

FURTHER COMBINATORICS AND APPLICATIONS OF TWO-TONED TILINGS

ROBERT DAVIS AND GREG SIMAY

ABSTRACT. Integer compositions, integer partitions, Fibonacci numbers, and generalizations of these have recently been shown to be interconnected via two-toned tilings of horizontal grids. In this article, we present refinements of two-toned tilings, describe functions which analyze them, and apply these to generalizations of integer compositions and partitions which interpolate between the two.

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1. INTRODUCTION

An n -tiling is an arrangement of *tiles* of sizes 1×1 through $1 \times n$ contained inside of a $1 \times n$ grid, covering the grid such that tiles may only intersect along common boundaries. We say the *length* of a $1 \times k$ tile is k . When n is understood or unimportant, we may simply call an n -tiling a *tiling*. The tiles may come in various colors; in this article, we consider white and red tiles satisfying certain conditions. The combinatorics of such tilings were initially explored in [1], where they were used to determine the number of compositions of an integer with at least or exactly p parts k , as well as general formulas for positively-indexed, generalized Fibonacci numbers.

Definition 1.1. *For nonnegative integers r and n , denote by $a(r, n)$ the number of ways to tile a $1 \times (n + r)$ grid using white tiles of any length (whose total length is n) and r indistinguishable red squares, i.e. tiles of length 1. Such a tiling is called a two-toned tiling of length $n + r$, or simply an $(n + r)$ -tiling when it is understood that the tiling is two-toned.*

If $n = 0$, then $a(r, 0)$ corresponds to a tiling using just the indistinguishable red squares, and $a(r, 0) = 1$. If $r = 0$, then $a(0, n)$ corresponds to a tiling by just the white tiles of lengths 1 to n , hence $a(0, n)$ is the number of compositions of n , i.e. $a(0, n) = 2^{n-1}$ for $n \geq 1$. Values of $a(r, n)$ for small choices of r and n are displayed in Table 1, along with known OEIS [11] sequences.

Using the red squares can allow one to compute the number of compositions of an integer where there are a specified number of specified parts. For example, $a(1, n)$ can denote the number of compositions of $n + k$ such that exactly one part is k and all remaining parts are at most n . Red squares can even represent a specified run of integers as long as none of the integers in the run are already represented by one or more of the white tiles. Indeed, Table 2 lists a variety of applications of two-toned tilings and related notation that will be addressed throughout this article.

$r \backslash n$	0	1	2	3	4	5	OEIS sequence
0	1	1	2	4	8	16	A001782
1	1	2	5	12	28	64	A045623
2	1	3	9	25	66	168	A058396
3	1	4	14	44	129	360	A062109
4	1	5	20	70	225	681	A169792
5	1	6	27	104	363	1182	A169793

TABLE 1. The numbers $a(r, n)$ for small values of r, n . The OEIS entry in row i corresponds to the sequence $\{a(i, n)\}_{n \geq 0}$.

Notation	Meaning
$a(r, n)$	Number of two-toned tilings of length $r + n$
$a(r, n, k)$	Number of ways to tile a $1 \times (n + r)$ grid using white tiles of lengths 1 to k (total length n) and r indistinguishable red squares
$a_s(r, n)$	The number of two-toned tilings of length $r + n + s$
$C_a(r, n)$	Number of parts of all compositions counted by $a(r, n)$
$C_b(n, k)$	Number of compositions of n where all parts k occur consecutively
$C_S(n)$	Number of compositions of n with parts from $S \subseteq [n]$
$C(n, \widehat{k})$	Number of compositions of n with no parts k
$C(n, m, \widehat{k})$	Number of $(n + m)$ -tilings using white tiles of any length except k
$C(n, [k])$	Number of compositions of n with no parts a multiple of k
$CF(n, k)$	Number of compositions of n with k frozen parts
$CF(n, [k])$	Number of compositions of n having multiples of k frozen
$C(n, \langle k_1, \dots, k_m \rangle)$	Number of compositions of n using only parts k_1, \dots, k_m
$E_p(n, k)$	Number of compositions of n with exactly p parts k
$E_p(n, m, k)$	Number of compositions of n with parts at most k having exactly p parts m
$E(n)$	Total number of parts over all compositions of n
$F(n, k, r)$	The r^{th} convolution of $\{F(j, k)\}_{j=1}^n$
$G(n, k)$	Number of compositions of n with largest part k
$G(n, k, r)$	Number of compositions of n with largest part k appearing exactly r times
$L(n, k)$	Number of compositions of n that have at least one instance of k as a part
$L(n, m, k)$	Number of compositions of n with parts at most k having at least one part m
$L_p(n, m, k)$	Number of compositions of n with parts at most k having at least p parts m
$m(r, n)$	Number of $(n + r)$ -tilings when r red squares combine palindromically with the palindromic white tile arrangements
$\text{negF}(n, k)$	The n^{th} negatively-indexed k -step Fibonacci number; see Definition 3.5
$R(n)$	Number of runs in all compositions of n
$R(n, k)$	Number of runs of k in all compositions of n
$R(n, k, l)$	Number of runs of k of length l over all compositions of n
$r(n, \{k\})$	Number of runs in all compositions of n with largest part k
$r(n, j, \{k\})$	Number of runs of j in all compositions of n having parts at most k
$S(n, k)$	Total parts k over all compositions of n

TABLE 2. Collection of notation used throughout the article.

$s \backslash n$	0	1	2	3	4
0	1	3	9	25	66
1	1	4	13	38	104
2	1	5	18	56	160
3	1	6	24	80	240
4	1	7	31	111	351
5	1	8	39	150	501
6	1	9	48	198	699
7	1	10	58	256	955
8	1	11	69	325	1280

TABLE 3. The numbers $a_s(2, n)$ for small values of s, n .

$a_0(0, n)$	$a_1(1, n)$	$a_2(2, n)$	$a_3(3, n)$	$a_4(4, n)$	$a_5(5, n)$	$a_6(6, n)$	$a_7(7, n)$
<u>A058396</u>	<u>A049611</u>	<u>A001793</u>	<u>A001788</u>	<u>A055580</u>	<u>A055581</u>	<u>A055582</u>	<u>A055583</u>

TABLE 4. OEIS sequences for $a_r(r, n)$ when $r \leq 7$.

2. IDENTITIES FOR $a(r, n)$

While there are several explicit expressions for $a(r, n)$ given in [1, 4] the recurrence relation is among the most convenient. By [1, Identity 1], $a(r, n)$ has the recurrence

$$a(r, n) = a(r-1, n) + 2a(r, n-1) - a(r-1, n-1)$$

for $r, n > 1$ with the initial conditions

$$a(r, n) = \begin{cases} 1 & \text{if } r \geq 0, n = 0, \\ 2^{n-1} & \text{if } r = 0, n \geq 1. \end{cases}$$

For example, $a(5, 5) = a(4, 5) + 2a(5, 4) - a(4, 4) = 1182$.

From the recurrence for $a(r, n)$, it is a straightforward induction argument to show that for fixed $r \geq 0$,

$$(2.1) \quad \sum_{n \geq 0} a(r, n)x^n = \left(\frac{1-x}{1-2x} \right)^{r+1}.$$

We can extend this generating function to consider both r and n by setting

$$A(x, y) = \sum_{r \geq 0} \sum_{n \geq 0} a(r, n)x^r y^n.$$

It quickly follows that

$$A(x, y) = \sum_{r \geq 0} \left(\frac{1-y}{1-2y} \right)^{r+1} x^r = \frac{1-y}{1-2y-x+xy}.$$

From these calculations, we see that for fixed r , $a(r, n)$ is the r -th convolution of the sequence of compositions of n . That is, for $r > 0$,

$$a(r, n) = \sum_{j=0}^n a(r-1, n-j)a(0, j).$$

Summations of $a(r, n)$ also have important applications, which in [1] motivated the following definition.

Definition 2.1. Let $s \geq 0$. Denote by $a_s(r, n)$ the number of two-toned tilings of a $1 \times (n+r+s)$ grid with r red squares, with the restriction that the last s tiles must be white.

Note that $a_0(r, n) = a(r, n)$. Values of $a_s(2, n)$ for small values of s and n are given in Table 3, and OEIS entries for $\{a_r(r, n)\}_{n \geq 0}$ for small r are listed in Table 5. By [1, Identity 6], we also have

$$a_s(r, n) = \sum_{i=0}^n a_{s-1}(r, i),$$

from which

$$\sum_{n \geq 0} a_s(r, n)x^n = \frac{1}{(1-x)^s} \left(\frac{1-x}{1-2x} \right)^{r+1}$$

follows. If $r = s$, then

$$\sum_{n \geq 0} a_r(r, n)x^n = \frac{1-x}{(1-2x)^{r+1}},$$

from which we can algebraically extract

$$a_r(r, n) = 2^{n-1} \left(\binom{n+r}{r} + \binom{n+r-1}{r-1} \right),$$

which is consistent with [1, Identity 9]. If $s = r + 1$, then

$$\sum_{n \geq 0} a_{r+1}(r, n)x^n = \frac{1}{(1-2x)^{r+1}},$$

from which we can also algebraically extract

$$a_{r+1}(r, n) = 2^n \binom{n+r}{r}.$$

Now, by [1, Identity 6], we may write a new recurrence for $a(r, n)$:

$$a(r, n) = a_1(r, n-1) + a(r-1, n).$$

Using [1, Identity 8], we then conclude that

$$a_s(r, n) = \sum_{j=0}^n \binom{n-1+s}{j-1+s} \binom{r+j}{r}.$$

The following conjectured identity has a similar flavor to these identities, but a proof has remained elusive.

Conjecture 2.2. *For all $s, r, n \geq 1$, we have*

$$a_s(r, n) = 2^{n-r-1+s} \sum_{j=0}^{r+1-s} \binom{r+1-s}{j} \binom{n+r-j}{n}.$$

Note that, as an immediate consequence of this conjecture, setting $s = 0$ recovers [1, Identity 5], that is,

$$a(r, n) = 2^{n-r-1} \sum_{j=0}^r \binom{r+1}{j} \binom{n+r-j}{n}.$$

3. APPLICATIONS

In this section we will present several applications of $(n+r)$ -tilings to compositions of n with various restrictions. These applications all make use of the function $a_s(r, n)$ in some manner, either explicitly or implicitly.

3.1. k -step Fibonacci numbers, positively and negatively indexed.

Definition 3.1. The n^{th} k -step Fibonacci number, denoted $F(n, k)$, is defined as

$$F(n, k) = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } n = 1, \\ \sum_{j=1}^k F(n-j, k) & \text{if } n \geq 2. \end{cases}$$

For example, the 3-step Fibonacci numbers begin $\dots, 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \dots$. Of course, setting $k = 2$ recovers the usual Fibonacci numbers. It is easy to verify that $F(j, k) = 2^{j-2}$ for $2 \leq j \leq k$.

Benjamin et. al [1, Identity 10] gave a combinatorial proof of the following result involving the k -step Fibonacci numbers. Here, we provide a generating function proof.

Theorem 3.2. For $k \geq 0$ and $n \geq -1$,

$$F(n+1, k) = \sum_{j \geq 0} (-1)^j a_j(j, n-j(k+1)).$$

Proof. We know that the generating function for $F(n+1, k)$ is $(1-x-x^2-\dots-x^k)^{-1}$. So, we examine the generating function for the right side of the desired equation: by reindexing, we get

$$\begin{aligned} \sum_{m \geq 0} \sum_{j \geq 0} (-1)^j a_j(j, m-j(k+1)) x^m &= \sum_{l \geq 0} (-1)^l \frac{1-x}{(1-2x)^{l+1}} x^{l(k+1)} \\ &= \frac{1-x}{1-2x} \sum_{l \geq 0} \left(\frac{-x^{k+1}}{1-2x} \right)^l \\ &= \frac{1-x}{1-2x} \left(\frac{1}{1 + \frac{x^{k+1}}{1-2x}} \right). \end{aligned}$$

From here, routine elementary algebra simplification shows that the resulting generating function is the same as that of $F(n+1, k)$. \square

From the expression for $a_r(r, n)$, Benjamin et al. provided an explicit formula [1, Identity 9] for $F(n+1, k)$. For $n, k \geq 1$,

$$F(n+1, k) = \sum_{r=0}^{\lfloor n/(k+1) \rfloor} (-1)^r \binom{n-rk}{r} \frac{n-rk+r}{n-rk} 2^{n-rk-r-1}.$$

Thus, $a_r(r, n)$ has the following recurrence relation.

Proposition 3.3. For $r, n \geq 1$,

$$a_r(r, n) = 2a_r(r, n-1) + a_{r-1}(r-1, n).$$

Proof. We know that the generating function for $a_r(r, n)$ can be written as

$$\frac{(1-x)^{r+1}}{(1-2x)^{r+1}(1-x)^r}.$$

So, the generating function of the right hand side of the desired identity is

$$2x \frac{(1-x)^{r+1}}{(1-2x)^{r+1}(1-x)^r} + \frac{(1-x)^r}{(1-2x)^r(1-x)^{r-1}}.$$

Routine algebraic simplifications show that this reduces to the generating function for $a_r(r, n)$, as desired. \square

Values of $a_r(r, n)$ for small r, n are displayed in Table 5. Soon we will see that the sequences along the diagonals in the table (an example of which is bolded) arise in an interesting manner.

$r \backslash n$	0	1	2	3	4	5	6
0	1	1	2	4	8	16	32
1	1	3	8	20	48	112	
2	1	5	18	56	160		
3	1	7	32	120			
4	1	9	50				
5	1	10					
6	1						

TABLE 5. Values of $a_r(r, n)$ for small choices of r and n .

Remark 3.4. We pause here to identify a number of connections between the numbers $a_s(r, n)$ and other results in the literature. It appears that the rows of Table 3 are related to p -ascent sequences as defined in [10]. Namely, $\{a_1(2, n)\}_{n \geq 0}$ appears to coincide with 3-ascent sequences; see OEIS sequence [A049611](#). The sequence $\{a_4(2, n)\}_{n \geq 0}$ appears to be a Björner-Welker sequence [2], providing Betti numbers of certain manifolds. The sequence $\{a_6(2, n)\}_{n \geq 0}$ appears to give the popularity of the pattern 231 in permutations of $[n]$; see [13] and OEIS sequence [A055581](#).

In Table 5, $\{a_1(1, n)\}_{n \geq 0}$ arises in [9, Proposition 41] as $\binom{1, n}{1, 2}$, which counts what they call $(n+1)$ -insets of a certain set X . It also appears that $\{a_3(2, n)\}_{n \geq 0}$ of the previous table arises as $\binom{0, n}{k, 2}$ in the same notation. Notably, all of Table 5 appears to be exactly the unsigned coefficients Chebyshev polynomials of the first kind; see OEIS sequence [A081277](#) and [8]. We invite the reader to establish bijections between two-toned tilings (and other objects studied within this article) and the objects listed above.

We will now consider a subtle but significant variation of the k -step Fibonacci numbers.

Definition 3.5. For a positive integer k , the negatively-indexed n^{th} k -step Fibonacci numbers is

$$\text{negF}(n, k) = \text{negF}(n-1, k) + \cdots + \text{negF}(n-k, k)$$

for any n , using the initial conditions $\text{negF}(n, k) = 0$ for $n = 0, -1, -2, \dots, -(k-2)$ and $\text{negF}(1, k) = 1$ for all k .

So, we have $\text{negF}(n, k) = F(n, k)$ whenever n is nonnegative, but $\text{negF}(n, k)$ may be nonzero for negative values of n . For example, with $k = 3$, a portion of the values $\text{negF}(n, 3)$ is

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
$\text{negF}(n, 3)$	-8	4	1	-3	2	0	-1	1	0	0	1

Theorem 3.6. For integers $n, k \geq 1$, let $n+1 = km + r$ where $0 \leq r < k$. Then

$$\text{negF}(-(n+1), k) = \sum_{j \geq 0} (-1)^{r-jk} a_{r+jk}(r+jk, m-r-j(k+1)).$$

Proof. We approach this problem by defining the sequence $\{b_i\}_{i \geq 0}$ by setting $\text{negF}(n, k) = b_{-n+1}$ and finding the generating function for $\{b_i\}_{i \geq 0}$. In this formulation, our new sequence satisfies

1. $b_0 = 0$,
2. $b_1 = \cdots = b_{k-1} = 0$, and
3. $b_i = b_{i-k} - \sum_{m=1}^{k-1} b_{i-m}$ for $i \geq k$.

Through standard algebraic arguments, we obtain

$$\sum_{i \geq 0} b_i x^i = \frac{1-x^k}{1-2x^k+x^{k+1}}.$$

By directly computing the generating function of the right hand side of the desired equality, we obtain the same expression. The claim then follows. \square

Note that positively-indexed classical Fibonacci numbers, i.e. when $k = 2$, obey the equation

$$F(n + 1, 2) = \sum_{i \geq 0} (-1)^i a_i(i, n - 3i),$$

while negatively-indexed Fibonacci numbers obey the equations

$$\text{negF}(-(n + 1), 2) = \sum_{i \geq 0} a_{2i}(2i, m - 3i)$$

if $n = 2m - 1$, and

$$\text{negF}(-(n + 1), 2) = \sum_{i \geq 0} (-1)^{i+1} a_{2i+1}(2i + 1, m - (3i + 1))$$

if $n = 2m$. Putting these together, we see that $\text{negF}(n, 2) = F(-n, 2)$ when $n < 0$. However, as k increases, $F(n, k)$ approaches 2^{n-1} for each n , while in $\text{negF}(n, k)$ for $n < 0$, the sequences $\{a_{j-i}(j - i, i)\}_{i=0}^j$ appear for each $j \geq 1$, separated by increasingly-large strings of 0s. The strings of consecutive nonzero integers are exactly those that are on the diagonals in Table 5.

3.2. Convolutions of the k -step Fibonacci sequence applied to compositions. Recall that if $\{s_i\}_{i \geq 0}$ is a sequence of real numbers, its r^{th} convolution is the sequence of coefficients of

$$\left(\sum_{k \geq 0} s_k x^k \right)^{r+1}$$

expressed in the standard vector space basis $1, x, x^2, \dots$ over \mathbb{R} . Let $\{F(n, k, r)\}_{n \geq 0}$ denote the r -th convolution of the k -step Fibonacci sequence $\{F(j, k)\}_{j=1}^n$, that is, $F(n, k, r)$ is the coefficient of x^k in

$$\left(\sum_{i=0}^n F(i, k) x^i \right)^{r+1}.$$

To illustrate,

$$\begin{aligned} F(n, k, 0) &= F(n, k), \\ F(n, k, 1) &= \sum_{j=1}^n F(n + 1 - j, k) F(j, k), \text{ and} \\ F(n, k, r) &= \sum_{j=1}^n F(n + 1 - j, k, r - 1) F(j, k, r - 1). \end{aligned}$$

Obtaining expressions for $F(n + 1, k, r)$ with $k \geq 1$ motivates the following definition.

Definition 3.7. For nonnegative integers r and n and $1 \leq k \leq n$, let $a(r, n, k)$ denote the number of ways to tile a $1 \times (n + r)$ grid using white tiles of lengths 1 to k (with total length n) and r indistinguishable red squares. Moreover, define

$$a_s(r, n, k) = \sum_{j=0}^n a_{s-1}(r, j, k)$$

where $a_0(r, n, k) = a(r, n, k)$ for all r, n, k .

Note that if $n = 0$, then $a(r, 0, k)$ corresponds to a tiling of just the indistinguishable red squares, so that $a(r, 0, k) = 1$. If $r = 0$, then $a(0, n, k)$ corresponds to a tiling of just the white tiles of lengths 1 to k , which we know to be $F(n + 1, k)$; note how, as $k \rightarrow \infty$, we recover $a(0, n) = C(n)$, the number of compositions of n .

Now, from the definition of two-toned tilings, we directly have the following recurrence.

Proposition 3.8. For all $n, k, r \geq 0$,

$$a(r, n, k) = \sum_{j=1}^k a(r, n - j, k).$$

Furthermore, we will extend equation (2.1). By a routine generating function exercise from Proposition 3.8, we obtain the following.

Proposition 3.9. *For all r and k we have the generating function*

$$\sum_{n \geq 0} a(r, n, k) x^n = \left(\frac{1-x}{1-2x+x^{k+1}} \right)^{r+1}.$$

It is routine to show, for example by [12, Chapter 2, Rule 3], that the generating function $f_{r,k}(n)$ in the proof of Proposition 3.9 is the same as the generating function for $\{F(n+1, k, r)\}_{n \geq 0}$. This gives us the following corollary.

Corollary 3.10. *For all $n, k, r \geq 0$, we have $a(r, n, k) = F(n+1, k, r)$.*

Although the recurrence relation for $a(r, n, k)$ given in Proposition 3.8 provides the most efficient method of computation known thus far, $a(r, n, k)$ can also be expressed as an alternating sum of summed $a_s(r, n)$ quantities.

Proposition 3.11. *For all $n, k, r \geq 0$,*

$$F(n+1, k, r) = a(r, n, k) = \sum_{j \geq 0} (-1)^j \binom{r+j}{r} a_j(r+j, n-j(k+1)).$$

Proof. Note that the generating function for the right side of the identity is

$$\sum_{n \geq 0} \left(\sum_{j \geq 0} (-1)^j \binom{r+j}{r} a_j(r+j, n-j(k+1)) \right) x^n,$$

which simplifies to

$$\sum_{i \geq 0} (-1)^i \binom{r+1}{i} \left(\frac{1-x}{1-2x} \right)^{r+1+i} \frac{x^{i(k+1)}}{(1-x)^i}.$$

Continuing, we get

$$\begin{aligned} \left(\frac{1-x}{1-2x} \right)^{r+1} \sum_{i \geq 0} (-1)^i \binom{r+1}{i} \left(\frac{1-x}{1-2x} \right)^i \frac{x^{i(k+1)}}{(1-x)^i} &= \left(\frac{1-x}{1-2x} \right)^{r+1} \frac{1}{\left(1 + \frac{(1-x)x^{k+1}}{(1-2x)(1-x)} \right)^{r+1}} \\ &= \left(\frac{1-x}{1-2x+x^{k+1}} \right)^{r+1}, \end{aligned}$$

This is exactly the generating function for $F(n+1, k, r)$. □

In [1], the $a(r, n)$ functions were used to compute the number of compositions of n with least or exactly p instances of a given part. The convoluted k -step Fibonacci sequences allow these quantities to be computed when the parts are at most k .

Definition 3.12. *For positive integers m, n, k with $m \leq k$, let $L(n, m, k)$ denote the number of compositions of n with parts at most k such that at least one part is m .*

Table 6 gives values of $F(n+1, 3, r)$ for small choices of n, r . We note that the sequence $\{F(4, 3, r)\}_{r \geq 0}$ has multiple existing combinatorial interpretations, described within the OEIS entry [A000096](#).

Proposition 3.13. *For all m, n, k with $m \leq k$,*

$$L(n, m, k) = \sum_{j \geq 1} (-1)^{j-1} F(n+1-jm, k, j).$$

Proof. First fix a positive integer j and consider $F(n+1-jm, k, j)$. By Proposition 3.11, $F(n+1-jm, k, j) = a(j, n-jm, k)$, the right side of which counts the number of $((n-jm)+j)$ -tilings whose white tiles have length at most k . By replacing each red square with a pink tile of length m , the resulting tiling has length $(n-jm)+jm = n$. Each of these correspond to a composition of n with parts at most k where at least one part has length m . The result follows from applying inclusion-exclusion. □

$j \backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	2	4	7	13	24	44	81
1	1	2	5	12	26	56	118	244	
2	1	3	9	25	63	153	359		
3	1	4	14	44	125	336			
4	1	5	20	70	220				
5	1	6	27	104					
6	1	7	35						
7	1	8							
8	1								

TABLE 6. Values of $F(n+1, 3, r)$ for small choices of n, r .

Next we consider a refinement of $L(n, m, k)$.

Definition 3.14. For $p, k, n \geq 1$ with $m \leq k$, let $L_p(n, m, k)$ denote the number of compositions of n with parts at most k such that there are at least p parts m .

With this notation, $L_1(n, m, k) = L(n, m, k)$. Further, the next proposition follows from the proof of [1, Identity 12].

Proposition 3.15. For all $p, k, n \geq 1$ with $m \leq k$,

$$L_p(n, m, k) = \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p-1} F(n+1 - jm, k, j).$$

Next, we consider compositions having exactly a particular number of parts.

Definition 3.16. For $p, m, n, k \geq 1$ and $m \leq k$, let $E_p(n, m, k)$ denote the number of compositions of n with parts at most k having exactly p parts k .

Proposition 3.17. For $p, m, n, k \geq 1$ with $m \leq k$,

$$E_p(n, m, k) = \sum_{j \geq p} (-1)^{j-p} \binom{j}{p} F(n+1 - jm, k, j).$$

Proof. The equality follows from the observation

$$E_p(n, m, k) = L_p(n, m, k) - L_{p+1}(n, m, k)$$

and from Pascal's identity for binomial coefficients. \square

We note that the previous proposition was also implicitly established in the proof of [1, Identity 13].

Let $S(n, k)$ denote the total number of parts k over all compositions of n . For example, $S(2, 4) = 5$ since there are three compositions of 4 in which 2 appears once and one composition in which 2 appears twice.

Proposition 3.18. For $1 \leq k < n$, $S(n, k) = 2^{n-2}(n+1)$.

Proof. It has been shown [5] that

$$(3.1) \quad S(n, k) = 2^{n-k-2}(n-k+3).$$

By [1, Identity 9], $a_1(1, n) = 2^{n-2}(n+1)$, and hence

$$(3.2) \quad S(n, k) = a(1, n-k).$$

With this in mind, [1, Identity 13] provides a route for establishing (3.2) and therefore (3.1). The number of times k is a part in compositions having exactly p copies of k is $kE_p(n, k)$. So,

$$S(n, k) = \sum_{j \geq 1} jE_j(n, k).$$

Applying [1, Identity 13] to each instance of $E_j(n, k)$, we get

$$S(n, k) = \sum_{j=1}^n a(1, n-j) = a_1(1, n-1) = 2^{n-2}(n+1),$$

as desired. \square

For the next definition, let $\lambda = (\lambda_1, \dots, \lambda_m)$ be an integer composition. A *run* in λ is a subsequence $\lambda_i, \lambda_{i+1}, \dots, \lambda_{i+l}$ such that

$$\lambda_{i-1} \neq \lambda_i = \lambda_{i+1} = \dots = \lambda_{i+l} \neq \lambda_{i+l+1},$$

using the convention $\lambda_0 = \lambda_{m+1} = 0$. The *length* of the run is l . So, for example, $(2, 2, 2, 4, 1, 1, 2)$ has four runs: one of length three, one of length two, and two of length one.

Definition 3.19. For positive integers j, k, n with $j \leq k \leq n$, let $r(n, j, \{k\})$ denote the number of runs of j , irrespective of length of the run, in all compositions of n whose parts are at most k . Further, let $r(n, \{k\})$ denote the total number of runs over all compositions of n whose parts are at most k .

Proposition 3.20. For $1 \leq j \leq k \leq n$,

$$r(n, j, \{k\}) = F(n+1-j, k, 1) - F(n+1-2j, k, 1).$$

Proof. If $j > n-j$, then all runs of j are of length one. The part j may then be represented by a red square in a tiling of n , and is combined with white squares of lengths less than j . The number of such tilings is given by $a(1, n-j, k)$, which is equal to $F(n+1-j, k, 1)$. Also note that if $j > n-j$, then $a(1, n-2j, k) = 0$.

If $j \leq n-j$, then some of the white tiles have length j , thereby being otherwise indistinguishable from the red square representing j . Therefore, some of the n -tilings merely increase the length of existing runs. The number of such instances is $a(1, n-2j, k) = F(n+1-2j, k, 1)$ since increasing the length of a run has the same effect. Hence, $r(n, j, k) = F(n+1-j, k, 1) - F(n+1-2j, k, 1)$. \square

Proposition 3.21. For all $1 \leq k \leq n$,

$$r(n, \{k\}) = \sum_{j \geq 0} F(n-2j, k, 1).$$

Proof. By definition,

$$r(n, \{k\}) = r(n, 1, \{k\}) + \dots + r(n, k, \{k\}).$$

By Proposition 3.20,

$$r(n, j, \{k\}) = F(n+1-j, k, 1) - F(n+1-2j, k, 1).$$

Thus,

$$r(n, \{k\}) = \sum_{j \geq 0} (F(n-j, k, 1) - F(n-1-2j, k, 1)),$$

which simplifies to the desired sum. \square

Definition 3.22. Denote by $C(n, \widehat{k})$ the number of compositions of n for which k is not a part.

From [3, Theorem 1, Equation (1)], we have the following recurrence:

$$C(n, \widehat{k}) = 2C(n-1, \widehat{k}) + C(n-k-1, \widehat{k}) - C(n-k, \widehat{k}).$$

Consequently, the generating function for $C(n, \widehat{k})$ for fixed k is

$$\frac{1-x}{1-2x+x^k-x^{k+1}}.$$

Recall that $C(n)$ denotes the number of compositions of n . Let $L(n, k)$ denote the number of compositions of n that have at least one instance of k as a part. It follows immediately that

$$C(n, \widehat{k}) = C(n) - L(n, k).$$

But $C(n) = a(0, n)$ and [1, Identity 11] gives

$$L(n, k) = \sum_{j \geq 1} (-1)^{j-1} a(j, n-jk)$$

for $n, k \geq 1$. This leads to the following result directly.

Proposition 3.23. For $n, k \geq 1$,

$$C(n, \widehat{k}) = \sum_{j \geq 0} (-1)^j a(j, n - jk).$$

Let $S = \{s_1, \dots, s_m\}$ be a subset of $[n]$, and let $C_S(n)$ denote the number of compositions of n whose unique parts are in S . Then the generating function for $C_S(n)$ is

$$\sum_{n \geq 0} C_S(n) x^n = \frac{1}{1 - x^{s_1} - \dots - x^{s_m}}.$$

From the above equation, it follows that the generating function for $C(n, \widehat{k})$ is

$$\frac{1}{1 + x^k - \sum_{i \geq 0} x^i}.$$

Theorem 3.24. Recall that $F(n - 1, 2)$ is the $(n - 1)$ -st (classical) Fibonacci number. Then

$$F(n - 1, 2) = \sum_{j \geq 0} (-1)^j a(j, n - j).$$

Proof. This follows by setting $k = 1$ in Proposition 3.23. □

Note that if $k > n/2$, then

$$C(n, \widehat{k}) = a(0, n) - a(1, n - k) = 2^{n-1} - 2^{n-k}(n - k + 3).$$

For $r, n, k \geq 0$, let $C(n, m, \widehat{k})$ denote the number of $(n + m)$ -tilings using white tiles of any length except k .

Theorem 3.25. For $n, m, k \geq 0$,

$$\begin{aligned} C(n, m, \widehat{k}) &= E_m(n + mk, k) \\ &= \sum_{j \geq m} (-1)^{j-m} \binom{j}{m} a(j, n - k(j - m)). \end{aligned}$$

Proof. Consider $C(n, m, \widehat{k})$. The white tiles correspond to tiles of any length except k . The red tiles cannot correspond to a nonnegative part already represented by a white tile, but they can correspond to k . So, there is a bijection between the two-toned tilings of $C(n, m, \widehat{k})$ and the number of compositions of $n + mk$ having exactly m copies of k , which is given by $E_m(n + mk, k)$. The summation arises from substituting m for p and $n + mk$ for n . □

To illustrate this result, we have the following special cases:

$$\begin{aligned} C(n, 0, \widehat{k}) &= a(0, n) - a(1, n - k) + a(2, n - 2k) - \dots \\ C(n, 1, \widehat{k}) &= a(1, n) - a(2, n - k) + 3a(3, n - 2k) - \dots \\ C(n + 2k, 2, \widehat{k}) &= a(2, n) - 3a(3, n - k) + 6a(4, n - 2k) - \dots \end{aligned}$$

3.3. The largest and second-largest parts of compositions of n . Let $G(n, k)$ denote the number of compositions of n having k as the largest part, and let $C(n, \langle 1, \dots, k \rangle)$ denote the number of compositions of n whose parts are at most k . It is clear that

$$C(n, \langle 1, \dots, k \rangle) - C(n, \langle 1, \dots, k - 1 \rangle) = G(n, k).$$

Further, by conditioning on the final part, we get

$$C(n, \langle 1, \dots, k \rangle) = \sum_{j=1}^k C(n - j, \langle 1, \dots, k \rangle),$$

from which it follows that $\sum_{n \geq 0} C(n, \langle 1, \dots, k \rangle) x^n = \frac{1-x}{1-2x+x^{k+1}}$. Therefore, $C(n, \langle 1, \dots, k \rangle) = F(n + 1, k)$, and

$$G(n, k) = F(n + 1, k) - F(n + 1, k - 1).$$

It follows that

$$\begin{aligned} \sum_{n \geq 0} G(n, k) x^n &= \frac{x^{k-1}}{(1-x-\dots-x^k)(1-x-\dots-x^{k-1})} \\ &= \frac{x^{k-1}(1-x)^2}{(1-2x+x^{k+1})(1-2x+x^k)}. \end{aligned}$$

Thus, $G(n+k-1, k)$ is the convolution of $F(n+1, k)$ and $F(n+1, k-1)$.

The next result relates $F(n, k)$ to the r th convolutions of $F(n, k-1)$.

Proposition 3.26. *For all n, k ,*

$$F(n, k) = \sum_{j \geq 0} F(n - jk, k - 1, j).$$

Proof. By considering the generating function of the right side, we get

$$\begin{aligned} \sum_{m \geq 0} \left(\sum_{j \geq 0} F(n - jk, k - 1, j) \right) x^m &= \sum_{m \geq 0} \frac{x^{mk}}{(1-x-x^2-\dots-x^{k-1})^{m+1}} \\ &= \frac{1}{1-x-x^2-\dots-x^{k-1}} \cdot \frac{1}{1-\frac{x^k}{1-x-x^2-\dots-x^{k-1}}} \\ &= \frac{1}{1-x-x^2-\dots-x^k}, \end{aligned}$$

which we know to be the generating function of $F(n, k)$. \square

Extending the definition of G , let $G(n, k, r)$ denote the number of compositions of n having k as the largest part exactly r times.

Proposition 3.27. *For $n, k, r \geq 0$, we have*

$$G(n, k, r) = F(n + 1 - kr, k - 1, r).$$

Proof. The value of $G(n, k, r)$ can be represented as a two-toned tiling by concatenating an $(n - kr)$ -tiling using white tiles of length at most $k - 1$ with a kr -tiling using just red squares. The number of ordered tiling arrangements is therefore $a(r, n - kr, k - 1)$, which is equal to $F(n + 1 - kr, k - 1, r)$. \square

Note that the previous result allows us to state that

$$G(n, k) = \sum_{r \geq 1} G(n, k, r).$$

3.4. Frozen parts.

Definition 3.28. *Let $1 \leq j \leq n$. A j -partially-ordered composition of n is any equivalence class of compositions of n under the relation $\lambda \sim \mu$ if and only if*

$$(\lambda_1, \dots, \lambda_j, \lambda_{\sigma(j+1)}, \dots, \lambda_{\sigma(k)}) = \mu$$

for some permutation $\sigma \in \mathfrak{S}_{\{j+1, \dots, k\}}$. The parts $\lambda_1, \dots, \lambda_j$ are called the frozen parts of the composition.

For example, $(8, 7, 9, 6, 4, 3, 2, 1)$ and $(8, 7, 9, 6, 1, 2, 3, 4)$ are the same 4-partially-ordered composition but are distinct 3-partially-ordered compositions. Moreover, it is clear that j -partially-ordered compositions could be defined by freezing any j of the parts; we only consider freezing the first j parts as a matter of computational simplicity. Notice that the 0-partially-frozen compositions are just the ordinary compositions of n , while the n -partially-frozen compositions are just the partitions of n .

Recall that $CF(n, k)$ denotes the number of compositions of n with part k frozen, and that $C(n, \widehat{k})$ denotes the number of compositions of n with no part k .

Proposition 3.29. *For all n and k ,*

$$CF(n, k) = \sum_{j \geq 0} C(n - jk, \widehat{k}).$$

Proof. The compositions of n can be grouped into compositions that have no copies of k , one copy of k , and so forth through j copies of k . Because the jk are frozen, they can be treated as being appended to the compositions of $n - jk$ having no k . The result follows. \square

Building off of $C(n, \langle 1, \dots, k \rangle)$, given nonnegative integers n, k_1, \dots, k_m with $1 \leq m \leq n$, let $C(n, \langle k_1, \dots, k_m \rangle)$ denote the number of compositions of n consisting only of parts k_1, \dots, k_m . For example, $C(5, \langle 1, 2, 5 \rangle) = 9$ since the compositions are the distinct permutations of (5) , $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$. Note as well that $C(n, \langle 1, \dots, k \rangle) = F(n + 1, k)$.

Proposition 3.30. *For all n, k ,*

$$CF(n, k) = C(n, \langle 1, \dots, k, 2k \rangle).$$

Proof. From [3, Theorem 1], the generating function for $C(n, \widehat{k})$ is

$$\sum_{n \geq 0} C(n, \widehat{k})x^n = \frac{1 - x}{1 - 2x + x^k - x^{k+1}}.$$

By Proposition 3.29, we can compare the generating functions and use routine algebra to conclude that

$$\begin{aligned} \sum_{n \geq 0} CF(n, k)x^n &= (1 + x^k + x^{2k} + \dots) \sum_{n \geq 0} C(n, \widehat{k})x^n \\ &= \frac{1}{1 - x^k} \cdot \frac{1 - x}{1 - 2x + x^k - x^{k+1}} \\ &= \frac{1}{1 - x - x^2 - \dots - x^k - x^{2k}}. \end{aligned}$$

\square

Proposition 3.31. *For all n, k ,*

$$CF(n, k) = \sum_{j \geq 0} F(n + 1 - 2kj, k, j).$$

Proof. By Proposition 3.30, $CF(n, k)$ consists of three things: compositions having only parts 1 through k ; compositions with parts from $1, \dots, k, 2k$; and compositions from parts $2k$ only. In the first case, there are $F(n + 1, k) = F(n + 1, k, 0)$ compositions. The compositions of the second type can be counted by $a(j, n - 2kj, k)$, corresponding to a tiling where part $2k$ has the role of the red square. The compositions of the third type consist of j copies of $2k$ only, which itself corresponds to j red squares. This is counted by $a(j, 2kj - 2kj, k) = a(j, 0, k) = 1$. With $a(r, n, k) = F(n + 1, k, r)$, the theorem follows. \square

3.5. Formulas involving the parts making up the compositions of n . The two-toned tiling functions provide a quick way to evaluate what might otherwise be obscure quantities.

Theorem 3.32. *Let $1 \leq j \leq n$. If each part j used in the compositions of n is replaced by the compositions of j , then the total number of resulting compositions is $a_1(2, n - 1)$.*

Proof. The number of times j is used in the compositions of n is $a(1, n - j)$. If the compositions of j are substituted for j then the resulting number of compositions is $2^{j-1}a(1, n - j)$, which is equal to $a(1, n - j)a(0, j)$. The total number of these compositions is

$$\sum_{j=1}^n a_1(1, n - j)a(0, j).$$

The corresponding generating function is

$$\left(\frac{1}{1 - x} \right) \left(\frac{1 - x}{1 - 2x} \right)^2 \left(\frac{1 - x}{1 - 2x} \right) = \frac{1}{1 - x} \left(\frac{1 - x}{1 - 2x} \right)^3,$$

which corresponds to $a_1(2, n - 1)$. \square

Proposition 3.33. *Let $1 \leq j \leq n$. If each part j used in the compositions of n is replaced by the parts used in the compositions of j , then the number of resulting parts is $a_1(3, n - 1)$.*

Proof. The number of times j is used in the compositions of n is $a(1, n - j)$. If the number of parts used in the compositions of j are substituted for j then the resulting number of parts is $a(1, n - j)a_1(1, n - j)$. The total number of these parts is

$$\sum_{j=1}^n a(1, n - j)a_1(1, n - j),$$

which has corresponding generating function

$$\left(\frac{1-x}{1-2x}\right)^2 \left(\frac{1}{1-x}\right)^2 \left(\frac{1-x}{1-2x}\right) = \frac{1}{1-x} \left(\frac{1-x}{1-2x}\right)^4.$$

This corresponds to $a_1(3, n - 1)$. □

Definition 3.34. For $r, n \geq 1$, let $C_a(r, n)$ denote the number of parts of all compositions counted by $a(r, n)$.

Proposition 3.35. For all r, n ,

$$C_a(r, n) = (r + 1)a_1(r + 1, n - 1) + ra_0(r, n).$$

Proof. Consider first the number of red tiles needed to form the compositions of $a(r, n)$. Each composition of n has r red tiles, so the number of red tiles needed is $ra(r, n)$.

Now consider the number of white tiles needed to form the compositions of $a(r, n)$. For a composition of $a(r, n)$ consisting of exactly j white tiles, the total number of parts is $j \binom{r+j}{r} \binom{n-1}{j-1}$, and the total number of white tiles is

$$\sum_{j=1}^n j \binom{r+j}{r} \binom{n-1}{j-1} = (r + 1)a_1(r + 1, n - 1).$$

□

Next, we denote by $C_b(n, k)$ the number of compositions of n where all parts k must occur consecutively. For example, $C_b(4, 1) = 7$ since the compositions counted are (4) , $(3, 1)$, $(1, 3)$, $(2, 2)$, $(2, 1, 1)$, $(1, 1, 2)$, $(1, 1, 1, 1)$. We will refine $C_b(n, k)$ by $C_b(n, k, p)$, which denotes the number of compositions of n having exactly p parts k , which must occur consecutively. So, $C_b(4, 1, 2) = 2$ since the compositions counted are $(2, 1, 1)$ and $(1, 1, 2)$.

Proposition 3.36. For all n, k , and p ,

$$C_b(n, k, p) = C(1, n - pk, \widehat{k}) = E_1(n - (p - 1)k, k).$$

Proof. Consider the compositions of n in which we insist the parts that are k are blocked together exactly p times. The quantity $n - pk$ then corresponds to the parts not equal to k . Because all of the k s are consecutive, the compositions are in bijection with compositions with a single k . Thus, in terms of an $(n + r)$ -tiling, the (k, \dots, k) is in bijection with a single red square. So, $C_b(n, k, p) = C(1, n - pk, \widehat{k})$.

Next, recall that the single red square appearing in $C(1, n - pk, \widehat{k})$ can represent k , which leads to the interpretation that $C(1, n - pk, \widehat{k})$ also counts the compositions of $n - pk + k = n - (p - 1)k$ having exactly one k . By definition, these are enumerated by $E_1(n - (p - 1)k, k)$. □

Proposition 3.37. For all n and k ,

$$C_b(n, k) = C(n, \widehat{k}) + \sum_{j \geq 0} E_1(n - jk, k).$$

Proof. The number of compositions with no k s is $C(n, \widehat{k})$. Additionally, the number of compositions for which the j copies of k are all consecutive is $E_1(n - (j - 1)k, k)$. Summing these cases and adjusting indices gives the result. □

The following follows directly from inclusion-exclusion, so the proof is omitted.

Proposition 3.38. For all n, k, p , we have

$$C_b(n, k, p) = \sum_{j \geq 1} (-1)^{j+1} ja(j, n - k(p + j - 1)).$$

Let $C(n, [k])$ denote the number of compositions of n with no parts a multiple of k . For example, $C(4, [2]) = 3$, counting the compositions $(3, 1)$, $(1, 3)$, and $(1, 1, 1, 1)$.

Proposition 3.39. *For all n and k ,*

$$C(n, [k]) = F(n + 1, k) - F(n + 1 - k, k).$$

Proof. The generating function for the right side of the desired equation is

$$\frac{1 - x^k}{1 - x - x^2 - \dots - x^k},$$

whereas the generating function for the left hand side is

$$\frac{1}{1 - \left(\sum_{i \geq 1} x^i\right) + \left(\sum_{j \geq 1} x^{jk}\right)}.$$

After simplifying the latter expression, we obtain the former. \square

We now return to runs of parts. Recall that given a composition $\lambda = (\lambda_1, \dots, \lambda_k)$, a *run* in λ is a subsequence of the form

$$\lambda_{i-1} \neq \lambda_i = \lambda_{i+1} = \dots = \lambda_j \neq \lambda_{j+1}$$

for some $1 \leq i \leq j \leq k$, using the convention $\lambda_0 = \lambda_{k+1} = 0$. Let $R(n, k)$ denote the total number of runs consisting of part k over all compositions of n , without regard to the length of the run.

Proposition 3.40. *For all n and k , $R(n, k) = a(1, n - k) - a(1, n - 2k)$.*

Proof. If $k > n - k$, all runs of k have length one. The part k can be represented by a red square, combining with white squares representing nonnegative integers less than k . The numbers of such $(n + r)$ -tilings is given by $a(1, n - k)$. Also note that if $k > n - k$, then $a(1, n - 2k) = 0$.

If $k \leq n - k$, then some of the white tiles have length k , thereby being indistinguishable from the red square representing k . Therefore, some of the $(n + r)$ -tilings merely increase the length of existing runs. The number of such instances is $a(1, n - 2k)$ because increasing the length of a run has the same effect. Hence, $a(1, n - k) - a(1, n - 2k) = R(n, k)$. \square

Corollary 3.41. *For all n and k ,*

$$R(n, k) = 2^{n-k-2}(n - k + 3) - 2^{n-2k-2}(n - 2k + 3).$$

Extending $R(n, k)$, let $R(n)$ denote the total number of all runs over all compositions of n .

Corollary 3.42. *For all n ,*

$$R(n) = \sum_{k \geq 1} a(1, n - (2k - 1)).$$

Proof. The result follows from observing that

$$R(n) = \sum_{k \geq 1} R(n, k) = \sum_{k \geq 1} a(1, n - k) - a(1, n - 2k).$$

In this sum, all terms of the form $a(1, n - 2k)$ are cancelled, leaving only those of the form $a(1, n - (2k + 1))$, as desired. \square

Let $E(n)$ denote the total number of parts used in all compositions of n . This is well-known to be $(n + 1)2^{n-2}$, but what is more interesting here is that we can relate $E(n)$ to other aspects of compositions. For example, note that $C(n) \leq R(n) \leq E(n)$. In fact, this relationship can be made more explicit.

Lemma 3.43. *The number of parts making up the compositions of n is equal to the number of runs of n and of $n - 1$. In other words, $E(n) = R(n) + R(n - 1)$.*

Proof. This follows entirely algebraically. Observe that

$$\begin{aligned} E(n) &= a_1(1, n-1) \\ &= \sum_{i \geq 1} a(1, n-i) \\ &= \left(\sum_{i \geq 1} a(1, n-2i+1) \right) + \left(\sum_{i \geq 1} a(1, n-2i) \right) \\ &= R(n) + R(n-1), \end{aligned}$$

as desired. \square

Next we will refine R by letting $R(n, k, l)$ denote the number of runs of k of length l over all compositions of n .

Conjecture 3.44. For all n, k, l ,

$$R(n, k, l) = a(1, n - kl) - 2a(1, n - (l+1)k) + a(1, n - (l+2)k).$$

3.6. Pell numbers. Recall that the sequence of *Pell numbers* is $\{P(n)\}_{n \geq 0}$ where $P(0) = 0$, $P(1) = 1$, and $P(n) = 2P(n-1) + P(n-2)$ for $n \geq 2$. The Pell numbers have many combinatorial interpretations, such as the sequence of denominators of the continued fraction for $\sqrt{2}$, and the number of 132-avoiding two-stack-sortable permutations [6]. We prove the following proposition using a generating function argument, but in light of the combinatorial descriptions of both the Pell numbers and the functions $a_s(n, k)$, we invite the reader to provide a combinatorial proof.

Proposition 3.45. For all n ,

$$P(n) = \sum_{i \geq 0} a_{2i}(2i+1, n-4i)$$

Proof. Using the recurrence for $P(n)$, is it straightforward to show that $P(n)$ has generating function

$$\sum_{n \geq 0} P(n)x^n = \frac{1}{1-2x-x^2}.$$

Let

$$u = \frac{1-x}{1-2x}.$$

Then the generating function of the right side of the desired identity is

$$\begin{aligned} \sum_{n \geq 0} u^2 \left(\frac{ux^{2n}}{1-x} \right)^{2n} &= \frac{(1-x)^2}{(1-2x)^2 - x^4} \\ &= \frac{1}{1-2x-x^2}, \end{aligned}$$

as desired. \square

3.7. Applications to palindromes. In [3], the authors found a number of generating functions and recurrence relations for palindromic quantities, and provided extensive data. The $a(r, n)$ functions can be used to provide formulas for this data. First, let $m(r, n)$ denote the number of $(n+r)$ -tilings when r red squares combine palindromically with the palindromic white tile arrangements.

To illustrate, consider the four palindromes of 6 without a central part: 33, 1221, 2112, and 111111. Combining two red squares, which we will represent by r in the palindromes, with the corresponding tilings of the palindromes, we result in twelve total palindromes:

$$\begin{array}{ll} 3rr3 & r1221r \\ r33r & r2112r \\ 12rr21 & 111rr111 \\ 1r22r1 & 11r11r11 \\ 21rr12 & 1r1111r1 \\ 2r11r2 & r111111r \end{array}$$

$r \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	1	2	2	4	4	8	8	16	16
1	1	0	1	0	2	0	4	0	8	0
2	1	1	3	3	8	8	20	20	48	48
3	1	0	2	0	5	0	12	0	28	0
4	1	1	4	4	13	13	38	38	104	104
5	1	0	3	0	9	0	25	0	66	0
6	1	1	5	5	19	19	63	63	192	192
7	1	0	4	0	14	0	44	0	129	0
8	1	1	6	6	26	26	96	96	321	321

TABLE 7. $m(r, n)$ for small choices of r and n .

Recall that a white tile of length k can be represented by the positive integer k . The red square can be represented by any integer whose length is not duplicated by any of the white spaces.

Proposition 3.46. *For all r and n ,*

$$\begin{aligned} m(2r, 2n) &= m(2r, 2n + 1) = a_1(r, \lfloor n/2 \rfloor) \\ m(2r + 1, 2n) &= a_0(r, n) \\ m(2r, 2n + 1) &= 0 \end{aligned}$$

Proof. We will treat each case separately. First let $N = 2n$ and $R = 2r$. The palindrome compositions of N are the compositions of $n - k$ paired with their mirror images and central part $2k$ (or no central part if $k = 0$), for some $k = 0, \dots, n$. This is counted by

$$a(r, n) + a(r, n - 1) + \dots + a(r, 0) = a_1(r, n).$$

So, $m(R, N) = a_1(r, n)$.

Now suppose $N = 2n$ and $R = 2r + 1$. In this case, there is an unpaired red square. It cannot be combined with an existing central part since that would disrupt the palindromicity. However, the square can serve as a lone central part. Thus, the number of these is $a_0(r, n)$. Putting this together with the case of $N = 2n$ and $R = 2r$, we obtain the first two inequalities in the statement of our result.

If $N = 2n + 1$ and $R = 2r$, then all of the palindromic compositions will have a central part. Thus, counting each $a(r, l)$ $l = 0, \dots, n$, we end up with $a_1(r, n)$.

Finally, suppose $N = 2n + 1$ and $R = 2r + 1$. In this case, all of the palindromic compositions have a central part. Also, there must be unpaired one square. However, it is impossible for these to simultaneously exist, so no such compositions exist, completing the proof. \square

Remark 3.47. *We again pause to mention connections with the existing literature. The sequence $\{m(3, 2n)\}_{n \geq 0}$ appears to coincide with $C_n^{2,2}$ of [7], counting certain equivalence classes of objects enumerated by the Catalan numbers. Multiple rows of Table 7 suggest that they are the coefficients of $C_2(x)^k$ for various k , as described in [7, Corollary 3.16]. We have seen $\{m(4, 2n)\}_{n \geq 0}$ before, as $\{a_1(2, n)\}_{n \geq 0}$. Again, we invite the reader to establish bijections among the objects counted by these sequences and the Catalan-like objects found in [7].*

Definition 3.48. *For nonnegative integers n , let $\text{Pal}(n)$ denote the number of palindromic compositions of n and let $\text{Pal}(n, \hat{k})$ denote the number of palindromes of n with no part k .*

It is not difficult to argue that $\text{Pal}(2n) = \text{Pal}(2n + 1) = 2^n$, and observe that $a_1(0, n) = 2^n$.

Theorem 3.49. *If n and k are nonnegative integers,*

$$\text{Pal}(n, k) = \sum_{j \geq 0} (-1)^j m(j, n - 2j).$$

In particular, if n and k have the same parity, then

$$\text{Pal}(n, k) = \sum_{j \geq 1} (-1)^j (a_1(j, n - jk) - a(j, n - (j + 1)k)),$$

and if n and k have different parity, then

$$\text{Pal}(n, k) = \sum_{j \geq 0} (-1)^j a_1(j, n - 2k).$$

Proof. First consider the case where n and k are both even, say $k = 2j$ for some integer j . Any palindromic composition of n must then have either no central part (i.e. a central part of 0) or an even central part. That is, a palindromic composition of n is of the form

$$(c_1, c_2, \dots, c_s, \ell, c_s, \dots, c_2, c_1)$$

where ℓ is a nonnegative even integer and (c_1, \dots, c_s) is a composition of $n' = \frac{1}{2}(n - \ell)$ with no parts k . The number of such compositions is $c(n', \widehat{k})$. By ranging over all possible ℓ , and then subtracting the instance where $\ell = k$, we obtain the desired formula. The formula involving $m(\cdot, \cdot)$ follows as a consequence.

The remaining cases (where n and k both odd or have different parity) follow similarly and therefore their details are omitted. \square

The above result has the following immediate corollary.

Corollary 3.50. *The number of palindromic compositions of n having at least one part k is*

$$\text{Pal}(n) - \text{Pal}(n, \widehat{k}) = \sum_{j \geq 1} (-1)^{j-1} (m(2j - 1, n - (2j - 1)k) + m(2j, 2jk)).$$

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DEPARTMENT OF MATHEMATICS, COLGATE UNIVERSITY, 13 OAK DR., HAMILTON, NY 13346
E-mail address: rdavis@colgate.edu

E-mail address: gregsimay@yahoo.com