The longest increasing subsequence in involutions avoiding 3412 and another pattern

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Abstract

In this note, we study the mean length of the longest increasing subsequence of a uniformly sampled involution that avoids the pattern 3412 and another pattern.

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1 Introduction

In this paper we study the longest increasing subsequence of involutions avoiding 3412 and another pattern. A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of length n is defined as an arrangement of the elements of the set $[n] := \{1, 2, \cdots, n\}$. A permutation σ is called an *involution* if $\sigma = \sigma^{-1}$, where $\sigma_i^{-1} = j$ if and only if $\sigma_j = i$. We use notations S_n and I_n to denote, respectively, the set of all permutations and the set of all involutions of length n. A subsequence of $\sigma \in S_n$ is defined as a sequence $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ where $1 \le i_1 < i_2 < \cdots < i_k \le n$. The subsequence is called an increasing subsequence if $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}$.

For any permutation σ, there is at least one longest increasing subsequence. We denote the length of this subsequence by $L_n(\sigma)$. The celebrated Ulam's problem is concerned with the asymptotic behavior, as n tends to infinity, of the expectation of $L_n(\sigma)$ when σ is chosen uniformly from S_n [1, 11]. The classical Ulam's problem has been extended and generalized in various directions [13, 14]. In particular, asymptotic behavior of the distribution of the longest increasing subsequence of random involutions is the topic of [2, 7].

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Variations of Ulam's problem have been considered also for permutations in S_n avoiding certain patterns [3, 8, 9, 10]. For permutations $\pi = \pi_1\pi_2\cdots\pi_k \in S_k$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, we say that σ contains *pattern* π if there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that

$$\sigma_{i_s} < \sigma_{i_t} \quad \text{if and only if} \quad \pi_s < \pi_t \quad \text{for all} \quad 1 \le s, t \le k.$$

For instance, the permutation 15243 contains 321 as a pattern because it has the subsequences $*5*43$, and 543 matches the pattern 321. If σ does not contain π as a pattern, then we say that σ *avoids* π or σ is a π-*avoiding* permutation. We denote by $S_n(\pi)$ and $I_{\mathrm{n}}(\pi)$, respectively, the sets of π-avoiding permutations and π-avoiding involutions of $[n]$.

The goal of this paper is to study Ulam's problem in the context of involutions in I_{n} avoiding 3412 and another pattern. In [4] Egge connected generating functions for various subsets of $I_{\mathrm{n}}(3412)$ with continued fractions and Chebyshev polynomials of the second kind, and gave a recursive formula for computing them. The formula exploits a bijection between $I_{\mathrm{n}}(3412)$ and Motzkin paths established in [6]. Many of the results in [4] are concerned with statistics of decreasing subsequences of involutions in $I_{\mathrm{n}}(3412)$. Later, Egge and Mansour [5] extended the results in [4] to certain bivariate generating functions involving statistics of two-cycles in involutions. In this paper we extend the method of [4, 5] to certain bivariate generating functions involving the statistic $L_n(\sigma)$, and use it as a tool for studying the Ulam's problem for such pattern-restricted involutions.

For a given set of patterns T, let $I_{\mathrm{n}}(T) = \bigcap_{\tau\in T} I_{\mathrm{n}}(\tau)$ and denote by $P_{n,T}$ the uniform distribution on $I_{\mathrm{n}}(T)$. Thus, the probability of choosing any $\sigma \in I_{\mathrm{n}}(T)$ under $P_{n,T}$ is $\frac{1}{|I_{\mathrm{n}}(T)|}$, where $|\cdot|$ is the size of the set. We use the notations $E_{n,T}(\,\cdot\,)$ and $\mathrm{Var}_{n,T}(\,\cdot\,)$ to denote, respectively, the expectation and the variance operators under $P_{n,T}$. We use the shortcut L_n to denote the random variable $L_n(\sigma)$, where $\sigma \in S_n$ is a random permutation sampled uniformly from $I_n(T)$.

Throughout the paper, we write $a_n \sim b_n$ to indicate that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. We have:

Theorem 1.1. *Consider L_n on $I_{\mathrm{n}}(T)$ under the uniform probability measure. Then we have the following:*

(i) If $T = \{3412\}$, then $E_{n,T}(L_n) = \frac{4n}{9}$.

(ii) If $T = \{3412, 123\}$, then $E_{n,T}(L_n) = \frac{n^2/2+3/4+(-1)^n/4}{n^2/4+7/8+(-1)^n/8} \sim 2$.

(iii) If $T = \{3412, 213\}$ or $T = \{3412, 132\}$, then $E_{n,T}(L_n) \sim \frac{n}{\sqrt{5}}$.

(iv) If $T = \{3412, 321\}$, then $E_{n,T}(L_n) \sim \frac{3+\sqrt{5}}{5+\sqrt{5}}n$.

(v) If $T = \{3412, 123\cdots k\}$ for some $k \ge 1$, then $E_{n,T}(L_n) \sim k-1$.

(vi) If $T = \{3412, 4123\}$, then,

$$E_{n,T}(L_n) \sim \frac{1}{457}(198\alpha^3 - 246\alpha^2 - 131\alpha + 299)n \approx 0.454689799955\cdots n.$$

Here α is the complex root of smallest absolute value of the polynomial $3x^4 - 3x^3 - x^2 + 3x - 1$.

τ	$H_\tau(x,q)=\sum_{n\geq 0}\sum_{\sigma\in I_n(3412,\tau)} x^n q^{L_n(\sigma)}$	$E_{n,T}=E_{n,T}(L_n), V_{n,T}=\mathrm{Var}_{n,T}(L_n)$ for $T=\{3412,\tau\}$
1234	$1+\frac{x}{(1-x)}q+\frac{x^2}{(1-x)^3(1+x)}q^2+\frac{x^3(x^2+1)}{(1-x)^5(x+1)^2}q^3$	$E_{n,T}\sim 3$, $V_{n,T}\sim\frac{12}{n^2}$
$1243, 2134, 1324$	$1+\frac{qx(x^4+(1+(q-2)x^2)(1-xq))}{(1-qx-x^2)^2(1-x)}$	$E_{n,T}\sim\frac{n}{\sqrt{5}}$, $V_{n,T}\sim\frac{4}{5\sqrt{5}}n$
$1342, 1423$ $2314, 3124$	$\frac{(q-1)x^3+x^2+x-1}{x^3-x^2-(1+q)x+1}$	$E_{n,T}\sim\frac{(3-2\alpha)(\alpha+1)}{7}n$, $V_{n,T}\sim\frac{-7\alpha^2+5\alpha+10}{49}n$, where $\alpha^3-\alpha^2-2\alpha+1=0$, $\alpha\approx 0.44504$
$1432, 3214, 2143, 4231$	$\frac{1-x}{1-x-qx}$	$E_{n,T}\sim\frac{n}{2}$, $V_{n,T}\sim\frac{1}{4}n$
$2341, 4123$	$\frac{1}{1-\frac{xq}{1-x}-\frac{x^4q^2}{(1-x)^3(1+x)}}$	$E_{n,T}\sim\frac{198\alpha^3-246\alpha^2-131\alpha+299}{457}n$, $V_{n,T}\sim\frac{28800\alpha^3-7157\alpha^2-8959\alpha+47230}{208849}n$, where $3\alpha^4-3\alpha^3-\alpha^2+3\alpha-1=0$, $\alpha\approx 0.45209$
$2413, 3142$	$\frac{1-xq-x^2(q-1)-\sqrt{(1-xq-x^2(q-1))^2-4x^2}}{2x^2}$	$E_{n,T}\sim\frac{4n}{9}$, $V_{n,T}\sim\frac{4n}{27}$
$2431, 3241$ $4132, 4213$	$\frac{1-qx-x^2}{q^2x^3+(q^2-q-1)x^2-2qx+1}$	$E_{n,T}\sim\frac{(\alpha+1)(\alpha+2)}{7}n$, $V_{n,T}\sim\frac{-7\alpha^2-4\alpha+13}{49}n$ where $\alpha^3-\alpha^2-2\alpha+1=0$, $\alpha\approx 0.44504$
$3421, 4312$	$\frac{1-(q+1)x}{(1-(q+1)x-qx^2)(1-qx)}$	$E_{n,T}\sim\frac{n}{2}$ $V_{n,T}\sim\frac{\sqrt{2}}{8}n$
4321	$\frac{1-qx}{q(q-1)x^3+q(q-1)x^2-2qx+1}$	$E_{n,T}\sim\frac{5}{8}n$, $V_{n,T}\sim\frac{7}{64}n$

Table 1: The list of the generating functions and asymptotic values of the mean and variance of the length of the longest increasing subsequence for uniformly random involutions from $\mathrm{I_n}(3412,\tau)$ with $\tau\in S_4$.

(vii) If $T=\{3412,4321\}$, *then* $E_{n,T}(L_n)\sim\frac{5n}{8}$.

Since 3412 contains the patterns 231 and 312, we have

$$I_n(3412,231)=I_n(231) \qquad \text{and} \qquad I_n(3412,312)=I_n(312).$$

As shown in section 3.2 of [8], $E_{n,T}(L_n)=\frac{n+1}{2}$ for $T=\{3412,231\}$ and $T=\{3412,312\}$. Thus, Theorem 1.1 covers all possible cases for $\mathrm{I_n}(3412,\tau)$ with $\tau\in S_3$.

Using similar arguments we also obtained the asymptotic of $E_{n,T}(L_n)$ and $\mathrm{Var}_{n,T}(L_n)$ for all possible cases $\mathrm{I_n}(3412,\tau)$ with $\tau\in S_4$. We summarize these results in Table 1, without explicit calculations for the sake of space.

The rest of the paper is organized as follows. In Section 2 we consider $\mathrm{I_n}(3412)$ and prove part (i) of Theorem 1.1. In Section 3 we consider $\mathrm{I_n}(3412,\tau)$ with various patterns τ and prove the rest of Theorem 1.1.

2 Longest increasing subsequences in $I_n(3412)$

For $\rho \in S_k$ and $\sigma \in S_m$, we denote by $\rho \oplus \sigma$ their *direct sum*, which is a permutation in S_{k+m} given by $\rho_1 \cdots \rho_k(\sigma_1 + k) \cdots (\sigma_m + k)$. Similarly, we denote by $\rho \ominus \sigma$ the *skew sum* of ρ and σ, which is an element of S_{k+m} given by $(\rho_1 + m) \cdots (\rho_k + m)\sigma_1 \cdots \sigma_m$.

Our proofs make use of the following recursive structure of the involutions in $I_n(3412)$, for the details see [6, Remark 4.28] and [4, Proposition 2.9]:

Proposition 2.1. *Let $\rho \in I_n(3412)$. Then either*

(i) $\rho = 1 \oplus \rho'$ and $\rho' \in I_{n-1}(3412)$, or

(ii) $\rho = (1 \ominus \rho'' \ominus 1) \oplus \rho'$, where $\rho'' \in I_{m-2}(3412)$ and $\rho' \in I_{n-m}(3412)$ for some $m \geq 2$.

Proof of Theorem 1.1-(i). Let $H(x, q)$ be the generating function for the number of involutions in $I_n(3412)$ according to the length of the longest increasing subsequence. More precisely,

$$H(x,q) = \sum_{n\geq 0} \sum_{\sigma \in I_n(3412)} x^n q^{L_n(\sigma)}. \tag{1}$$

To obtain a closed form for $H(x, q)$, we partition $I_n(3412)$ as a union of the following four non-overlapping subsets, by virtue of Proposition 2.1:

(i) $\mathcal{I}_{n,1}$ - the set of the empty involution;

(ii) $\mathcal{I}_{n,2}$ - the set of the involutions in $I_n(3412)$ that start with 1;

(iii) $\mathcal{I}_{n,3}$ - the set of the involutions in $I_n(3412)$ that start with 21;

(iv) $\mathcal{I}_{n,4}$ - the set of the involutions in $I_n(3412)$ that can be written as $(1\ominus\sigma''\ominus 1)\oplus\sigma'$, where σ'' is a nonempty 3412-avoiding involution and σ' is any 3412-avoiding involution.

Adding together contributions of all the four sets, we obtain:

$$H(x,q) = \underbrace{1}_{\mathcal{I}_{n,1}} + \underbrace{xqH(x,q)}_{\mathcal{I}_{n,2}} + \underbrace{x^2qH(x,q)}_{\mathcal{I}_{n,3}} + \underbrace{x^2(H(x,q)-1)H(x,q)}_{\mathcal{I}_{n,4}}.$$

Hence,

$$H(x,q) = \frac{1 - xq - x^2(q-1) - \sqrt{(1 - xq - x^2(q-1))^2 - 4x^2}}{2x^2}.$$

Note that $H(x,1) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, which is the generating function for Motzkin numbers [4, 6]. Furthermore,

$$\frac{\partial}{\partial q} H(x,q)\Big|_{q=1} = -\frac{x+1}{2x} + \frac{1+x^2}{2x\sqrt{1-2x-3x^2}}.$$

Hence,

$$E_{n,3412}(L_n) = \frac{[x^n]\frac{\partial}{\partial q}H(x,q)\mid_{q=1}}{[x^n]H(x,1)} \sim \frac{\frac{2n\sqrt{3}}{9\sqrt{\pi n}n}3^{n+1}}{\frac{\sqrt{3}}{2\sqrt{\pi n}n}3^{n+1}} = \frac{4n}{9},$$

which completes the proof of Theorem 1.1-(i). □

3 Longest increasing subsequences in $I_{\rm n}(3412,\tau)$

In this section, we extend our arguments from $I_{\rm n}(3412)$ to $I_{\rm n}(3412,\tau)$ for various patterns τ. Toward this end, similar to (1), we define

$$H_\tau(x,q) = \sum_{n\geq 0} \sum_{\sigma\in I_n(3412,\tau)} x^n q^{L_n(\sigma)}.$$

More generally, for a collection of patterns T, we set

$$H_T(x,q) = \sum_{n\geq 0} \sum_{\sigma\in I_n(3412)\bigcap I_n(T)} x^n q^{L_n(\sigma)}.$$

When $T = \{\tau,\tau'\}$, for simplicity, we write $H_{\tau,\tau'}(x,q)$. We also set $H_\emptyset(x,q) := 0$ and let $H_{\tau/\tau'}(x,q) := H_\tau(x,q) - H_{\tau,\tau'}(x,q)$ denote the corresponding generating function for the involutions in $I_n(3412,\tau)$ that contain the pattern τ'.

We call a permutation *irreducible* if it cannot be represented as a direct sum of two nonempty permutations. It is easy to show that every permutation ρ can be written as a direct sum

$$\rho = \rho^{(1)} \oplus \rho^{(2)} \oplus \cdots \oplus \rho^{(k)},$$

where $\rho^{(1)},\ldots,\rho^{(k)}$ are nonempty irreducible permutations, uniquely determined by ρ. We next introduce a *bar operator* for permutations following [4].

Definition 3.1. *For $\rho\in S_m$, define $\overline{\rho}$ as follows:*

1. *$\overline{\emptyset} = \emptyset$ and $\overline{1} = \emptyset$.*
2. *If $m\geq 2$ and there exists a permutation σ such that $\rho = 1\ominus\sigma\ominus 1$, then $\overline{\rho}=\sigma$.*
3. *If $m\geq 2$ and there exists a permutation σ such that $\rho = 1\ominus\sigma$, and σ does not end with 1, then $\overline{\rho}=\sigma$.*
4. *If $m\geq 2$ and there exists a permutation σ such that $\rho = \sigma\ominus 1$, and ρ does not begin with m, then $\overline{\rho}=\sigma$.*
5. *If $m\geq 2$ and ρ does not begin with m, and it does not end with 1, then $\overline{\rho}=\rho$.*

Our main technical tool for calculating the corresponding generating functions for the classes $I_{\rm n}(3412,\tau)$ is the following extension of a result for $I_{\rm n}(3412)$ given by Corollary 5.6 in [4].

Proposition 3.2. *Suppose that $\tau = \tau^{(1)}\oplus\tau^{(2)}\oplus\cdots\oplus\tau^{(s)}$ is a direct sum of nonempty irreducible permutations $\tau^{(1)},\ldots,\tau^{(s)}$ such that $\tau^{(1)}$ is not a decreasing sequence. For $i\in[s]$, define*

$$\theta^{(i)} := \overline{\tau^{(1)}\oplus\cdots\oplus\tau^{(i)}} \qquad \textit{and} \qquad \theta^{<i>} := \tau^{(i)}\oplus\cdots\oplus\tau^{(s)}.$$

Then we have:

(i) If $\tau^{(1)} = 1$*, then*

$$H_\tau(x,q) = 1 + \frac{xq}{1-x} H_{\theta(<2>)}(x,q) + x^2 \sum_{i=2}^{s} \{H_{\theta^{(i)}/12}(x,q) - H_{\theta^{(i-1)}/12}(x,q)\} H_{\theta<i>}(x,q).$$

(ii) If $\tau^{(1)} = 21$*, then*

$$H_\tau(x,q) = 1 + xqH_\rho(x,q) + \frac{x^2 q}{1-x} H_{\theta<2>}(x,q) + x^2 \sum_{i=2}^{s} \{H_{\theta^{(i)}/12}(x,q) - \delta_{i>2} H_{\theta^{(i-1)}/12}(x,q)\} H_{\theta<i>}(x,q),$$

where δ_A *is one if* A *is true, and is zero otherwise.*

(iii) If $\tau^{(1)} = m(m-1)\cdots 1$ *with* $m \geq 3$*, then*

$$H_\tau(x,q) = 1 + (x + x^2 + \cdots + x^{m-1}) qH_\rho(x,q) + \frac{x^m q}{1-x} H_{\theta<2>}(x,q) + x^2 \sum_{i=1}^{s} \{H_{\theta^{(i)}/12}(x,q) - H_{\theta^{(i-1)}/12}(x,q)\} H_{\theta<i>}(x,q).$$

(iv) If $\tau^{(1)} \neq m(m-1)\cdots 1$ *and* $\rho^{(1)} \in S_m$ *with* $m \geq 3$*, then*

$$H_\tau(x,q) = 1 + \frac{xq}{1-x} H_\rho(x,q) + x^2 \sum_{i=1}^{s} \{H_{\theta^{(i)}/12}(x,q) - H_{\theta^{(i-1)}/12}(x,q)\} H_{\theta<i>}(x,q).$$

We will only prove parts (i) and (iv) of the proposition. The proofs of the other two cases are very similar, and therefore are omitted.

Proof of Proposition 3.2-(i). Assume first that $\tau^{(1)} = 1$. We partition the set $I_n(3412, \tau)$ into three non-overlapping subsets:
(i) $\mathcal{J}_{n,1}$ - the set of the empty involution;
(ii) $\mathcal{J}_{n,2}$ - the set of those involutions of the form $r(r-1)\cdots 1 \oplus \sigma'$ for some $r \geq 2$;
(iii) $\mathcal{J}_{n,3}$ - the set of those involutions which do not begin with a decreasing sequence.
It is easy to see that the involutions in the sets $\mathcal{J}_{n,1}$ and $\mathcal{J}_{n,2}$ contribute 1 and $\frac{xq}{1-x}H_\tau(x,y)$, respectively, to $H_\tau(x,y)$. To obtain the contribution of the involutions in the set $\mathcal{J}_{n,3}$, we first observe that in view of Proposition 2.1, all involutions in $\mathcal{J}_{n,3}$ can be written in the form $\sigma = (1 \ominus \sigma'' \ominus 1) \oplus \sigma'$ with σ'' that contains 12. Thus, the involutions in $\mathcal{J}_{n,3}$ that avoid $\tau^{(1)}$ contribute $x^2 H_{\theta^{(1)}/12}(x,q) H_\tau(x,q) = 0$. Furthermore, any involution in $\mathcal{J}_{n,3}$ that

contains $\tau^{(1)}$, avoids $\theta^{(i)}$ and contains $\theta^{(i-1)}$ for some $i = 2, 3, \ldots, s$. The total contribution of such involutions into $H_\tau(x,q)$ is equl to

$$x^2 \sum_{i=2}^{s} \left(H_{\theta^{(i)}/12}(x,q) - H_{\theta^{(i-1)}/12}(x,q) \right) H_{\theta^{<i>}}(x,q).$$

Adding together the contributions of $\mathcal{J}_{n,1}$, $\mathcal{J}_{n,2}$, and $\mathcal{J}_{n,3}$, we obtain the desired result. $\square$

Proof of Proposition 3.2-(iv). Suppose now that $\tau^{(1)} \neq m(m-1)\cdots 1$ and $\tau^{(1)} \in S_m$ with $m \geq 3$. We will consider again the partition $I_n(3412,\tau) = \bigcup_{k=1}^{3} \mathcal{J}_{n,k}$ defined in the course of the proof of part (i) of the proposition. It is easy to verify that in this case, $\mathcal{J}_{n,1}$ contributes 1 to $H_\tau(x,q)$, while permutations in the set $\mathcal{J}_{n,2}$ contribute $\frac{xq}{1-x} H_\tau(x,y)$. To obtain the contribution of $\mathcal{J}_{n,3}$, recall that by Proposition 2.1, all involutions in this set have the form $\sigma = (1 \ominus \sigma'' \ominus 1) \oplus \sigma'$ where σ'' contains 12. Thus, the involutions in $\mathcal{J}_{n,3}$ that avoid $\tau^{(1)}$ contribute $x^2 H_{\theta^{(1)}/12}(x,q) H_\tau(x,q)$, while the involutions in $\mathcal{J}_{n,3}$ that contain $\tau^{(1)}$ contribute

$$x^2 \sum_{i=1}^{s} \left(H_{\theta^{(i)}/12}(x,q) - H_{\theta^{(i-1)}/12}(x,q) \right) H_{\theta^{<i>}}(x,q).$$

Adding up all the contributing terms listed above, yields the desired result. $\square$

The rest of this section is divided into fives subsections, each one is concerned with $I_{\rm n}(3412,\tau)$ for a particular type of pattern τ and presents the proof of the corresponding part in Theorem 1.1.

3.1 $E_{n,T}(\boldsymbol{L}_n)$ on $\boldsymbol{I}_{\rm n}(3412,\tau)$ with $\tau \in S_2$

Note that the only involution in $I_{\rm n}(3412,12)$ is $n(n-1)\cdots 1$. Thus,

$$H_{12}(x,q) = 1 + \frac{xq}{1-x}. \tag{2}$$

Similarly, the only involution in $I_{\rm n}(3412,21)$ is $12\cdots n.$. Thus,

$$H_{21}(x,q) = \frac{1}{1-xq}.$$

3.2 $E_{n,T}(\boldsymbol{L}_n)$ on $\boldsymbol{I}_{\rm n}(3412,\tau)$ with $\tau \in S_3$

Proof of Theorem 1.1-(ii). An application of Proposition 3.2-(i) with $\tau = 1 \oplus 1 \oplus 1 = 123$ gives

$$\begin{aligned} H_{123}(x,q) = 1 + \frac{xq}{1-x} H_{12}(x,q) + x^2 (H_{12/12}(x,q) - H_{1/12}(x,q)) H_{12}(x,q) \\ + x^2 (H_{123/12}(x,q) - H_{12/12}(x,q)) H_1(x,q). \end{aligned}$$

It follows from (2) and the decomposition

$$H_{123/12}(x,q) = H_{123}(x,q) - H_{12}(x,q) \tag{3}$$

that

$$H_{123}(x,q) = 1 + \frac{xq}{1-x}\left(1 + \frac{xq}{1-x}\right) + x^2 H_{123}(x,q) - x^2\left(1 + \frac{xq}{1-x}\right).$$

Therefore,

$$H_{123}(x,q) = 1 + \frac{xq(1-x(1-q)-x^2+x^3)}{(1-x)^3(1+x)}.$$

Hence, for $T = \{3412, 123\}$ we have:

$$E_{n,T}(L_n) = \frac{[x^n]\frac{\partial}{\partial q}H_{123}(x,q) \mid_{q=1}}{[x^n]H_{123}(x,1)} = \frac{n^2/2 + 3/4 + (-1)^n/4}{n^2/4 + 7/8 + (-1)^n/8} \sim 2.$$

□

Proof of Theorem 1.1-(iii). Proposition 3.2-(i) implies that for $\tau = 1 \oplus 21 = 132$,

$$H_{132}(x,q) = 1 + \frac{xq}{1-x}H_{21}(x,q) + x^2(H_{132/12}(x,q) - H_{1/12}(x,q))H_{21}(x,q).$$

Using (3) and the fact that $H_{1/12}(x,q) = 0$, we get

$$H_{132}(x,q) = \frac{1-x^2(1-q)}{1-xq-x^2}.$$

Therefore, for $T = \{3412, 132\}$ we have:

$$E_{n,T}(L_n) \sim \frac{n}{\sqrt{5}}.$$

We next apply Proposition 3.2-(ii) to $\tau = 21 \oplus 1 = 213$, to get

$$H_{213}(x,q) = 1 + xqH_{213}(x,q) + \frac{x^2q}{1-x} + x^2(H_{213}(x,q) - H_{12}(x,q))H_1(x,q).$$

It follows then from (2) that

$$H_{213}(x,q) = \frac{1-x^2(1-q)}{1-xq-x^2}.$$

Hence, for $T = \{3412, 213\}$ we have: $E_{n,T}(L_n) \sim \frac{n}{\sqrt{5}}$. □

Proof of Theorem 1.1-(iv). An application of Proposition 3.2-(iii) to $\tau = 321$ yields:

$$H_{321}(x,q) \quad = 1 + (x+x^2)qH_{321}(x,q) + x^2(H_1(x,q) - H_{1,12}(x,q))H_{321}(x,q).$$

Since $H_1(x,q) = H_{1,12}(x,q) = 1$, this implies that

$$H_{321}(x,q) = \frac{1}{1-qx-qx^2}.$$

Thus, for $T = \{3412, 321\}$ we have $E_{n,T}(L_n) \sim \frac{3+\sqrt{5}}{5+\sqrt{5}}n$. □

3.3 $E_{n,T}(\boldsymbol{L}_n)$ on $\boldsymbol{I}_{\mathrm{n}}(3412,\tau)$ with $\tau = 12\cdots k$

Proof of Theorem 1.1-(v). Let $F_k(x,q) := H_{12\cdots k}(x,q)$. Applying Proposition 3.2 to the permutation $\tau = 12\cdots k$ with $k \geq 1$, we obtain:

$$F_k(x,q) = 1 + \frac{xq}{1-x}F_{k-1}(x,q) + x^2\sum_{i=3}^{k}(F_i(x,q) - F_{i-1}(x,q))F_{k-i+1}(x,q).$$

Let $F(x,q;y) := \sum_{k\geq 1} F_k(x,q)y^k$. Multiplying both sides of the above recurrence equation by y^k, summing over $k \geq 1$, and using the fact that $F_0(x,q) = 0$ and $F_1(x,q) = 1$, we obtain:

$$\begin{aligned} F(x,q;y) = {} & \frac{y}{1-y} + \frac{xqy}{1-x}F(x,q;y) + \frac{x^2}{y}F(x,q;y)F(x,q;y) - x^2yH_{12}(x,q)F(x,q;y) \\ & - x^2F(x,q;y) - x^2F(x,q;y)F(x,q;y) + x^2yF(x,q;y). \end{aligned}$$

Taking (2) into account and solving for $F(x,y;q)$, we obtain:

$$\begin{aligned} F(x,q;y) &= \frac{y}{1-y} + \frac{(1-qyx-(1+qy)x^2 - \sqrt{(1-qyx-(qy+1)x^2)^2 - 4q(1+x)x^3y})y}{2x^2(1-y)} \\ &= \frac{y}{1-y} + \frac{qxy^2}{(1-y)(1-x-qyx)}C\left(\frac{qx^3y}{(1+x)(1-x-qyx)^2}\right), \end{aligned}$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Substituting a series representation of the generating function from A001263 in [12], we obtain that

$$F(x,q;y) = \frac{y}{1-y} + \frac{1}{1-y}\sum_{j\geq 0}\sum_{i=1}^{j+1}\frac{\frac{1}{i}\binom{j-1}{i-1}\binom{j}{i-1}x^{2i+j-1}}{(1-x)^{2j+1}(1+x)^j}q^{j+1}y^{j+2}.$$

Therefore, for $k \geq 2$ we have:

$$[y^k]F(x,q;y) = 1 + \sum_{j=0}^{k-2}\sum_{i=1}^{j+1}\frac{\frac{1}{i}\binom{j-1}{i-1}\binom{j}{i-1}x^{2i+j-1}}{(1-x)^{2j+1}(1+x)^j}q^{j+1}. \tag{4}$$

Hence, for all $k \geq 2$, using the usual bracket notation for coefficient extraction,

$$[x^ny^k]F(x,1;y) \sim \frac{1}{(k-1)2^{k-2}(2k-4)!}\binom{2k-4}{k-2}n^{2k-4}$$

and

$$[x^ny^k]\frac{\partial}{\partial q}F(x,q;y)\Big|_{q=1} \sim \frac{1}{2^{k-2}(2k-4)!}\binom{2k-4}{k-2}n^{2k-4},$$

which yields the result in Theorem 1.1-(v). □

3.4 $E_{n,T}(\boldsymbol{L}_n)$ on $\boldsymbol{I}_{\rm n}(3412,\tau)$ with $\tau = k12\cdots(k-1)$

Proof of Theorem 1.1-(vi). Let $G_k(x,q) := H_{k12\cdots(k-1)}(x,q)$. Applying Proposition 3.2-(iv) to $\tau = k12\cdots(k-1)$ with $k \geq 3$, we obtain that

$$G_k(x,q) = 1 + \frac{xq}{1-x}G_k(x,q) + x^2(F_{k-1}(x,q) - F_2(x,q))G_k(x,q),$$

which in view of (2) leads to

$$G_k(x,q) = \frac{1}{1 - \frac{xq}{1-x} - x^2(F_{k-1}(x,q) - 1 - \frac{xq}{1-x})}.$$

Taking (4) into account, we arrive to the following result:

Lemma 3.3. *For $k \geq 3$,*

$$H_{k12\cdots(k-1)}(x,q) = \frac{1}{1 - \frac{xq}{1-x} - x^2\sum_{j=1}^{k-3}\sum_{i=1}^{j+1}\frac{\frac{1}{i}\binom{j-1}{i-1}\binom{j}{i-1}x^{2i+j-1}}{(1-x)^{2j+1}(1+x)^j}q^{j+1}}.$$

For example, $H_{4123}(x,q) = \frac{1}{1-xq/(1-x)-x^4q^2/((1-x)^3(1+x))}$. Let α be the root of smallest absolute value of the polynomial $3x^4 - 3x^3 - x^2 + 3x - 1$. Thus $\alpha \approx 0.45208778430$, and for $T = \{3412, 4123\}$ we have:

$$\begin{aligned} E_{n,T}(L_n) &= \frac{[x^n]\frac{\partial}{\partial q}H_{4123}(x,q)\mid_{q=1}}{[x^n]H_{4123}(x,1)} \\ &\sim \frac{1}{457}(198\alpha^3 - 246\alpha^2 - 131\alpha + 299)n \approx 0.454689799955\cdots n. \end{aligned}$$

This completes the proof of Theorem 1.1-(vi). □

3.5 $E_{n,T}(\boldsymbol{L}_n)$ on $\boldsymbol{I}_{\rm n}(3412,\tau)$ with $\tau = k(k-1)\cdots 1$

Proof of Theorem 1.1-(vii). Let $F_k(x,q) := H_{k(k-1)\cdots 1}(x,q)$. Applying Proposition 3.2 to the permutation $\tau = k(k-1)\cdots 1$ with $k \geq 3$, we see that

$$\begin{aligned} F_k(x,q) &= 1 + (x + x^2 + \cdots + x^{k-1})qF_k(x,q) \\ &\quad + x^2(F_{k-2}(x,q) - 1 - (x + x^2 + \cdots + x^{k-3})q)F_k(x,q). \end{aligned}$$

Thus,

$$F_k(x,q) = \frac{1}{1-qx-(q-1)x^2-x^2F_{k-2}(x,q)} \qquad (5)$$

with $F_1(x,q) = 1$ and $F_2(x,q) = \frac{1}{1-qx}$. Iterating this equation, one can obtain an expression for $F_k(x,q)$ in the form of finite continued fractions. Alternatively, $F_k(x,q)$ can be expressed in terms of Chebyshev polynomials.

Recall that Chebyshev polynomials of the second kind can be defined as the solution to the recursion

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

with initial conditions $U_0(t) = 1$ and $U_1(t) = 2t$. Using this recursion and induction, one can derive the following result from (5).

Lemma 3.4. *For all* $k \geq 1$,

$$H_{(2k+1)(2k)\cdots 1}(x,q) = \frac{U_{k-1}\left(\frac{1-qx-(q-1)x^2}{2x}\right) - xU_{k-2}\left(\frac{1-qx-(q-1)x^2}{2x}\right)}{x\left(U_k\left(\frac{1-qx-(q-1)x^2}{2x}\right) - xU_{k-1}\left(\frac{1-qx-(q-1)x^2}{2x}\right)\right)}$$

and

$$H_{(2k+2)(2k+1)\cdots 1}(x,q) = \frac{\frac{1-xq}{x}U_{k-1}\left(\frac{1-qx-(q-1)x^2}{2x}\right) - U_{k-2}\left(\frac{1-qx-(q-1)x^2}{2x}\right)}{x\left(\frac{1-xq}{x}U_k\left(\frac{1-qx-(q-1)x^2}{2x}\right) - U_{k-1}\left(\frac{1-qx-(q-1)x^2}{2x}\right)\right)}.$$

We remark that the results in Lemma 3.4 with $q = 1$ recover formulas (7) and (8) in [4] for ordinary generating functions for the number of involutions avoiding 3412 and $k(k-1)\cdots 1$.

An application of the lemma with $k = 1$ yields for $T = \{3412, 4321\}$:

$$E_{n,T}(L_n) \quad = \frac{[x^n]\frac{\partial}{\partial q}H_{4321}(x,q)|_{q=1}}{[x^n]H_{4321}(x,1)} \sim \frac{5}{8}n,$$

which completes the proof of Theorem 1.1-(vii). □

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