. Since

$$\frac{\alpha_n(t)}{(1 - t)^{n+1}} = \sum_{m=0}^{\infty} [x^n] a^m(x) t^m = [x^n] \frac{1}{1 - ta(x)},$$

then

$$\sum_{n=0}^{\infty} \alpha_n(t) x^n = \frac{1 - t}{1 - ta(x(1 - t))}. \tag{1}$$

Example 2.3. This example is a generalization of the examples considered in [11]. Denote $\tilde{\alpha}_0(x) = 1$, $\tilde{\alpha}_n(x) = (1/x)\alpha_n(x)$. If

$$[n, \searrow]\left(1, \frac{x}{1 + \varphi x + \beta x^2}\right) = \frac{\alpha_n(x)}{(1 - x)^{n+1}},$$

then

$$\tilde{\alpha}_n(x) = [n, \rightarrow]\left(\frac{1}{1 + \varphi x + \beta x^2}, \frac{\beta x^2}{1 + \varphi x + \beta x^2}\right).$$

Really,

$$1 + \sum_{n=1}^{\infty} \tilde{\alpha}_n(t) x^n = \left(\frac{1}{1 + \varphi x + \beta x^2}, \frac{\beta x^2}{1 + \varphi x + \beta x^2}\right) \frac{1}{1 - tx} = 1 + \frac{-\varphi x - \beta(1 - t) x^2}{1 + \varphi x + \beta(1 - t) x^2},$$

$$\sum_{n=0}^{\infty} \alpha_n(t) x^n = 1 + t \sum_{n=1}^{\infty} \tilde{\alpha}_n(t) x^n = \frac{1 + \varphi(1 - t) x + \beta(1 - t)^2 x^2}{1 + \varphi x + \beta(1 - t) x^2},$$

that corresponds to the formula (1).

Let $_{(\beta)}a(x)$ is the generalized binomial series:

$$_{(\beta)}a^{\varphi}(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + n\beta} \binom{\varphi + n\beta}{n} x^n;$$

$$_{(0)}a(x) = 1 + x, \qquad _{(1)}a(x) = \frac{1}{1-x}, \qquad _{(2)}a(x) = \frac{1 - \sqrt{1-4x}}{2x},$$

$$_{(-1)}a(x) = \frac{1 + \sqrt{1+4x}}{2}, \qquad _{(1/2)}a(x) = \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right)^2.$$

Denote

$$[n, \searrow]\left(1, x_{(\beta)}a(x)\right) = \frac{_{(\beta)}\alpha_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \frac{m}{m+n\beta}\binom{m+n\beta}{n}x^m,$$

$$[n, \rightarrow]\left(1, _{(\beta)}a(x) - 1\right) = _{(\beta)}v_n(x).$$

Theorem 2.5.

$$_{(\beta)}\alpha_n(x) = \frac{1}{n}\sum_{m=0}^{n}\binom{n(1-\beta)}{m-1}\binom{n\beta}{n-m}x^m, \qquad n > 0. \qquad (2)$$

Proof. We take into account the property of generalized binomial series: $_{(\beta)}a(x) - 1 = x_{(\beta)}a^\beta(x)$. Then

$$_{(\beta)}v_n(x) = [n, \rightarrow]\left(1, x_{(\beta)}a^\beta(x)\right) = \sum_{m=0}^{n}\frac{m}{n}\binom{n\beta}{n-m}x^m.$$

We use the factorial representation of binomial coefficients, i.e., we prove the theorem for positive integers β. By polynomial argument (binomial coefficients under consideration are polynomials in β) this is equivalent to the general proof. Since

$$[m, \rightarrow]V_n^{-1} = \sum_{i=0}^{m}\binom{m-n-1}{m-i}x^i = \sum_{i=0}^{m}(-1)^{m-i}\binom{n-i}{m-i}x^i,$$

then

$$[x^m]\,_{(\beta)}\alpha_n(x) = [x^m]V_n^{-1}\,_{(\beta)}v_n(x) =$$

$$= \sum_{i=0}^{m}(-1)^{m-i}\binom{n-i}{m-i}\frac{i}{n}\binom{n\beta}{n-i}\frac{(n\beta-n+m)!}{(n\beta-n+m)!} =$$

$$= \frac{1}{n}\binom{n\beta}{n-m}\sum_{i=0}^{m}(-1)^{m-i}i\binom{n\beta-n+m}{m-i} =$$

$$= \frac{1}{n}\binom{n\beta}{n-m}(-1)^{m-1}\binom{n\beta-n+m-2}{m-1} = \frac{1}{n}\binom{n\beta}{n-m}\binom{n(1-\beta)}{m-1}. \qquad \square$$

Note that

$$_{(0)}\alpha_n(x) = x^n, \qquad _{(1)}\alpha_n(x) = x, \qquad _{(1/2)}\alpha_{2n}(x) = \frac{1}{2}(1+x)x^n.$$

Since $_{(1-\beta)}a(x) = _{(\beta)}a^{-1}(-x)$, then $_{(1-\beta)}\alpha_n(x) = xJ_{n(\beta)}\alpha_n(x)$.

3 Generalized Narayana polynomials

Constructive relationships between the ordinary and exponential Riordan matrices exist. Particular manifestations of these relationships resemble the details of construction, the general plan of which is a secret for us. Following [14] – [16], we will consider some of such manifestations associated with the numerator polynomials. Since

$$[n, \searrow](b(x), xa(x)) = \sum_{m=0}^{\infty}\frac{s_n(m)}{n!}x^m,$$

then

$$[n, \searrow](b(x), xa(x))_E = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \frac{s_n(m)}{n!} x^m = \sum_{m=0}^{\infty} \frac{[m+1]_n s_n(m)}{n!} x^m.$$

If $a_1 \neq 0$, then $[x+1]_n s_n(x)$ is the polynomial of degree $2n$, in general case – of degree $\leq 2n$, so that

$$\sum_{m=0}^{\infty} \frac{[m+1]_n s_n(m)}{n!} x^m = \frac{h_n(x)}{(1-x)^{2n+1}}, \qquad h_n(x) = \frac{(2n)!}{n!} U_{2n}[x+1]_n s_n(x).$$

Since $[-m]_n = 0$ when $m = 0, 1, \ldots, n-1$, then in accordance with the Theorem 2.2.

$$\sum_{m=0}^{\infty} \frac{[-m]_n s_n(-m-1)}{n!} x^m = \frac{(-1)^{2n} J_{2n} h_n(x)}{(1-x)^{2n+1}} = \frac{(-1)^{2n} x^n J_n h_n(x)}{(1-x)^{2n+1}},$$

i.e., $h_n(x)$ is the polynomial of degree $\leq n$. Since $[x^{2n}][x+1]_n s_n(x) = b_0(a_1)^n$, then in accordance with the Theorem 2.3.

$$h_n(1) = b_0(a_1)^n \frac{(2n)!}{n!}.$$

If $b(x) = 1$, $a(x) = (1-x)^{-1}$, then

$$h_n(x) = (n+1)! N_n(x) = (1-x)^{2n+1} \sum_{m=0}^{\infty} [m+1]_n \binom{m+n-1}{n} x^m,$$

where

$$N_0(x) = 1, \qquad N_n(x) = \frac{1}{n} \sum_{m=0}^{n} \binom{n}{m-1} \binom{n}{n-m} x^m$$

are the Narayana polynomials. In this connection, we will called polynomials $h_n(x)$ the generalized Narayana polynomials.

We introduce the matrices F_n:

$$F_n = \frac{(2n)!}{n!} U_{2n}([x+1]_n, x) I_n, \qquad F_n^{-1} = \frac{n!}{(2n)!} ([x+1]_n, x)^{-1} U_{2n}^{-1} I_n,$$

$$F_n x^p = \frac{(2n)!}{n!} U_{2n} x^p [x+1]_n = (1-x)^{2n+1} \sum_{m=0}^{\infty} m^p \binom{m+n}{n} x^m,$$

$$F_n^{-1} x^p = \frac{n!}{(2n)!} (x)_p [x+n+1]_{n-p}.$$

For example,

$$F_1 = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 3 \\ 1 & -3 & 9 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 4 & 4 & 4 \\ 3 & -8 & 12 & 52 \\ -1 & 4 & -16 & 64 \end{pmatrix};$$

$$F_1^{-1} = \frac{1}{2!} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad F_2^{-1} = \frac{2!}{4!} \begin{pmatrix} 12 & 0 & 0 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad F_3^{-1} = \frac{3!}{6!} \begin{pmatrix} 120 & 0 & 0 & 0 \\ 74 & 20 & -4 & 2 \\ 15 & 9 & 3 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then $F_n s_n(x) = h_n(x)$.

Example 3.1. We explain the identity

$$F_n[x + n + 1]_n = \frac{(2n)!}{n!}.$$

Matrix $\left((xa(x))', xa(x)\right)_E$ is similar to the matrix $(a(x), xa(x))$ in the sense that $[n, \searrow]\left((xa(x))', xa(x)\right)_E = (1/x)[n, \searrow](1, xa(x))_E$, $n > 0$. This is a consequence of the identities

$$[x^n]\left(1 + x(\log a(x))'\right) a^m(x) = \frac{m+n}{m}[x^n] a^m(x),$$

$$[x^n](xa(x))' a^m(x) = \frac{m+n+1}{m+1}[x^n] a^{m+1}(x).$$

It also shows that

$$[n, \rightarrow]\left((xa(x))', \log a(x)\right)_E = (x + n + 1)\tilde{u}_n(x+1), \qquad n > 0,$$

where $\tilde{u}_n(x) = (1/x)u_n(x)$. If $a(x) = C(x)$, where $C(x)$ is the Catalan series, then

$$\tilde{u}_n(x) = [x + n + 1]_{n-1}, \qquad (x + n + 1)\tilde{u}_n(x+1) = [x + n + 1]_n.$$

Hence, numerator polynomials of the matrix $(1, xC(x))_E$ are the monomials $((2n)!/n!) x$.

Theorem 3.1.

$$F_n E^{n+1}(1, -x) F_n^{-1} = (-1)^n J_n.$$

Proof.

$$E^{n+1}(1, -x)(x)_p[x + n + 1]_{n-p} = (-x - n - 1)_p[-x]_{n-p} = (-1)^n(x)_{n-p}[x + n + 1]_p,$$

or

$$E^{n+1}(1, -x) F_n^{-1} = (-1)^n F_n^{-1} J_n. \qquad \square$$

Denote $(1, xa(x))^{-1} = (1, x\bar{a}(x))$.

Theorem 3.2. *Polynomials* $(-1)^n J_n h_n(x)$ *are the numerator polynomials of the matrix* $\left(b(x\bar{a}(x))(x\bar{a}(x))', x\bar{a}(x)\right)_E$.

Proof. By the Theorem 3.1.

$$(-1)^n J_n h_n(x) = F_n s_n(-x - n - 1).$$

We will imagine an infinite table whose kth row, $k = 0, \pm1, \pm2, \ldots$, has the generating function

$$b(x) a^k(x) = \sum_{n=0}^{\infty} \frac{s_n(k)}{n!} x^n.$$

If the row numbers of the table decrease from top to bottom, then, according to the Lagrange inversion theorem, the kth descending diagonal of table has the generating function

$$b(x\bar{a}(x))\left(1 + x(\log \bar{a}(x))'\right)\left(\frac{1}{\bar{a}(x)}\right)^k = \sum_{n=0}^{\infty} \frac{s_n(k - n)}{n!} x^n.$$

Hence,

$$b(x\bar{a}(x))(x\bar{a}(x))'(\bar{a}(x))^k = \sum_{n=0}^{\infty} \frac{s_n(-k - n - 1)}{n!} x^n. \qquad \square$$

Denote

$$[n, \searrow](1, xa(x))_E = \frac{\varphi_n(x)}{(1-x)^{2n+1}}, \qquad [n, \searrow](1, xa(x))_E^{-1} = \frac{\varphi_n^{[-1]}(x)}{(1-x)^{2n+1}}.$$

Then
$$\varphi_n^{[-1]}(x) = (-1)^n x J_n \varphi_n(x), \qquad n > 0.$$
Hence, if the matrix $(1, xa(x))$ is the pseudo-involution [3], i.e., $(1, xa(x))^{-1} = (1, xa(-x))$, then $\varphi_n(x) = x J_n \varphi_n(x)$.

We will find a specific generating function for the sequence of polynomials $\varphi_n(t)$. Since

$$\frac{\varphi_n(t)}{(n+1)!(1-t)^{2n+1}} = \sum_{m=0}^{\infty} \frac{1}{n+1} \binom{n+m}{m} [x^n] a^m(x) t^m =$$

$$= \frac{1}{n+1} [x^n] \frac{1}{(1-ta(x))^{n+1}} = [x^n] b(x),$$

where according to the Lagrange inversion theorem

$$b(x) = \frac{1}{1 - ta(xb(x))}, \qquad (1, xb(x)) = (1, x(1 - ta(x)))^{-1},$$

then

$$\sum_{n=0}^{\infty} \varphi_n(t) \frac{x^n}{(n+1)!} = (1-t) b\left(x(1-t)^2\right).$$

Example 3.2.

$$a(x) = \frac{1}{1-x}, \qquad b(x) = \frac{1 + x - t - \sqrt{1 - 2x - 2t - 2xt + x^2 + t^2}}{2x},$$

$$\sum_{n=0}^{\infty} \varphi_n(t) \frac{x^n}{(n+1)!} = \sum_{n=0}^{\infty} N_n(t) x^n = \frac{1 + x(1-t) - \sqrt{1 - 2x(1+t) + x^2(1-t)^2}}{2x}.$$

Generating functions of the second-order Euler numbers which are the coefficients of numerator polynomials of the matrices $(1, e^x - 1)_E$, $(e^x, e^x - 1)_E$ are considered in [17].

4   Connection matrices

We introduce the matrices $S_n = F_n U_n^{-1}$, $S_n^{-1} = U_n F_n^{-1}$. For example,

$$S_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \qquad S_2 = 2! \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & 3 & 6 \end{pmatrix}, \qquad S_3 = 3! \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & 4 & 0 & 0 \\ 9 & 12 & 10 & 0 \\ 1 & 4 & 10 & 20 \end{pmatrix},$$

$$S_4 = 4! \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 16 & 5 & 0 & 0 & 0 \\ 36 & 30 & 15 & 0 & 0 \\ 16 & 30 & 40 & 35 & 0 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix};$$

$$S_1^{-1} = \frac{1}{2!} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \qquad S_2^{-1} = \frac{2!}{4!} \begin{pmatrix} 6 & 0 & 0 \\ -8 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix}, \qquad S_3^{-1} = \frac{3!}{6!} \begin{pmatrix} 20 & 0 & 0 & 0 \\ -45 & 5 & 0 & 0 \\ 36 & -6 & 2 & 0 \\ -10 & 2 & -1 & 1 \end{pmatrix},$$

$$S_4^{-1} = \frac{4!}{8!} \begin{pmatrix} 70 & 0 & 0 & 0 & 0 \\ -224 & 14 & 0 & 0 & 0 \\ 280 & -28 & 14/3 & 0 & 0 \\ -160 & 20 & -16/3 & 2 & 0 \\ 35 & -5 & 5/3 & -1 & 1 \end{pmatrix}.$$

Then $S_n g_n(x) = h_n(x)$.

Theorem 4.1.

$$S_n = V_n^{-1} C_n V_n, \qquad C_n x^p = \frac{(n+p)!}{p!} x^p.$$

Proof. We use the Theorem 2.4. and the identities

$$U_n^{-1} V_n^{-1} x^p = \frac{n!}{p!} (x)_p, \qquad U_{n+p}^{-1} x^p = (x)_p [x+1]_n.$$

Then

$$F_n U_n^{-1} V_n^{-1} x^p = \frac{(2n)!}{n!} U_{2n} \frac{n!}{p!} (x)_p [x+1]_n =$$

$$= \frac{(2n)!}{p!} (1-x)^{n-p} \frac{(n+p)!}{(2n)!} U_{n+p} (x)_p [x+1]_n = \frac{(n+p)!}{p!} (1-x)^{n-p} x^p,$$

or

$$S_n V_n^{-1} = V_n^{-1} C_n. \qquad \square$$

For example,

$$S_3 = 3! \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 4.2.

$$S_n x^p = \frac{(n+p)!\,(n-p)!}{n!} \sum_{m=p}^{n} \binom{n}{m-p} \binom{n}{n-m} x^m.$$

Proof.

$$[m, \rightarrow] V_n^{-1} = \sum_{i=0}^{m} \binom{m-n-1}{m-i} x^i = \sum_{i=0}^{m} (-1)^{m-i} \binom{n-i}{m-i} x^i,$$

$$C_n V_n x^p = \sum_{i=p}^{n} \frac{(n+i)!}{i!} \binom{n-p}{i-p} x^i,$$

$$[x^m] V_n^{-1} C_n V_n x^p = \sum_{i=p}^{m} (-1)^{m-i} \binom{n-i}{m-i} \frac{(n+i)!}{i!} \binom{n-p}{i-p} =$$

$$= \frac{(n+p)!\,(n-p)!}{(n-m)!\,m!} \sum_{i=p}^{m} (-1)^{m-i} \binom{n+i}{i-p} \binom{m}{i} =$$

$$= \frac{(n+p)!\,(n-p)!}{(n-m)!\,m!} (-1)^{m-p} \binom{m-n-p-1}{m-p} = \frac{(n+p)!\,(n-p)!}{n!} \binom{n}{m-p} \binom{n}{n-m}. \square$$

Theorem 4.3.

$$S_n^{-1} x^p = \frac{p!\,(n-p)!}{(2n)!} \sum_{m=p}^{n} \binom{-n}{m-p} \binom{2n}{n-m} x^m.$$

Proof.

$$[x^m] V_n^{-1} C_n^{-1} V_n x^p = \sum_{i=p}^{m} (-1)^{m-i} \binom{n-i}{m-i} \frac{i!}{(n+i)!} \binom{n-p}{i-p} =$$

$$= \frac{p!\,(n-p)!}{(n-m)!\,(n+m)!} \sum_{i=p}^{m} (-1)^{m-i} \binom{i}{i-p} \binom{n+m}{m-i} =$$

$$= \frac{p!\,(n-p)!}{(n-m)!\,(n+m)!}(-1)^{m-p}\binom{n+m-p-1}{m-p} = \frac{p!\,(n-p)!}{(2n)!}\binom{-n}{m-p}\binom{2n}{n-m}.\ \square$$

Example 4.1. Numerator polynomials of the Pascal matrix are equal to one. Hence, numerator polynomials of the matrix $|e^x|^{-1}P\,|e^x|$ are the polynomials $S_n x^0 = n!^B N_n(x)$, where $^B N_n(x)$ are the so-called Narayana polynomials of type B:

$$^B N_n(x) = \sum_{m=0}^{n}\binom{n}{m}^2 x^m = (1-x)^{2n+1}\sum_{m=0}^{\infty}\binom{m+n}{n}^2 x^m.$$

Pascal matrix belongs to a subgroup of the Riordan group formed by the matrices of the form $(a(x), xa(x))$ and called the Bell subgroup. Numerator polynomials with the property $h_n(x) = J_n h_n(x)$ will be called symmetric.

Theorem 4.4. *In the exponential Bell subgroup, only the matrices $|e^x|^{-1}P^\varphi\,|e^x|$ have the symmetric numerator polynomials.*

Proof. Let $h_n(x)$ are the numerator polynomials of the matrix $(a(x), xa(x))_E$. Since $a(x\bar{a}(x)) = 1/\bar{a}(x)$, then by the Theorem 3.2. the polynomials $(-1)^n J_n h_n(x)$ are the numerator polynomials of the matrix

$$\left(1 + x(\log \bar{a}(x))', x\bar{a}(x)\right)_E = \left(1 + x(\log a(x))', xa(x)\right)_E^{-1}.$$

Thus, the necessary condition for the symmetry of numerator polynomials is the condition $a(x) = 1 + x(\log a(x))'$. It comes down to the condition $a^2(x) = (xa(x))'$, or $\sum_{m=0}^{n} a_{n-m} a_m = (n+1)\,a_n$, that is feasible only in the case $a_n = (a_1)^n$. $\square$

nth numerator polynomial of the matrix A (i.e., ordinary or exponential Riordan matrix) will be denoted by $[P_n(x)]\,A$.

Example 4.2. Since $[P_n(x)]\,(1+x, x(1+x)) = x^{n-1}$, $n > 0$, then

$$[P_n(x)]\,(1+x, x(1+x))_E = S_n x^{n-1} = \frac{(2n)!}{n!2}(1+x)\,x^{n-1}:$$

$$(1+x, x(1+x))_E = |e^x|^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 6 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}|e^x|.$$

Since

$$\left(1 + x(\log(1+x))', x(1+x)\right)_E^{-1} = \left(1 + x(\log C(-x))', xC(-x)\right)_E,$$

then

$$[P_n(x)]\left(1 + x(\log C(x))', xC(x)\right)_E = \frac{(2n)!}{n!2}(1+x):$$

$$\left(1 + x(\log C(x))', xC(x)\right)_E = |e^x|^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 6 & 3 & 1 & 0 & 0 & \cdots \\ 35 & 20 & 10 & 4 & 1 & 0 & \cdots \\ 126 & 70 & 35 & 15 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}|e^x|.$$

Example 4.3. Since $[P_n(x)]\left((xC(x))', xC(x)\right)_E = (2n)!/n!$, then

$$[P_n(x)]\left((xC(x))', xC(x)\right) = \frac{(2n)!}{n!}S_n^{-1}x^0 = \sum_{m=0}^{n}\binom{-n}{m}\binom{2n}{n-m}x^m:$$

$$\left((xC(x))', xC(x)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 20 & 10 & 4 & 1 & 0 & 0 & \cdots \\ 70 & 35 & 15 & 5 & 1 & 0 & \cdots \\ 252 & 126 & 56 & 21 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Respectively,

$$(-1)^n \sum_{m=0}^{n} \binom{2n}{m}\binom{-n}{n-m} x^m = [P_n(x)]\left(1 + x(\log C(x))', xC^{-1}(x)\right):$$

$$\left(1 + x(\log C(x))', xC^{-1}(x)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 1 & -1 & 1 & 0 & 0 & \cdots \\ 35 & 4 & 0 & -2 & 1 & 0 & \cdots \\ 126 & 15 & 1 & 0 & -3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $_{(\beta)}a(x)$ is the generalized binomial series. Denote

$$[n, \searrow]\left(1, x_{(\beta)}a(x)\right)_E = \frac{_{(\beta)}\varphi_n(x)}{(1-x)^{2n+1}} = \sum_{m=0}^{\infty} \frac{m}{m+\beta n}\binom{m+\beta n}{n}[m+1]_n x^m.$$

Theorem 4.5.

$$_{(\beta)}\varphi_n(x) = \frac{(n+1)!}{n}\sum_{m=0}^{n}\binom{n(2-\beta)}{m-1}\binom{n\beta}{n-m}x^m, \qquad n > 0. \tag{3}$$

Proof. We prove the theorem by analogy with the proof of Theorem 2.5.

$$[x^m]\,_{(\beta)}\varphi_n(x) = [x^m]\,S_{n(\beta)}\alpha_n(x) = [x^m]\,V_n^{-1}C_{n(\beta)}v_n(x) =$$

$$= \sum_{i=0}^{m}(-1)^{m-i}\binom{n-i}{m-i}\frac{(n+i)!}{i!}\frac{i}{n}\binom{n\beta}{n-i}\frac{(n\beta - n + m)!}{(n\beta - n + m)!} =$$

$$= \frac{(n+1)!}{n}\binom{n\beta}{n-m}\sum_{i=0}^{m}(-1)^{m-i}\binom{n+i}{i-1}\binom{n\beta - n + m}{m-i} =$$

$$= \frac{(n+1)!}{n}\binom{n\beta}{n-m}(-1)^{m-1}\binom{n\beta - 2n + m - 2}{m-1} = \frac{(n+1)!}{n}\binom{n\beta}{n-m}\binom{n(2-\beta)}{m-1}.\square$$

Note that

$$_{(0)}\varphi_n(x) = \frac{(2n)!}{n!}x^n, \qquad _{(1)}\varphi_n(x) = (n+1)!N_n(x), \qquad _{(2)}\varphi_n(x) = \frac{(2n)!}{n!}x.$$

Since $\left(1, x_{(\beta)}a(x)\right)^{-1} = \left(1, x_{(\beta-1)}a^{-1}(x)\right)$, $_{(\beta-1)}a^{-1}(-x) = {}_{(2-\beta)}a(x)$, then $_{(2-\beta)}\varphi_n(x) = xJ_{n(\beta)}\varphi_n(x)$.

5 Generalized Lagrange series

Let $_{(\beta)}a(x)$ is the generalized binomial series. We agree to denote $_{(0)}a(x) = a(x)$. Then $\left(1, xa^{-\beta}(x)\right)^{-1} = \left(1, x_{(\beta)}a^{\beta}(x)\right)$, $\left(1, x_{(\beta)}a^{\varphi}(x)\right)^{-1} = \left(1, x_{(\beta-\varphi)}a^{-\varphi}(x)\right)$. We extend the generalization that underlies this construction to any formal power series $a(x)$, $a_0 = 1$. Let $\left(1, xa^{-1}(x)\right)^{-1} = (1, xb(x))$. Denote $[x^n]a^m(x) = a_n^{(m)}$. For Riordan matrices, two forms of the Lagrange inversion formula

$$f(x) = f_0 + \sum_{n=1}^{\infty} \frac{x^n}{a^n(x)} \frac{1}{n} \left[x^{n-1}\right] f'(x) a^n(x),$$

$$\frac{f(x)}{1 - x(\log a(x))'} = \sum_{n=0}^{\infty} \frac{x^n}{a^n(x)} \left[x^n\right] f(x) a^n(x),$$

where $f(x)$ is an arbitrary series, take the form

$$[n, \to]\left(1, xa^{-1}(x)\right)^{-1} = \sum_{m=0}^{n} \frac{m}{n} a_{n-m}^{(n)} x^m, \qquad n > 0;$$

$$[n, \to]\left(1 - x(\log a(x))', xa^{-1}(x)\right)^{-1} = \sum_{m=0}^{n} a_{n-m}^{(n)} x^m.$$

Hence,

$$[x^n] b^m(x) = \frac{m}{m+n} [x^n] a^{m+n}(x) = [x^n]\left(1 - x(\log a(x))'\right) a^{m+n}(x),$$

$$[x^n]\left(1 + x(\log b(x))'\right) b^m(x) = \frac{m+n}{m} [x^n] b^m(x) = [x^n] a^{m+n}(x),$$

$$\left(1 - x(\log a(x))', xa^{-1}(x)\right)^{-1} = \left(1 + x(\log b(x))', xb(x)\right).$$

Denote $\left(1, xa^{-\beta}(x)\right)^{-1} = \left(1, x_{(\beta)}a^{\beta}(x)\right)$. Then

$$[x^n]_{(\beta)}a^{\beta m}(x) = \frac{\beta m}{\beta m + \beta n} [x^n] a^{\beta m + \beta n}(x),$$

$$_{(\beta)}a^{\varphi}(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + \beta n} \frac{u_n(\varphi + \beta n)}{n!} x^n, \qquad u_n(x) = [n, \to]\left(1, \log a(x)\right)_E.$$

Let $(1, \log a(x))^{-1} = (1, q(x))$. Then

$$\left(1, \log_{(\beta)}a(x)\right) = \left(1, x_{(\beta)}a^{\beta}(x)\right)\left(1, \log a(x)\right),$$

$$\left(1, \log_{(\beta)}a(x)\right)^{-1} = (1, q(x))\left(1, xa^{-\beta}(x)\right) = \left(1, q(x) e^{-\beta x}\right).$$

Denote.

$$_{(\beta)}u_n(x) = [n, \to]\left(1, \log_{(\beta)}a(x)\right)_E, \qquad _{(\beta)}q_n(x) = \left(1, \log_{(\beta)}a(x)\right)_E^{-1} x^n.$$

Since $\left(1, q(x) e^{-\beta x}\right)_E x^n = \left(e^{-\beta n x}, q(x)\right)_E x^n$, then the system of identities take place:

$$_{(\beta)}u_n(x) = \frac{x}{x + n\beta} u_n(x + n\beta), \qquad _{(\beta)}q_n(x) = \frac{1}{1 + n\beta x} q_n\left(\frac{x}{1 + n\beta x}\right),$$

$$\sum_{n=0}^{\infty} {}_{(\beta)}u_n(\varphi) \, _{(\beta)}q_n(x) = \frac{1}{1 - \varphi x}.$$

Series $_{(\beta)}a(x)$ for integers $\beta = k$ (denoted by $S_k(x)$) were introduced in [18]. In [19], these series, called generalized Lagrange series, are considered in connection with the Riordan matrices. They have the following visual interpretation. We will imagine the table $\{b(x), a^\varphi(x)\}_0$ whose kth row, $k = 0, \pm 1, \pm 2, \ldots$, has the generating function $b(x) a^{\varphi k}(x)$. Lagrange inversion formula connects the rows of the table with its ascending and descending diagonals in a certain way. For definiteness, we assume that $\varphi > 0$ and the table row numbers decrease from top to bottom. We replace the kth row of the table $\{b(x), a^\varphi(x)\}_0$ with the kth ascending diagonal, the resulting table will be denoted by $\{b(x), a^\varphi(x)\}_1$. We perform the same operation with the table $\{b(x), a^\varphi(x)\}_1$, the result will be denoted by $\{b(x), a^\varphi(x)\}_2$, etc. Tables obtained in a similar way for descending diagonals are numbered with negative numbers. Then it turns out that the kth row of the table $\{b(x), a^\varphi(x)\}_v$ has the generating function

$$b\left(x_{(v\varphi)}a^{v\varphi}(x)\right)\left(1 + x\left(\log_{(v\varphi)}a^{v\varphi}(x)\right)'\right)_{(v\varphi)}a^{\varphi k}(x),$$

that can be written as the pair of mutually inverse identities:

$$\{b(x), a^\varphi(x)\}_v = \left\{b\left(x_{(v\varphi)}a^{v\varphi}(x)\right)\left(1 + x\left(\log_{(v\varphi)}a^{v\varphi}(x)\right)'\right), {}_{(v\varphi)}a^\varphi(x)\right\}_0,$$

$$\left\{b\left(x_{(v\varphi)}a^{v\varphi}(x)\right)\left(1 + x\left(\log_{(v\varphi)}a^{v\varphi}(x)\right)'\right), {}_{(v\varphi)}a^\varphi(x)\right\}_{-v} = \{b(x), a^\varphi(x)\}_0.$$

Operators mapping rows of one table to rows of another, and vice versa, correspond to the matrices

$$\left(1 + x\left(\log_{(\beta)}a^\beta(x)\right)', x_{(\beta)}a^\beta(x)\right), \quad \left(1 + x\left(\log a^{-\beta}(x)\right)', xa^{-\beta}(x)\right), \quad \beta = v\varphi.$$

In the Riordan matrices theory, generalized Lagrange series do not have a generally accepted notation and are present implicitly in the form of various constructions. These constructions are presented in the papers [20] – [26].

6   Polynomials $_{(\beta)}g_n(x)$.

Let $[n, \to](b(x), \log a(x))_E = s_n(x)$. Then by definition of the series $_{(\beta)}a(x)$

$$[n, \to]\left(b\left(x_{(\beta)}a^\beta(x)\right)\left(1 + x\left(\log_{(\beta)}a^\beta(x)\right)'\right), \log_{(\beta)}a(x)\right)_E = s_n(x + \beta n).$$

Denote

$$[P_n(x)]\left(b\left(x_{(\beta)}a^\beta(x)\right)\left(1 + x\left(\log_{(\beta)}a^\beta(x)\right)'\right), x_{(\beta)}a(x)\right) = {}_{(\beta)}g_n(x).$$

We introduce the matrices $G_n^\beta = U_n E^{n\beta} U_n^{-1}$. Then $G_n^\beta g_n(x) = {}_{(\beta)}g_n(x)$. For example,

$$G_1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 6 & 3 & 1 \\ -8 & -3 & 0 \\ 3 & 1 & 0 \end{pmatrix}, \qquad G_3 = \begin{pmatrix} 20 & 10 & 4 & 1 \\ -45 & -20 & -6 & 0 \\ 36 & 15 & 4 & 0 \\ -10 & -4 & -1 & 0 \end{pmatrix},$$

Since $[x]_{(\beta)}a(x) = a_1$, then $_{(\beta)}g_n(1) = g_n(1)$; therefore, the sum of elements of each column of the matrix G_n^β is equal to one.

Theorem 6.1.

$$G_n^{-\beta} = J_n G_n^\beta J_n.$$

Proof. Since $E^\varphi(1, -x) = (1, -x) E^{-\varphi}$, then by the Theorem 2.1.

$$J_n U_n E^{n\beta} U_n^{-1} J_n = U_n E (1, -x) E^{n\beta} E (1, -x) U_n^{-1} = U_n E^{-n\beta} U_n^{-1}. \qquad \square$$

Thus,

$$G_1^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \qquad G_2^{-1} = \begin{pmatrix} 0 & 1 & 3 \\ 0 & -3 & -8 \\ 1 & 3 & 6 \end{pmatrix}, \qquad G_3^{-1} = \begin{pmatrix} 0 & -1 & -4 & -10 \\ 0 & 4 & 15 & 36 \\ 0 & -6 & -20 & -45 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

Theorem 6.2.
$$G_n^\beta = V_n^{-1} \left((1+x)^{n\beta}, x \right)^T V_n.$$

Proof. Since

$$n! \, |e^x| \, V_n U_n x^p = [p, \rightarrow] (1, e^x - 1)_E,$$
$$(1/n!) \, U_n^{-1} V_n^{-1} |e^x|^{-1} x^p = [p, \rightarrow] (1, \log(1+x))_E,$$
$$(1, \log(1+x))_E \left(e^{n\beta}, x \right)_E (1, e^x - 1)_E = \left((1+x)^{n\beta}, x \right)_E,$$

then

$$V_n U_n E^{n\beta} U_n^{-1} V_n^{-1} = \left((1+x)^{n\beta}, x \right)^T I_n. \qquad \square$$

For example,

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 6.3.
$$G_n^\beta x^p = \sum_{m=0}^{n} \binom{-n\beta + p}{m} \binom{n\beta + n - p}{n - m} x^m.$$

Proof. We will use the factorial representation of binomial coefficients. Then

$$\left((1+x)^{n\beta}, x \right)^T V_n x^p = \sum_{i=0}^{n} \binom{n\beta + n - p}{n - i} x^i,$$

$$[m, \rightarrow] V_n^{-1} = \sum_{i=0}^{m} \binom{m - n - 1}{m - i} x^i = \sum_{i=0}^{m} (-1)^{m-i} \binom{n - i}{m - i} x^i,$$

$$[x^m] V_n^{-1} \left((1+x)^{n\beta}, x \right)^T V_n x^p =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{n - i}{m - i} \binom{n\beta + n - p}{n - i} \frac{(n\beta + m - p)!}{(n\beta + m - p)!} =$$

$$= \binom{n\beta + n - p}{n - m} \sum_{i=0}^{m} (-1)^{m-i} \binom{n\beta + m - p}{m - i} =$$

$$= \binom{n\beta + n - p}{n - m} (-1)^m \binom{n\beta + m - p - 1}{m} = \binom{n\beta + n - p}{n - m} \binom{-n\beta + p}{m}. \qquad \square$$

We introduce the matrices $X_n = V_n^{-1}(x, x)^T V_n$. We find:

$$X_n x^0 = \frac{1 - x - (1 - x)^{n+1}}{x}, \qquad X_n x^p = x^{p-1} (1 - x).$$

Then

$$G_n^\beta = (I_n + X_n)^{n\beta} = \sum_{m=0}^{n} \binom{n\beta}{m} X_n^m.$$

For example,

$$G_3 = I_3 + 3 \begin{pmatrix} 3 & 1 & 0 & 0 \\ -6 & -1 & 1 & 0 \\ 4 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 \end{pmatrix} + 3 \begin{pmatrix} 3 & 2 & 1 & 0 \\ -8 & -5 & -2 & 1 \\ 7 & 4 & 1 & -2 \\ -2 & -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \end{pmatrix},$$

$$G_3^{-1} = I_3 - 3 \begin{pmatrix} 3 & 1 & 0 & 0 \\ -6 & -1 & 1 & 0 \\ 4 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 \end{pmatrix} + 6 \begin{pmatrix} 3 & 2 & 1 & 0 \\ -8 & -5 & -2 & 1 \\ 7 & 4 & 1 & -2 \\ -2 & -1 & 0 & 1 \end{pmatrix} - 10 \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

Thus, $I_n + X_n = G_n^{1/n} = U_n E U_n^{-1}$. For example,

$$G_2^{1/2} = \begin{pmatrix} 3 & 1 & 0 \\ -3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad G_3^{1/3} \begin{pmatrix} 4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad G_4^{1/4} = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ -10 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note the identities that follow from the Theorem 2.4.

$$\left((1-x)^{-m}, x\right) G_n^\beta \left((1-x)^m, x\right) I_{n-m} = G_{n-m}^{\frac{n\beta}{n-m}}.$$

$$\left((1-x)^{-m}, x\right) G_n^{1/n} \left((1-x)^m, x\right) I_{n-m} = G_{n-m}^{1/(n-m)}.$$

Example 6.1. Let $_{(\beta)}a(x)$ is the generalized binomial series. Then

$$[P_n(x)](1, xa(x)) = x^n, \qquad [P_n(x)]\left(1, x_{(1)}a(x)\right) = x,$$

$$[P_n(x)](a(x), xa(x)) = x^{n-1}, \qquad [P_n(x)]\left(_{(1)}a(x), x_{(1)}a(x)\right) = 1.$$

Hence,

$$G_n^\beta x^n = [P_n(x)]\left(1 + x\left(\log_{(\beta)}a^\beta(x)\right)', x_{(\beta)}a(x)\right),$$

$$G_n^\beta x = [P_n(x)]\left(1 + x\left(\log_{(\beta+1)}a^\beta(x)\right)', x_{(\beta+1)}a(x)\right),$$

$$G_n^\beta x^{n-1} = [P_n(x)]\left(_{(\beta)}a(x)\left(1 + x\left(\log_{(\beta)}a^\beta(x)\right)'\right), x_{(\beta)}a(x)\right),$$

$$G_n^\beta x^0 = [P_n(x)]\left(_{(\beta+1)}a(x)\left(1 + x\left(\log_{(\beta+1)}a^\beta(x)\right)'\right), x_{(\beta+1)}a(x)\right).$$

Since $G_n^\beta x^0 = G_n^{\beta+1} x^n$, then the identity is manifested here:

$$_{(\beta)}a(x)\left(1 + x\left(\log_{(\beta)}a^{\beta-1}(x)\right)'\right) = 1 + x\left(\log_{(\beta)}a^\beta(x)\right)'.$$

To match the identities $G_n^{-\beta} x^0 = J_n G_n^\beta x^n$, $G_n^{-\beta} x = J_n G_n^\beta x^{n-1}$ with the definition of polynomials $(-1)^n J_n g_n(x)$, we use the identity $_{(1-\beta)}a(x) = _{(\beta)}a^{-1}(-x)$.

7 Polynomials $_{(\beta)}h_n(x)$.

Denote

$$[P_n(x)]\left(b\left(x_{(\beta)}a^\beta(x)\right)\left(1+x\left(\log_{(\beta)}a^\beta(x)\right)'\right),x_{(\beta)}a(x)\right)_E={}_{(\beta)}h_n(x).$$

We introduce the matrices $H_n^\beta=F_nE^{n\beta}F_n^{-1}$. Then $H_n^\beta h_n(x)={}_{(\beta)}h_n(x)$. For example,

$$H_1=\frac{1}{2}\begin{pmatrix}3&1\\-1&1\end{pmatrix},\quad H_2=\frac{1}{6}\begin{pmatrix}15&5&1\\-12&2&4\\3&-1&1\end{pmatrix},\quad H_3=\frac{1}{20}\begin{pmatrix}84&28&7&1\\-108&-4&15&9\\54&-6&-1&9\\-10&2&-1&1\end{pmatrix}.$$

Sum of elements of each column of the matrix H_n^β is equal to one for the same reason as for the matrix G_n^β.

Theorem 7.1.

$$H_n^{-\beta}=J_nH_n^\beta J_n.$$

Proof. By the Theorem 3.1.

$$J_nF_nE^{n\beta}F_n^{-1}J_n=F_nE^{n+1}(1,-x)E^{n\beta}E^{n+1}(1,-x)F_n^{-1}=F_nE^{-n\beta}F_n^{-1}.\quad\square$$

Matrix H_n^β can be represented as

$$H_n^\beta=S_nG_n^\beta S_n^{-1}=V_n^{-1}C_n\left((1+x)^{n\beta},x\right)^TC_n^{-1}V_n.$$

For example,

$$H_3=\begin{pmatrix}1&0&0&0\\-3&1&0&0\\3&-2&1&0\\-1&1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&4&0&0\\0&0&10&0\\0&0&0&20\end{pmatrix}\begin{pmatrix}1&3&3&1\\0&1&3&3\\0&0&1&3\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{1}{4}&0&0\\0&0&\frac{1}{10}&0\\0&0&0&\frac{1}{20}\end{pmatrix}\begin{pmatrix}1&0&0&0\\3&1&0&0\\3&2&1&0\\1&1&1&1\end{pmatrix}.$$

Denote

$$t_n(\varphi|\beta,x)=\sum_{m=0}^n\binom{\varphi}{m}\binom{\beta}{n-m}x^m.$$

Theorem7. 2.

$$H_n^\beta x^p=\sum_{m=p}^n\binom{n+m}{m}^{-1}\binom{n-p}{n-m}(1-x)^{n-m}t_m(-n\beta+n+m|n\beta,x).$$

Proof. If $c_p(x)$ is a polynomial of degree $p\le n$, then $V_n^{-1}c_p(x)=(1-x)^{n-p}V_p^{-1}c_p(x)$. Then

$$\frac{1}{n!}[x^m]V_p^{-1}C_n\left((1+x)^{n\beta},x\right)^Tx^p=$$

$$=\sum_{i=0}^m(-1)^{m-i}\binom{p-i}{m-i}\binom{n\beta}{p-i}\binom{n+i}{i}\frac{(n\beta+m-p)!}{(n\beta+m-p)!}=$$

$$=\binom{n\beta}{p-m}\sum_{i=0}^m(-1)^{m-i}\binom{n+i}{i}\binom{n\beta+m-p}{m-i}=$$

$$=\binom{n\beta}{p-m}(-1)^m\binom{n\beta+m-p-n-1}{m}=\binom{n\beta}{p-m}\binom{-n\beta+n+p}{m},$$

$$V_n^{-1} C_n \left((1+x)^{n\beta}, x \right)^T x^p = n! (1-x)^{n-p} t_p \left(-n\beta + n + p | n\beta, x \right).$$

It remains to add that

$$C_n^{-1} V_n x^p = \frac{1}{n!} \sum_{m=p}^{n} \binom{n+m}{m}^{-1} \binom{n-p}{n-m} x^m. \qquad \square$$

In particular,

$$H_n^\beta x^n = \binom{2n}{n}^{-1} \sum_{m=0}^{n} \binom{-n\beta + 2n}{m} \binom{n\beta}{n-m} x^m,$$

$$H_n^\beta x^0 = \binom{2n}{n}^{-1} \sum_{m=0}^{n} \binom{-n\beta}{m} \binom{n\beta + 2n}{n-m} x^m.$$

Example 7.1. Let $_{(\beta)}a(x)$ is the generalized binomial series. Then, considering that $(1, xa(x))^{-1} = (1, x_{(2)}a(-x))$,

$$[P_n(x)] (1, xa(x))_E = S_n x^n = \frac{(2n)!}{n!} x^n, \quad [P_n(x)] \left(1, x_{(2)}a(x)\right)_E = \frac{(2n)!}{n!} x,$$

$$[P_n(x)] \left((xa(x))', xa(x) \right)_E = \frac{(2n)!}{n!} x^{n-1}, \quad [P_n(x)] \left((x_{(2)}a(x))', x_{(2)}a(x) \right)_E = \frac{(2n)!}{n!},$$

$$[P_n(x)] (a(x), xa(x))_E = S_n x^{n-1} = \frac{(2n)!}{n!2} (1+x) x^{n-1},$$

$$[P_n(x)] \left(1 + x \left(\log_{(2)} a(x) \right)', x_{(2)}a(x) \right)_E = \frac{(2n)!}{n!2} (1+x).$$

Hence,

$$\frac{(2n)!}{n!} H_n^\beta x^n = [P_n(x)] \left(1 + x \left(\log_{(\beta)} a^\beta(x) \right)', x_{(\beta)}a(x) \right)_E,$$

$$\frac{(2n)!}{n!} H_n^\beta x^0 = \frac{(2n)!}{n!} H_n^{\beta+2} x^n = [P_n(x)] \left(1 + x \left(\log_{(\beta+2)} a^{\beta+2}(x) \right)', x_{(\beta+2)}a(x) \right)_E,$$

$$\frac{(2n)!}{n!} H_n^\beta x = [P_n(x)] \left(1 + x \left(\log_{(\beta+2)} a^\beta(x) \right)', x_{(\beta+2)}a(x) \right)_E,$$

$$\frac{(2n)!}{n!} H_n^\beta x^{n-1} = [P_n(x)] \left(_{(\beta)}a(x) \left(1 + x \left(\log_{(\beta)} a^{\beta+1}(x) \right)' \right), x_{(\beta)}a(x) \right)_E,$$

$$\frac{(2n)!}{n!2} H_n^\beta (1+x) x^{n-1} = [P_n(x)] \left(_{(\beta)}a(x) \left(1 + x \left(\log_{(\beta)} a^\beta(x) \right)' \right), x_{(\beta)}a(x) \right)_E,$$

$$\frac{(2n)!}{n!2} H_n^\beta (1+x) = [P_n(x)] \left(1 + x \left(\log_{(\beta+2)} a^{\beta+1}(x) \right)', x_{(\beta+2)}a(x) \right)_E.$$

Note that

$$\binom{2n-1}{n-1} H_n^\beta (1+x) x^{n-1} = (1-x) \sum_{m=0}^{n-1} \binom{-n\beta + 2n - 1}{m} \binom{n\beta}{n-1-m} x^m +$$

$$+ \sum_{m=0}^{n} \binom{-n\beta + 2n}{m} \binom{n\beta}{n-m} x^m = \sum_{m=0}^{n} \binom{-n\beta + 2n - 1}{m} \binom{n\beta + 1}{n-m} x^m,$$

$$\binom{2n-1}{n-1} H_n^\beta (1+x) = \sum_{m=0}^{n} \binom{-n\beta + 1}{m} \binom{n\beta + 2n - 1}{n-m} x^m.$$

To match the identities $H_n^{-\beta} x^0 = J_n H_n^\beta x^n$, $H_n^{-\beta} x = J_n H_n^\beta x^{n-1}$ with the definition of polynomials $(-1)^n J_n h_n(x)$, we use the identities $(1, x_{(\beta)}a(x))^{-1} = (1, x_{(\beta-1)}a^{-1}(x))$, $_{(\beta-1)}a^{-1}(-x) = _{(2-\beta)}a(x)$.

8   Matrices $\tilde{U}_n$, $\tilde{F}_n$

Let, as before, $[P_n(x)](1, xa(x)) = \alpha_n(x)$, $[n, \to](1, \log a(x))_E = u_n(x)$. For the sequence of polynomials $c_n(x)$ such that $c_0(0) = 1$, $c_n(0) = 0$ we denote: $\tilde{c}_0(x) = 1$, $\tilde{c}_n(x) = (1/x)c_n(x)$. We introduce the matrices $\tilde{U}_n$, $n > 0$:

$$\tilde{U}_n = (x, x)^T U_n(x, x), \qquad \tilde{U}_n^{-1} = (x, x)^T U_n^{-1}(x, x),$$

$$\tilde{U}_n x^p = \frac{1}{n!}(1 - x)^{n-1-p}\tilde{A}_{p+1}(x), \quad \tilde{U}_n^{-1} x^p = (x - 1)_p[x + 1]_{n-p-1}, \quad 0 \le p < n.$$

For example,

$$\tilde{U}_4 = \frac{1}{4!}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 3 & 11 \\ 3 & -1 & -3 & 11 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \qquad \tilde{U}_4^{-1} = \begin{pmatrix} 6 & -2 & 2 & -6 \\ 11 & -1 & -1 & 11 \\ 6 & 2 & -2 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then $\tilde{U}_n \tilde{u}_n(x) = \tilde{\alpha}_n(x)$. Since

$$\frac{\tilde{\alpha}_n(x)}{(1 - x)^{n+1}} = \sum_{m=0}^{\infty} \frac{u_n(m + 1)}{n!}x^m, \qquad \tilde{\alpha}_n(x) = U_n E u_n(x),$$

then the matrices $\tilde{U}_n$, $\tilde{U}_n^{-1}$ can be represented as

$$\tilde{U}_n = U_n E(x, x) I_{n-1}, \qquad \tilde{U}_n^{-1} = (x, x)^T E^{-1} U_n^{-1} I_{n-1}.$$

We introduce the matrices $\tilde{J}_n = J_{n-1}$, $\tilde{I}_n = I_{n-1}$, $\tilde{V}_n = V_{n-1}$. Then

$$\tilde{U}_n^{-1}\tilde{V}_n^{-1} x^p = \frac{n!}{(p+1)!}\sum_{m=0}^{p} s(p+1, \ m+1) x^m, \quad \tilde{V}_n \tilde{U}_n x^p = \frac{1}{n!}\sum_{m=0}^{p} m! S(p+1, \ m+1) x^m.$$

For example, $\tilde{U}_4^{-1}\tilde{V}_4^{-1}$, $\tilde{V}_4 \tilde{U}_4$:

$$4!\begin{pmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3!} & 0 \\ 0 & 0 & 0 & \frac{1}{4!} \end{pmatrix}, \qquad \frac{1}{4!}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3! & 0 \\ 0 & 0 & 0 & 4! \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 8.1.
$$\tilde{U}_n(1, -x)\tilde{U}_n^{-1} = (-1)^{n-1}\tilde{J}_n.$$

Proof.

$$(1, -x)(x - 1)_p[x + 1]_{n-p-1} = (-x - 1)_p[-x + 1]_{n-p-1} = (-1)^{n-1}(x - 1)_{n-p-1}[x + 1]_p. \ \square$$

Example 8.1. This example illustrates the advantage of the matrices $\tilde{U}_n$ over the matrices U_n in certain cases. Denote

$$[P_n(x)](1, xa^m(x)) = \alpha_n^{(m)}(x), \qquad W_{(n,m)} = \tilde{U}_n(m, mx)\tilde{U}_n^{-1}.$$

Then $W_{(n,m)}\tilde{\alpha}_n(x) = \tilde{\alpha}_n^{(m)}(x)$. We build the matrix $(a(x), x)_m$ by the rule $[n, \to](a(x), x)_m = [mn + m - 1, \to](a(x), x)$. For example,

$$(a(x), x)_2 = \begin{pmatrix} a_1 & a_0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & \cdots \\ a_7 & a_6 & a_5 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (a(x), x)_3 = \begin{pmatrix} a_2 & a_1 & a_0 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & \cdots \\ a_8 & a_7 & a_6 & a_5 & \cdots \\ a_{11} & a_{10} & a_9 & a_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 8.2.

$$W_{(n,m)} = \left(w_m^{n+1}(x), x\right)_m \tilde{I}_n, \qquad w_m^{n+1}(x) = \left(\frac{1 - x^m}{1 - x}\right)^{n+1}.$$

Proof. Since

$$[x^p]\, \frac{\tilde{\alpha}_n^{(m)}(x)}{(1-x)^{n+1}} = \left[x^{mp+m-1}\right] \frac{\tilde{\alpha}_n(x)}{(1-x)^{n+1}}, \qquad \frac{\tilde{\alpha}_n(x)}{(1-x)^{n+1}} = \frac{w_m^{n+1}(x)\, \tilde{\alpha}_n(x)}{(1 - x^m)^{n+1}} = \sum_{r=0}^{m-1} \frac{x^r c_r(x)}{(1 - x^m)^{n+1}},$$

$$c_r(x) = \sum_{p=0}^{\infty} \left(\left[x^{mp+r}\right] w_m^{n+1}(x)\, \tilde{\alpha}_n(x)\right) x^{mp},$$

then

$$\frac{\tilde{\alpha}_n^{(m)}(x^m)}{(1 - x^m)^{n+1}} = \frac{c_{m-1}(x)}{(1 - x^m)^{n+1}}, \qquad [x^p]\, \tilde{\alpha}_n^{(m)}(x) = \left[x^{mp+m-1}\right] w_m^{n+1}(x)\, \tilde{\alpha}_n(x),$$

$$\tilde{\alpha}_n^{(m)}(x) = \left(w_m^{n+1}(x), x\right)_m \tilde{\alpha}_n(x). \qquad \square$$

For example, $W_{(1,2)} = (2)$, $W_{(1,3)} = (3)$, $W_{(1,4)} = (4)$,

$$W_{(2,2)} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad W_{(3,2)} = \begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix}, \qquad W_{(4,2)} = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 10 & 10 & 5 & 1 \\ 1 & 5 & 10 & 10 \\ 0 & 0 & 1 & 5 \end{pmatrix},$$

$$W_{(2,3)} = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}, \qquad W_{(3,3)} = \begin{pmatrix} 10 & 4 & 1 \\ 16 & 19 & 16 \\ 1 & 4 & 10 \end{pmatrix}, \qquad W_{(4,3)} = \begin{pmatrix} 15 & 5 & 1 & 0 \\ 51 & 45 & 30 & 15 \\ 15 & 30 & 45 & 51 \\ 0 & 1 & 5 & 15 \end{pmatrix},$$

$$W_{(2,4)} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}, \qquad W_{(3,4)} = \begin{pmatrix} 20 & 10 & 4 \\ 40 & 44 & 40 \\ 4 & 10 & 20 \end{pmatrix}, \qquad W_{(4,4)} = \begin{pmatrix} 35 & 15 & 5 & 1 \\ 155 & 135 & 101 & 65 \\ 65 & 101 & 135 & 155 \\ 1 & 5 & 15 & 35 \end{pmatrix}.$$

Since $\alpha_n^{(m)}(1) = m^n \alpha_n(1)$, then the sum of elements of each column of the matrix $W_{(n,m)}$ is equal to m^n. Note the identities

$$W_{(n,m)} \tilde{A}_n(x) = m^n \tilde{A}_n(x), \qquad W_{(n,m)} \tilde{J}_n = \tilde{J}_n W_{(n, m)}, \qquad W_{(n,m)} W_{(n,p)} = W_{(n,mp)},$$

$$\left((1 - x)^{-p}, x\right) W_{(n,m)} \left((1 - x)^p, x\right) \tilde{I}_{n-p} = W_{(n-p,m)}.$$

For example,

$$\begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 10 & 4 & 1 \\ 16 & 19 & 16 \\ 1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = 27 \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 20 & 10 & 4 \\ 40 & 44 & 40 \\ 4 & 10 & 20 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (2).$$

Since $(1, a(x) - 1)(1, (1 + x)^m - 1) = (1, a^m(x) - 1)$, then the matrix $W_{(n,m)}$ can be represented as

$$W_{(n,m)} = \tilde{V}_n^{-1} \left(\frac{(1+x)^m - 1}{x}, (1+x)^m - 1 \right)^T \tilde{V}_n.$$

For example,

$$\begin{pmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{pmatrix} = \frac{1}{3!} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Matrices $(1/m^n)\left(W_{(n,m)}\right)^T$ are known as amazing matrices [27, p.156]. They find application in various fields of mathematics [28] – [31].

Denote $[P_n(x)](1, xa(x))_E = \varphi_n(x)$. We introduce the matrices $\tilde{F}_n$:

$$\tilde{F}_n = (x, x)^T F_n(x, x), \qquad \tilde{F}_n^{-1} = (x, x)^T U_n^{-1}(x, x),$$

$$\tilde{F}_n x^p = (1 - x)^{2n+1} \sum_{m=0}^{\infty} (m+1)^{p+1} \binom{m+n+1}{n} x^m,$$

$$\tilde{F}_n^{-1} x^p = \frac{n!}{(2n)!} (x - 1)_p [x + n + 1]_{n-p-1}, \qquad 0 \le p < n.$$

For example,

$$\tilde{F}_4 = 5 \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & 3 & 15 & 39 \\ 3 & -9 & 9 & 171 \\ -1 & 5 & -25 & 125 \end{pmatrix}, \qquad \tilde{F}_4^{-1} = \frac{4!}{8!} \begin{pmatrix} 210 & -30 & 10 & -6 \\ 107 & 19 & -13 & 11 \\ 18 & 10 & 2 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then $\tilde{F}_n \tilde{u}_n(x) = \tilde{\varphi}_n(x)$. Since (see Example 3.1.)

$$\frac{\tilde{\varphi}_n(x)}{(1-x)^{2n+1}} = \sum_{m=0}^{\infty} \binom{n+m}{m} (m+n+1) \tilde{u}_n(m+1) x^m, \quad \tilde{\varphi}_n(x) = F_n E(x+n) \tilde{u}_n(x),$$

then the matrices $\tilde{F}_n$, $\tilde{F}_n^{-1}$ can be represented as

$$\tilde{F}_n = F_n E(x+n, x) I_{n-1} = F_n(x+n+1, x) E I_{n-1}, \quad \tilde{F}_n^{-1} = (x+n, x)^{-1} E^{-1} F_n^{-1} I_{n-1}.$$

Note that since

$$[x^n] F_n x^p (x+n+1) = [x^n] F_n E x^p (x+n) = 0, \qquad p < n;$$

$$F_n x^p = (1-x)^{2n+1} \sum_{m=0}^{\infty} m^p \binom{m+n}{n} x^m, \quad F_n E I_n x^p = (1-x)^{2n+1} \sum_{m=0}^{\infty} (m+1)^p \binom{m+n}{n} x^m,$$

then the identities for the nth elements of columns of the matrices F_n, $F_n E I_n$ are manifested here:

$$\sum_{m=0}^{n} (-1)^{n-m} \binom{2n+1}{n-m} m^p \binom{m+n}{n} = (-1)^{n+p} (n+1)^p,$$

23

$$\sum_{m=0}^{n} (-1)^{n-m} \binom{2n+1}{n-m} (m+1)^p \binom{m+n}{n} = (-1)^{n+p} n^p, \qquad p \le n.$$

Theorem 8.3.
$$\tilde{F}_n E^n (1, -x) \tilde{F}_n^{-1} = (-1)^{n-1} \tilde{J}_n.$$

Proof.

$$E^n (1, -x) (x-1)_p [x+n+1]_{n-p-1} = (-x-n-1)_p [-x+1]_{n-p-1} =$$

$$= (-1)^{n-1} (x-1)_{n-p-1} [x+n+1]_p. \qquad \square$$

We introduce the matrices $\tilde{S}_n = \tilde{V}_n \tilde{C}_n \tilde{V}_n$, $\tilde{C}_n x^p = ((n+p+1)!/(p+1)!\,) x^p$. Then $\tilde{S}_n \tilde{\alpha}_n (x) = \tilde{\varphi}_n (x)$.

9 Polynomials $_{(\beta)}\alpha_n (x)$, $_{(\beta)}\varphi_n (x)$

Denote $[P_n (x)] \left(1, x_{(\beta)} a (x)\right) =_{(\beta)} \alpha_n (x)$. We introduce the matrices $A_n^\beta = \tilde{U}_n E^{n\beta} \tilde{U}_n^{-1}$. Then $A_n^\beta \tilde{\alpha}_n (x) = _{(\beta)}\tilde{\alpha}_n (x)$. For example,

$$A_2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 5 & 5/2 & 1 \\ -6 & -2 & 0 \\ 2 & 1/2 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 14 & 7 & 3 & 1 \\ -28 & -35/3 & -10/3 & 0 \\ 20 & 22/3 & 5/3 & 0 \\ -5 & -5/3 & -1/3 & 0 \end{pmatrix}.$$

Theorem 9.1.
$$A_n^{-\beta} = \tilde{J}_n A_n^\beta \tilde{J}_n.$$

Proof. By the Theorem 8.1.

$$\tilde{J}_n \tilde{U}_n E^{n\beta} \tilde{U}_n^{-1} \tilde{J}_n = \tilde{U}_n (1, -x) E^{n\beta} (1, -x) \tilde{U}_n^{-1} = \tilde{U}_n E^{-n\beta} \tilde{U}_n^{-1}. \qquad \square$$

Note the identity

$$\left((1-x)^{-m}, x\right) A_n^\beta \left((1-x)^m, x\right) \tilde{I}_{n-m} = A_{n-m}^{\frac{n\beta}{n-m}}.$$

Let D is the matrix of derivation operator. We introduce the diagonal matrix $\tilde{D} = D(x, x)$, $\tilde{D} x^n = (n+1) x^n$.

Theorem 9.2.
$$A_n^\beta = \tilde{V}_n^{-1} \tilde{D} \left((1+x)^{n\beta}, x\right)^T \tilde{D}^{-1} \tilde{V}_n.$$

Proof. Since
$$n! \, |e^x| \, \tilde{D}^{-1} \tilde{V}_n \tilde{U}_n x^p = [p, \rightarrow] (e^x, e^x - 1)_E,$$

$$(1/n!\,) \tilde{U}_n^{-1} \tilde{V}_n^{-1} |e^x|^{-1} \tilde{D} x^p = [p, \rightarrow] \left((1+x)^{-1}, \log(1+x)\right)_E,$$

$$\left((1+x)^{-1}, \log(1+x)\right)_E \left(e^{n\beta}, x\right)_E (e^x, e^x - 1)_E = \left((1+x)^{n\beta}, x\right)_E,$$

then
$$\tilde{V}_n \tilde{U}_n E^{n\beta} \tilde{U}_n^{-1} \tilde{V}_n^{-1} = \tilde{D} \left((1+x)^{n\beta}, x\right)^T \tilde{D}^{-1} \tilde{I}_n. \qquad \square$$

For example,

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 9.3.

$$T_n^\beta x^p = \sum_{m=p}^{n-1} \frac{1}{(m+1)} \binom{n-1-p}{n-1-m} (1-x)^{n-m-1} t_m \left(-n\beta + m + 1 | n\beta, x\right).$$

Proof. In this case $p = 0, 1, \ldots, n-1$. Then

$$[x^m] \tilde{V}_{p+1}^{-1} \tilde{D}\left((1+x)^{n\beta}, x\right)^T x^p =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{p-i}{m-i} \binom{n\beta}{p-i} (i+1) \frac{(n\beta + m - p)!}{(n\beta + m - p)!} =$$

$$= \binom{n\beta}{p-m} \sum_{i=0}^{m} (-1)^{m-i} (i+1) \binom{n\beta + m - p}{m-i} =$$

$$= \binom{n\beta}{p-m} (-1)^m \binom{n\beta + m - p - 2}{m} = \binom{n\beta}{p-m} \binom{-n\beta + p + 1}{m},$$

$$\tilde{V}_n^{-1} \tilde{D}\left((1+x)^{n\beta}, x\right)^T x^p = (1-x)^{n-p-1} t_p \left(-n\beta + p + 1 | n\beta, x\right).$$

$$\tilde{D}^{-1} \tilde{V}_n x^p = \sum_{m=p}^{n-1} \frac{1}{(m+1)} \binom{n-1-p}{n-1-m} x^m. \qquad \square$$

In particular,

$$A_n^\beta x^{n-1} = \frac{1}{n} \sum_{m=0}^{n-1} \binom{n(1-\beta)}{m} \binom{n\beta}{n-1-m} x^m, \quad A_n^\beta x^0 = \frac{1}{n} \sum_{m=0}^{n-1} \binom{-n\beta}{m} \binom{n(1+\beta)}{n-1-m} x^m,$$

that corresponds to the formula (2).

Denote $[P_n(x)] \left(1, x_{(\beta)} a(x)\right)_E =_{(\beta)} \varphi_n(x)$. We introduce the matrices $T_n^\beta = \tilde{F}_n E^{n\beta} \tilde{F}_n^{-1}$. Then $T_n^\beta \tilde{\varphi}_n(x) = {}_{(\beta)} \tilde{\varphi}_n(x)$. For example,

$$T_2 = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad T_3 = \frac{1}{5} \begin{pmatrix} 12 & 4 & 1 \\ -9 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix}, \quad T_4 = \frac{1}{14} \begin{pmatrix} 55 & 55/3 & 5 & 1 \\ -66 & 0 & 10 & 6 \\ 30 & -6 & 0 & 6 \\ -5 & 5/3 & -1 & 1 \end{pmatrix}.$$

Theorem 9.4.

$$T_n^{-\beta} = \tilde{J}_n T_n^\beta \tilde{J}_n.$$

Proof. By the Theorem 8.3.

$$\tilde{J}_n \tilde{F}_n E^{n\beta} \tilde{F}_n^{-1} \tilde{J}_n = \tilde{F}_n E^n (1, -x) E^{n\beta} E^n (1, -x) \tilde{F}_n^{-1} = \tilde{F}_n E^{-n\beta} \tilde{F}_n^{-1}. \qquad \square$$

Matrix T_n^β can be represented as

$$T_n^\beta = \tilde{S}_n A_n^\beta \tilde{S}_n^{-1} = \tilde{V}_n^{-1} \tilde{C}_n \tilde{D}\left((1+x)^{n\beta}, x\right)^T \tilde{D}^{-1} \tilde{C}_n^{-1} \tilde{V}_n,$$

$$\tilde{C}_n \tilde{D} x^p = (n+1)! \binom{n+1+p}{p} x^p.$$

For example,

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 56 \end{pmatrix} \begin{pmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{21} & 0 \\ 0 & 0 & 0 & \frac{1}{56} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 9.5.

$$T_n^\beta x^p = \sum_{m=p}^{n-1} \binom{n+1+m}{m}^{-1} \binom{n-1-p}{n-1-m} (1-x)^{n-m-1} t_m \left(-n\beta + n + m + 1 | n\beta, x\right).$$

Proof.

$$\frac{1}{(n+1)!} [x^m] \tilde{V}_{p+1}^{-1} \tilde{C}_n \tilde{D} \left((1+x)^{n\beta}, x\right)^T x^p =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{p-i}{m-i} \binom{n\beta}{p-i} \binom{n+1+i}{i} \frac{(n\beta+m-p)!}{(n\beta+m-p)!} =$$

$$= \binom{n\beta}{p-m} \sum_{i=0}^{m} (-1)^{m-i} \binom{n+1+i}{i} \binom{n\beta+m-p}{m-i} =$$

$$= \binom{n\beta}{p-m} (-1)^m \binom{n\beta+m-p-n-2}{m} = \binom{n\beta}{p-m} \binom{-n\beta+n+p+1}{m},$$

$$\tilde{V}_n^{-1} \tilde{C}_n \tilde{D} \left((1+x)^{n\beta}, x\right)^T x^p = (n+1)!(1-x)^{n-p-1} t_p \left(-n\beta+n+p+1 | n\beta, x\right),$$

$$\tilde{D}^{-1} \tilde{C}_n^{-1} \tilde{V}_n x^p = \frac{1}{(n+1)!} \sum_{m=p}^{n-1} \binom{n+1+m}{m}^{-1} \binom{n-1-p}{n-1-m} x^m. \qquad \square$$

In particular,

$$T_n^\beta x^{n-1} = \binom{2n}{n-1}^{-1} \sum_{m=0}^{n-1} \binom{n(2-\beta)}{m} \binom{n\beta}{n-1-m} x^m,$$

$$T_n^\beta x^0 = \binom{2n}{n-1}^{-1} \sum_{m=0}^{n-1} \binom{-n\beta}{m} \binom{n(2+\beta)}{n-1-m} x^m.$$

that corresponds to the formula (3).

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