Fibonacci Plays Billiards

Elwyn Berlekamp and Richard Guy

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Abstract

A *chain* is an ordering of the integers 1 to n such that adjacent pairs have sums of a particular form, such as squares, cubes, triangular numbers, pentagonal numbers, or Fibonacci numbers. For example 4 1 2 3 5 form a Fibonacci chain while 1 2 8 7 3 12 9 6 4 11 10 5 form a triangular chain. Since 1 + 5 is also triangular, this latter forms a triangular *necklace*. A search for such chains and necklaces can be facilitated by the use of paths of billiard balls on a rectangular or other polygonal billiard table.

At the July, 2002 Combinatorial Games Conference in Edmonton we found Yoshiyuki Kotani looking for values of n which would enable him to arrange the numbers 1 to n in a chain so that adjacent links summed to a perfect cube. Part of such a chain might be

$$\ldots \quad 61 \quad 3 \quad 5 \quad 22 \quad 42 \quad \ldots$$

He had seen the corresponding problem asked for squares. Later Ed Pegg informed us that this latter problem, with squares and with $n = 15$, was proposed by Bernardo Recaman Santos, of Colombia, at the 2000 World Puzzle Championship. More recently this has appeared as Puzzle 30 in [6].

$$(16\rightarrow)9\rightarrow7\rightarrow2\leftarrow14\rightarrow11\rightarrow5\rightarrow4\leftarrow12\leftarrow13\rightarrow3\leftarrow6\leftarrow10\leftarrow15\rightarrow1\leftarrow8(\leftarrow17)$$

Figure 1: Solution(s) to Recaman's problem for $n = 15, 16, 17$.

This inspired Joe Kisenwether to ask for the numbers 1 to 32 to be arranged as a necklace whose neighboring beads add to squares (Figure 2).

The extension to cubes was suggested by Nob Yoshigahara. The least n for such a chain or necklace may be greater than 300. But it seems certain that

4	21	28	8	1	15	10	26	23
32								2
17								14
19								22
30								27
6								9
3								16
13								20
12	24	25	11	5	31	18	7	29

Figure 2: A necklace with adjacent pairs of beads adding to squares.

such chains and necklaces can be found for all sufficiently large n, and for any other powers or polynomials, e.g., figurate numbers of various kinds; see Figure 3.

				3				
			7		12			
		8				9		
	2						6	
1		5		10		11		4

Figure 3: A necklace with adjacent pairs of beads adding to triangular numbers.

So we asked about more rapidly divergent sequences. For powers of 2, it is not possible to connect chains of odd numbers to chains of even numbers, and there are similar difficulties with powers of larger numbers.

However, the corresponding problem with neighbors summing to Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$, has a better balanced solution.

We can draw a graph with the numbers 1 to n as vertices and edges joining pairs whose sum is a Fibonacci number: for $n = 11$, this is Figure 4. The arrows are drawn from the larger to the smaller number to emphasize that the larger number is not part of the graph unless the smaller is already present. From the graph we can read off 1 2; 1 2 3; 4 1 2 3; 4 1 2 3 5; 4 1 7 6 2 3 5; 4 1 7 6 2 3 5 8; 9 4 1 7 6 2 3 5 8 and 9 4 1 7 6 2 11 10 3 5 8. We can also verify that 6 and 10 can't be included in a chain unless some larger number is also present (in the former case 4, 5 and 6 are monovalent vertices and all three can't be ends of the chain; in the latter case, 8, 9 and 10). Evidently the Law of Small Numbers is at work. Six and ten are the only numbers which are not powers of primes. Is there some connexion with projective planes? No, but the Law

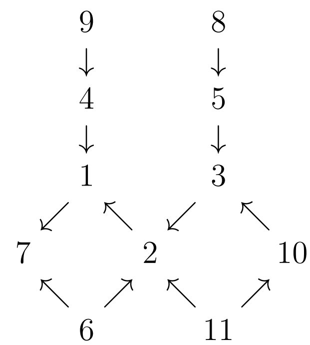

Figure 4: Graph whose adjacencies are Fibonacci sums

of Small Numbers is indeed at work, but the villains are 9 and 11.

Theorem 1. *There is a chain formed with the numbers 1 to n with each adjacent pair adding to a Fibonacci number, just if* $n = 9$, *11, or* F_k *or* $F_k - 1$, *where* F_k *is a Fibonacci number with* $k \geq 4$. *The chain is essentially unique.*

Proof. For $n \leq 11$ ($k = 4, 5, 6$) this follows from Figure 2. If $k = 7$, then $12 = F_7 - 1$ can be appended to the 11-chain, forming a 4-circuit; also, $F_7 = 13$ can be appended at the other end, as shown in Figure 5.

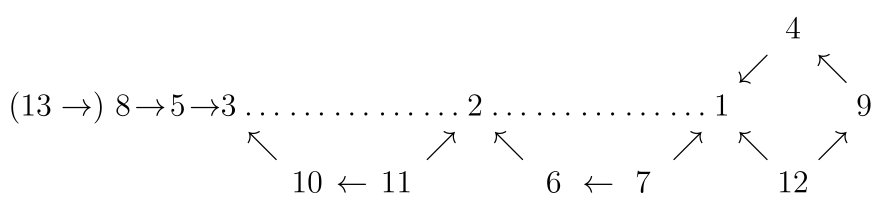

Figure 5: Ball and chain for 12 or 13.

Although 2 is adjacent to 1 and 3, the chain for 12 or 13 is essentially unique, except that the right tail may be 12 or 4 for either chain. None of the Fibonacci chains that we have seen will form a necklace; nor will any others.

The rest of the proof is by induction, but the comparatively simple pattern is made more difficult to describe by the fact that only every third Fibonacci number is even.

Balls and chains occur just for $F_{3m+1} - 1$ and F_{3m+1} with $m \geq 1$; other cases are simple chains. The chain 1—2—3 can be thought of as the "zeroth ball" (Figure 6).

There are no chains for $n = 14$, 15, 16, 17, 18 or 19, since, when we successively append these numbers to the graph, the first three are monovalent vertices, as

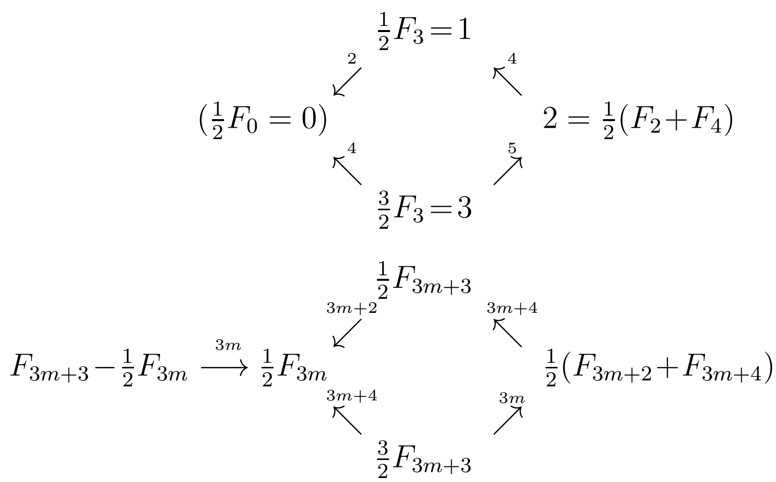

Figure 6: Zeroth ball and general ball. Small numbers above the arrows are ranks of Fibonacci numbers to which pairs of linked numbers sum.

also is 17 ($=\frac{1}{2}F_9$), though this last can be accommodated by breaking the ball and allowing 17 to become an end of the chain. When we adjoin 18 & 19 they repectively allow 16 & 15 to become bivalent, but a chain is not reachieved until we append $F_8 - 1 = 20$ at 1 & 14.

Note that all the partitions (5&3, 2&6, 7&1) of $F_6 = 8$ into two distinct parts have been bypassed by the partitions of $F_9 = 34$ into parts of size less than $F_8 = 21$, which itself can then be appended to form a new tail to the chain. Because F_9 is even, as is every third Fibonacci number, $\frac{1}{2}F_9 = 17$ can only be appended to 4 ($=\frac{1}{2}F_6$).

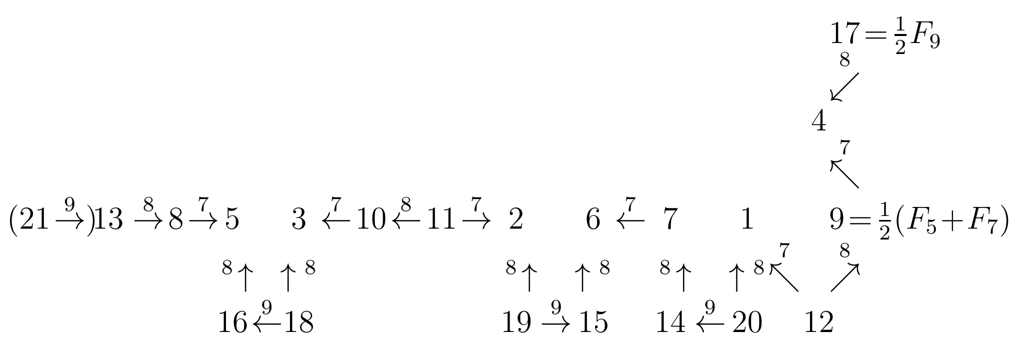

Figure 7: Fibonacci chains for $F_8 - 1 = 20$ and $F_8 = 21$.

If we continue, we find that a chain cannot again be achieved until we have replaced the six partitions of $F_7 = 13$ by links of partitions of $F_{10} = 55$ into two parts of size at most $F_9 - 1 = 33$ (Figure 8).

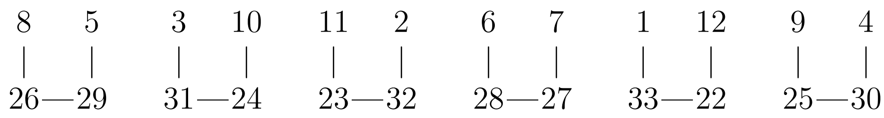

Figure 8: Links extending the chain to $F_9 - 1 = 33$.

$F_9 = 34$ can then be appended to $21 = F_8$ to make a new tail to the chain.

The next chain is for $F_{10} - 1 = 54$, obtained by appending links of partitions of $F_{11} = 89$ into parts of size at most 54: 54—35, 53—36, ..., 45—44 to the ten partitions 1—20, 2—19, ..., 10—11, of $F_8 = 21$. The chain for $F_{10} = 55$ can be formed by appended it at the end $F_9 = 34$.

Note that the link —51—38— **need not immediately** replace the end link, —4—17, of the chain, but the latter can remain as part of a new ball, the case $m = 2$ of Figure 6, until we wish to append $\frac{1}{2}F_{12} = 72$, which we will do when forming the 88- and 89-chains.

We have seen several stages of the induction. In Figure 5 the numbers between $F_5 = 5$ and $F_6 = 8$ and F_6 itself are appended, as also are the numbers between $F_6 = 8$ and $F_7 = 13$ and 13 itself. In Figures 7 and 8, the numbers between F_k and F_{k+1} are appended for $k = 7$ and 8 respectively. Note that in the former $\frac{1}{2}F_{k+2} = 17$ is appended to $\frac{1}{2}F_{k-1} = 4$.

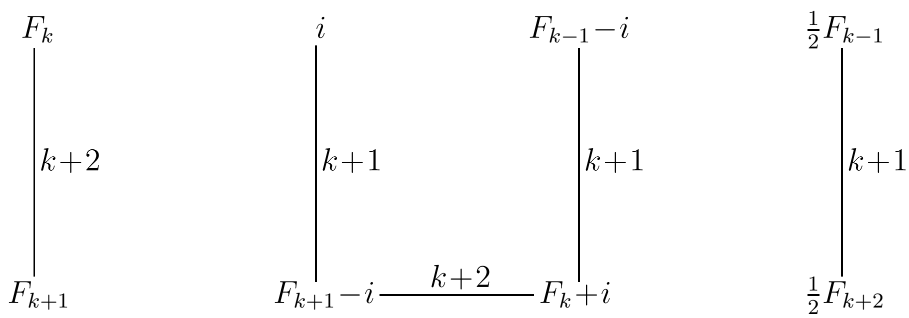

Figure 9: Extending $F_k - 1$ and F_k chains to those for $F_{k+1} - 1$ and F_{k+1} The appendage on the right is required only when $k = 3m + 1$.

Generally, as in Figure 9, we append the pairs of numbers $F_k + i$, $F_{k+1} - i$ for $1 \le i \le \frac{1}{2}(F_{k-1} - 1)$, except that, when $k = 3m + 1$, $\frac{1}{2}(F_{k-1} - 1)$ is not an integer and we have a new tail, $\frac{1}{2}F_{k+2}$, which **is** an integer, appended to $\frac{1}{2}F_{k-1}$. $\square$

These last numbers are denominators of the convergents to the continued fraction for $\sqrt{5}$, sequence A001076 in Neil Sloane's Online Encyclopedia of Integer

Sequences [5].

The proof can be made much more perspicuous with billiards diagrams, which will also throw light on the other kinds of chain in which we are interested.

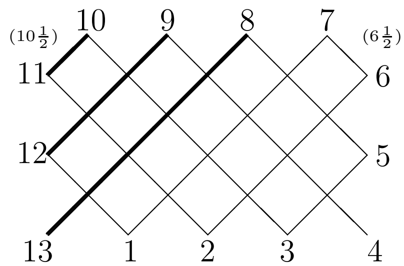

Figure 10: Fibonacci plays billiards. The thick upward paths connect 21-sums. The other upward paths connect 8-sums. The down paths connect 13-sums.

Figure 10 is equivalent to Figure 5. The 'ball' may be achieved by connecting the Fibonacci sum 1—+—4 = 5.

This billiard table viewpoint is useful for depicting long chains whose adjacent pair-sums all lie in a set of only three or four elements. If successive corners are at a, b, c, d, where $a < b < c < d$, then the semi-perimeter must be $c - a = d - b$, and the perimeter is $P = 2(c - a) = 2(d - b)$. One side must be $b - a = d - c$, and the other must be $c - b = a - d \pmod{P}$. Viewed along the 45 degree path taken by the billiard ball, each integer along the side of the table has valence 2, and each integer in a corner has valence 1. Hence, if the corners include 2 integers (called pockets) and 2 non-integers, then the path beginning at either pocket must eventually terminate in the other pocket.

Figure 11 shows a rectangle of perimeter 21, whose corners are at $a = 2$, $b = 6.5$, $c = 12.5$, $d = 17$. The sequence between pockets (thick lines) is 2, 11, 14, 20, 5, 8, 17. This sequence fails to reach many of the other integers along the perimeter, which lie in the following cycle: 1, 3, 10, 15, 19, 6, 7, 18, 16, 9, 4, 21, 13, 12, 1. The question of which rectangular billiard tables yield a single covering path and which yield a degeneracy of this sort is answered by the following lemma.

Lemma. Let A, B, C, D, be positive integers such that $A < B < C < D$ and $C - A = D - B$. Let $a = A/2$; $b = B/2$; $c = C/2$, and $d = D/2$. Further suppose that exactly two of a, b, c, d are integers, so that the corresponding billiard table has two corner pockets. Then the 45 degree path between the pockets touches **all** of the integers along the perimeter just if the rectangle's double-sides, $B - A$ and $C - B$, are relatively prime.

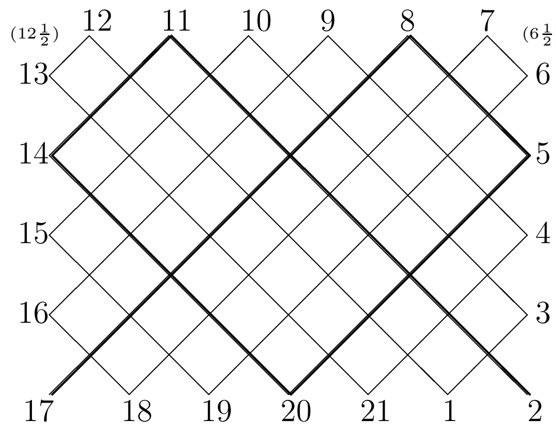

Figure 11: A billiard table with $A = 4$, $B = 13$, $C = 25$, $D = 34$ and perimeter $P = 21$. The double-sides $B - A = 9$ and $C - B = 12$ are not relatively prime.

Note. In Figure 11 the sides are $6.5 - 2 = 4.5$ and $12.5 - 6.5 = 6$, so the double-sides are 9 and 12. They have a common factor of 3. So we could color each integer of shape $3k + 2$ and both pockets would be colored. Every integer along the ball's path would then also be colored. In general, this argument shows that a degeneracy occurs whenever the double-sides are not relatively prime.

Proof of non-degeneracy. If the double-sides are prime to each other, and hence to the perimeter $P = C - A = D - B$, so that, mod P, $A \equiv C$ and $D \equiv B$, then consider any two integers separated by exactly one bounce along the ball's path. If the bounce is at x, these integers, mod P, are at $A - x$ and $B - x$, and the distance between them is $B - A \equiv D - C$ if measured in one direction mod P, or $A - B \equiv C - D \equiv A - D \equiv C - B$ if measured in the other direction. But since $B - A$ is a double-side, which is relatively prime to P, it follows that the sequence, obtained by looking at **alternate** bounce-points along the ball's path, cannot cycle back to itself, mod P, without first reaching a pocket. Since this is true for all values of x, the ball-path from one pocket to the other must go through every integer point on the rectangle's perimeter.

We can take three corners of a rectangle as the halves of any three consecutive Fibonacci numbers (recall that the corners are allowed to be half-integers). The perimeter of this rectangle will be the middle of these three Fibonacci numbers. Since any pair of adjacent Fibonacci numbers is relatively prime, the path from pocket to pocket is complete.

Square chains. For the 'square' chains and necklaces which we mentioned at the outset, Ed Pegg and Edwin Clark have verified that there are chains for $n =$ 15, 16, 17, 23, 25 to 31 and necklaces (and hence chains) for $n = 32$ upwards.

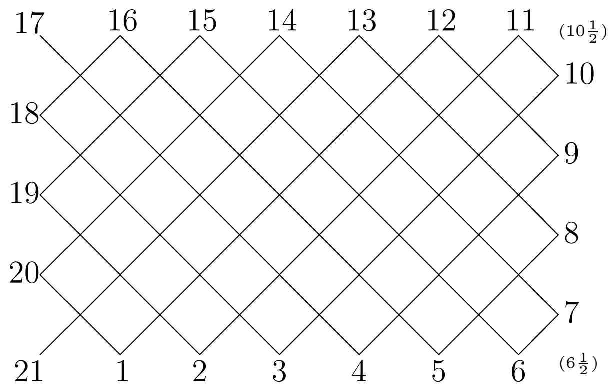

Figure 12: A billiard table giving a Fibonacci chain of length $P = 21$.

The existence problem was solved quite recently; more in the appendix at the end.

The billiards technique allows us to construct arbitrarily large specimens. Figure 13 shows how our billiard table technique can be used to find a 'square' chain of length 16.

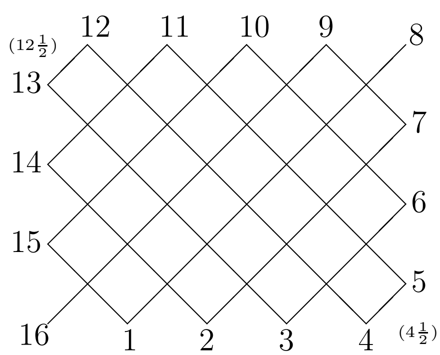

Figure 13: A billiard table with $A = 9$, $B = 16$, $C = 25$ (all squares), and $D = 32$ (half a square) and perimeter $P = 16$.

We may delete 16, or append 17, giving the 15-, 16- and 17-chains of Figure 1.

It is possible to accommodate other numbers by using billiard tables with more than four corners! Figure 14 shows such a table with corners at 4.5, 8.5, 9, 12.5, 24.5, and 32. The corner at 8.5 is reflex; the others are right. The perimeter is 39. There are two pockets: a conventional corner pocket at 32, and a side pocket at 9. The path between these two pockets is complete.

Square necklaces. In order to connect the two pockets and make a necklace, we must be sure that they sum to a square. Two half-squares summing to a

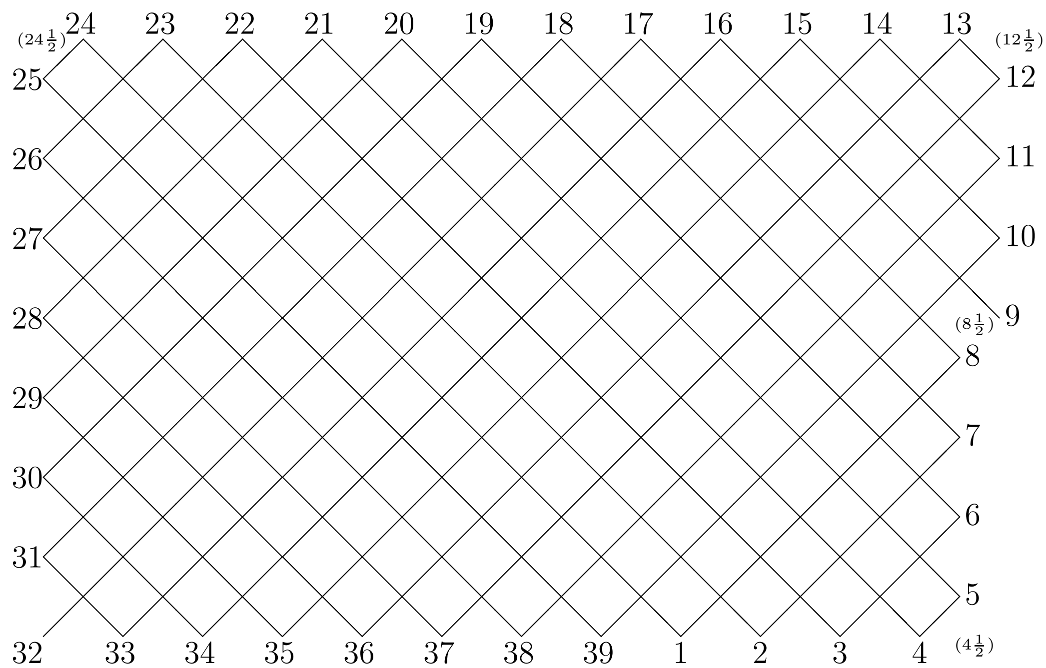

Figure 14: A billiard table with six corners.

square are provided by the parametric equation

$$\left((r+s)^2 - 2r^2\right)^2 + \left((r+s)^2 - 2s^2\right)^2 = 2\left(r^2 + s^s\right)^2$$

For example, $1^2 + 7^2 = 2 \cdot 5^2$. We multiply the solution by 6 to get the parity right and to avoid the sides having a common factor of 3. $42^2 - 6^2 = 2^6 \cdot 3^3$ can be arranged as the difference of two odd squares, which are not multiples of 3, in just two different ways, $43^2 - 11^2$ and $433^2 - 431^2$. Billiard tables with half these squares as corners have perimeters 1728 and 185725. Their double-sides, $(5 \cdot 17, 31 \cdot 53)$ and $(2^6 \cdot 3^3, 11 \cdot 43 \cdot 389)$ are coprime, so the chains contain every integer on the perimeter. Moreover, the ends of the chains are $\frac{1}{2}6^2$ and $\frac{1}{2}42^2$) which sum to 30^2 so that they may be joined to form necklaces.

Here are some small square necklaces. The bold numbers are $6x$, $6y$.

r, s	$x^2 + y^2 = 2z^2$ x, y	corners are half the squares of:	double sides are coprime	perimeter P
2,1	1,7	**6**,11,**42**,43	85,1643	1728
3,2	7,17	**42**,**102**,119,151	3757,8640	12397
4,3	17,31	**102**,**186**,197,251	4213,24192	28405
7,3	1,41	**6**,23,**246**,247	493,59987	60480
7,5	23,47	109,**138**,269,**282**	7163,53317	60480
5,4	31,49	**186**.**294**,373,437	51840,52693	104533
7,3	1,41	**6**,**246**,397,467	60480,97093	157573
2,1	1,7	**6**,**42**,431,433	1728,183997	185725
5,3	7,23	**42**,**138**,859,869	17280,718837	736117
7,3	1,41	**6**,**246**,2153,2167	60480,4574893	4635373
3,2	7,17	**42**,**102**,2159,2161	8640,4650877	4659517
5,4	31,49	**186**,**294**,2587,2597	51840,6606133	6657973
5,3	7,23	**42**,**138**,4319,4321	17280,18634717	18651997
4,3	17,31	**102**,**186**,6047,6049	24192,36531613	36555805
7,5	23,47	**138**,**282**,15119,15121	60480,228504637	228565117

Of course, if one looked for square chains by putting halves of odd squares at the corners of a billiard table, then, by Theorem 0 of number theory, namely that odd squares are congruent to 1 mod 8, we would find that our tour broke up into four separate loops, those containing 0 and 1, −1 and 2, −2 and 3, and those containing −3 and 4 modulo 8. However, we are able to make a single necklace, by breaking the loops at places which sum to a square on other loops. For example, the billiard table with corners at 4.5, 24.5, 40.5 and 60.5 yields four 18-loops which may be connected to form a 72-necklace as follows

. . . 1 — 3 . . . 6 — 10 . . . 71 — 29 . . . 52 —48 . . .

where the dots represent the other 16 members of each of the four loops.

More generally, if the odd squares are $(s-2r)^2$, $(s+2r)^2$, $(2s-r)^2$ and $(2s+r)^2$, we will have $n = 3(s^2 - r^2)$. In order that the point 1 is on an edge adjacent to the smallest square, we must have $s \geq r + \sqrt{(9r^2-1)/2}$.

Cubic chains. The billiard table with corners at {62.5, 171.5, 256, 365} has perimeter 387. The sides are relatively prime, so the path between the pockets is complete. The adjacent pair-sums are 125, 343, 512 and 730. In pursuit of a chain all of whose pair-sums are cubes, we move the corner from 365 to 364.5, and insert a new reflex corner at 386.5 and a side pocket at 387. A detailed calculation reveals that the path between the pockets at 387 and 256 is complete, so we then have a cubic chain among the numbers from 1 through 387. This chain uses only the cubes 125, 343, 512 and 729.

By deleting the endpoint at 387 we obtain a cubic chain among the numbers from 1 through 386. Since each of our Fibonacci chains also has a pocket at its highest number, we can similarly delete that maximum number and obtain a Fibonacci chain among the numbers from 1 to $F_k - 1$, for any $k > 3$. We leave the reader to design billiard tables with extra corners to accommodate such numbers.

No doubt, in answer to Nob Yoshigara's question, cubic chains and necklaces exist for all sufficiently large n, but not for $n < 295$. When $n = 295$ the graph has just two monovalent vertices, at 216 and 256, which have to form the tails of a chain, but it cannot be completed. We can construct a **cubic necklace** if we can find a number which is the sum of two odd cubes in two different ways. If the cubes are $a^3 + d^3 = b^3 + c^3$, then we also need that $a^3 < c^3 - b^3$ (to make sure the necklace includes all the numbers from 1 on) and that $\gcd(c^3 - b^3, b^3 - a^3) = 2$ (else the necklace will split up into smaller necklaces). The smallest try is $23^3 + 163^3 = 121^3 + 137^3$, but the relevant gcd is 14 and we have 7 small necklaces each of length 114256 instead of a single necklace of length 799792. Fortunately, Andrew Bremner observes that $21^3 + 257^3 = 167^3 + 231^3$ where $167^3 - 21^3 = 2 \cdot 13 \cdot 31 \cdot 73 \cdot 79$ and $231^3 - 167^2 = 2^6 \cdot 119827$ have gcd 2, so that if we put halves of these four odd cubes at the corners of a billiard table, we will have a cubic necklace of length the latter number, 7668928. Surely there are smaller ones.

Triangular chains exist for $n = 2$ and probably for all $n \geq 9$. Necklaces appear to exist for $n \geq 12$, except for $n = 14$. We would like to see proofs of these statements, which we have verified to $n = 70$. It is easy to find arbitrarily large triangular chains, by taking numbers which are the sum of two triangular numbers in two different ways. If the triangular numbers $A < B < C < D$ are odd and not all multiples of three (in fact two will have to be multiples of 3 and two of them congruent to 1 mod 3), then, by placing their halves at the corners of a billiard table, we will have a **triangular necklace** of length $C - A$, provided that the sides of the table are coprime, and that $A < C - B$ (else we will lose some of the beads from the beginning of the necklace).

Here are some triangular necklaces.

corners are half the triangular numbers:	sides are coprime	perimeter; # of beads
1, 15, 91, 105	7, 38	90
55, 153, 253, 351	49, 50	198
91, 231, 325, 465	47, 70	234
15, 253, 465, 703	106, 119	450
21, 55, 561, 595	17, 253	540
45, 153, 595, 703	54, 221	550
91, 253, 741, 903	81, 244	650
253, 703, 1035, 1485	166, 225	782
3, 325, 903, 1225	161, 289	900
325. 703, 1275, 1653	189, 286	950
45, 91, 1035, 1081	23, 472	990
465, 703, 1653, 1891	119, 475	1188
171, 1225, 1431, 2485	103, 527	1260
45, 325, 1431, 1711	140, 553	1386
1, 55, 1431, 1485	27, 688	1430
45, 1035, 1711, 2701	338, 495	1666
1, 435, 1711, 2145	217, 638	1710
171, 703, 1953, 2485	266, 625	1782
91, 153, 1891, 1953	31, 869	1800
55, 1485, 1891, 3321	203, 715	1836
105, 595, 2211, 2701	245, 808	2106
15, 231, 2485, 2701	108, 1127	2470
91, 1485, 2701, 4095	608, 697	2610
55, 435, 2701, 3081	190, 1133	2646
21, 595, 3081, 3655	287, 1243	3060
3, 325, 3081, 3403	161, 1378	3078
171, 253, 3321, 3403	41, 1584	3250
1, 91, 4005, 4095	45, 1957	4004

The existence of 'triangular triples', such as —29—91—62—, —44—92—61—, —27—93—78— in which each pair sums to a triangular number, enable us to expand the 90-necklace at the head of the last list, to 91-, 92- and 93-necklaces, as in Figure 15.

In the same way we can insert —101—199—152— and —100—200—53— into the 198-necklace which is the second in the list.

Pentagonal chains, i.e., those in which adjacent links sum to the pentagonal numbers, 1, 2, 5, 7, 12, 15, ..., $\frac{1}{2}n(3n \pm 1)$, appear to exist for all $n \geq 4$ (e.g., 1—4—3—2) and necklaces for all $n \geq 9$. E.g., –6–1–4–8–7–5–2–3–9–6– or

This has been checked to $n = 49$. Here are some other necklaces.

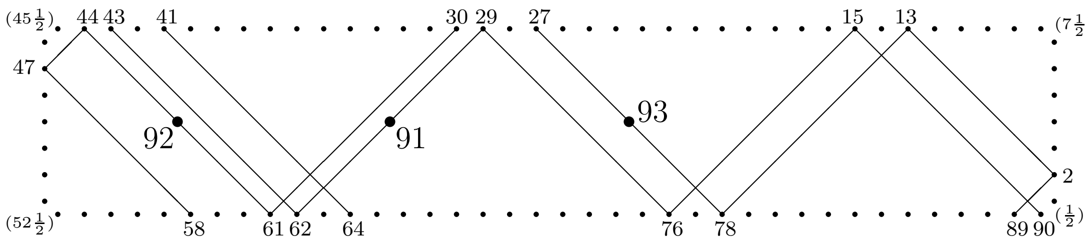

Figure 15: Expanding a 'triangular' 90-necklace by one, two or three beads.

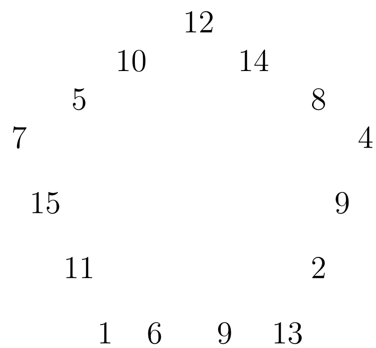

Figure 16: A necklace with adjacent pairs of beads adding to pentagonal numbers.

corners are half the pentagonal numbers:	sides are coprime	perimeter; # of beads
15, 35, 57, 77	10, 11	42
1, 7, 51, 57	3, 22	50
7, 35, 117, 145	14, 41	110
35, 77, 145, 187	21, 34	110
15, 117, 145, 247	51, 14	130
57, 155, 247, 345	49, 46	190
7, 145, 287, 425	69, 71	280
1, 15, 287, 301	7, 136	286
7, 51, 301, 345	22, 125	294
7. 77, 425, 495	35, 174	418

Prime chains have been considered from time to time [3, 4], but as in all cases except the Fibonacci numbers and the Lucas numbers, existence proofs for all large enough n are elusive.

Theorem 2. *There is a chain formed with the numbers 1 to n with each* *adjacent pair adding to a Lucas number, just if* $n = 5$, *or* L_k *or* $L_k - 1$, *where* L_k *is a Lucas number with* $k \geq 2$ ($L_2 = 3$, $L_3 = 4$, $L_{n+1} = L_n + L_{n-1}$). *The chain is essentially unique.*

The proof can follow either of the methods used for Theorem 1.

There are corresponding theorems for sequences satisfying the same recurrence relation. For example, the chains that can be formed using the numbers 4, 5, 9, 14, 23, 37, ... have length one of those numbers, or one less than one of them.

Appendix on square necklaces.

In the sixteen years since this paper was written, one author has collected square necklaces for $32 \leq n \leq 252$. They are not unique. Figure 17 shows a pair of necklaces for $n = 40$.

1	3	6	19	30	34	15	10	39	25	24
8										40
17										9
32										16
4										33
21										31
28										18
36										7
13										2
12										23
37	27	22	14	35	29	20	5	11	38	26

3	6	19	30	34	15	10	39	25	24	40
22										9
27										16
37										33
12										31
13										18
36										7
28										29
21										20
4										5
32	17	8	1	35	14	2	23	26	38	11

Figure 17: A pair of square necklaces for $n = 40$.

At a recent MathFest presentation by the other author, a member of the audience claimed to have used a computer to find square necklaces for $32 \leq n \leq 1000$.

We were delighted to learn that the problem was recently solved by Robert Gerbicz; see the Mersenne Forum blog thread [1]. Square necklaces exist for any length of the form $n = (71 * 25^k - 1)/2$ with $k \geq 0$. A generalization of this construction proves the existence of square necklaces of any length $n \geq 32$ and square chains of any length $n \geq 25$. Gerbicz's C code for generating square necklaces is available for download [2].

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