

CLASSIFICATION OF HIGHER WIDE SUBCATEGORIES FOR HIGHER AUSLANDER ALGEBRAS OF TYPE A

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ABSTRACT. A subcategory \mathcal{W} of an abelian category is called wide if it is closed under kernels, cokernels, and extensions. Wide subcategories are of interest in representation theory because of their links to other homological and combinatorial objects, established among others by Ingalls–Thomas and Marks–Šťovíček.

If $d \geq 1$ is an integer, then Jasso introduced the notion of d -abelian categories, where kernels, cokernels, and extensions have been replaced by longer complexes. Wide subcategories can be generalised to this situation.

Important examples of d -abelian categories arise as the d -cluster tilting subcategories $\mathcal{M}_{n,d}$ of $\text{mod } A_n^{d-1}$, where A_n^{d-1} is a higher Auslander algebra of type A in the sense of Iyama. This paper gives a combinatorial description of the wide subcategories of $\mathcal{M}_{n,d}$ in terms of what we call non-interlacing collections.

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1. INTRODUCTION

There has recently been considerable interest in wide subcategories of abelian categories, which are full subcategories closed under kernels, cokernels, and extensions. It was shown in [IT, thm. 1.1] that in the category of finite dimensional representations of a finite, acyclic quiver Q , wide subcategories have strong links to other objects from homological algebra. For instance, there is a bijection between wide subcategories and torsion classes. If Q is of extended Dynkin type, then there are also links to combinatorial objects, such as a bijection to the noncrossing partitions associated with Q . The bijection between wide subcategories and torsion classes was extended to a large class of module categories over finite dimensional algebras in [MS, cor. 3.11].

For an integer $d \geq 1$, the notion of d -abelian categories was introduced in [J] as an analogue of abelian categories from the point of view of higher dimensional Auslander-Reiten theory. In

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a d -abelian category, kernels, cokernels and extensions are replaced by longer complexes, see [J, def. 3.1]. The prototypical example of a d -abelian category is a d -cluster tilting subcategory of the module category of a finite dimensional algebra. Algebras of global dimension d admitting d -cluster tilting subcategories with finitely many indecomposables are called d -representation finite [IO] and have been extensively studied as a source of d -abelian categories. One of the features of a d -representation finite algebra is that its module category has a unique d -cluster tilting subcategory.

An algebra is 1-representation finite if and only if it is representation finite and hereditary. So if Q is a Dynkin quiver, then the path algebra of Q is 1-representation finite. For $d > 1$, the first known d -representation finite algebras are the higher Auslander algebras of type A introduced by Iyama in [I2]. They are constructed recursively starting from path algebras of Dynkin type A with linear orientation. Then for each d , one takes the algebras obtained for $d - 1$ and passes to their corresponding higher Auslander algebras. The structure of the corresponding d -abelian categories was described in detail in [OT]. Recently, higher Auslander algebras of type A have been used in [DJL] as a tool to study partially wrapped Fukaya categories of symmetric products of the unit disc with finitely many stops.

The notion of wide subcategories was generalised to d -abelian categories in [HJV, def. 2.11]. A general theory of such wide subcategories was developed in [HJV], and some simple examples were worked out in [HJV, sec. 7]. In this paper we use the theory of [HJV] and [OT] to describe combinatorially all wide subcategories of the d -abelian categories which arise from higher Auslander algebras of type A .

The main result is Theorem 3.1, and the description is in terms of what we call non-interlacing collections. It has as a special case the bijection in [IT, thm. 1.1] between classic wide subcategories and non-crossing partitions in Dynkin type A .

The d -cluster tilting subcategory \mathcal{M} of the module category of a d -representation finite algebra also gives rise to a d -cluster tilting subcategory \mathcal{C} of its bounded derived category. The category \mathcal{C} has the structure of a $(d + 2)$ -angulated category in the sense of [GKO], and there is a strong interplay between the d -abelian structure of \mathcal{M} and the $(d + 2)$ -angulated structure of \mathcal{C} . In certain situations this can even be used to study $(d + 2)$ -angulated categories in a more general setting (see for instance [JJ]).

In [B] it is shown that for a hereditary algebra A there is a bijection between classic wide subcategories of the module category of A and wide subcategories of its bounded derived category. This was generalised in [F] to the setting described above, i.e., there is a bijection between the wide subcategories of \mathcal{M} and the wide subcategories of \mathcal{C} . In particular, this can be applied to higher Auslander algebras of type A . In other words, combining our results with [F] we obtain a classification of wide subcategories of the $(d + 2)$ -angulated categories associated to higher Auslander algebras of type A .

2. PRELIMINARIES

2.1. Conventions. Let K be a field. All categories and functors are assumed to be K -linear. In all contexts we denote the K -dual $\text{Hom}_K(-, K)$ by D .

Let \mathcal{M} be a category and $M, N \in \mathcal{M}$. A diagram $M \leftrightarrow \cdots \leftrightarrow N$, where \leftrightarrow represents a non-zero morphism in either direction is called a walk from M to N . We call a category connected if any two non-zero objects are connected by a walk.

Let \mathcal{M} be an additive category. By an additive subcategory $\mathcal{W} \subseteq \mathcal{M}$ we mean a full subcategory closed under direct sums and summands. For a collection of objects \mathcal{C} in \mathcal{M} we denote by $\text{add } \mathcal{C}$ the smallest additive subcategory of \mathcal{M} containing \mathcal{C} .

We call \mathcal{M} Krull-Schmidt if each object decomposes into a direct sum of finitely many indecomposable objects and each indecomposable object has local endomorphism algebra. For such a category we denote by $\text{Rad}_{\mathcal{M}}$ its Jacobson radical, i.e.,

$$\text{Rad}_{\mathcal{M}}(M, N) = \{f \in \mathcal{M}(M, N) \mid 1_M - g \circ f \text{ is invertible for all } g \in \mathcal{M}(N, M)\}.$$

The square of the Jacobson radical is

$$\text{Rad}_{\mathcal{M}}^2(M, N) = \{f \circ g \mid f \in \text{Rad}_{\mathcal{M}}(U, N), g \in \text{Rad}_{\mathcal{M}}(M, U) \text{ for some } U \in \mathcal{M}\}.$$

Next assume that \mathcal{M} is skeletally small, Hom-finite and $\mathcal{M}(M, M)/\text{Rad}_{\mathcal{M}}(M, M) = K$ for all indecomposable $M \in \mathcal{M}$. We then define the quiver Q of \mathcal{M} as follows. As vertices Q_0 choose a set of representatives of the isomorphism classes of indecomposable objects in \mathcal{M} . For $M, N \in Q_0$ we choose as arrows from M to N a subset of $\text{Rad}_{\mathcal{M}}(M, N)$, that gives a K -basis of $\text{Rad}_{\mathcal{M}}(M, N)/\text{Rad}_{\mathcal{M}}^2(M, N)$.

By an algebra we mean a finite dimensional K -algebra. For an algebra A we denote the category of finitely generated right A -modules by $\text{mod } A$. By A -module we always mean an object in $\text{mod } A$. We use the terminology of quivers, quiver representations, and path algebras following the conventions in [ASS].

Let \mathcal{W} be an additive subcategory of $\text{mod } A$. By a \mathcal{W} -resolution of $M \in \text{mod } A$ we mean an exact sequence

$$0 \rightarrow W_m \rightarrow \cdots \rightarrow W_0 \rightarrow M \rightarrow 0,$$

where $m \geq 0$ and $W_i \in \mathcal{W}$.

2.2. d -abelian categories and wide subcategories. In this section we introduce some basic results for d -abelian categories. We mainly rely on [J], in which d -abelian categories were first introduced. We then recall the notion of a wide subcategory of a d -abelian category following [HJV].

Definition 2.1. Let \mathcal{M} be an additive category and

$$\mathbb{E} : M_{d+1} \xrightarrow{f} M_d \rightarrow \cdots \rightarrow M_1 \xrightarrow{g} M_0$$

a sequence in \mathcal{M} .

(1) We call

$$M_{d+1} \xrightarrow{f} M_d \rightarrow \cdots \rightarrow M_1$$

a d -kernel of g if

$$0 \rightarrow \mathcal{M}(M, M_{d+1}) \xrightarrow{f \circ -} \mathcal{M}(M, M_d) \rightarrow \cdots \rightarrow \mathcal{M}(M, M_0)$$

is exact for all $M \in \mathcal{M}$.

(2) We call

$$M_d \rightarrow \cdots \rightarrow M_1 \xrightarrow{g} M_0$$

a d -cokernel of f if

$$0 \rightarrow \mathcal{M}(M_0, M) \xrightarrow{- \circ g} \mathcal{M}(M, M_1) \rightarrow \cdots \rightarrow \mathcal{M}(M, M_d)$$

is exact for all $M \in \mathcal{M}$.

(3) If both (1) and (2) are satisfied we call \mathbb{E} a d -exact sequence (or a d -extension of M_0 by M_{d+1}).

(4) We say that \mathcal{M} is d -abelian if it is idempotent split, every morphism admits a d -kernel and d -cokernel, and every monomorphism f respectively epimorphism g fits into a d -exact sequence of the form \mathbb{E} .

There is a natural equivalence relation on d -extensions introduced in [J], which we now recall.

Definition 2.2. Let \mathcal{M} be a d -abelian category. We call two d -extensions

$$\mathbb{E} : X \rightarrow E_d \rightarrow \cdots \rightarrow E_1 \rightarrow Y$$

and

$$\mathbb{E}' : X \rightarrow E'_d \rightarrow \cdots \rightarrow E'_1 \rightarrow Y$$

equivalent if there is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & E_d & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & Y \\ \parallel & & \downarrow & & & & \downarrow & & \parallel \\ X & \longrightarrow & E'_d & \longrightarrow & \cdots & \longrightarrow & E'_1 & \longrightarrow & Y \end{array}$$

It follows from [J, Proposition 4.10] that this indeed does define an equivalence relation.

Next we introduce the notion of wide subcategories.

Definition 2.3. Let \mathcal{M} be a d -abelian category. We call an additive subcategory $\mathcal{W} \subseteq \mathcal{M}$ *wide* if the following conditions hold.

- (1) Every morphism $f : M \rightarrow N$ in \mathcal{W} admits a d -kernel and d -cokernel in \mathcal{M} with terms in \mathcal{W} .
- (2) Every d -extension

$$\mathbb{E} : X \rightarrow E_d \rightarrow \cdots \rightarrow E_1 \rightarrow Y$$

in \mathcal{M} with $X, Y \in \mathcal{W}$ is equivalent to a d -extension

$$\mathbb{E} : X \rightarrow E'_d \rightarrow \cdots \rightarrow E'_1 \rightarrow Y$$

with $E'_i \in \mathcal{W}$ for all $1 \leq i \leq d$.

For a class of objects \mathcal{C} in \mathcal{M} we denote the smallest wide subcategory of \mathcal{M} containing \mathcal{C} by $\text{wide } \mathcal{C}$.

A difference between abelian and d -abelian categories is that d -kernels and d -cokernels are not unique up to isomorphism. Similarly, equivalent d -extensions may be non-isomorphic. However, it is shown in [J] that uniqueness holds if we replace isomorphism of complexes by homotopy equivalence.

Next we consider the case when \mathcal{M} is a Krull-Schmidt category. We start by showing that representatives of d -kernels, d -cokernels and d -extensions can be chosen in a certain minimal way that is unique.

Proposition 2.4. *Let \mathcal{M} be a d -abelian Krull-Schmidt category and $X, Y \in \mathcal{M}$.*

- (1) *Let $f \in \mathcal{M}(X, Y)$. Then there is a d -kernel*

$$K_d \xrightarrow{k_d} \cdots \xrightarrow{k_2} K_1 \rightarrow X$$

of f such that $k_i \in \text{Rad}_{\mathcal{M}}(K_i, K_{i-1})$ for all $2 \leq i \leq d$. Moreover, this d -kernel appears as a direct summand (in the category of \mathcal{M} -complexes) of any other d -kernel of f .

- (2) *Let $f \in \mathcal{M}(X, Y)$. Then there is a d -cokernel*

$$Y \rightarrow C_d \xrightarrow{c_d} \cdots \xrightarrow{c_2} C_1$$

of f such that $c_i \in \text{Rad}_{\mathcal{M}}(C_i, C_{i-1})$ for all $2 \leq i \leq d$. Moreover, this d -cokernel appears as a direct summand (in the category of \mathcal{M} -complexes) of any other d -cokernel of f .

- (3) *In every equivalence class of d -extensions of Y by X there is a representative*

$$\mathbb{E} : X \rightarrow E_d \xrightarrow{e_d} \cdots \xrightarrow{e_2} E_1 \rightarrow Y$$

such that $e_i \in \text{Rad}_{\mathcal{M}}(E_i, E_{i-1})$ for all $2 \leq i \leq d$. Moreover, \mathbb{E} is a direct summand (in the category of \mathcal{M} -complexes) of every other equivalent d -extension.

Proof. (1) Let

$$K_d \xrightarrow{k_d} \cdots \xrightarrow{k_2} K_1 \xrightarrow{k_1} X$$

be a d -kernel of f such that the number of indecomposable direct summands of $\bigoplus_{i=1}^d K_i$ is minimal. If k_i is not a radical morphism for some $2 \leq i \leq d$, then we may write $K_i = U \oplus \tilde{K}_i$ and $K_{i-1} = U \oplus \tilde{K}_{i-1}$ such that $k_i = 1_U \oplus \tilde{k}_i$ for some $\tilde{k}_i : \tilde{K}_i \rightarrow \tilde{K}_{i-1}$ where U is indecomposable. It follows that replacing $k_i : K_i \rightarrow K_{i-1}$ in the d -kernel by $k'_i : K'_i \rightarrow K'_{i-1}$, gives a d -kernel

with smaller total number of indecomposable direct summands in its terms, which contradicts minimality.

Next let

$$K'_d \xrightarrow{k'_d} \cdots \xrightarrow{k'_2} K'_1 \xrightarrow{k'_1} X$$

be another d -kernel of f . Using the defining property of d -kernels we obtain the commutative diagrams

$$\begin{array}{ccccccc} K_d & \xrightarrow{k_d} & K_{d-1} & \xrightarrow{k_{d-1}} & \cdots & \xrightarrow{k_2} & K_1 & \xrightarrow{k_1} & X \\ \downarrow a_d & & \downarrow a_{d-1} & & & & \downarrow a_1 & & \parallel \\ K'_d & \xrightarrow{k'_d} & K'_{d-1} & \xrightarrow{k'_{d-1}} & \cdots & \xrightarrow{k'_2} & K'_1 & \xrightarrow{k'_1} & X \end{array}$$

and

$$\begin{array}{ccccccc} K'_d & \xrightarrow{k'_d} & K'_{d-1} & \xrightarrow{k'_{d-1}} & \cdots & \xrightarrow{k'_2} & K'_1 & \xrightarrow{k'_1} & X \\ \downarrow b_d & & \downarrow b_{d-1} & & & & \downarrow b_1 & & \parallel \\ K_d & \xrightarrow{k_d} & K_{d-1} & \xrightarrow{k_{d-1}} & \cdots & \xrightarrow{k_2} & K_1 & \xrightarrow{k_1} & X. \end{array}$$

Now consider $c_i = 1_{K_i} - b_i \circ a_i \in \mathcal{M}(K_i, K_i)$. Using the property of d -kernels we find morphisms h_i in the diagram below

$$\begin{array}{ccccccc} K_d & \xrightarrow{k_d} & K_{d-1} & \xrightarrow{k_{d-1}} & \cdots & \xrightarrow{k_2} & K_1 & \xrightarrow{k_1} & X \\ \downarrow c_d & \swarrow h_{d-1} & \downarrow c_{d-1} & \swarrow h_{d-2} & & & \downarrow c_1 & & \downarrow 0 \\ K_d & \xrightarrow{k_d} & K_{d-1} & \xrightarrow{k_{d-1}} & \cdots & \xrightarrow{k_2} & K_1 & \xrightarrow{k_1} & X \end{array}$$

such that

$$\begin{cases} c_1 = k_2 \circ h_1, \\ c_i = k_{i+1} \circ h_i + h_{i-1} \circ k_i, & 2 \leq i \leq d-1, \\ c_d = h_{d-1} \circ k_d. \end{cases}$$

In particular, $c_i \in \text{Rad}_{\mathcal{M}}(K_i, K_i)$ and so $b_i \circ a_i = 1_{K_i} - c_i$ is an isomorphism. Hence we may write $1_{K_i} = ((b_i \circ a_i)^{-1} b_i) \circ a_i$. The claim follows as \mathcal{M} has split idempotents.

(2) This is dual to (1).

(3) Existence of \mathbb{E} follows in the same way as in (1). Now let

$$\mathbb{E}' : X \xrightarrow{e'_{d+1}} E'_d \xrightarrow{e'_d} \cdots \xrightarrow{e'_2} E'_1 \xrightarrow{e'_1} Y$$

be an equivalent d -extension. Then there is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{e_{d+1}} & E_d & \xrightarrow{e_d} & \cdots & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & Y \\ \parallel & & \downarrow a_d & & & & \downarrow a_1 & & \parallel \\ X & \xrightarrow{e'_{d+1}} & E'_d & \xrightarrow{e'_d} & \cdots & \xrightarrow{e'_2} & E'_1 & \xrightarrow{e'_1} & Y. \end{array}$$

By [J, Proposition 4.10], there is also a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{e'_{d+1}} & E'_d & \xrightarrow{e'_d} & \cdots & \xrightarrow{e'_2} & E'_1 & \xrightarrow{e'_1} & Y \\ \parallel & & \downarrow b_d & & & & \downarrow b_1 & & \parallel \\ X & \xrightarrow{e_{d+1}} & E_d & \xrightarrow{e_d} & \cdots & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & Y. \end{array}$$

As in (1) we proceed to show that $c_i = 1_{E_i} - b_i \circ a_i \in \text{Rad}_{\mathcal{M}}(E_i, E_i)$ by constructing a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{e_{d+1}} & E_d & \xrightarrow{e_d} & \cdots & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & Y \\ \downarrow 0 & \swarrow h_d & \downarrow c_d & \swarrow h_{d-1} & & \swarrow h_1 & \downarrow c_1 & & \downarrow 0 \\ X & \xrightarrow{e_{d+1}} & E_d & \xrightarrow{e_d} & \cdots & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & Y \end{array}$$

such that

$$\begin{cases} c_1 = e_2 \circ h_1, \\ c_i = e_{i+1} \circ h_i + h_{i-1} \circ e_i, & 2 \leq i \leq d, \\ 0 = h_d \circ e_{d+1}. \end{cases}$$

By the defining property of d -cokernels, the last condition implies that h_d factors through e_d and so is a radical morphism. Hence $c_i \in \text{Rad}_{\mathcal{M}}(E_i, E_i)$ for all $1 \leq i \leq d$ and the claim follows as in (1). \square

Definition 2.5. We call the d -kernels, d -cokernels and d -extensions appearing in Proposition 2.4 *minimal*. By the Krull-Schmidt property it follows immediately that they are unique up to isomorphism.

A consequence of Proposition 2.4 is the following characterisation of wide subcategories.

Corollary 2.6. Let \mathcal{M} be a d -abelian Krull-Schmidt category and $\mathcal{W} \subseteq \mathcal{M}$ a full subcategory closed under direct sums and direct summands. Then \mathcal{W} is wide if and only if the following conditions hold.

- (1) For any morphism $f : M \rightarrow N$ in \mathcal{W} , the terms of its minimal d -kernel and d -cokernel in \mathcal{M} lie in \mathcal{W} .
- (2) For any $X, Y \in \mathcal{W}$, every minimal d -extension of Y by X in \mathcal{M} has terms in \mathcal{W} .

The following result about d -kernels will be useful to compute wide subcategories.

Proposition 2.7. Let \mathcal{M} be a d -abelian Krull-Schmidt category, $M \in \mathcal{M}$ indecomposable and $K_d \rightarrow \cdots \rightarrow K_1 \xrightarrow{k} M$ be a d -kernel of a non-zero morphism $M \xrightarrow{f} N$. If $U \in \mathcal{M}$ is indecomposable and there exists $g \in \mathcal{M}(U, M) \setminus \text{Rad}_{\mathcal{M}}^2(U, M)$ such that $f \circ g = 0$, then U appears as a summand in K_1 .

Proof. Assume that U does not appear as a summand in K_1 . Since $f \circ g = 0$ we may write $g = k \circ h$ for some $h \in \mathcal{M}(U, K_1) = \text{Rad}_{\mathcal{M}}(U, K_1)$. Since f is non-zero and M is indecomposable, $k \in \text{Rad}_{\mathcal{M}}(K_1, M)$ and so $g \in \text{Rad}_{\mathcal{M}}^2(U, M)$, which is a contradiction. \square

2.3. Wide subcategories of d -cluster tilting subcategories. Let A be a finite dimensional algebra and $\text{mod } A$ the category of finitely generated right A -modules. We recall the the definition of d -cluster tilting from [I2] (see also [HJV, Section 2])

Definition 2.8. Let \mathcal{M} be a functorially finite subcategory of $\text{mod } A$. We say that \mathcal{M} is d -cluster tilting if

$$\begin{aligned} \mathcal{M} &= \{X \in \text{mod } A \mid \text{Ext}_A^i(X, M) = 0 \text{ for all } M \in \mathcal{M}, 1 \leq i \leq d-1\} \\ &= \{X \in \text{mod } A \mid \text{Ext}_A^i(M, X) = 0 \text{ for all } M \in \mathcal{M}, 1 \leq i \leq d-1\}. \end{aligned}$$

As mentioned in the introduction it is shown in [J] that any d -cluster tilting subcategory \mathcal{M} is d -abelian. Moreover, the following characterisation of d -kernels, d -cokernels and d -exact sequences in \mathcal{M} holds.

Proposition 2.9. Let $\mathcal{M} \subseteq \text{mod } A$ be d -cluster tilting. Then \mathcal{M} is d -abelian. Let

$$M_{d+1} \xrightarrow{f} M_d \rightarrow \cdots \rightarrow M_1 \xrightarrow{g} M_0$$

be a sequence in \mathcal{M} .

(1) The sequence $M_{d+1} \rightarrow M_d \rightarrow \cdots \rightarrow M_1$ is a d -kernel of g if and only if

$$0 \rightarrow M_{d+1} \rightarrow M_d \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$$

is exact in $\text{mod } A$.

(2) The sequence $M_d \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$ is a d -cokernel of f if and only if

$$M_{d+1} \rightarrow M_d \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow 0$$

it is exact in $\text{mod } A$.

(3) The sequence $M_{d+1} \rightarrow M_d \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$ is d -exact if and only if

$$0 \rightarrow M_{d+1} \rightarrow M_d \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow 0$$

is exact in $\text{mod } A$.

(4) Let $X, Y \in \mathcal{M}$. There is a bijection from the set of equivalence classes of d -extensions of Y by X in \mathcal{M} to $\text{Ext}_A^d(Y, X)$ that sends the equivalence class of a d -extension

$$0 \rightarrow X \rightarrow E_d \rightarrow \cdots \rightarrow E_1 \rightarrow Y \rightarrow 0$$

to its Yoneda-class

$$[0 \rightarrow X \rightarrow E_d \rightarrow \cdots \rightarrow E_1 \rightarrow Y \rightarrow 0].$$

Proof. The fact that \mathcal{M} is d -abelian is shown in [J, Theorem 3.16].

The statements (1), (2), (3) are well-known and easily shown using the fact that $\text{Ext}_A^i(M, M') = 0$ for all $M, M' \in \mathcal{M}$ and $1 \leq i \leq d$.

Statement (4) follows from [I1, Proposition A.1]. \square

It will be convenient to construct wide subcategories from several smaller wide subcategories. For this purpose the following result is useful.

Proposition 2.10. *Let $\mathcal{M} \subseteq \text{mod } A$ be d -cluster tilting and $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{M}$ wide subcategories. If*

$$\text{Hom}_A(X_1, X_2) = \text{Hom}_A(X_2, X_1) = \text{Ext}_A^d(X_1, X_2) = \text{Ext}_A^d(X_2, X_1) = 0$$

for all $X_1 \in \mathcal{W}_1, X_2 \in \mathcal{W}_2$. Then $\text{add}\{\mathcal{W}_1, \mathcal{W}_2\} \subseteq \mathcal{M}$ is wide.

Proof. Let $X, Y \in \text{add}\{\mathcal{W}_1, \mathcal{W}_2\}$. Then we may write $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ for some $X_1, Y_1 \in \mathcal{W}_1$ and $X_2, Y_2 \in \mathcal{W}_2$

Let $f : X \rightarrow Y$ be a morphism. By assumption we may write $f = f_1 \oplus f_2$ for some $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$. Since \mathcal{W}_1 and \mathcal{W}_2 are wide there are d -kernels respectively d -cokernels of f_1 and f_2 with terms in $\text{add}\{\mathcal{W}_1, \mathcal{W}_2\}$. Taking their direct sums gives a d -kernel respectively d -cokernel of f .

Next consider the natural map

$$\text{Ext}_A^d(Y_1, X_1) \oplus \text{Ext}_A^d(Y_2, X_2) \rightarrow \text{Ext}_A^d(Y, X)$$

defined by the biadditivity of Ext_A^d . It maps $([\mathbb{E}_1], [\mathbb{E}_2])$ to $[\mathbb{E}_1 \oplus \mathbb{E}_2]$ and is bijective since $\text{Ext}_A^d(Y_1, X_2) = \text{Ext}_A^d(Y_2, X_1) = 0$. Hence, by Proposition 2.9, any d -extension of Y by X is equivalent to $\mathbb{E}_1 \oplus \mathbb{E}_2$ for some d -extensions \mathbb{E}_1 in \mathcal{W}_1 and \mathbb{E}_2 in \mathcal{W}_2 . \square

To compute wide subcategories of d -cluster tilting subcategories, we will apply the following result.

Theorem 2.11. [HJV, Theorem B] *Let A be a finite dimensional algebra and \mathcal{M} a d -cluster tilting subcategory of $\text{mod } A$. Let $\mathcal{W} \subseteq \mathcal{M}$ be an additive subcategory. Let $P \in \mathcal{W}$ be a module and set $B = \text{End}_A(P)$, so that P becomes a B - A -bimodule. Assume the following:*

(1) *As an A -module P has finite projective dimension.*

(2) *$\text{Ext}_A^i(P, P) = 0$ for all $i \geq 1$.*

(3) *Each $W \in \mathcal{W}$ admits an $\text{add } P$ -resolution*

$$0 \rightarrow P_m \rightarrow \cdots \rightarrow P_0 \rightarrow W \rightarrow 0, \quad P_i \in \text{add } P.$$

(4) *$\text{Hom}_A(P, \mathcal{W}) \subseteq \text{mod } B$ is d -cluster tilting.*

Then \mathscr{W} is a wide subcategory of \mathscr{M} and there is an equivalence of categories

$$- \otimes_B P : \text{Hom}_A(P, \mathscr{W}) \rightarrow \mathscr{W}.$$

2.4. Higher Auslander algebras of type A . Next we introduce a description of the higher Auslander algebras of type A by quivers and relations. Our notation differs slightly from the one in [OT].

We begin by introducing some combinatorial data that is needed to make our definitions.

Definition 2.12. Throughout we fix two integers n and d with $n \geq 1$ and $d \geq 0$.

- (1) Set $N_{n,d} := \{1, \dots, n+d\}$.
- (2) Set $\mathcal{V}_{n,d} := \{(x_0, x_1, \dots, x_d) \in N_{n,d}^{d+1} \mid x_0 < x_1 < \dots < x_d\}$.
- (3) For each $k \in N_{n,d}$, define the partial functions

$$\sigma_k^+ : \mathcal{V}_{n,d} \rightarrow \mathcal{V}_{n,d} \quad \text{and} \quad \sigma_k^- : \mathcal{V}_{n,d} \rightarrow \mathcal{V}_{n,d}$$

by $\sigma_k^\pm(x) = y$, where

$$y_i = \begin{cases} x_i \pm 1 & \text{if } i = k, \\ x_i & \text{if } i \neq k, \end{cases}$$

whenever such $y \in \mathcal{V}_{n,d}$ exists.

- (4) Define the relations \preceq and \prec on $\mathcal{V}_{n,d}$ by

$$x \preceq y \quad \text{if and only if} \quad x_0 \leq y_0 < x_1 \leq y_1 < \dots < x_d \leq y_d,$$

and

$$x \prec y \quad \text{if and only if} \quad x_0 < y_0 \leq x_1 < y_1 \leq \dots \leq x_d < y_d.$$

- (5) Let $x, y \in \mathcal{V}_{n,d}$. We say that x and y interlace in case $x \preceq y$, $y \preceq x$, $x \prec y$ or $y \prec x$ hold.
- (6) Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{V}_{n,d}$. We say that \mathcal{X} and \mathcal{Y} interlace in case there are $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that x and y interlace.
- (7) For $S \subseteq N_{n,d}$ we let \mathcal{X}_S be the set of all $x \in \mathcal{V}_{n,d}$ such that $x_i \in S$ for all i . We say that $S, S' \subseteq N_{n,d}$ interlace in case \mathcal{X}_S and $\mathcal{X}_{S'}$ interlace.
- (8) For $x \in \mathcal{V}_{n,d}$ we set $S_x = \{x_0, \dots, x_{n+d}\} \subseteq N_{n,d}$. Note in particular that $\mathcal{X}_{S_x} = \{x\}$.

Next define a quiver $Q^{n,d}$ with vertices $\mathcal{V}_{n,d}$ and arrows $\alpha_k^x : x \rightarrow \sigma_k^+(x)$, for all $k \in N_{n,d}$ and $x \in \mathcal{V}_{n,d}$ such that $\sigma_k^+(x)$ is defined.

Let $k, l \in N_{n,d}$ be two distinct elements and $x \in \mathcal{V}_{n,d}$ be such that $y \in \mathcal{V}_{n,d}$, where

$$y_i = \begin{cases} x_i \pm 1 & \text{if } i \in \{k, l\}, \\ x_i & \text{if } i \notin \{k, l\}, \end{cases}$$

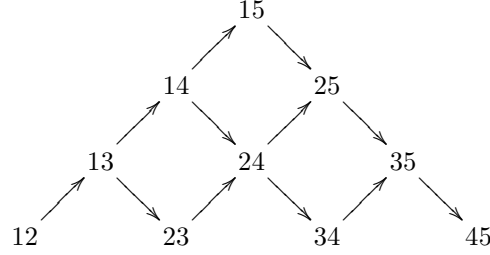
We introduce a relation ρ_{kl}^x from x to y defined by

$$\rho_{kl}^x = \begin{cases} \alpha_k^x \alpha_l^{\sigma_k^+(x)} - \alpha_l^x \alpha_k^{\sigma_l^+(x)} & \text{if both } \sigma_k^+(x) \text{ and } \sigma_l^+(x) \text{ are defined,} \\ \alpha_k^x \alpha_l^{\sigma_k^+(x)} & \text{if } \sigma_k^+(x) \text{ is defined but } \sigma_l^+(x) \text{ is undefined,} \\ \alpha_l^x \alpha_k^{\sigma_l^+(x)} & \text{if } \sigma_l^+(x) \text{ is defined but } \sigma_k^+(x) \text{ is undefined,} \end{cases}$$

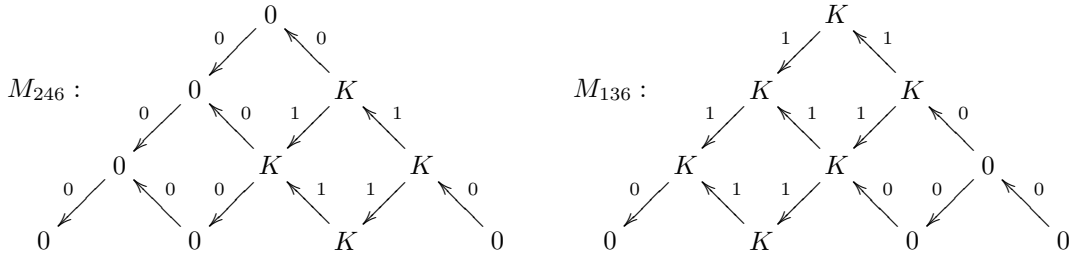
and let $I_{n,d}$ be the ideal in $KQ^{n,d}$ generated by all ρ_{kl}^x .

The algebra $A_n^d = (KQ^{n,d}/I_{n,d})^{\text{op}}$ is called a higher Auslander algebra of type A . Note that if $d = 0$, then A_n^d is just a path algebra of Dynkin type A_n , which strictly should not be called a higher Auslander algebra. We include it so that for $d > 0$ we may say that A_n^d is the higher Auslander algebra of A_n^{d-1} . To explain why this is so we define an indecomposable A_n^{d-1} -module M_x for each $x \in \mathcal{V}_{n,d}$. As a representation M_x assigns the vector space K to all vertices $y \in \mathcal{V}_{n,d-1}$ such that $x_i \leq y_i < x_{i+1}$ for all $0 \leq i \leq d-1$. To all other vertices M_x assigns the zero vector space. Moreover, all arrows $K \rightarrow K$ act as the identity, while other arrows (by necessity) act as zero.

Example 2.13. Let us consider $n = 4$ and $d = 2$. Then the quiver $Q^{4,1}$ is



The modules M_{246} and M_{136} are given as representations of $(Q^{4,1})^{\text{op}}$ as follows



Theorem 2.14. [I2][OT, Section 3] Let n and d be positive integers. Then A_n^{d-1} has global dimension d and admits a unique basic d -cluster tilting module

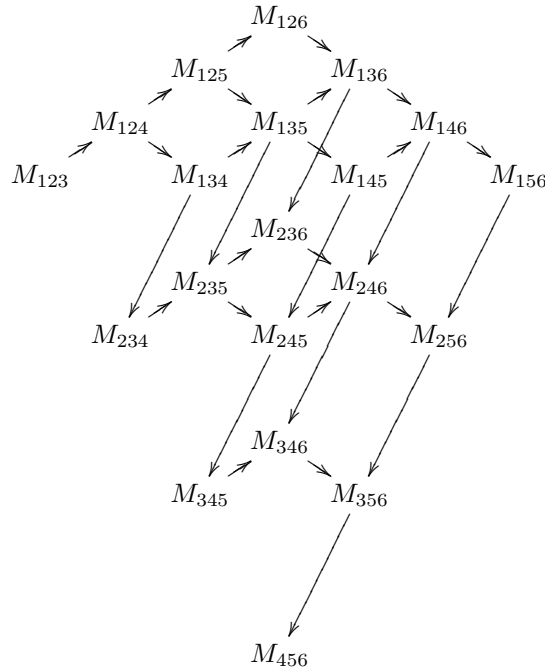
$$M = \bigoplus_{x \in \mathcal{V}_{n,d}} M_x.$$

Moreover, $\text{End}_{A_n^{d-1}}(M)$ is isomorphic to $A_{n,d}$.

We let $\mathcal{M}_{n,d} = \text{add}\{M\}$, where M is the d -cluster tilting module in Theorem 2.14. Then $\mathcal{M}_{n,d}$ is d -abelian. Our aim is to classify the wide subcategories of $\mathcal{M}_{n,d}$.

The isomorphism from $A_{n,d}$ to $\text{End}_{A_n^{d-1}}(M)$ comes from realising the quiver of $\mathcal{M}_{n,d}$ as $Q^{n,d}$. We illustrate this in one example.

Example 2.15. Consider the case $n = 4$ and $d = 2$. Below is the quiver of $\mathcal{M}_{4,2}$ (i.e, $Q^{4,2}$).



Note in particular, that there is a path from M_{136} to M_{246} , the modules appearing in Example 2.13. Considering the relations defining $A_{4,2}$, this path should correspond to a nonzero morphism $\phi : M_{136} \rightarrow M_{246}$. Indeed, such a ϕ is easy to find. As a morphism of representations we may define it by $\phi_{24} = \phi_{25} = 1_K$ and $\phi_{ij} = 0$ for all other indices ij . Compare this with the fact that $(1, 3, 6) \preceq (2, 4, 6)$.

As in the above example, there is in general an obvious bijection between the indecomposables in $\mathcal{M}_{n,d}$ and $Q_0^{n,d}$. Moreover, morphisms corresponding to the arrows in $Q_0^{n,d}$ are easy to write down. In fact morphisms and extensions between indecomposables in $\mathcal{M}_{n,d}$ have been computed in [OT]. Here we recall some of their results rewritten in our notation.

Theorem 2.16. [OT, Theorem 3.6]

- (1) Let $x \in \mathcal{V}_{n,d-1}$ and $x' = (1, x_0 + 1, \dots, x_{d-1} + 1) \in \mathcal{V}_{n,d}$. Then $e_x A_n^{d-1} = M_{x'}$.
- (2) Let $y \in \mathcal{V}_{n,d-1}$ and $y' = (y_0, \dots, y_{d-1}, n + d) \in \mathcal{V}_{n,d}$. Then $D(A_n^{d-1} e_y) = M_{y'}$.
- (3) For $x, y \in \mathcal{V}_{n,d}$ we have

$$\dim_K \operatorname{Hom}_{A_n^{d-1}}(M_x, M_y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{else.} \end{cases}$$

- (4) For $x, y \in \mathcal{V}_{n,d}$ we have

$$\dim_K \operatorname{Ext}_{A_n^{d-1}}^d(M_y, M_x) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{else.} \end{cases}$$

Note that in particular Theorem 2.16(3) implies that $\operatorname{End}_{A_n^{d-1}}(M_x) = K$, which verifies that M_x is indeed indecomposable.

By Theorem 2.16(3) we find that if $x \preceq y$, then any path from x to y in $Q^{n,d}$ gives a basis of $\operatorname{Hom}_{A_n^{d-1}}(M_x, M_y)$. Similarly, Theorem 2.16(4) can be made more explicit.

Theorem 2.17. [OT, Theorem 3.8] *If $x \preceq y$, then there is an exact sequence*

$$\mathbb{E}_{xy} : 0 \rightarrow M_x \rightarrow E_d \rightarrow \dots \rightarrow E_1 \rightarrow M_y \rightarrow 0,$$

where $E_k = \bigoplus_z M_z$, taken over all $z \in \mathcal{V}_{n,d}$ such that $z_i \in \{x_i, y_i\}$ for all i and

$$|\{i \mid z_i = x_i\}| = k.$$

Note that $x \preceq y$ implies $x_i \neq y_i$ for all i , and so $|\{i \mid z_i = x_i\}| = k$ may be replaced with

$$|\{i \mid z_i = y_i\}| = d - k + 1$$

in the above condition. Moreover, we may deduce the following result which is useful for computing wide subcategories

Proposition 2.18. *If $x \preceq y$, then $M_z \in \operatorname{wide}\{M_x, M_y\}$ for any $z \in \mathcal{V}_{n,d}$ that satisfies $z_i \in \{x_i, y_i\}$ for all i .*

Proof. Each module M_z satisfying $z_i \in \{x_i, y_i\}$ for all i , appears as a summand in exactly one term of \mathbb{E}_{xy} . Hence the morphisms in \mathbb{E}_{xy} are all radical morphisms. The claim now follows from Corollary 2.6 and Proposition 2.9. \square

Corollary 2.19. *Let $x \in \mathcal{V}_{n,d}$ and $s \in N_{n,d}$ such that $s < x_0$. Then there is an exact sequence*

$$0 \rightarrow M_{x^s} \rightarrow \dots \rightarrow M_{x^0} \rightarrow M_x \rightarrow 0$$

where $x^i = (s, x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathcal{V}_{n,d}$. In particular, if $s = 1$, this is the minimal projective resolution of M_x .

Proof. The sequence is precisely $\mathbb{E}_{x^d x}$ from Theorem 2.17. If $s = 1$, then each M_{x^i} is projective by Theorem 2.16(1). \square

3. WIDE SUBCATEGORIES

3.1. Main result. In this section we classify wide subcategories of $\mathcal{M}_{n,d}$. We start by introducing the basic building blocks for such subcategories. Let $S \subseteq N_{n,d}$ and recall the subset $\mathcal{X}_S \subseteq \mathcal{V}_{n,d}$ from Definition 2.12(7). We set

$$\mathcal{W}_S = \text{add}\{M_x \mid x \in \mathcal{X}_S\} \subseteq \mathcal{M}_{n,d}.$$

Since $S_x = \{x\}$ (see Definition 2.12(8)), we have in particular that $\mathcal{W}_{S_x} = \text{add}\{M_x\}$.

Note that \mathcal{W}_S is non-zero if and only if $|S| \geq d+1$. In that case we call S admissible. In the interest of brevity a set of admissible subsets of $N_{n,d}$ is called a collection. A collection is called non-interlacing if it has no two distinct members that interlace. With this terminology we are now ready to state the main result of our paper.

Theorem 3.1. *Let n and d be positive integers. Then there is a bijection*

$$\{\text{non-interlacing collections of subsets of } N_{n,d}\} \rightarrow \{\text{wide subcategories of } \mathcal{M}_{n,d}\}$$

that sends a collection Σ to $\text{add}\{\mathcal{W}_S \mid S \in \Sigma\}$.

Remark 3.2. Note that by Theorem 2.16, two admissible subsets $S, S' \subseteq N_{n,d}$ interlace if and only if there are $M_x \in \mathcal{W}_S$ and $M_{x'} \in \mathcal{W}_{S'}$ such that one of the following conditions hold:

$$\begin{aligned} \text{Hom}_{A_n^{d-1}}(M_x, M_{x'}) \neq 0, & \quad \text{Ext}_{A_n^{d-1}}^d(M_x, M_{x'}) \neq 0, \\ \text{Hom}_{A_n^{d-1}}(M_{x'}, M_x) \neq 0, & \quad \text{Ext}_{A_n^{d-1}}^d(M_{x'}, M_x) \neq 0. \end{aligned}$$

In the remaining sections we will prove Theorem 3.1.

3.2. Injectivity. In this section we show that the map in Theorem 3.1 is well-defined and injective. We begin by showing that categories of the form \mathcal{W}_S are wide.

Proposition 3.3. *Let $S \subseteq N_{n,d}$ be admissible. Then*

- (1) *The subcategory $\mathcal{W}_S \subseteq \mathcal{M}_{n,d}$ is wide.*
- (2) *Let $n' = |S| - d$ and $\iota : N_{n',d} \rightarrow S$ be the unique order preserving bijection. Then there is an equivalence from $\mathcal{M}_{n',d}$ to \mathcal{W}_S sending $M_{x'}$ to $M_{\iota(x')}$, where $\iota(x') = (\iota(x'_0), \dots, \iota(x'_d))$.*
- (3) *In particular, the terms of d -kernels, d -cokernels and d -extensions can be computed for \mathcal{W}_S as in $\mathcal{M}_{n',d}$ using ι .*

Proof. We apply Theorem 2.11. Let $s = \min S$. Set

$$P = \bigoplus_{x \in \mathcal{X}_S, x_0 = s} M_x.$$

Since A_n^{d-1} has global dimension d the projective dimension of P is at most d . Theorem 2.16(4) implies $\text{Ext}_{A_n^{d-1}}^d(P, P) = 0$. The exact sequence in Corollary 2.19 provides an $\text{add}\{P\}$ -resolution for each M_x , with $x \in \mathcal{X}_S$.

We claim that $\text{End}_{A_n^{d-1}}(P)$ is isomorphic to $A_{n'}^{d-1}$. To see this note that ι gives a bijection between the vertices of $Q^{n',d-1}$ and the indecomposable summands of P by sending (x'_0, \dots, x'_{d-1}) to M_x , where

$$x = (s, \iota(x'_0 + 1), \iota(x'_1 + 1), \dots, \iota(x'_{d-1} + 1)).$$

Given the descriptions of $A_{n'}^{d-1}$ and $\mathcal{M}_{n,d}$ by quivers and relations, it is readily checked that this bijection extends to an isomorphism from $A_{n'}^{d-1}$ to $\text{End}_{A_n^{d-1}}(P)$.

Hence $\text{mod End}_{A_n^{d-1}}(P)$ has a unique d -cluster tilting subcategory that we may identify with $\mathcal{M}_{n',d}$. Next we claim that under this identification $\text{Hom}_{A_n^{d-1}}(P, -)$ sends $M_x \in \mathcal{W}_S$ to $M_{\iota^{-1}(x)}$. If $x_0 = s$, then $M_x \in \text{add}\{P\}$ and the claim is immediate. Otherwise consider the $\text{add}\{P\}$ -resolution of M_x given in Corollary 2.19. Applying $\text{Hom}_{A_n^{d-1}}(P, -)$ to this resolution we find the minimal projective resolution of $M_{\iota^{-1}(x)} \in \mathcal{M}_{n',d}$ and the claim follows. Hence $\text{Hom}_{A_n^{d-1}}(P, \mathcal{W}_S)$ is the d -cluster tilting subcategory of $\text{mod End}_{A_n^{d-1}}(P)$.

Thus Theorem 2.11 applies and the equivalence claimed in part (2) is given by the functor $-\otimes_{\text{End}_{A_n^{d-1}}(P)} P$. \square

Corollary 3.4. *The map in Theorem 3.1 is well-defined and injective.*

Proof. For a non-interlacing collection Σ we need to show that $\text{add}\{\mathcal{W}_S \mid S \in \Sigma\}$ is wide. By Proposition 3.3(1), each \mathcal{W}_S is wide. Moreover, since the sets S are non-interlacing, there are no non-trivial morphisms or d -extensions between modules in \mathcal{W}_S and $\mathcal{W}_{S'}$ for $S \neq S'$ (see Remark 3.2). It follows from Proposition 2.10 that $\text{add}\{\mathcal{W}_S \mid S \in \Sigma\}$ is wide.

To show injectivity consider a wide subcategory \mathcal{W} in the image of the map in Theorem 3.1. Let $\mathcal{P}_{\mathcal{W}}$ be the poset of all admissible sets S satisfying $\mathcal{W}_S \subseteq \mathcal{W}$ ordered by inclusion. Now write $\mathcal{W} = \text{add}\{\mathcal{W}_S \mid S \in \Sigma\}$ for a non-interlacing collection $\Sigma = \{S_1, \dots, S_l\}$. We claim that Σ equals the set of maximal elements in $\mathcal{P}_{\mathcal{W}}$. This implies injectivity as we can recover Σ from \mathcal{W} .

To show the claim first note that for $i \neq j$ we have $S_i \not\subseteq S_j$ as S_i and S_j do not interlace. It remains to show for all $S \in \mathcal{P}_{\mathcal{W}}$ that $S \subseteq S_i$ for some i . To do this note that \mathcal{W}_S is connected and satisfies

$$\mathcal{W}_S \subseteq \text{add}\{\mathcal{W}_{S_1}, \dots, \mathcal{W}_{S_l}\}.$$

Since Σ is non-interlacing there is no walk in \mathcal{W} connecting some $M_x \in \mathcal{W}_{S_i}$ with some $M_y \in \mathcal{W}_{S_j}$ for $i \neq j$, and so $\mathcal{W}_S \subseteq \mathcal{W}_{S_i}$ for some i , which implies $S \subseteq S_i$. \square

3.3. Surjectivity. It remains to show that the map in Theorem 3.1 is surjective. We will do this using several Lemmas, each stating that a certain wide subcategory is of the form \mathcal{W}_S for some admissible set S . Lemmas 3.5, 3.6 and 3.7 take care of certain special cases that have to be dealt with separately.

Lemma 3.5. *Assume $n \geq 2$. Let $l \geq d + 2$ and $s_1 < \dots < s_l$ be elements in $N_{n,d}$. Set $S = \{s_1, \dots, s_{l-1}\}$ and $S' = \{s_2, \dots, s_l\}$. Then*

$$\text{wide}\{\mathcal{W}_S, \mathcal{W}_{S'}\} = \mathcal{W}_{S \cup S'}.$$

Proof. We apply Proposition 3.3(3) to $S \cup S'$. Hence we may assume that $S = \{1, \dots, n+d-1\}$ and $S' = \{2, \dots, n+d\}$ so that $\mathcal{W}_{S \cup S'} = \mathcal{M}_{n,d}$. By Proposition 2.18 any module $M_z \in \mathcal{M}_{n,d}$ satisfies $M_z \in \text{wide}\{M_x, M_y\}$ for some $M_x \in \mathcal{W}_S$ and $M_y \in \mathcal{W}_{S'}$. The claim follows. \square

Lemma 3.6. *We have*

$$\text{wide}\{M_{(1,2,\dots,d+1)}, \mathcal{W}_{\{2,3,\dots,n+d\}}\} = \mathcal{M}_{n,d}.$$

Proof. The proof is by induction on n . The case $n = 1$ is trivial. Assume $n > 1$. By induction hypothesis

$$\text{wide}\{M_{(1,2,\dots,d+1)}, \mathcal{W}_{\{2,3,\dots,n+d-1\}}\} = \mathcal{W}_{\{1,2,\dots,n+d-1\}}.$$

Since $\mathcal{W}_{\{2,3,\dots,n+d-1\}} \subseteq \mathcal{W}_{\{2,3,\dots,n+d\}}$ we have

$$\text{wide}\{M_{(1,2,\dots,d+1)}, \mathcal{W}_{\{2,3,\dots,n+d\}}\} \supseteq \text{wide}\{\mathcal{W}_{\{1,2,\dots,n+d-1\}}, \mathcal{W}_{\{2,3,\dots,n+d\}}\}.$$

Now applying Lemma 3.5 with $S = \{1, 2, \dots, n+d-1\}$ and $S' = \{2, 3, \dots, n+d\}$ we get

$$\text{wide}\{M_{(1,2,\dots,d+1)}, \mathcal{W}_{\{2,3,\dots,n+d\}}\} \supseteq \mathcal{W}_{S \cup S'} = \mathcal{M}_{n,d}.$$

\square

Lemma 3.7. *Let $x, x' \in N_{n,d}$ such that $x_k \neq x'_k$ for some k and $x_i = x'_i$ for all $i \neq k$. Then*

$$\text{wide}\{M_x, M_{x'}\} = \mathcal{W}_{S_x \cup S_{x'}}$$

Proof. Without loss of generality assume $x_k < x'_k$. Apply Corollary 2.19 to

$$y = (x_1, x_2, \dots, x_k, x'_k, x_{k+1}, \dots, x_d)$$

with $s = x_0$ to obtain the exact sequence

$$\mathbb{E}: \quad 0 \rightarrow M_{y^d} \rightarrow \dots \rightarrow M_{y^0} \rightarrow M_y \rightarrow 0.$$

Then $y^k = x$ and $y^{k-1} = x'$ (for the extreme case $k = 0$, this should be interpreted as $y = y^{-1} = x'$).

Extracting the morphism $M_{y^k} \rightarrow M_{y^{k-1}}$ we get a non-zero morphism $f : M_{x'} \rightarrow M_x$. Moreover, we may view the sequence \mathbb{E} as the concatenation of the map f with its minimal d -kernel and d -cokernel. In particular, the terms of the sequence all belong to $\text{wide}\{M_x, M_{x'}\}$. But the terms are exactly the indecomposable modules in $\mathscr{W}_{S_x \cup S_{x'}}$. \square

In general the d -kernels and d -cokernels of morphisms $M_x \rightarrow M_y$ are not as easy to compute as in the proof of Lemma 3.7. Fortunately we will not need complete information about such d -kernels and d -cokernels to prove our results. More precisely we will make do with the following statement.

Lemma 3.8. *Let $f : M_x \rightarrow M_y$ be a non-zero morphism.*

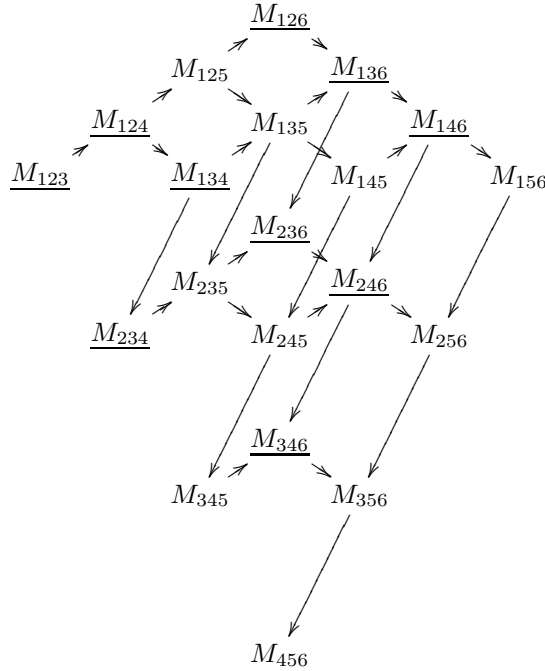
- (1) *If $x_{k-1} < y_{k-1} < x_k$, then for $x' = (x_0, \dots, x_{k-1}, y_{k-1}, x_{k+1}, \dots, x_d)$ the module $M_{x'}$ appears as a summand in the term K_1 in any d -kernel $K_d \rightarrow \dots \rightarrow K_1 \rightarrow M_x$ of f . In particular, $M_{x'} \in \text{wide}\{M_x, M_y\}$ and there is a non-zero morphism $M_{x'} \rightarrow M_x$.*
- (2) *If $y_k < x_{k+1} < y_{k+1}$, then for $y' = (y_0, \dots, y_{k-1}, x_{k+1}, y_{k+1}, \dots, y_d)$ the module $M_{y'}$ appears as a summand in the term C_1 of any d -cokernel $M_y \rightarrow C_1 \rightarrow \dots \rightarrow C_d$ of f . In particular, $M_{y'} \in \text{wide}\{M_x, M_y\}$ and there is a non-zero morphism $M_y \rightarrow M_{y'}$.*

Proof. We only prove (1) as (2) is dual.

The d -kernels of f may be computed in the wide subcategory \mathscr{W}_S , where $S = S_x \cup S_y$. By Proposition 3.3(3), we may assume that $S = N_{n,d}$ and $\mathscr{W}_S = \mathscr{M}_{n,d}$. Then $x' = \sigma_k^-(x)$ and so there is an arrow in Q_n^d from x' to x . Hence there is a corresponding morphism $g \in \text{Rad}_{\mathscr{M}_{n,d}}(M_{x'}, M_x) \setminus \text{Rad}_{\mathscr{M}_{n,d}}^2(M_{x'}, M_x)$. Notice that $x' \not\leq y$ and so $\text{Hom}_{A_n^{d-1}}(M_{x'}, M_y) = 0$ by Theorem 2.16(3). In particular, $f \circ g = 0$ and the claim follows from Proposition 2.7. \square

Before continuing with the proof we illustrate the utility of Lemma 3.8 in an example.

Example 3.9. Let $n = 4$ and $d = 2$ as in Example 2.15 and consider $\mathscr{W} = \text{wide}\{M_{136}, M_{246}\}$. Since $\{1, 3, 6\}$ and $\{2, 4, 6\}$ interlace, we must have $\mathscr{W} = \mathscr{W}_{\{1,2,3,4,6\}}$ in order for Theorem 3.1 to hold. Indeed, if $\mathscr{W} = \text{add}\{\mathscr{W}_S \mid S \in \Sigma\}$ for some non-interlacing collection Σ , we must have $\{1, 3, 6\} \subseteq S$ and $\{2, 4, 6\} \subseteq S$ for some common $S \in \Sigma$ and so the smallest possible collection is $\Sigma = \{\{1, 2, 3, 4, 6\}\}$. Below is the quiver of $\mathscr{M}_{4,2}$ with the indecomposables in $\mathscr{W}_{\{1,2,3,4,6\}}$ underlined.



We verify that $\mathscr{W} = \mathscr{W}_{\{1,2,3,4,6\}}$ does indeed hold by applying Lemma 3.8 repeatedly.

Considering the 2-kernel of the nonzero morphism $M_{136} \rightarrow M_{246}$ we find that $M_{126}, M_{134} \in \mathscr{W}$. Considering the 2-cokernel we find $M_{346} \in \mathscr{W}$. Similarly, the 2-cokernel of the nonzero morphism $M_{134} \rightarrow M_{136}$ gives $M_{146} \in \mathscr{W}$.

Next we consider the nonzero morphism $M_{126} \rightarrow M_{136}$ and get $M_{123}, M_{236} \in \mathscr{W}$. This allows us to consider $M_{236} \rightarrow M_{246}$, which gives $M_{234} \in \mathscr{W}$, and then $M_{134} \rightarrow M_{234}$, which gives $M_{124} \in \mathscr{W}$.

Hence all 10 indecomposables in $\mathscr{W}_{\{1,2,3,4,6\}}$ lie in \mathscr{W} .

We now continue with the general case. The following Lemma is the main tool in the proof of Theorem 3.1 and can be thought of as a generalisation of Example 3.9. The proof is similar to the strategy of Example 3.9 in that we successively build up more and more objects in a certain wide subcategory.

Lemma 3.10. *Let $x \in N_{n,d}$ and $S \subseteq N_{n,d}$ admissible. If*

$$\mathrm{Hom}_{A_n^{d-1}}(\mathscr{W}_S, M_x) \neq 0 \quad \text{or} \quad \mathrm{Hom}_{A_n^{d-1}}(M_x, \mathscr{W}_S) \neq 0,$$

then $\mathrm{wide}\{\mathscr{W}_S, M_x\} = \mathscr{W}_{S \cup S_x}$.

Proof. Recall from Theorem 2.16(3) that

$$\dim_K \mathrm{Hom}_{A_n^{d-1}}(M_x, M_y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{else.} \end{cases}$$

We will use this freely throughout the proof to characterise when $\mathrm{Hom}_{A_n^{d-1}}(M_x, M_y) = 0$.

Set $\mathscr{W} = \mathrm{wide}\{\mathscr{W}_S, M_x\}$. It is enough to show the inclusion $\mathscr{W}_{S \cup S_x} \subseteq \mathscr{W}$. Note that the claim is trivial if $S_x \subseteq S$. We proceed in several steps.

Step 1: First consider the case when S_x contains only one element a that is not in S , i.e., $S \cup S_x = S \cup \{a\}$. We use induction on $|S|$. If $|S| = d+1$, then \mathscr{W}_S consists of a single indecomposable $M_{x'}$ and the claim follows by Lemma 3.7.

Next assume that $|S| > d+1$. By Proposition 3.3(3), we may assume $S \cup \{a\} = N_{n,d}$ so that $\mathscr{W}_{S \cup S_x} = \mathscr{M}_{n,d}$. We proceed with a case by case analysis depending on the value of a . Our strategy in each case is to produce more and more elements of \mathscr{W} to finally reach $\mathscr{M}_{n,d}$.

Step 1.1: Assume $a = 1$, i.e., $S = \{2, \dots, n+d\}$. Then $x_0 = 1$ and we have $\mathrm{Hom}_{A_n^{d-1}}(\mathscr{W}_S, M_x) = 0$ so that

$$\mathrm{Hom}_{A_n^{d-1}}(M_x, \mathscr{W}_S) \neq 0$$

which implies $x_1 \geq 3$. Next let $x' \in N_{n,d}$ be such that $x'_0 = 1$, $x'_1 \geq 3$, $M_{x'} \in \mathscr{W}$ and the value of $\sum_{i=1}^d x'_i$ is as small as possible. We claim that

$$x' = (1, 3, 4, \dots, d+2).$$

Otherwise there is some $b \geq 3$ satisfying $x'_{k-1} < b < x'_k$ for some $k \geq 1$. Set

$$y = \begin{cases} (2, x'_1, \dots, x'_{k-2}, b, x'_k, \dots, x'_d), & \text{if } k \geq 2, \\ (b, x'_1, \dots, x'_d), & \text{if } k = 1. \end{cases}$$

Then $M_y \in \mathscr{W}_S$ and applying Lemma 3.8(1) to a non-zero morphism $M_{x'} \rightarrow M_y$ we get that $M_{x''} \in \mathrm{wide}\{M_{x'}, M_y\} \subseteq \mathscr{W}$ for

$$x'' = (1, x'_1, \dots, x'_{k-1}, b, x'_{k+1}, \dots, x'_d)$$

contradicting the minimality of $\sum_{i=1}^d x'_i$.

Next set

$$y' = (2, 3, \dots, d+2).$$

Then $M_{y'} \in \mathscr{W}$ and applying Lemma 3.7 to $M_{x'}$ and $M_{y'}$ we find that

$$M_{(1,2,\dots,d+1)} \in \mathrm{wide}\{M_{x'}, M_{y'}\} \subseteq \mathscr{W}.$$

Finally by Lemma 3.6,

$$\mathscr{M}_{n,d} = \mathrm{wide}\{M_{(1,2,\dots,d+1)}, \mathscr{W}_{\{2,3,\dots,n+d\}}\} \subseteq \mathscr{W}.$$

Step 1.2: Assume $a = n + d$, i.e., $S = \{1, \dots, n + d - 1\}$. This is similar to Step 1.1 and the proof is omitted.

Step 1.3: Assume $1 < a < n + d$, i.e., $S = \{1, \dots, a - 1, a + 1, \dots, n + d\}$. Then $x_k = a$ for some k . The condition

$$\mathrm{Hom}_{A_n^{d-1}}(\mathcal{W}_S, M_x) \neq 0 \quad \text{or} \quad \mathrm{Hom}_{A_n^{d-1}}(M_x, \mathcal{W}_S) \neq 0,$$

implies that $x_{k-1} < a - 1$ or $x_{k+1} > a + 1$. We treat these inequalities in three separate substeps.

Step 1.3.1: Assume both $x_{k-1} < a - 1$ and $x_{k+1} > a + 1$ hold. In particular $1 \leq k \leq d - 1$. Set

$$x' = (x_0, \dots, x_{k-1}, a - 1, x_{k+1}, \dots, x_d).$$

Then $M_{x'} \in \mathcal{W}_S \subseteq \mathcal{W}$ and applying Lemma 3.7 to M_x and $M_{x'}$ we find that $M_{x''} \in \mathcal{W}$ for

$$x'' = (x_1, \dots, x_{k-2}, a - 1, a, x_{k+1}, \dots, x_d).$$

Set

$$y = (x_1, \dots, x_{k-2}, a - 1, a + 1, x_{k+1}, \dots, x_d)$$

and $S' = S \setminus \{1\}$. Then $M_{x''} \in \mathcal{W}_{S' \cup \{a\}}$ and $M_y \in \mathcal{W}_{S'}$. Moreover, $\mathrm{Hom}_{A_n^{d-1}}(M_{x''}, M_y) \neq 0$ and so by induction hypothesis

$$\mathrm{wide}\{M_{x''}, \mathcal{W}_{S'}\} = \mathcal{W}_{S' \cup \{a\}} = \mathcal{W}_{\{2, \dots, n+d\}}.$$

Hence

$$\mathcal{W} \supseteq \mathrm{wide}\{M_{x''}, \mathcal{W}_{S'}, \mathcal{W}_S\} = \mathrm{wide}\{\mathcal{W}_{\{2, \dots, n+d\}}, \mathcal{W}_S\}.$$

Set

$$z = \begin{cases} (1, 3, \dots, a - 1, a + 1, \dots, d + 3), & \text{if } a \leq d + 2, \\ (1, 3, \dots, d + 2), & \text{if } a > d + 2. \end{cases}$$

Then $M_z \in \mathcal{W}_S$ and $\mathrm{Hom}_{A_n^{d-1}}(M_z, \mathcal{W}_{\{2, \dots, n+d\}}) \neq 0$. By Step 1.1

$$\mathrm{wide}\{\mathcal{W}_{\{2, \dots, n+d\}}, \mathcal{W}_S\} = \mathcal{M}_{n,d}.$$

Step 1.3.2: Assume $x_{k-1} < a - 1$ and $x_{k+1} = a + 1$. Since $|S| > d + 1$ there is some $b \in S \setminus (S_x \cup \{a - 1\})$. Choose b as small as possible. There are two cases to consider: $b < a - 1$ and $b > a + 1$.

Step 1.3.2.1: Assume $b < a - 1$. First we replace x with something more suitable. For this purpose let $x' \in N_{n,d}$ be such that $x'_i = x_i$ for all $i \geq k$, $x'_{k-1} < a - 1$, $M_{x'} \in \mathcal{W}$ and $\sum_{i=1}^{k-1} x'_i$ as large as possible. Notice that such x' exists since x is a candidate. We claim that $x'_0 > 1$. Otherwise there is $b' < a - 1$ such that $x'_l < b' < x'_{l+1}$ for some $0 \leq l \leq k - 1$. Set

$$y = \begin{cases} (x'_0, \dots, x'_l, b', x'_{l+2}, \dots, x'_{k-1}, a - 1, x'_{k+1}, \dots, x'_d), & \text{if } l < k - 1, \\ (x'_0, \dots, x'_l, b', x'_{k+1}, \dots, x'_d), & \text{if } l = k - 1. \end{cases}$$

Then $M_y \in \mathcal{W}_S \subseteq \mathcal{W}$ and applying Lemma 3.8(2) to a non-zero morphism $M_y \rightarrow M_{x'}$ we find that $M_{x''} \in \mathcal{W}$ for

$$x'' = (x'_0, \dots, x'_{l-1}, b', x'_{l+1}, \dots, x'_d)$$

contradicting the maximality of $\sum_{i=1}^{k-1} x'_i$.

Now set

$$y' = (x'_0, \dots, x'_{k-1}, a - 1, x'_{k+1}, \dots, x'_d).$$

Then $\mathrm{Hom}_{A_n^{d-1}}(M_{y'}, M_{x'}) \neq 0$ and $M_{y'} \in \mathcal{W}_{S'}$ for $S' = S \setminus \{1\}$. As in step 1.3.1 we get by induction that

$$\mathrm{wide}\{M_{x'}, \mathcal{W}_{S'}\} = \mathcal{W}_{S' \cup \{a\}} = \mathcal{W}_{\{2, \dots, n+d\}}$$

and Step 1.1 yields

$$\mathcal{W} \supseteq \mathrm{wide}\{M_{x'}, \mathcal{W}_{S'}, \mathcal{W}_S\} = \mathcal{M}_{n,d}.$$

Step 1.3.2.2: Assume $b > a + 1$. As before we replace x with something more suitable. Let $x' \in N_{n,d}$ be such that $x'_i = x_i$ for all $i \leq k$, $M_{x'} \in \mathcal{W}$ and $\sum_{i=k+1}^d x'_i$ as large as possible. Note that there is a b' such that $x'_l < b' < x'_{l+1}$ for some $l \geq k$ or $x'_d < b'$.

We claim that $l = k$ so that Step 1.3.1 can be applied after replacing x by x' . To show this we assume $l > k$ and reach a contradiction.

Let

$$x'' = (x'_0, \dots, x'_{k-1}, a-1, x'_{k+1}, \dots, x'_d)$$

and apply Lemma 3.7 to $M_{x'}$ and $M_{x''}$ to obtain $M_y \in \mathcal{W}$ for

$$y = (x'_0, \dots, x'_{k-1}, a-1, a, x'_{k+2}, \dots, x'_d).$$

Next set

$$y' = (x'_0, \dots, x'_{k-1}, a-1, x'_{k+1}, x'_{k+2}, \dots, x'_{l-1}, b', x'_{l+1}, \dots, x'_d).$$

Then $M_{y'} \in \mathcal{W}_S \subseteq \mathcal{W}$. Applying Lemma 3.8(2) to a non-zero morphism $M_y \rightarrow M_{y'}$ we obtain $M_{y''} \in \mathcal{W}$ for

$$y'' = (x'_0, \dots, x'_{k-1}, a, x'_{k+1}, x'_{k+2}, \dots, x'_{l-1}, b', x'_{l+1}, \dots, x'_d).$$

But then $\sum_{i=k+1}^d y''_i > \sum_{i=k+1}^d x'_i$, which is a contradiction.

Step 1.3.3: Assume $x_{k-1} = a-1$ and $x_{k+1} > a+1$. This is similar to step 1.3.2 and the proof is omitted.

Step 2: Now consider the general case. We proceed by induction on $m = |S_x \setminus S|$. Note that the cases $m = 0$ and $m = 1$ have already been proved. Thus consider the case $m > 1$. We assume $\text{Hom}_{A_n^{d-1}}(M_x, \mathcal{W}_S) \neq 0$ (the case $\text{Hom}_{A_n^{d-1}}(\mathcal{W}_S, M_x) \neq 0$ is similar). Then there exists y with $S_y \subseteq S$ such that there is a non-zero morphism $f : M_x \rightarrow M_y$. Hence $x \preceq y$. Since $m > 1$ there must be a $1 \leq k \leq d$ such that $x_{k-1} \notin S$. In particular $y_{k-1} \neq x_{k-1}$ and so

$$x_{k-1} < y_{k-1} < x_k.$$

Hence by Lemma 3.8(1), we get that $M_{x'} \in \mathcal{W}$ for

$$x' = (x_0, \dots, x_{k-1}, y_{k-1}, x_{k+1}, \dots, x_d).$$

Moreover, there is a non-zero morphism $g : M_{x'} \rightarrow M_x$. Now $x_{k-1} < y_{k-1} < x_k$ also means that $x_{k-1} < x'_k < x_k$ so by Lemma 3.8(2) we find that $M_{x''} \in \mathcal{W}$ for

$$x'' = (x_0, \dots, x_{k-2}, y_{k-1}, x_k, x_{k+1}, \dots, x_d).$$

Notice that x'' is obtained from x by replacing x_{k-1} (which is not in S) by y_{k-1} (which is in S). In particular, $|S_{x''} \setminus S| < m$ and setting $S' = S \cup S_{x''}$ we get by induction that $\mathcal{W}_{S'} \subseteq \text{wide}\{\mathcal{W}_S, M_{x''}\}$ and so $\mathcal{W}_{S'} \subseteq \mathcal{W}$. On the other hand $\text{Hom}_{A_n^{d-1}}(M_x, \mathcal{W}_{S'}) \neq 0$ and $S_x \setminus S' = \{x_{k-1}\}$ so by Step 1 we get that $\mathcal{W}_{S' \cup S_x} \subseteq \text{wide}\{\mathcal{W}_{S'}, M_x\} \subseteq \mathcal{W}$. But $S' \cup S_x = S \cup S_x$ so the claim follows. \square

Lemma 3.11. *Let \mathcal{W} be a wide subcategory, $M_x \in \mathcal{W}$ and $\mathcal{W}_S \subseteq \mathcal{W}$ for some admissible set S that is maximal (with respect to inclusion) with this property. If there is $M_y \in \mathcal{W}_S$ such that there is a walk of morphisms in \mathcal{W} connecting M_x and M_y then $M_x \in \mathcal{W}_S$.*

Proof. The claim follows in case the walk consists of a single morphism by maximality and applying Lemma 3.10. Propagating along an arbitrary walk gives the general result. \square

Proposition 3.12. *The map in Theorem 3.1 is surjective.*

Proof. Let \mathcal{W} be a wide subcategory. Consider again the poset $\mathcal{P}_{\mathcal{W}}$ of all admissible subsets S satisfying $\mathcal{W}_S \subseteq \mathcal{W}$ ordered by inclusion. Let Σ be the collection of maximal elements in $\mathcal{P}_{\mathcal{W}}$. Since every module $M_x \in \mathcal{W}$ lies in some $\mathcal{W}_S \subseteq \mathcal{W}$ (e.g., $S = S_x$) it follows that $\mathcal{W} = \text{add}\{\mathcal{W}_S \mid S \in \Sigma\}$. It remains to show that Σ is non-interlacing. We do this by showing that if $S, S' \in \Sigma$ interlace, then $\mathcal{W}_S = \mathcal{W}_{S'}$, and so $S = S'$. Note that if $S, S' \in \Sigma$ interlace, then by Remark 3.2, there are $M_x \in \mathcal{W}_S$ and $M_{x'} \in \mathcal{W}_{S'}$ such that there is either a non-zero morphism connecting M_x and $M_{x'}$ or a non-trivial d -extension in \mathcal{W} with endpoints M_x and $M_{x'}$ (in some order) as described in Theorem 2.17. In either case there is a walk in \mathcal{W} from M_x to $M_{x'}$. Moreover, since \mathcal{W}_S and $\mathcal{W}_{S'}$ are connected there is in fact a walk in \mathcal{W} from any indecomposable in \mathcal{W}_S to any indecomposable in $\mathcal{W}_{S'}$. The claim now follows by Lemma 3.11. \square

This completes the proof of Theorem 3.1.

3.4. Number of wide subcategories. Finally, we briefly discuss the number of wide subcategories of $\mathcal{M}_{n,d}$, which we denote by $w_{n,d}$. For $d = 1$ it is well-known that the numbers $w_{n,1}$ are Catalan numbers:

$$w_{n,1} = \frac{1}{n+2} \binom{2n+2}{n+1}.$$

For $n = 1$ we have $w_{1,d} = 2$ since $\mathcal{M}_{n,d}$ has precisely one indecomposable in this case.

For $n = 2$ the category $\mathcal{M}_{n,d}$ has $d + 1$ indecomposables

$$M_1 = M_{1,2,\dots,d,d+1}, M_2 = M_{1,2,\dots,d,d+2}, \dots, M_d = M_{1,3,\dots,d+1,d+2}, M_{d+1} = M_{2,3,\dots,d+1,d+2}.$$

Hence a wide subcategory of $\mathcal{M}_{n,d}$ is determined by a subset of $\{M_1, \dots, M_{d+1}\}$, which is naturally encoded as word of length $d + 1$ in letters 0, 1. Such a word corresponds to a wide subcategory if and only if it avoids any occurrence of 11 when read cyclically or is the word 11...1. Counting the number of such words is straightforward. In fact, $(w_{2,d})_d$ appears as A001612 in [OEIS] and satisfies the recurrence relation

$$w_{2,d} = w_{2,d-1} + w_{2,d-2} - 1.$$

For general n and d , we have not found an easy formula $w_{n,d}$. However, due to the simple combinatorial description of wide subcategories in Theorem 3.1 it is straightforward to write an algorithm that computes the number $w_{n,d}$ for any $n \geq 1, d \geq 1$. For instance one may generate each Σ by iteratively attaching admissible sets S that do not interlace. Running such an algorithm on a computer one finds the following values for small n and d .

Some values of $w_{n,d}$

d	$w_{1,d}$	$w_{2,d}$	$w_{3,d}$	$w_{4,d}$	$w_{5,d}$	$w_{6,d}$	$w_{7,d}$	$w_{8,d}$
1	2	5	14	42	132	429	1,430	4,862
2	2	8	47	374	4,083	62,824	1,376,012	42,579,642
3	2	12	237	16,830	4,597,078			
4	2	19	1,724	3,499,884				
5	2	30	17,934					
6	2	48	273,092					
7	2	77	5,732,137					

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