

GENERALIZED FISHBURN NUMBERS AND TORUS KNOTS

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ABSTRACT. Andrews and Sellers recently initiated the study of arithmetic properties of Fishburn numbers. In this paper, we prove prime power congruences for generalized Fishburn numbers. These numbers are the coefficients in the $1 - q$ expansion of the Kontsevich-Zagier series $\mathcal{F}_t(q)$ for the torus knots $T(3, 2^t)$, $t \geq 2$. The proof uses a strong divisibility result of Ahlgren, Kim and Lovejoy and a new “strange identity” for $\mathcal{F}_t(q)$.

1. INTRODUCTION

The Fishburn numbers $\xi(n)$ are the coefficients in the formal power series expansion

$$F(1 - q) =: \sum_{n \geq 0} \xi(n)q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \dots \quad (1.1)$$

where $F(q) := \sum_{n \geq 0} (q)_n$ is the Kontsevich-Zagier “strange” series [28] and

$$(a_1, a_2, \dots, a_j)_n = (a_1, a_2, \dots, a_j; q)_n := \prod_{k=1}^n (1 - a_1 q^{k-1})(1 - a_2 q^{k-1}) \dots (1 - a_j q^{k-1})$$

is the standard q -hypergeometric notation, valid for $n \in \mathbb{N} \cup \{\infty\}$. Here, the moniker “strange” is used as $F(q)$ does not converge on any open subset of \mathbb{C} , but is well-defined when q is a root of unity (where it is finite) and when q is replaced by $1 - q$ as in (1.1). The Fishburn numbers are of interest for their numerous combinatorial variants (see A022493 in [23]), asymptotics [24, 28] and arithmetic properties [1, 3, 7, 8, 25]. In their marvelous paper, Andrews and Sellers [3] proved congruences for $\xi(n)$ modulo primes which were then extended to prime powers [1, 25]. For example, we have

$$\xi(5^r n - 1) \equiv \xi(5^r n - 2) \equiv 0 \pmod{5^r}, \quad (1.2)$$

$$\xi(7^r n - 1) \equiv 0 \pmod{7^r} \quad (1.3)$$

and

$$\xi(11^r n - 1) \equiv \xi(11^r n - 2) \equiv \xi(11^r n - 3) \equiv 0 \pmod{11^r} \quad (1.4)$$

for all natural numbers r and n . Our interest in this paper lies in the knot theoretic interpretation of $F(q)$ as it leads to a natural generalization of the coefficients $\xi(n)$.

Let K be a knot and $J_N(K; q)$ be the usual colored Jones polynomial, normalized to be 1 for the unknot. As a knot invariant, the colored Jones polynomial plays the lead role in many open

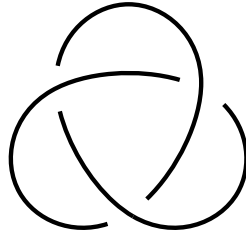
Date: February 4, 2020.

2010 Mathematics Subject Classification. 33D15, 57K16.

Key words and phrases. Generalized Fishburn numbers, colored Jones polynomial, torus knots, congruences.

problems in quantum topology. The sequence $\{J_N(K; q)\}_{N \in \mathbb{N}}$ appears to encode many subtle geometric and topological properties of the knot K at a remarkably deep level. For example, the Volume Conjecture [15, 21, 22] relates the value at $\zeta_N = e^{2\pi i/N}$ of the N th colored Jones polynomial (or, equivalently, the N th Kashaev invariant) of a knot to its hyperbolic volume, the Strong Slope Conjecture [6, 14] says that the maximal and minimal degrees of $J_N(K; q)$ in q contain information about essential surfaces in knot exteriors and the AJ Conjecture [5] connects the recurrence relation for $J_N(K; q)$ to the A -polynomial of the knot [4], a plane curve describing the $SL(2, \mathbb{C})$ character variety of the knot complement. Explicit formulas for $J_N(K; q)$ in terms of q -hypergeometric series are also of importance and have been proven for various families of knots [9, 11, 12, 17, 18, 19, 20, 26, 27]. For example, if $T(3, 2) = T(2, 3)$ is the right-handed torus knot as in Figure 1, then [9, 17]

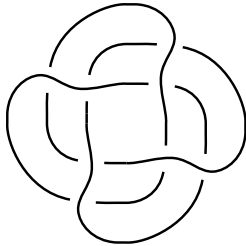
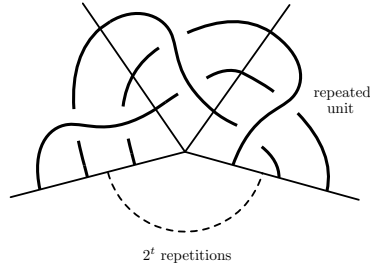
$$J_N(T(3, 2); q) = q^{1-N} \sum_{n \geq 0} q^{-nN} (q^{1-N})_n.$$

FIGURE 1. $T(3, 2)$

Observe that the Kontsevich-Zagier series $F(q)$ matches the colored Jones polynomial for $T(3, 2)$ at roots of unity, that is, for $q = \zeta_N$

$$\zeta_N F(\zeta_N) = J_N(T(3, 2); \zeta_N).$$

Consider the family of torus knots $T(3, 2^t)$ for $t \geq 2$ as in Figures 2 and 3.

FIGURE 2. $T(3, 4)$ FIGURE 3. $T(3, 2^t)$

In this case, a q -hypergeometric expression for the colored Jones polynomial has been computed, namely (see page 41, Théorème 3.2 in [16])

$$\begin{aligned}
J_N(T(3, 2^t); q) &= (-1)^{h''(t)} q^{2^t-1-h'(t)-N} \sum_{n \geq 0} (q^{1-N})_n q^{-Nnm(t)} \\
&\times \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ j_\ell \equiv 1 \pmod{m(t)}}} (-q^{-N})^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\
&\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}
\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}
h''(t) &= \begin{cases} \frac{2^t-1}{3} & \text{if } t \text{ is even,} \\ \frac{2^t-2}{3} & \text{if } t \text{ is odd,} \end{cases} & h'(t) &= \begin{cases} \frac{2^t-4}{3} & \text{if } t \text{ is even,} \\ \frac{2^t-5}{3} & \text{if } t \text{ is odd,} \end{cases} & a(t) &= \begin{cases} \frac{2^{t-1}+1}{3} & \text{if } t \text{ is even,} \\ \frac{2^{t+1}}{3} & \text{if } t \text{ is odd,} \end{cases} \\
m(t) &= 2^{t-1}, \quad I(*) \text{ is the characteristic function and}
\end{aligned}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the q -binomial coefficient. We note that the $t = 2$ case of (1.5) recovers equation (16) in [11]. We now define the *Kontsevich-Zagier series for torus knots* as

$$\begin{aligned}
\mathcal{F}_t(q) &= (-1)^{h''(t)} q^{-h'(t)} \sum_{n \geq 0} (q)_n \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ j_\ell \equiv 1 \pmod{m(t)}}} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\
&\times \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}.
\end{aligned} \tag{1.6}$$

As with the original Kontsevich-Zagier series, $\mathcal{F}_t(q)$ converges in a similar fashion and thus we may write

$$\mathcal{F}_t(1-q) =: \sum_{n \geq 0} \xi_t(n) q^n.$$

For example,

$$\mathcal{F}_2(1-q) = 1 + 3q + 11q^2 + 50q^3 + 280q^4 + 1890q^5 + \dots$$

and

$$\mathcal{F}_3(1-q) = 1 + 7q + 49q^2 + 420q^3 + 4515q^4 + 59367q^5 + \dots$$

In addition, we have

$$\zeta_N^{2^t-1} \mathcal{F}_t(\zeta_N) = J_N(T(3, 2^t); \zeta_N).$$

The purpose of this paper is to illustrate how a recent result of Ahlgren, Kim and Lovejoy [2] combined with a new “strange identity” for $\mathcal{F}_t(q)$ allow one to prove prime power congruences

akin to (1.2)–(1.4) for the *generalized Fishburn numbers* $\xi_t(n)$. For natural numbers s and $t \geq 2$ and the periodic function

$$\chi_t(n) = \chi_{3 \cdot 2^{t+1}}(n) := \begin{cases} 1 & \text{if } n \equiv 2^{t+1} - 3, 3 + 2^{t+2} \pmod{3 \cdot 2^{t+1}}, \\ -1 & \text{if } n \equiv 2^{t+1} + 3, 2^{t+2} - 3 \pmod{3 \cdot 2^{t+1}}, \\ 0 & \text{otherwise,} \end{cases} \quad (1.7)$$

we define the set

$$S_{t,\chi_t}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}} \pmod{s} \text{ where } \chi_t(n) \neq 0 \right\}.$$

Our main result is now the following.

Theorem 1.1. *If $p \geq 5$ is a prime and $j \in \{1, 2, \dots, p-1 - \max S_{t,\chi_t}(p)\}$, then*

$$\xi_t(p^r n - j) \equiv 0 \pmod{p^r}$$

for all natural numbers r , n and $t \geq 2$.

One can check that $S_{2,\chi_2}(5) = \{0, 2, 3\}$, $S_{2,\chi_2}(17) = \{0, 2, 3, 4, 7, 8, 9, 11, 14\}$, $S_{3,\chi_3}(7) = \{0, 2, 3, 4\}$ and $S_{3,\chi_3}(13) = \{0, 2, 5, 6, 7, 8, 11\}$. Thus, by Theorem 1.1, we have

$$\begin{aligned} \xi_2(5^r n - 1) &\equiv 0 \pmod{5^r}, \\ \xi_2(17^r - 1) &\equiv \xi_2(17^r - 2) \equiv 0 \pmod{17^r}, \\ \xi_3(7^r n - 1) &\equiv \xi_3(7^r n - 2) \equiv 0 \pmod{7^r} \end{aligned}$$

and

$$\xi_3(13^r n - 1) \equiv 0 \pmod{13^r}$$

for all natural numbers r and n .

The paper is organized as follows. In Section 2, we recall the main result from [2] and then record some preliminaries, including the new “strange identity”. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we discuss some possibilities for future work.

2. PRELIMINARIES

Our first step is to recall the setup from [2]. Let \mathcal{F} be a function of the form

$$\mathcal{F}(q) = \sum_{n \geq 0} (q)_n f_n(q) \quad (2.1)$$

where $f_n(q) \in \mathbb{Z}[q]$. For positive integers s and N , consider the partial sum

$$\mathcal{F}(q; N) := \sum_{n=0}^N (q)_n f_n(q)$$

and its s -dissection

$$\mathcal{F}(q; N) = \sum_{i=0}^{s-1} q^i A_{\mathcal{F},s}(N, i, q^s)$$

where $A_{\mathcal{F},s}(N, i, q) \in \mathbb{Z}[q]$. Now, consider the partial theta functions

$$P_{a,b,\chi}^{(\nu)}(q) := \sum_{n \geq 0} n^\nu \chi(n) q^{\frac{n^2-a}{b}} \quad (2.2)$$

where $\nu \in \{0, 1\}$, $a \geq 0$, $b > 0$ are integers and $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ is a function satisfying the following two conditions:

$$\chi(n) \neq 0 \text{ only if } \frac{n^2 - a}{b} \in \mathbb{Z} \quad (2.3)$$

and for each root of unity ζ ,

$$\text{the function } n \rightarrow \zeta^{\frac{n^2-a}{b}} \chi(n) \text{ is periodic and has mean value zero.} \quad (2.4)$$

Finally, define the set $S_{a,b,\chi}(s)$ by

$$S_{a,b,\chi}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - a}{b} \pmod{s} \text{ where } \chi(n) \neq 0 \right\}.$$

We can now state the main result in [2].

Theorem 2.1. *Suppose that \mathcal{F} and $P_{a,b,\chi}^{(\nu)}$ are functions as in (2.1) and (2.2) and for each root of unity ζ , we have the asymptotic expansion*

$$P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim \mathcal{F}(\zeta e^{-t})$$

as $t \rightarrow 0^+$. Suppose that s and N are positive integers and $i \notin S_{a,b,\chi}(s)$. Then

$$(q)_{\lambda(N,s)} \mid A_{\mathcal{F},s}(N, i, q)$$

where $\lambda(N, s) := \lfloor \frac{N+1}{s} \rfloor$.

We now turn to our situation and record the following result. Throughout, we assume that $t \geq 2$.

Proposition 2.2. *The periodic function χ_t (as defined by (1.7)) satisfies (2.3) and (2.4).*

Proof. A straightforward calculation using (1.7) confirms (2.3) with $a = (2^{t+1} - 3)^2$ and $b = 3 \cdot 2^{t+2}$. Suppose ζ is a root of unity of order M and define

$$\psi_t(n) = \zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} \chi_t.$$

Note that ψ_t has period $M(3 \cdot 2^{t+1})$. We now claim that

$$\sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) = 0. \quad (2.5)$$

Suppose M is even. By (1.7), we obtain

$$\chi_t(n + 3 \cdot 2^t M) = \chi_t(n) \quad (2.6)$$

for all n . We also have

$$\zeta^{\frac{(n+3 \cdot 2^t M)^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} = \zeta^{\frac{M(n+3 \cdot 2^{t-1} M^2)}{2}} \zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}}. \quad (2.7)$$

As ψ_t is supported on odd integers, we can assume that n is odd. Then $M(n + 3 \cdot 2^{t-1} M^2)$ is an odd multiple of M and so (2.7) equals $-\zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}}$. This fact and (2.6) imply

$$\sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) = \sum_{n=1}^{M(3 \cdot 2^t)} (\psi_t(n) + \psi_t(n + 3 \cdot 2^t M)) = 0.$$

Now, suppose M is odd. We break up (2.5) into four sums, one for each congruence class modulo $3 \cdot 2^{t+1}$ in (1.7). Specifically,

$$\begin{aligned} \sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) &= \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+2} + 3)m + 2^t + 2 + 1} - \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+2} - 3)m + 2^t - 1} \\ &\quad + \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+1} - 3)m} - \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+1} + 3)m + 2}. \end{aligned} \quad (2.8)$$

Let $i_{t,M} = 2^{-t}$ be the unique solution to $2^t i \equiv 1 \pmod{M}$. Performing the shift $m \rightarrow m + i_{t,M}$ (respectively, $m \rightarrow m - i_{t,M}$) to the second (respectively, fourth) sum in (2.8) followed by a routine simplification implies (2.5). The result now follows. \square

For χ_t as in (1.7), consider the series

$$H_t(x) = H_t(x, q) = \sum_{n \geq 0} \chi_t(n) q^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} x^{\frac{n - (2^{t+1} - 3)}{2}}.$$

Proposition 2.3. *Let $h(t) = 2^t - 2$. We have*

$$\begin{aligned} H_t(x) &= (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} \sum_{n \geq 0} (x)_{n+1} x^{nm(t)} \\ &\quad \times \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ \pmod{m(t)}}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\ &\quad \times \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[\begin{matrix} n + I(\ell \leq k) \\ j_\ell \end{matrix} \right]. \end{aligned} \quad (2.9)$$

Proof. Let $f_t(x)$ denote the right-hand side of (2.9). Then $f_t(x)$ satisfies the difference equation (see (3.3.11) in [16])

$$f_t(x) = 1 - q^2 x^3 - q^{2^t - 1} x^{2^t} + q^{3+2^t} x^{3+2^t} + q^{5 \cdot 2^t - 3} x^{3 \cdot 2^t} f_t(q^2 x). \quad (2.10)$$

One can directly verify that $H_t(x)$ also satisfies (2.10) using (1.7). \square

Recall that the Kontsevich-Zagier series $F(q)$ satisfies the “strange identity”

$$F(q) \text{ “} = \text{”} - \frac{1}{2} \sum_{n \geq 1} n \left(\frac{12}{n} \right) q^{\frac{n^2-1}{24}}$$

where “ = ” means that the two sides agree to all orders at every root of unity (for further details, see Sections 2 and 5 in [28]) and $\left(\frac{12}{*} \right)$ is the quadratic character of conductor 12. We now prove the following new strange identity for the Kontsevich-Zagier series for torus knots.

Proposition 2.4. *We have*

$$\mathfrak{F}_t(q) \text{ “} = \text{”} - \frac{1}{2} \sum_{n \geq 0} n \chi_t(n) q^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}}. \quad (2.11)$$

Proof. For ease of notation, let $'$ denote the condition

$$3 \sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \pmod{m(t)}$$

occurring in the second sum in (1.5) on the j_ℓ 's,

$$v = v(j_1, \dots, j_{m(t)-1}) := \frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}$$

and \bar{c} the reduction of an integer c modulo $m(t)$. The following identity¹ implies (2.11) upon setting $q = \zeta e^{-t}$, then letting $t \rightarrow 0^+$:

$$\begin{aligned} & \frac{1}{2} \sum_{n \geq 0} n \chi_t(n) q^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} - \frac{2^{t+1} - 3}{2} (q^{2^{t+1}}, q^{2^t-1}, q^{2^t+1}; q^{2^{t+1}})_\infty (q^2, q^{2^{t+2}-2}; q^{2^{t+2}})_\infty \\ &= (-1)^{h''(t)+1} q^{-h'(t)} \sum_{n \geq 0} \left[(q)_n - (q)_\infty \right] \\ & \quad \times \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \left[\begin{matrix} n + I(\ell \leq k) \\ j_\ell \end{matrix} \right] \\ &+ (-1)^{h''(t)+1} q^{-h'(t)} (q)_\infty \left(\sum_{i=1}^{\infty} \frac{q^i}{1 - q^i} \right) \sum_{n \geq 0} b_{n,t}(q) + (-1)^{h''(t)} q^{-h'(t)} (q)_\infty \sum_{n \geq 0} (n - h(t)) b_{n,t}(q) \end{aligned} \quad (2.12)$$

where

$$b_{n,t}(q) = a_{n,t}(q) - a_{n-1,t}(q) \quad (2.13)$$

and

¹The $t = 2$ case of (2.12) gives an alternative (and corrected) version of Proposition 5 in [13].

$$a_{n,t}(q) = \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \prod_{\ell=1}^{m(t)-1} \left[\frac{n - \sum_{\ell=1}^{m(t)-1} j_\ell - \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}}{m(t)} + I(\ell \leq \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}) \right]. \quad (2.14)$$

To prove (2.12), we begin by rewriting (2.9) as

$$\begin{aligned} H_t(x) &= (-1)^{h''(t)} q^{-h'(t)} (1-x) \sum_{n \geq 0} \left[(qx)_n - (qx)_\infty \right] x^{nm(t)} \\ &\times \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{-h(t) + \sum_{\ell=1}^{m(t)-1} j_\ell} q^v \\ &\times \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[n + I(\ell \leq k) \right] + (qx)_\infty (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} (1-x) M_t(x, q) \end{aligned} \quad (2.15)$$

where

$$M_t(x, q) := \sum_{n \geq 0} x^{nm(t)} \sum'_{j_1, \dots, j_{m(t)-1}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[n + I(\ell \leq k) \right]. \quad (2.16)$$

We now claim that

$$(1-x)M_t(x, q) = \sum_{n \geq 0} b_{n,t}(q) x^n \quad (2.17)$$

where $b_{n,t}(q)$ is given by (2.13) and (2.14). To see (2.17), we first write

$$\sum_{\ell=1}^{m(t)-1} j_\ell = m(t) \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}. \quad (2.18)$$

We now have that $M_t(x, q)$ equals

$$\begin{aligned} &\sum'_{j_1, \dots, j_{m(t)-1}} \sum_{k=0}^{m(t)-1} \sum_{n \geq 0} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{m(t) \left(n + \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor \right) + k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}} q^v \prod_{\ell=1}^{m(t)-1} \left[n + I(\ell \leq k) \right] \\ &\quad \text{(using (2.16) and (2.18))} \\ &= \sum'_{j_1, \dots, j_{m(t)-1}} \sum_{k=0}^{m(t)-1} \sum_{n \geq 0} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{m(t)n + k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}} q^v \prod_{\ell=1}^{m(t)-1} \left[n - \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor + I(\ell \leq k) \right] \\ &\quad \left(\text{letting } n \rightarrow n - \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{m(t)-1} \sum'_{j_1, \dots, j_{m(t)-1}} \sum_{n \equiv k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell} \pmod{m(t)}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^n q^v \prod_{\ell=1}^{m(t)-1} \left[\frac{n-k-\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} + I(\ell \leq k) \right] \\
&\quad \left(\text{letting } n \rightarrow \frac{n-k-\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \text{ and using (2.18)} \right) \\
&= \sum_{n \geq 0} \sum'_{j_1, \dots, j_{m(t)-1}} x^n (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \prod_{\ell=1}^{m(t)-1} \left[\frac{n-\sum_{\ell=1}^{m(t)-1} j_\ell - \overline{(n-\sum_{\ell=1}^{m(t)-1} j_\ell)}}{m(t)} + I(\ell \leq \overline{n-\sum_{\ell=1}^{m(t)-1} j_\ell}) \right] \\
&= \sum_{n \geq 0} a_{n,t}(q) x^n.
\end{aligned}$$

Thus, we obtain (2.17) via (2.13). We now substitute (2.17) into (2.15), then take the derivative of both sides of (2.15) with respect to x , set $x = 1$ and apply Watson's quintuple product identity

$$\sum_{k \in \mathbb{Z}} q^{\frac{k(3k-1)}{2}} x^{3k} (1-xq^k) = (q, x, qx^{-1})_\infty (qx^2, qx^{-2}; q^2)_\infty \quad (2.19)$$

to the resulting left-hand side with $q \rightarrow q^{2^{t+1}}$ and $x = q^{2^t-1}$ to obtain (2.12). This proves the result. \square

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Let p be a prime ≥ 5 and consider the truncation $\mathcal{F}_t(q, N)$ of (1.6) and its p -dissection

$$\mathcal{F}_t(q, N) = \sum_{i=0}^{p-1} q^i A_{\mathcal{F}_t, p}(N, i, q^p).$$

We have

$$\mathcal{F}_t(1-q, N) = \sum_{i \in S_{t, \chi_t}(p)} (1-q)^i A_{\mathcal{F}_t, p}(N, i, (1-q)^p) + \sum_{i \notin S_{t, \chi_t}(p)} (1-q)^i A_{\mathcal{F}_t, p}(N, i, (1-q)^p).$$

By Theorem 2.1, Propositions 2.2 and 2.4 and the fact that $(1-q^k)^n \mid (q)_m$ for $m \geq kn$, we can write

$$\mathcal{F}_t(1-q, N) = \sum_{i \in S_{t, \chi_t}(p)} (1-q)^i A_{\mathcal{F}_t, p}(N, i, (1-q)^p) + (1-(1-q)^p)^n \sum_{i \notin S_{t, \chi_t}(p)} (1-q)^i g_i(q)$$

for n such $N \geq np^2 - 1$ and $g_i(q) \in \mathbb{Z}[q]$. By Lemma 3.3 in [25],

$$(1-(1-q)^p)^n \equiv O(q^{pn-(p-1)(r-1)}) \pmod{p^r}$$

and so

$$\mathcal{F}_t(1 - q, N) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(N, i, (1 - q)^p) + O(q^{pn - (p-1)(r-1)}) \pmod{p^r}. \quad (3.1)$$

The sum in (3.1) is an integral linear combination of terms of the form $(1 - q)^{i+lp}$ for $i \in S_{t, \chi_t}(p)$ and natural numbers l . Thus, it suffices to show that

$$\binom{i + lp}{p^r n - j} \equiv 0 \pmod{p^r} \quad (3.2)$$

for all $j \in \{1, 2, \dots, p-1 - \max S_{t, \chi_t}(p)\}$. The condition on j implies that $j < p - i$ and so (3.2) follows from Lemma 3.4 in [25]. The result now follows after letting $n \rightarrow \infty$ in (3.1). \square

4. FURTHER COMMENTS

There are several avenues for further study. First, one could investigate combinatorial descriptions and asymptotic properties for the numbers $\xi_t(n)$. Second, the colored Jones polynomial of a knot K satisfies a *cyclotomic expansion* of the form

$$J_N(K; q) = \sum_{n \geq 0} (q^{1+N})_n (q^{1-N})_n C_n(K; q),$$

where the *cyclotomic coefficients* $C_n(K; q)$ are Laurent polynomials independent of N [10]. It would be highly desirable to find the cyclotomic expansion for the torus knots $T(3, 2^t)$, $t \geq 2$. For $t = 1$, the cyclotomic expansion has been found by Masbaum [20]. However, it is unclear if his techniques are sufficient when $t \geq 2$. They require finding a link whose components are unknotted, from which $T(3, 2)$ can be recovered by introducing twists into a single region of one of the components. Employing this process for $t \geq 2$ appears to require extending it to allow for multiple twist regions. Third, in relation to Proposition 2.4, a strange identity for the Kontsevich-Zagier series associated to the torus knots $T(2, 2t + 1)$ for $t \geq 1$ has also been computed (see (15) in [12]). Do strange-type identities exist for q -series associated to satellite or hyperbolic knots? Finally, as the Kontsevich-Zagier series $F(q)$ is a foundational example of a quantum modular form [29], it is natural to wonder if the same is true for $\mathcal{F}_t(q)$. This is the subject of forthcoming work.

ACKNOWLEDGEMENTS

The second author was partially funded by the Natural Sciences and Engineering Research Council of Canada. The fourth author would like to thank the Max-Planck-Institut für Mathematik for their support during the initial stages of this project, the Ireland Canada University Foundation for the James M. Flaherty Visiting Professorship award and McMaster University for their hospitality during his stay from May 17 to August 9, 2019. Finally, the first, third, sixth and seventh authors graciously thank the Fields Institute for the opportunity to take part in their 2019 Undergraduate Summer Research Program.

REFERENCES

- [1] S. Ahlgren, B. Kim, *Dissections of a “strange” function*, Int. J. Number Theory **11** (2015), no. 5, 1557–1562.
- [2] S. Ahlgren, B. Kim and J. Lovejoy, *Dissections of strange q -series*, Ann. Comb. **23** (2019), no. 3–4, 427–442.
- [3] G. E. Andrews, J. Sellers, *Congruences for the Fishburn numbers*, J. Number Theory **161** (2016), 298–310.
- [4] D. Cooper, M. Culler, H. Gillet, D. D. Long and P. B. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. **118** (1994), no. 1, 47–84.
- [5] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, 291–309, Geom. Topol. Monogr., **7**, Geom. Topol. Publ., Coventry, 2004.
- [6] S. Garoufalidis, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69.
- [7] F. G. Garvan, *Congruences and relations for r -Fishburn numbers*, J. Combin. Theory Ser. A **134** (2015), 147–165.
- [8] P. Guerzhoy, Z. Kent and L. Rolin, *Congruences for Taylor expansions of quantum modular forms*, Res. Math. Sci. **1** (2014), Art. 17, 17pp.
- [9] K. Habiro, *On the colored Jones polynomial of some simple links*, in: Recent progress toward the volume conjecture (Kyoto, 2000), Sūrikaiseikikenkyūsho Kōkyūroku **1172** (2000), 34–43.
- [10] K. Habiro, *A unified Witten-Reshetikhin-Turaev invariant for homology spheres*, Invent. Math. **171** (2008), no. 1, 1–81.
- [11] K. Hikami, *Difference equation of the colored Jones polynomial for torus knot*, Internat. J. Math. **15** (2004), no. 9, 959–965.
- [12] K. Hikami, *q -series and L -functions related to half-derivates of the Andrews-Gordon identity*, Ramanujan J. **11** (2006), no. 2, 175–197.
- [13] K. Hikami, A. Kirillov, *Hypergeometric generating function of L -function, Slater’s identities, and quantum invariant*, St. Petersburg Math. J. **17** (2006), no. 1, 143–156.
- [14] E. Kalfagianni, A. T. Tran, *Knot cabling and the degree of the colored Jones polynomial*, New York J. Math. **21** (2015), 905–941.
- [15] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997) no. 3, 269–275.
- [16] I. Konan, *Autour des q -séries, des formes modulaires quantiques et des nœuds toriques*, Master’s thesis, Université Denis Diderot - Paris 7, available at https://www.irif.fr/_media/users/konan/memoire.pdf
- [17] T. T. Q. Lê, *Quantum invariants of 3-manifolds: Integrality, splitting, and perturbative expansion*, Topology Appl. **127** (2003), no. 1–2, 125–152.
- [18] J. Lovejoy, R. Osburn, *The colored Jones polynomial and Kontsevich-Zagier series for double twist knots*, preprint available at <https://arxiv.org/abs/1710.04865>
- [19] J. Lovejoy, R. Osburn, *The colored Jones polynomial and Kontsevich-Zagier series for double twist knots, II*, New York J. Math. **25** (2019), 1312–1349.
- [20] G. Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebr. Geom. Topol. **3** (2003), 537–556.
- [21] H. Murakami, *An introduction to the volume conjecture*, Interactions between hyperbolic geometry, quantum topology and number theory, 1–40, Contemp. Math., **541**, Amer. Math. Soc., Providence, RI, 2011.
- [22] H. Murakami, J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), no. 1, 85–104.
- [23] The On-Line Encyclopedia of Integer Sequences, <https://oeis.org>
- [24] A. Stoimenow, *Enumeration of chord diagrams and an upper bound for Vassiliev invariants*, J. Knot Theory Ramifications **7** (1998), no. 1, 93–114.
- [25] A. Straub, *Congruences for Fishburn numbers modulo prime powers*, Int. J. Number Theory **11** (2015), no. 5, 1679–1690.
- [26] T. Takata, *A formula for the colored Jones polynomial of 2-bridge knots*, Kyungpook Math. J. **48** (2008), no. 2, 255–280.
- [27] K. Walsh, *Patterns and stability in the coefficients of the colored Jones polynomial*, Ph.D. thesis, University of California, San Diego, 2014.
- [28] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40** (2001), no. 5, 945–960.

- [29] D. Zagier, *Quantum modular forms*, *Quanta of maths*, 659–675, Clay Math. Proc., **11**, Amer. Math. Soc., Providence, RI, 2010.

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