# SOME RESULTS ON AVERAGE OF FIBONACCI AND LUCAS SEQUENCES 

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#### Abstract

The numerical sequence in which the $n$-th term is the average (that is, arithmetic mean) of the of the first $n$ Fibonacci numbers may be found in the OEIS (see A111035 ). An interesting question one might pose is which terms of the sequences $$
\begin{equation*} \left(\frac{1}{n} \sum_{i=1}^{n} F_{i}\right)_{n=1}^{\infty} \text { and }\left(\frac{1}{n} \sum_{i=1}^{n} L_{i}\right)_{n=1}^{\infty} \tag{0.1} \end{equation*}
$$ are integers? The average of the first three Fibonacci sequence, $(1+1+3) / 3=4 / 3$, is not whereas the average of the first 24 Fibonacci sequence, $\frac{\sum_{i=1}^{24} F_{i}}{24}=5058$ is. In this paper, we address this question and also present some properties of average Fibonacci and Lucas numbers using the Wall-Sun-Sun prime conjecture.


## 1. Introduction

Fibonacci numbers originally arose in a problem in Liber Abaci, published in 1202, which was one of the first texts to describe the Hindu-Arabic numeral system. Since then Fibonacci numbers have become one of the most popular sequences to study, appearing in a wealth of problems not only in enumerative combinatorics.
The Fibonacci numbers, denoted by $\left\{F_{n}\right\}_{n=0}^{\infty}$, are defined by the following recurrence relation

$$
\begin{equation*}
F_{n+1}=F_{n}+F_{n-1}, \tag{1.1}
\end{equation*}
$$

with initial values $F_{0}=0$ and $F_{1}=1$. The first elements of this sequence are given in (A000045) , as

$$
\begin{array}{llllllllll}
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \ldots
\end{array}
$$

The Fibonacci numbers are closely related to binomial coefficients; it is a well-known fact that they are given by the sums of the rising diagonal lines of Pascal's triangle (see [1])

$$
F_{n+1}=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-i}{i}
$$

The Lucas numbers $\left\{L_{n}\right\}_{n=0}^{\infty}$, are defined by the same recurrence relation as the Fibonacci numbers with different initial values (see A000032). In a similar, way the Lucas sequence

[^0]satisfies the recurrence relation
\[

$$
\begin{equation*}
L_{n+1}=L_{n}+L_{n-1}, \quad L_{0}=2, \quad L_{1}=1 \tag{1.2}
\end{equation*}
$$

\]

Fibonacci and Lucas numbers have been extensively studied. A closed form of the Binet's formula, for instance, can be expressed in terms of the characteristic roots of the Fibonacci sequence. Let $\alpha$ and $\beta$ denote the roots of the polynomial $x^{2}-x-1=0$, i.e., $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$.

$$
F_{n}=\frac{1}{\sqrt{5}}\left(\alpha^{n}-\beta^{n}\right) \quad \text { and } \quad L_{n}=\alpha^{n}+\beta^{n}
$$

These formulas can be extended to negative integers $n$ in a natural way with $F_{-n}=(-1)^{n-1} F_{n}$ and $L_{-n}=(-1)^{n} L_{n}$ for all $n$.

In this paper, we pose the following question with a slightly different formulation.
Problem 1.1. which terms of the sequences

$$
\begin{equation*}
\left(\frac{1}{n} \sum_{i=1}^{n} F_{i}\right)_{n=1}^{\infty} \text { and } \quad\left(\frac{1}{n} \sum_{i=1}^{n} L_{i}\right)_{n=1}^{\infty} \tag{1.3}
\end{equation*}
$$

are integers?
The average of the Fibonacci and Lucas sequences is not always an integer, for instance, Fibonacci sequence for $n=3$ have $\frac{1+1+2}{3}=\frac{4}{3}$. However, for $n=24$ for instance, we obtain an integer as the average. We will be interested in examining for which $n$ is this average is an integer.

The paper is organized as follows. First, in Section 2, we recall some well-known facts about the Fibonacci and Lucas numbers that will be used later. Section 3 includes the main theorem and finally, in Section 4, we gave further properties on the average of Fibonacci and Lucas numbers using the Wall-Sun-Sun prime conjecture.

## 2. Preliminaries

First, we give some identities and theorems about Fibonacci and Lucas numbers which we will make use of later[1].

$$
\begin{gather*}
2\left|F_{n} \Leftrightarrow 2\right| L_{n} \Leftrightarrow 3 \mid n ;  \tag{2.1}\\
F_{1}+F_{2}+F_{3}+\cdots F_{n}=F_{n+2}-1 ;  \tag{2.2}\\
L_{1}+L_{2}+L_{3}+\cdots L_{n}=L_{n+2}-3 ;  \tag{2.3}\\
F_{2 k}=F_{k+1}^{2}-F_{k-1}^{2} ;  \tag{2.4}\\
24 \mid F_{12 k} ;  \tag{2.5}\\
\text { for } n \geq 3 F_{n}\left|F_{m} \Leftrightarrow n\right| m . \tag{2.6}
\end{gather*}
$$

We also recall the following known formulas for Fibonacci and Lucas sequences [2].

Theorem 2.1. For any positive integer $k$, we have

$$
\begin{aligned}
& F_{4 k+2}-1=F_{2 k} L_{2 k+2} \\
& F_{4 k+3}-1=F_{2 k+2} L_{2 k+1} \\
& F_{4 k+4}-1=F_{2 k+3} L_{2 k+1} \\
& F_{4 k+5}-1=F_{2 k+2} L_{2 k+3} .
\end{aligned}
$$

Lemma 2.2. Let $n$ be a positive integer. Then

$$
L_{n} \equiv L_{n+6} \quad(\bmod 4)
$$

Proof. We know that the Lucas sequence satisfies the recurrence relation

$$
\begin{equation*}
L_{n}=L_{n-1}+L_{n-2} \tag{2.7}
\end{equation*}
$$

with the initial values $L_{0}=2$ and $L_{1}=1$. Consider the first values of this sequence. It is easy to see $L_{1} \equiv L_{7}(\bmod 4), \quad L_{2} \equiv L_{8}(\bmod 4), \quad L_{3} \equiv L_{9}(\bmod 4)$, and so on. The claim follows by induction on $n$ using the formula (2.7).

Lemma 2.3. Let $k$ be a be a positive integer. The sum of three consecutive Lucas numbers with odd index is divisible by 4, i.e.,

$$
L_{2 k+1}+L_{2 k+3}+L_{2 k+5} \equiv 0 \quad(\bmod 4)
$$

Proof. Consider the first values of the Lucas sequence that have odd index. The proof follows by induction on $k$ as in the proof of Lemma (2.2).

Lemma 2.4. Let $k$ be a positive integer. Then

$$
L_{2 k+2} \not \equiv 0 \quad(\bmod 4)
$$

Proof. We show that $L_{2 k+2} \equiv 2(\bmod 4)$ whenever the Lucas number $L_{2 k+2}$ is even. Using the well-known formulas for the Lucas sequence, we can write

$$
L_{2 k+2}-2=L_{2(k+1)}-2=\sum_{i=1}^{k+1} L_{2 i-1} .
$$

According (2.1), since $L_{2 k+2}$ is an even number when $3 \mid k+1$. Hence, for positive integers $k^{\prime}$, we have

$$
\sum_{i=1}^{k+1} L_{2 i-1}=\sum_{i=1}^{3 k^{\prime}} L_{2 i-1}
$$

We will show by induction on $k^{\prime}$ that

$$
\sum_{i=1}^{3 k^{\prime}} L_{2 i-1} \equiv 0 \quad(\bmod 4)
$$

For $k^{\prime}=1$ we have

$$
\sum_{i=1}^{3} L_{2 i-1}=16 \equiv 0 \quad(\bmod 4)
$$

Assume now that it holds for all positive integers less than some $k^{\prime} \geq 2$. For $k^{\prime}+1$ we have

$$
\sum_{i=1}^{3\left(k^{\prime}+1\right)} L_{2 i-1}=\sum_{i=1}^{3 k^{\prime}} L_{2 i-1}+\sum_{i=3 k^{\prime}+1}^{3 k^{\prime}+3} L_{2 i-1}
$$

Using the Lemma (2.3), $\sum_{i=3 k^{\prime}+1}^{3 k^{\prime}+3} L_{2 i-1} \equiv 0(\bmod 4)$, which completes the proof.

## 3. Main Theorems

In this section, we focus on the average of Fibonacci sequences (OEIS: A111035 ). Looking at the first few values of this sequence we see that most of these numbers are divisible by 24.
In our proof, we use often identity (2.2), which states that the sum of the first $n$ Fibonacci numbers is $F_{n+2}-1$. (this sequence also appears in the OEIS, see A000071).

Theorem 3.1. Let $n$ be a positive integer. There are infinitely many numbers such that

$$
n \mid \sum_{i=1}^{n} F_{i} .
$$

Proof. For a positive integer $k$, consider $n=4 k$ such that $4 k \mid \sum_{i=1}^{4 k} F_{i}$. Identity (2.2) and Theorem (2.1) give

$$
4 k \mid \sum_{i=1}^{4 k} F_{i}=F_{4 k+2}-1=F_{2 k} L_{2 k+2} .
$$

By Lemma (2.4), $4 k \nmid L_{2 k+2}$, which implies that $4 k \mid F_{2 k}$. Identity (2.4) thus yields

$$
4 \mid F_{k+1}^{2}-F_{k-1}^{2}
$$

This means that $F_{k+1}$ and $F_{k-1}$ are either both even, or they are both odd. It follows that $F_{k}$ is even, thus according identity (2.1) we may set $k=3 k^{\prime}$ and so $12 k^{\prime} \mid F_{6 k^{\prime}}$. Put $k^{\prime}=2 k^{\prime \prime}$, which gives $24 k^{\prime \prime} \mid F_{12 k^{\prime \prime}}$. According (2.5), the following holds

$$
\begin{equation*}
k^{\prime \prime} \mid F_{12 k^{\prime \prime}} \tag{3.1}
\end{equation*}
$$

Now, to conclude the proof, it is sufficient to show that there are infinitely many numbers such that $k^{\prime \prime} \mid F_{12 k^{\prime \prime}}$. Put $k^{\prime \prime}=F_{3 \ell}$, so $F_{3 \ell} \mid F_{12 F_{3 \ell}}$, by (2.6), $3 \ell \mid 12 F_{3 \ell}$. Since $3 \mid 12 F_{3 \ell}$, we have $\ell \mid F_{3 \ell}$.

Continuing this process we put $\ell=F_{3 d}$, with $d$ a positive integer. So, $F_{3 d} \mid F_{3 F_{3 d}}$, and this leads to $3 d \mid 3 F_{3 d}$, thus, $d \mid F_{3 d}$. Continuing this process infinitely many times, we have that $k^{\prime \prime}$ is equal to the following sequence

$$
k^{\prime \prime}=F_{3 F_{3 F_{3 F}}}
$$

In the proof of the Theorem (3.1), using the properties of the Fibonacci numbers, there are infinitely many numbers $k^{\prime \prime}$ such that $k^{\prime \prime} \mid F_{k^{\prime \prime}}$ and $F_{k^{\prime \prime}} \mid F_{12 k^{\prime \prime}}$, We have, there are infinity many numbers such that $k^{\prime \prime} \mid F_{12 k^{\prime \prime}}$.
In the following theorem, we proved the result of the Theorem (3.1) is true for Lucas sequence. First, we recall the following known formula for Lucas sequences (see [1], page 111).

Lemma 3.2. Let $m \geq n$ be positive integers. Then

$$
L_{m+n}-L_{m-n}=\left\{\begin{array}{lc}
L_{m} L_{n} & \text { if } n \text { is odd } \\
5 F_{m} F_{n} & \text { otherwise } .
\end{array}\right.
$$

Theorem 3.3. Let $n$ be a positive integer. There are infinitely many numbers such that

$$
n \mid \sum_{i=1}^{n} L_{i} .
$$

Proof. According Lemma (3.2), for $n=2 k$ and $m=2 k+2$ we have

$$
L_{m+n}-L_{m-n}=L_{4 k+2}-L_{2}=L_{4 k+2}-3=5 F_{2 k+2} F_{2 k}
$$

Identity (2.3) give

$$
\sum_{i=1}^{4 k} L_{i}=F_{4 k+2}-3=5 F_{2 k} F_{2 k+2}
$$

To conclude the proof, it is sufficient consider $n=4 k$ such that $4 k \mid \sum_{i=1}^{4 k} L_{i}=5 F_{2 k} F_{2 k+2}$. Now, with similar proof of the Theorem (3.1), there are infinitely many numbers $k$, such that $4 k \mid F_{2 k}$.

Theorem 3.4. Let $\alpha$ be a non-negative integer. Then

$$
\begin{equation*}
3.2^{\alpha+3} \mid \sum_{i=1}^{3.2^{\alpha+3}} F_{i}=F_{3.2^{\alpha+3}+2}-1 \tag{3.2}
\end{equation*}
$$

Proof. Let $k \geq 1$ be a positive integer. We will use the Fibonacci identities

$$
F_{4 k+2}-1=F_{2 k} L_{2 k+2} \text { and } F_{2 k}=F_{k} L_{k} .
$$

Therefore

$$
\begin{aligned}
F_{3.2^{\alpha+3}+2}-1 & =F_{3.2^{\alpha+2}} L_{3.2^{\alpha+2}+2} \\
& =F_{3.2^{\alpha+1}} L_{3.2^{\alpha+1}} L_{3.2^{\alpha+2}+2} \\
& =F_{3.2^{\alpha}} L_{3.2^{\alpha}} L_{3.2^{\alpha+1}} L_{3.2^{\alpha+2}+2} \\
& =\cdots \\
& =F_{3} L_{3} L_{6} \cdots L_{3.2^{\alpha}} L_{3.2^{\alpha+1}} L_{3.2^{\alpha+2}+2}
\end{aligned}
$$

But $F_{3}=2, L_{3}=4$ and each of $L_{6}, \cdots, L_{3.2^{\alpha+1}}$ are divisible by 2. Also, $L_{3.2^{\alpha+2}+2}$ is divisible by 3. This concludes the proof.

The following theorem is a generalization of Theorem (3.4).
Theorem 3.5. Let $\alpha, \beta$ and $\gamma$ be positive integers. Then

$$
\begin{equation*}
2^{\alpha+3} .3^{\beta+1} .5^{\gamma} \mid \sum_{i=1}^{2^{\alpha+3} .3^{\beta+1} .5^{\gamma}} F_{i} \tag{3.3}
\end{equation*}
$$

Proof. In Theorem (3.4), we proved the cases $\alpha=\beta=\gamma=0$. Now, let $\beta, \gamma>0$. Then

$$
4 \mid 2^{\alpha+3} \cdot 3^{\beta+1} \cdot 5^{\gamma}
$$

By the proof of Theorem (3.1), there are positive integers $k$ such that

$$
24 k \mid F_{12 k}
$$

By (2.5), it can be seen that,

$$
\begin{equation*}
k \mid F_{12 k} \tag{3.4}
\end{equation*}
$$

To avoid any confusion, we set $k=2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma}$. Now, by utilizing (3.4), we get

$$
2^{\alpha} .3^{\beta} .5^{\gamma} \mid F_{2^{\alpha+2} .3^{\beta+1} .5^{\gamma}}
$$

To conclude the proof we need the following three steps:
Step I. First, we shall prove that $2^{\alpha} \mid F_{3.2^{\alpha+2}}$. The proof of this step can be achieved by induction. For $\alpha=1$, the claim is true, since $2 \mid F_{3.8}=46368$. Suppose the claim is true for all positive integers less than $\alpha$. We show that the same is true for $\alpha+1$, that is $2^{\alpha+1} \mid 2 F_{3.2^{\alpha+3}}$. By multiplying to 2 , we can write

$$
2^{\alpha+1} \mid 2 F_{3.2^{\alpha+2}}
$$

Note that

$$
2 F_{3.2^{\alpha+2}} \mid F_{3.2^{\alpha+3}}
$$

Since $2 \mid F_{3}=2$ and $F_{3} \mid F_{3.2^{\alpha+3}}$, we have by way of identity (2.6) that

$$
F_{3.2^{\alpha+2}} \mid F_{3.2^{\alpha+3}}
$$

Hence, $2^{\alpha+1} \mid 2 F_{3.2^{\alpha+3}}$.
Step II. Next we show that $3^{\beta} \mid F_{2^{\alpha+2} .3^{\beta+1} .5^{\gamma}}$ by induction on $\beta$. For $\beta=1$, according (2.6), $F_{3} \mid F_{36.2^{\alpha} .5^{\gamma}}$. Assume the statement holds for all positive integers less than $\beta$. We need to show that the statement also holds for $\beta+1$. Multiplying both sides of the induction hypothesis by 3 gives

$$
3^{\beta+1} \mid 3 F_{2^{\alpha+2} .3^{\beta+1} .5 \gamma}
$$

To complete the proof of this step, it is sufficient to show

$$
3 F_{2^{\alpha+2.3^{\beta+1.5 \gamma}}} \mid F_{2^{\alpha+2.3^{\beta+2.5 \gamma}}}
$$

Much like in the previous step, we have $3 \mid F_{2^{\alpha+2} .3^{\beta+2} .5^{\gamma}}$ and

$$
F_{2^{\alpha+2} .3^{\beta+1.5 \gamma}} \mid F_{2^{\alpha+2} .3^{\beta+2} .5^{\gamma}},
$$

whence

$$
3^{\beta+1} \mid F_{2^{\alpha+2.3^{\beta+2} .5^{\gamma}}}
$$

which concludes step II.

Step III. Finally, using the same arguments of the previous steps, with induction on $\gamma$, we show $5^{\gamma} \mid F_{5 \gamma}$. For $\gamma=1$ it is easy to see that $5 \mid F_{5}=5$. By induction, we assume that the statement is true for all positive integers less than $\gamma$. We have to prove that the claim is true for $\gamma+1$. By multiplying the induction hypothesis by 5 , we obtain

$$
5^{\gamma+1} \mid 5 F_{5 \gamma}
$$

Now, we have to show

$$
5 F_{5 \gamma} \mid F_{5 \gamma+1}
$$

Again, since $5 \mid F_{5^{\gamma+1}}$ and $F_{5^{\gamma}} \mid F_{5^{\gamma+1}}$.

$$
5^{\gamma+1} \mid F_{5 \gamma+1}
$$

According the results of the previous steps, we have

$$
2^{\alpha} .3^{\beta} .5^{\gamma} \mid F_{2^{\alpha+2} .3^{\beta+1.5 \gamma}}
$$

Therefore,

$$
2^{\alpha+3} .3^{\beta+1} .5^{\gamma} \mid \sum_{i=1}^{2^{\alpha+2} .3^{\beta+1} .5^{\gamma}} F_{i}
$$

In the same manner of the Theorem (3.5), we obtain the following Theorem.
Theorem 3.6. Let $\alpha, \beta$ and $\gamma$ be positive integers. Then

$$
\begin{equation*}
2^{\alpha+3} .3^{\beta+1} .5^{\gamma} \mid \sum_{i=1}^{2^{\alpha+3} .3^{\beta+1} .5^{\gamma}} L_{i} . \tag{3.5}
\end{equation*}
$$

## 4. Prime numbers and the average of Fibonacci numbers

In this section, we discuss Theorem (3.1) for some special $n$. First, we consider the sum of the first $n$ Fibonacci numbers, when $n$ is a prime number, $p$. Before we present our results, we first recall some well-known definitions and theorems.

An integer $a$ is called a quadratic residue modulo $p$ (with $p>2$ ) if $p \not X a$ and there exists an integer $b$ such that $a \equiv b^{2}(\bmod p)$. Otherwise, it is called a non-quadratic residue modulo $p$. Legendre introduced the following practical notation:

$$
\left(\frac{a}{b}\right)= \begin{cases}+1 & \text { if } a \text { is a quadratic residue modulo } p  \tag{4.1}\\ 0 & \text { if } p \text { divides } a \\ -1 & \text { otherwise }\end{cases}
$$

We note that for a prime number $p$, and $b=5$, it is easy to see that the Legendre symbol, $\left(\frac{p}{5}\right)$, is equal to

$$
\left(\frac{p}{5}\right)= \begin{cases}+1 & \text { if } p \equiv \pm 1 \quad(\bmod 5)  \tag{4.2}\\ 0 & \text { if } p \equiv 0 \quad(\bmod 5) \\ -1 & \text { if } p \equiv \pm 2 \quad(\bmod 5)\end{cases}
$$

We shall denote by $\sigma(n)$ the rank of apparition of the Lucas sequence $L_{n}$, if it exists. According [3], the rank of apparition of the sequence $\left\{S_{n}\right\}$ is the smallest index $k$ such that $m \mid S_{k}$ for some non-zero element $S_{k}$, provided it exists. The rank of apparition of the Fibonacci sequence $\left\{F_{n}\right\}$ is denoted by $\rho(n)$. These numbers are sometimes called Fibonacci periods or Pisano periods. The initial values of $\rho(n)$ is given OEIS, A001602 .
The following lemma concerning Fibonacci and Lucas numbers is well-known (see [4], page $41-55)$.

Lemma 4.1. Let $n$ be a positive integer and $\rho(n)$ be the rank of apparition of the Fibonacci sequence. Then
i) $m \mid F_{n}$ if and only if $\rho(m) \mid n$;
ii) If $p \mid F_{n}$, then $p^{e} \mid F_{n p^{e-1}}$ for $e \geq 1$;
iii) $\left(F_{n}, L_{n}\right) \mid 2$.

The property (ii) of the Lemma (4.1) known as the law of apparition of Lucas sequences of the first kind, in general. We will also need the following lemma from [3].

Lemma 4.2. The odd prime power $p^{r}$ is a divisor of the Lucas sequence $\left\{L_{n}\right\}$ if and only if $\rho\left(p^{r}\right)$ is even. If $p^{r}$ is a divisor of the sequence $L_{n}$, then

$$
\sigma\left(p^{r}\right)=\frac{\rho\left(p^{r}\right)}{2}
$$

and

$$
p^{r} \left\lvert\, L_{n} \Longleftrightarrow n \equiv \frac{\rho\left(p^{r}\right)}{2} \quad\left(\bmod \rho\left(p^{r}\right)\right)\right.
$$

We are now ready to state our theorem.
Theorem 4.3. Let $p$ be an odd prime number. Then

$$
\begin{equation*}
p \not \backslash \sum_{i=1}^{p} F_{i} . \tag{4.3}
\end{equation*}
$$

Proof. Assume that $p$ is an odd prime number such that

$$
p \mid \sum_{i=1}^{p} F_{i} .
$$

We investigate the cases $p=4 k+1$ and $p=4 k+3$ separately.

Case I. Suppose that $p=4 k+1$. According (2.2) and Theorem (2.1), we have

$$
\begin{equation*}
p \mid \sum_{i=1}^{p} F_{i}=F_{4 k+3}-1=F_{2 k+2} L_{2 k+1} . \tag{4.4}
\end{equation*}
$$

Hence, $p \mid F_{2 k+2}$ or $p \mid L_{2 k+1}$. Suppose first that $p \mid F_{2 k+2}$. According Lemmas (4.1) and (4.2), there exists a positive integer $k^{\prime}$ such that

$$
k^{\prime}\left(p-\left(\frac{p}{5}\right)\right)=2 k+2
$$

This implies

$$
k^{\prime}\left(4 k+1-\left(\frac{p}{5}\right)\right)=2 k+2
$$

which is impossible, since $\left(\frac{p}{5}\right)= \pm 1$.
Now, suppose that $p \mid L_{2 k+1}$. From the identity $F_{2 n}=L_{n} F_{n}$ we have $p \mid F_{4 k+2}$. So, by Lemma (4.2), there exists a positive integer $k^{\prime \prime}$ such that

$$
\begin{equation*}
4 k+2=k^{\prime \prime}\left(\frac{p-\left(\frac{p}{5}\right)}{2}\right) \tag{4.5}
\end{equation*}
$$

which implies

$$
8 k+4=k^{\prime \prime}\left(4 k+1-\left(\frac{p}{5}\right)\right) .
$$

We have again a contradiction, since $\left(\frac{p}{5}\right)= \pm 1$.
Case II. Assume now that $p=4 k+3$. We follow the argument given in Case I. According (2.2) and Theorem (2.1), we can write

$$
\begin{equation*}
p \mid \sum_{i=1}^{p} F_{i}=F_{4(k+1)+1}-1=F_{2(k+1)} L_{2(k+1)+1} . \tag{4.6}
\end{equation*}
$$

The prime number $p=4 k+3$ must divide $F_{2(k+1)}$ or $L_{2(k+1)+1}$. Suppose first that $p \mid F_{2(k+1)}$. By Lemma (4.1), there exists a positive integer $k^{\prime}$ such that

$$
k^{\prime}\left(p-\left(\frac{p}{5}\right)\right)=2(k+1)
$$

Hence,

$$
k^{\prime}\left(4 k+3-\left(\frac{p}{5}\right)\right)=2 k+2
$$

which cannot hold, since $\left(\frac{p}{5}\right)= \pm 1$.
Now, let $p \mid L_{2(k+1)+1}$. According Lemma (4.2), there exists a positive integer $k^{\prime \prime}$ such that

$$
\begin{equation*}
4(k+1)+2=k^{\prime \prime}\left(\frac{p-\left(\frac{p}{5}\right)}{2}\right) \tag{4.7}
\end{equation*}
$$

which implies

$$
k^{\prime \prime}\left(4 k+3-\left(\frac{p}{5}\right)\right)=8(k+1)+4
$$

This is impossible, since $\left(\frac{p}{5}\right)= \pm 1$.
Proof of Theorem(??) follows immediately from the above two contradictions.

Consider now the first values of the sequence in OEIS A111035 :

$$
\begin{aligned}
& 1,2,24,48,72,77,96,120,144,192,216,240,288,319, \\
& 323,336,360,384,432,480,576,600,648,672,720,768, \\
& 864,960,1008,1080,1104,1152,1200,1224,1296,1320, \\
& 1344,1368,1440,1517,1536,1680,1728,1800,1920 \cdots .
\end{aligned}
$$

We see that each odd integer within this sequence is square-free, for example, $77=11 \times$ $7,319=11 \times 29, \quad 323=17 \times 19$ and $1517=37 \times 41$. By assuming the validity of the well-known conjecture concerning Wall-Sun-Sun (or Fibonacci-Wieferich) primes, we show in Theorem (4.5) below that if $n$ is an odd positive integer such that $n \mid \sum_{i=1}^{n} F_{i}$ then $n$ is squarefree.
We first state the Wall-Sun-Sun Prime Conjecture:
Conjecture 4.4 (Wall-Sun-Sun Prime Conjecture [6]). There are no prime numbers $p$ such that

$$
p^{2} \left\lvert\, F_{p-\left(\frac{p}{5}\right)} .\right.
$$

In 1992, the authors of [6] proved that if Wall-Sun-Sun prime number conjecture is true, then the Fermat equation $x^{p}+y^{p}=z^{p}$ has no integral solutions with $p \nmid x y z$. Empirically it has been observed that there are no Wall-Sun-Sun primes less than 100, 000, 000, 000, 000.

Theorem 4.5. Let $n$ be an odd positive integer. If $n \mid \sum_{i=1}^{n} F_{i}$, then $n$ is a square-free.
Proof. Let $n=p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}} \cdots p_{k}^{\alpha_{k}}$, where each $p_{i}$ is an odd prime and suppose $n \mid \sum_{i=1}^{n} F_{i}$. We will show that $\alpha_{1}=\alpha_{2}=\cdots=\alpha_{k}=1$.
By Theorem (2.1) and identity (2.2), for positive integers $a, b$, we have

$$
\begin{equation*}
n \mid \sum_{i=1}^{n} F_{i}=F_{a} L_{b} \tag{4.8}
\end{equation*}
$$

so for $1 \leq i \leq k$ we have $p_{i}^{\alpha_{i}} \mid F_{a} L_{b}$.
It follows from the above argument that $p_{i}^{\alpha_{i}} \mid F_{a}$ or $p_{i}^{\alpha_{i}} \mid L_{b}$ so suppose $p_{i}^{\alpha_{i}} \mid F_{a}$. Employing Lemma (4.1) and the Wall-Sun-Sun-prime conjecture (4.4), we have

$$
p_{i}^{2} X F_{p_{i}-\left(\frac{p_{i}}{5}\right)}
$$

which implies $p_{i}^{2} X F_{a}$.
Suppose now that $p_{i}^{\alpha} \mid L_{b}$. Again, according Lemma (4.2) and the Wall-Sun-Sun Prime Conjecture we have

$$
p_{i}^{2} \nmid L_{b} .
$$

It follows that $p_{i}^{2} X F_{a} L_{b}$ for $1 \leq i \leq k$, which implies that $\alpha_{1}=\alpha_{2}=\cdots=\alpha_{k}=1$, so we see immediately that $n$ is square-free.

We conclude this paper with interesting conjectures concerning averages of Fibonacci numbers.

Conjecture 4.6. For each positive integer $t$, there is positive integer like $n$ such that if

$$
n \mid \sum_{i=1}^{n} F_{i}, \quad \text { then } \quad n+t \mid \sum_{i=1}^{n+t} F_{i} .
$$

Conjecture 4.7. There are infinitely many pairs of positive integers $(n, n+1)$ such that

$$
n \mid \sum_{i=1}^{n} F_{i} \quad \text { and } \quad n+1 \mid \sum_{i=1}^{n+1} F_{i} .
$$

Using Mathematica we have empirically found all such pairs of integers up to 1000000. These pairs are:

$$
(1,2),(6479,6480),(11663,11664),(51983,51984),(196559,196560) .
$$

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