# Millions of Perrin pseudoprimes including A <br> FEW GIANTS 

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#### Abstract

The calculation of many and large Perrin pseudoprimes is a challenge. This is mainly due to their rarity. Perrin pseudoprimes are one of the rarest known pseudoprimes. In order to calculate many such large numbers, one needs not only a fast algorithm but also an idea how most of them are structured to minimize the amount of numbers one have to test.

We present a quick algorithm for testing Perrin pseudoprimes and develop some ideas on how Perrin pseudoprimes might be structured. This leads to some conjectures that still need to be proved.

We think that we have found well over $90 \%$ of all 20 -digit Perrin pseudoprimes. Overall, we have been able to calculate over 9 million Perrin pseudoprimes with our method, including some very large ones. The largest number found has 3101 digits. This seems to be a breakthrough, compared to the previously known just over 100,000 Perrin pseudoprimes, of which the largest have 20 digits.

In addition, we propose two new sequences that do not provide any pseudoprimes up to $10^{9}$ at all.


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## 1 Introduction

To motivate that it makes sense to deal with primes, it is best to quote Gauss [3]:

> The problem of distinguishing prime numbers from composite numbers, and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic. It has engaged the industry and wisdom of ancient and modern geometers to such an extent that it would be superfluous to discuss the problem at length. Nevertheless we must confess that all methods that
> have been proposed thus far are either restricted to very special cases or are so laborious and difficult that even for numbers that do not exceed the limits of tables constructed by estimable men, they try the patience of even the practiced calculator. And these methods do not apply at all to larger numbers.

Prime numbers are a very serious issue. We prefer dealing with pseudoprimes. Pseudoprimes are numbers that behave similar to primes.
Sometimes it is a big challenge to compute all or at least many or some very large pseudoprimes of a given type.
In this paper, we introduce a quick algorithm for the calculation of Perrin pseudoprimes. This is nothing special, there are already many fast algorithms. Similar to primes, also for pseudoprimes it is difficult to guess their structure. Therefore, in order to calculate all of them there is nothing left but to test every single number. This strongly limits the size of the numbers. It turns out, however, that the structure can be guessed for most of the pseudoprimes. This very much limits the range of potential numbers to be tested and makes it possible to calculate millions of them and even very large ones.
We do the following Notations:

- The set of all primes is denoted by $\mathbb{P}$.
- $a \mid b$ means $a$ divides $b$ or $b$ is divisible by $a$.
- We state some classical facts from number theory as theorems, omitting the proofs.


### 1.1 The Perrin sequence

Let us define a sequence (called Perrin sequence) $P_{n}$ recursively:

$$
\begin{aligned}
& P_{0}=3 \\
& P_{1}=0 \\
& P_{2}=2 \\
& P_{n}=P_{n-2}+P_{n-3}, n \geq 3
\end{aligned}
$$

and calculate the first entries:

$$
\left(P_{n}\right)_{n=0}^{\infty}=3,0,2,3,2,5,5,7,10,12,17,22,29,39,51,68,90,119, \ldots
$$

We observe: If $n$ is prime, we have $n \mid P_{n}$ and that goes on for a long time.
Anyone seeing this sequence for the first time is certainly quite surprised, since it is believed that there is no simple algorithm for calculating the primes.
The recursion law of this sequence was found in 1899 by Edouard Lucas. This sequence with the initial values given above, was first used by Raoul Perrin [7, 8].
Probably many mathematicians and amateur mathematicians have tried to answer the question of whether this sequence really only produces primes. Considering that already the number $P_{811}$ has 100 digits, one can imagine how difficult that has been.
The answer was not found until 1982, when Jeffrey Shallit (according to [8]) calculated the first two non-prime numbers - so-called Perrin pseudoprimes (PPP) - with a computer. Here they are: $271441=521 \cdot 521$ and $904631=7 \cdot 13 \cdot 9941 . P_{271441}$ has 33150 digits. Today it is known that there are infinitely many Perrin pseudoprimes [1]. Nevertheless, they are very rare, which makes their finding still difficult.
In this paper, we develop an effective algorithm for calculating Perrin pseudoprimes and present some numerical results that constitute, to our knoledge, right now the world's largest collection of Perrin pseudoprimes including the largest PPP.

## 2 Pseudoprimes

### 2.1 Iff- and if-Theorems

There are two kinds of theorems dealing with primes that can be used to test a given number on whether it is a prime.

1) Theorems like: $p \in \mathbb{P}$ if and only if property $A(p)$ holds.
2) Theorems like: $p \in \mathbb{P}$, then property $A(p)$ holds.

Theorems of the first kind are, for example

- Theorem: $p \in \mathbb{P} \Longleftrightarrow \forall k \in \mathbb{P}, k \leq \sqrt{p}: k \nmid p$
- Theorem (Wilson): $p \in \mathbb{P} \Longleftrightarrow p \mid 1 \cdot 2 \cdot 3 \cdots(p-1)+1$


## - Theorem:

$$
\begin{equation*}
p \in \mathbb{P} \Longleftrightarrow p \left\lvert\,\binom{ p}{k} \quad \forall k=1\right., \ldots, p-1 \tag{1}
\end{equation*}
$$

These theorems allow for deterministic tests. If for a given number $p$ the property $A(p)$ holds, then $p$ is prime.
Unfortunately, algorithms based on deterministic testing have high complexity, so far.
Theorems of the second kind state: If for a given number $p$ the property $A(p)$ holds, then $p$ can be prime or not. This is useful, if $p$ is prime with very high "probability". Testing $A(p)$ one can be "very sure" that $p$ is prime. Typically such kind of probabilistic tests are much faster (have a lower complexity) than deterministic ones. Thus, it is useful to create tests with a very small equivalence gap, the gap between if and iff.
Numbers $n$ that lie in this gap, i.e. $A(n)$ holds, but $n$ is composite, are called pseudoprimes with respect to property $A$.
One example, following immediately from (1) is:
Theorem: $p \in \mathbb{P} \Longrightarrow p \left\lvert\, \sum_{k=1}^{p-1} a_{k}\binom{p}{k}\right.$ for some given integers $a_{k}$.
It is clear that looking at a linear combination of binomial coefficients instead of all coefficients in detail, we loose information. This is just the equivalence gap. Looking at a given linear combination of binomial coefficients is faster than looking at every one in detail. The idea is to choose such coefficients $a_{k}$ so that the equivalence gap is small.

Here, we define some kind of probability (better frequency) for a pseudoprime test. Let $\pi(n)$ be the number of primes less than $n$ and $P(n)$ the number of pseudoprimes less than $n$ for a given pseudoprime test. By $W(n)=P(n) / \pi(n)$ we define the frequency of numbers incorrectly tested and call it error rate. Thus, the lower the error rate $W(n)$, the better the test.
Of course, it would be best if a test provided only a finite number of pseudoprimes. These would be calculated and stored in a database which allowed for a deterministic test, practically. Such a test is not yet known. In contrast, until now, for many pseudoprime number type, it has been proved sooner or later that there are infinitely many ones.

### 2.2 Fermat and Carmichael pseudoprimes

The simplest pseudoprimes are Fermat pseudoprimes. They are consequences of Fermat's little
Theorem: Given an integer $z \geq 2$. If $p \in \mathbb{P}$ then $p \mid z^{p}-z$.
Conversely, if a number $n$ for some $z$ satisfies $n \mid z^{n}-z$ but $n \notin \mathbb{P}, n$ is called Fermat ${ }_{z}$ pseudoprime.
Best known is the special case $z=2$ :
Theorem: If $p \in \mathbb{P}$ then $p \mid 2^{p}-2$.
A number $n \notin \mathbb{P}$ with $n \mid 2^{n}-2$ is called Fermat ${ }_{2}$ pseudoprime.

### 2.2.1 Fermat $_{2}$ pseudoprimes

Fermat's little Theorem for $z=2$ is an easy consequence of Theorem 1 .

Indeed, multiplying out $(a+b)^{n}$ with integers $a, b$ we get

$$
(a+b)^{n}=a^{n}+\binom{n}{1} a^{n-1} b+\binom{n}{2} a^{n-2} b^{2}+\binom{n}{3} a^{n-3} b^{3}+\cdots+b^{n}
$$

Therefore, defining

$$
f_{n}=(a+b)^{n}-a^{n}-b^{n}=\binom{n}{1} a^{n-1} b+\ldots+\binom{n}{n-1} a b^{n-1}
$$

we obtain the
Theorem: If $p \in \mathbb{P}$ then $p \mid f_{p}$.
The special case ( $a=b=1$ ) yields Fermat's little Theorem to the base $z=2$.

Let's calculate the first ones:

| $n$ | $2^{n}-2$ | $n \mid 2^{n}-2 ?$ | $n$ is prime? |  |
| ---: | ---: | ---: | ---: | ---: |
| 2 | 2 | yes! | yes! |  |
| 3 | 6 | yes! | yes! |  |
| 4 | 14 | no! | no! |  |
| 5 | 30 | yes! | yes! |  |
| 6 | 62 | no! | no! |  |
| 7 | 126 | yes! |  |  |
| 341 | $4479 \ldots$ (103 digits) | yes! | no! | $341=11 \cdot 31$ |
| 561 | $7547 \ldots$ (169 digits) | yes! | no! $561=3 \cdot 11 \cdot 17$ |  |
| 645 | $1459 \ldots$ (195 digits) | yes! | no! $645=3 \cdot 5 \cdot 43$ |  |

Up to 100000 we have 78 pseudoprimes and 9592 primes. Thus, we have $W\left(10^{5}\right)=0.00813178$.

### 2.2.2 Carmichael numbers

Instead of $z=2$ we can consider Fermat ${ }_{z}$ pseudoprimes with other bases. Maybe other bases provides fewer pseudoprimes? It turns out that $z=2$ is one of the best bases. Moreover, there are non-primes $n$ with $n \mid z^{n}-z$ for any base $z$, the so-called Carmichael numbers. 561 is the smallest one. The next ones are

| Carmichael number | factors |
| :---: | :---: |
| 561 | $3 \cdot 11 \cdot 17$ |
| 1105 | $5 \cdot 13 \cdot 17$ |
| 1729 | $7 \cdot 13 \cdot 19$ |
| 2465 | $5 \cdot 17 \cdot 29$ |
| 2821 | $7 \cdot 13 \cdot 31$ |
| 6601 | $7 \cdot 23 \cdot 41$ |
| 8911 | $7 \cdot 19 \cdot 67$ |
| 10585 | $5 \cdot 29 \cdot 73$ |


| Carmichael number | factors |
| :---: | :---: |
| 15841 | $7 \cdot 31 \cdot 73$ |
| 29341 | $13 \cdot 37 \cdot 61$ |
| 41041 | $7 \cdot 11 \cdot 13 \cdot 41$ |
| 46657 | $13 \cdot 37 \cdot 97$ |
| 52633 | $7 \cdot 73 \cdot 103$ |
| 62745 | $3 \cdot 5 \cdot 47 \cdot 89$ |
| 63973 | $7 \cdot 13 \cdot 19 \cdot 37$ |
| 75361 | $11 \cdot 13 \cdot 17 \cdot 31$ |

There are 16 Carmichael numbers up to 100000. Moreover, we have the following
Theorem: There are infinitely many Carmichael numbers [1].

### 2.3 General pseudoprimes

### 2.3.1 Sums of powers. Multinomial coefficients

Similar to binomial coefficients, there is a theorem for multinomial coefficients:
Theorem: $\left.p \in \mathbb{P} \Longleftrightarrow p\right|_{i!~} ^{i!j!k!}, \forall i, j, k$ with $0<i+j+k=p$
From this, we conclude the following
Theorem: $p \in \mathbb{P} \Longrightarrow p \left\lvert\, \sum_{0<i+j+k=p} a_{i j k} \frac{p!}{i!j!k!}\right.$. for some integer coefficients $a_{i j k}$.
From this, multiplying out $(a+b+c)^{n}$ with integers $a, b, c$ we conclude the
Theorem: Given a sequence

$$
f_{n}=(a+b+c)^{n}-a^{n}-b^{n}-c^{n}=\sum_{0<i+j+k=n} \frac{n!}{i!j!k!} a^{i} b^{j} c^{k}
$$

Then, $p \in \mathbb{P}$ implies $p \mid f_{p}$.
Similarly we get the
Theorem: Given integers $a_{1}, \ldots, a_{k}$. Build the sequence

$$
\begin{equation*}
f_{n}=\left(a_{1}+a_{2}+\ldots+a_{k}\right)^{n}-\left(a_{1}^{n}+a_{2}^{n}+\ldots+a_{k}^{n}\right) \tag{2}
\end{equation*}
$$

Then, $p \in \mathbb{P}$ implies $p \mid f_{p}$.
The example $a_{i}=1$ yields $f_{n}=k^{n}-k$, Fermat's little theorem in the general case.
Perrin's sequence is given in a recurrent way. Here, we recall the important connection between polynomials and recurrence sequences.

### 2.3.2 Polynomials and recurrence sequences

A linear recurrence sequence (or linear difference equation) of order $k$ is a sequence $\left(h_{n}\right)_{n=0}^{\infty}$ defined in the following way:
Given $k$ numbers $c_{1}, \ldots, c_{k}$ set

$$
\begin{equation*}
h_{n}=c_{1} h_{n-1}+c_{2} h_{n-2}+\ldots+c_{k} h_{n-k} . \tag{3}
\end{equation*}
$$

Together with $k$ initial conditions $h_{0}, h_{1}, \ldots, h_{k-1}$ such a sequence is uniquely determined.
Obviously, if $c_{1}, \ldots, c_{k}$ and $h_{0}, h_{1}, \ldots, h_{k-1}$, are integers, then $h_{n}$ is an integer for all $n$.
There is a remarkable connection between such sequences and polynomials of degree $k$. If we put $h_{n}=x^{n}$ and multiply by $x^{k-n}$, we get an algebraic equation for the roots of a polynomial formed from the coefficients of the sequence

$$
\begin{equation*}
Q(x)=-x^{k}+c_{1} x^{k-1}+c_{2} x^{k-2}+\ldots+c_{k-1} x+c_{k} . \tag{4}
\end{equation*}
$$

This polynomial has $k$ - in general complex - roots $x_{1}, \ldots, x_{k}$. For simplicity, we assume that the roots are different.

Set

$$
\begin{equation*}
g_{n}=b_{1} x_{1}^{n}+b_{2} x_{2}^{n}+\ldots+b_{k} x_{k}^{n}, \tag{5}
\end{equation*}
$$

with some coefficients $b_{1}, \ldots, b_{k}$. Solve the system of $k$ linear equations $h_{i}=g_{i}, i=0, \ldots, k-1$ with respect to the unknown $b_{j}$. This is always uniquely solvable, because the corresponding matrix is the Vandermonde matrix $\left(x_{i}^{j}\right)$. Its determinant does not vanish if the roots $x_{i}$ are different, as required.
Theorem: For any $n \geq 0$ we have $g_{n}=h_{n}$.
This is easily proved, since we have $Q\left(x_{i}\right)=0$ for $i=1, \ldots, k$.
The opposite is also true:
Theorem: Given $k$ different complex numbers $x_{1}, \ldots, x_{k}$ and $k$ real numbers $b_{1}, \ldots, b_{k}$. Calculate the first entries $h_{0}, \ldots, h_{k-1}$ of some sequence $\left(h_{n}\right)$ by the right-hand side of (5) and compile a polynomial (4) from it's roots $x_{1}, \ldots, x_{k}$

$$
Q(x)=-\left(x-x_{1}\right) \cdots\left(x-x_{k}\right)=-x^{k}+(-1)^{k+1}\left(x_{1}+\ldots+x_{k}\right) x^{k-1}+\ldots
$$

Then, the sequence (3), given in a recurrent way is exactly the sequence (5), given explicitely.
Thus, we have a one-to-one correspondence between the linear recurrence sequence (3) and the sum of powers (5).
This can be applied to Perrin's sequence.

### 2.3.3 Perrin's sequence, given explicitely

Starting with the sequence

$$
\begin{aligned}
& P_{0}=3 \\
& P_{1}=0 \\
& P_{2}=2 \\
& P_{n}=P_{n-2}+P_{n-3}, n \geq 3
\end{aligned}
$$

at first, we compile the polynomial from the coefficients

$$
Q(x)=-x^{3}+x+1
$$

Its roots are

$$
\begin{aligned}
a & =1.32472 \ldots \\
b & =-0.662359 \ldots+0.56228 \ldots i \\
c & =-0.662359 \ldots-0.56228 \ldots i
\end{aligned}
$$

Set $h_{n}=a^{n}+b^{n}+c^{n}$ (since $a+b+c=0$ ). The first entries are

$$
\begin{aligned}
& h_{0}=a^{0}+b^{0}+c^{0}=3 \\
& h_{1}=a^{1}+b^{1}+c^{1}=0 \\
& h_{2}=a^{2}+b^{2}+c^{2}=(a+b+c)^{2}-2(a b+b c+c a)=0-2(-1)=2
\end{aligned}
$$

Thus, the sequences $P_{n}$ and $h_{n}$ coincide.

The theorem

$$
p \in \mathbb{P} \quad \Longrightarrow \quad p \mid P_{p}=a^{p}+b^{p}+c^{p}
$$

does not follow from this, immediately, since $a, b, c$ are not integers. We have to answer two questions:

- When is $f_{n}=(a+b+c)^{n}-a^{n}-b^{n}-c^{n}$ an integer sequence?
- When does $\left(p \in \mathbb{P} \Longrightarrow p \mid f_{p}\right)$ hold?


### 2.3.4 When is $f_{n}=(a+b+c)^{n}-a^{n}-b^{n}-c^{n}$ an integer?

For any $n$, the expression $(a+b+c)^{n}-a^{n}-b^{n}-c^{n}$ is a symmetric polynomial in $a, b$ and $c$. Theorem: Any symmetric polynomial can be expressed in terms of elementary symmetric polynomials.
Here, these are

$$
A_{1}=a+b+c, A_{2}=a b+b c+c a, A_{3}=a b c
$$

which are the coefficients of a polynomial with roots $a, b, c$.
Calculating, for example, the first entries, we get

$$
\begin{aligned}
(a+b+c)^{0}-a^{0}-b^{0}-c^{0} & =-2 \\
(a+b+c)^{1}-a^{1}-b^{1}-c^{1} & =0 \\
(a+b+c)^{2}-a^{2}-b^{2}-c^{2} & =2 A_{2} \\
(a+b+c)^{3}-a^{3}-b^{3}-c^{3} & =3 A_{1} A_{2}-3 A_{3} \\
(a+b+c)^{4}-a^{4}-b^{4}-c^{4} & =4 A_{1}^{2} A_{2}-4 A_{1} A_{3}-2 A_{2}^{2}
\end{aligned}
$$

Hence, $f_{n}$ is integer if $a, b, c$ are roots of a polynomial with integer coefficients.

### 2.3.5 When does $\left(p \in \mathbb{P} \Longrightarrow p \mid f_{p}\right)$ hold?

We have

$$
f_{n}=(a+b+c)^{n}-\left(a^{n}+b^{n}+c^{n}\right)=\sum_{0<i+j+k=n} \frac{n!}{i!j!k!} a^{i} b^{j} c^{k}
$$

and $\left.p \in \mathbb{P} \Longrightarrow p\right|_{\frac{p!}{i!j!k!},}, \forall i, j, k$ with $0<i+j+k=n$.
$\frac{n!}{i!j!k!}$ does not change by a permutation of $i, j, k$. It can be lifted out.

$$
\sum_{0<i+j+k=n} \frac{n!}{i!j!k!} a^{i} b^{j} c^{k}=\sum_{0<i \leq j \leq k} \frac{n!}{i!j!k!} \sum_{\pi(i, j, k)} a^{i} b^{j} c^{k}
$$

$\sum_{\pi(i, j, k)} a^{i} b^{j} c^{k}$ is again a symmetric polynomial and so it is an integer if $a, b, c$ are roots of an polynomial with integer coefficients.
Hence, if $a, b, c$ are roots of a polynomial with integer coefficients, and $f_{n}=(a+b+c)^{n}-\left(a^{n}+\right.$ $b^{n}+c^{n}$ ), then $p \in \mathbb{P} \Longrightarrow p \mid f_{p}$.

### 2.3.6 The recurrent calculation of the sequence

From the polynomial $Q(x)$ it is easy to compile the recurrent relation

$$
g_{n}=a_{1} f_{n-1}+a_{2} f_{n-2}+a_{3} f_{k-3}+\ldots+a_{k} f_{n-k}
$$

corresponding to the explicit expression

$$
g_{n}=x_{1}^{n}+\ldots+x_{k}^{n} .
$$

From this explicit expression we have to calculate the initial values $g_{0}, \ldots, g_{k-1}$. Then, we have

$$
f_{n}=g_{n}-a_{1}^{n}
$$

Actually, this is practicable if $a_{1}=0$ (like in the Perrin case) or $a_{1}= \pm 1$. In other cases, $a_{1}^{n}$ increases rapidly and it is better to look on

$$
f_{n}=\left(x_{1}^{n}+\ldots+x_{k}^{n}\right)-\left(x_{1}+\ldots+x_{k}\right)^{n}
$$

as on a sum of $k+1$ powers. This corresponds to a sequence of order $k+1$, having a corresponding polynomial with the $k+1$ roots $x_{1}, \ldots, x_{k}, a_{1}=x_{1}+\ldots+x_{k}$. This polynomial is

$$
\begin{aligned}
G(x) & =-\left(x-x_{1}\right) \cdots\left(x-x_{1}\right)\left(x-x_{1}-\ldots-x_{k}\right)=Q(x)\left(x-a_{1}\right)= \\
& =-x^{k+1}+2 a_{1} x^{k}+\sum_{i=1}^{k-1}\left(a_{i+1}-a_{1} a_{i}\right) x^{k-i}-a_{1} a_{k}
\end{aligned}
$$

### 2.3.7 The main theorem

Connecting the last facts together, we finally obtain the
Main Theorem: Given a polynomial of degree $k$

$$
Q(x)=-x^{k}+a_{1} x^{k-1}+a_{2} x^{k-2}+a_{3} x^{k-3}+\ldots+a_{k-1} x+a_{k}
$$

with integer coefficients $a_{i} \in \mathbb{Z}$ and (maybe complex) roots $x_{1}, \ldots, x_{k}$. Then, the sequence

$$
f_{n}=\left(x_{1}^{n}+\ldots+x_{k}^{n}\right)-\left(x_{1}+\ldots+x_{k}\right)^{n}
$$

is an integer sequence and it holds $\quad p \in \mathbb{P} \Longrightarrow p \mid f_{p}$.
The sequence $f_{n}$ can be calculated in a recurrent way from an order $k$-recurrent relation

$$
g_{n}=a_{1} f_{n-1}+a_{2} f_{n-2}+a_{3} f_{k-3}+\ldots+a_{k} f_{n-k}
$$

by $f_{n}=g_{n}-a_{1}^{n}$ or directly from an order $(k+1)$-recurrent relation

$$
f_{n}=2 a_{1} f_{n-1}+\sum_{i=1}^{k-1}\left(a_{i+1}-a_{1} a_{i}\right) f_{n-i-1}-a_{1} a_{k} f_{n-k-1}
$$

We can conclude that any polynomial with integer coefficients is cantidate to generate pseudoprimes.

## 3 Numerical algorithms

To calculate pseudoprimes, at first we have to calculate $f_{n}$ by a recurrent or explicit expression and then we test whether $n \mid f_{n}$.
The recurrence relation seems to be very fast, with some additions for every number. Unfortunately, the entries $f_{n}$ grow very fast. For the Perrin sequence we have $P_{n} \sim 1.32472 \ldots{ }^{n}$ (the largest root). Thus, $P_{271441}$ has 33150 decimal digits, $P_{99607901521441}$ - the 17-th Perrin pseudoprime has $12,164,524,642,561$ decimal digits requiring $\sim 5$ TByte to store it.
The same problem arises with the explicit expression. We have to calculate $x_{j}^{n}$ considering a huge number of digits to get an integer in the end. But this is necessary to check the remainder of $f_{n}$ when divided by $n$.

The only useful method is to carry out all operations modulo $n$. This will save us from the usage of the huge numbers $f_{n}$. We can still use the recurrence relation but for every new number we have to start at the very beginning of the sequence, since calculating $f_{n} \bmod n$, we cannot use the result to calculate $f_{n+1} \bmod n+1$.
Even doing so, this is still a problem if we want (and we want!) to deal with large numbers $n$ having, say, 100 digits. Note, this is the number of digits of the index, not of the sequence member!
Thus, if $n=10^{100}$ we need a fast algorithm for $10^{100}$ additions of numbers like $10^{100}$ (all done modulo $n$ ).
Clearly, this has to be an algorithm with logarithmic complexity. This can be done in pursuing following steps:

- We can calculate $k$ entries of the sequence at once, using matrix powers.
- The $n$-th power of a matrix can be performed in $\log _{b} n$ operations using the decomposition of $n$ with respect to a fixed basis and Horner's method.
- In some special cases - and the Perrin sequence is such a case - the calculation can be further simplified.


### 3.1 Matrix powers instead of additions

Given a recurrence sequence of order $k$

$$
\begin{equation*}
f_{n}=c_{k-1} f_{n-1}+c_{k-2} f_{n-2}+\ldots+c_{0} f_{n-k} \tag{6}
\end{equation*}
$$

with initial values

$$
\begin{equation*}
F_{0}:=\left(f_{0}, \ldots, f_{k-1}\right) \tag{7}
\end{equation*}
$$

The $k$-th entry

$$
f_{k}=c_{k-1} f_{k-1}+c_{k-2} f_{k-2}+\ldots+c_{0} f_{0}
$$

is a linear combination of the initial values and so are all entries, for example the $k+1$-th entry

$$
\begin{aligned}
f_{k+1} & =c_{k-1} f_{k}+c_{k-2} f_{k-1}+\ldots+c_{0} f_{1}= \\
& =c_{k-1}\left(c_{k-1} f_{k-1}+c_{k-2} f_{k-2}+\ldots+c_{0} f_{0}\right)+c_{k-2} f_{k-1}+c_{k-3} f_{k-2}+\ldots+c_{0} f_{1}= \\
& =\left(c_{k-1}^{2}+c_{k-2}\right) f_{k-1}+\left(c_{k-1} c_{k-2}+c_{k-3}\right) f_{k-2}+\ldots+\left(c_{k-1} c_{1}+c_{0}\right) f_{1}+c_{k-1} c_{0} f_{0}
\end{aligned}
$$

Writing all the entries $F_{1}:=\left(f_{k}, \ldots, f_{2 k-1}\right)$ as linear combinations of $F_{0}=\left(f_{0}, \ldots, f_{k-1}\right)$, we can compile a matrix $\mathbf{A}$ and write $F_{1}=\mathbf{A} F_{0}$, i.e.,

$$
\left(\begin{array}{c}
f_{k} \\
f_{k+1} \\
\vdots \\
f_{2 k-1}
\end{array}\right)=\left(\begin{array}{cccc}
c_{0} & c_{1} & \cdots & c_{k-1} \\
c_{k-1} c_{0} & c_{k-1} c_{1}+c_{0} & \cdots & c_{k-1}^{2}+c_{k-2} \\
\vdots & \vdots & \vdots & \vdots
\end{array}\right)\left(\begin{array}{c}
f_{0} \\
f_{1} \\
\vdots \\
f_{k-1}
\end{array}\right)
$$

This is an equivalent description of (6), (7).
In the special case $k=3$ we have

$$
\left(\begin{array}{c}
f_{3} \\
f_{4} \\
f_{5}
\end{array}\right)=\left(\begin{array}{ccc}
c_{0} & c_{1} & c_{2} \\
c_{0} c_{2} & c_{0}+c_{1} c_{2} & c_{2}^{2}+c_{1} \\
c_{0} c_{2}^{2}+c_{0} c_{1} & c_{1}^{2}+c_{2}^{2} c_{1}+c_{0} c_{2} & c_{2}^{3}+2 c_{1} c_{2}+c_{0}
\end{array}\right)\left(\begin{array}{c}
f_{0} \\
f_{1} \\
f_{2}
\end{array}\right)
$$

It follows $F_{m}=\mathbf{A}^{m} F_{0}$ for $F_{m}=\left(f_{m k}, f_{m k+1}, \ldots, f_{(m+1) k-1}\right)$. Thus, if we want to know $f_{n}$, we have to divide $n$ by $k$ with remainder, i.e., to write $n=m k+i$ with $i=0, \ldots, k-1$ and calculate $\mathbf{A}^{m}$. Instead of additions we have to calculate the power of a matrix. This can be done very effectively.

### 3.2 Horner's method instead of matrix powers

We have to calculate $\mathbf{A}^{m}$ for a given matrix $\mathbf{A}$. Let $m=a_{0} b^{j}+\ldots+a_{j-1} b+a_{j}$ be the decomposition of $m$ to base $b$ with $a_{0}>0$ and $b>a_{i} \geq 0$. Then, calculating the polynomial $a_{0} b^{j}+\ldots+a_{j-1} b+a_{j}$ with Horner's method, iteratively

$$
a_{0} b^{j}+\ldots+a_{j-1} b+a_{j}=\left(\left(\left(\left(a_{0} b\right) b+a_{1}\right) b+a_{2}\right) b+\ldots+a_{j}\right)
$$

we conclude

$$
\mathbf{A}^{m}=\mathbf{A}^{a_{0} b^{j}+\ldots+a_{j-1} b+a_{j}}=\left(\left(\left(\left(\left(\mathbf{I}^{b} \mathbf{A}^{a_{0}}\right)^{b} \mathbf{A}^{a_{1}}\right)^{b} \mathbf{A}^{a_{2}}\right)^{b} \cdots\right) \mathbf{A}^{a_{j}}\right)
$$

The vector

$$
\left(\mathbf{A}_{0}, \mathbf{A}_{1}, \ldots, \mathbf{A}_{b-1}\right)=\left(\mathbf{A}^{0}, \mathbf{A}^{1}, \mathbf{A}^{2}, \ldots, \mathbf{A}^{b-1}\right)
$$

can be calculated and stored in advance. The calculation runs especially effectively if $b$ itself is a power of 2 . For practicle purposes we used $b=2,4,8$.

### 3.3 A fast algorithm for the Perrin sequence

The following algorithm was written in 1982 by Frank Bauernöppel and Uwe Kaufmann [2] in Berlin.
1st step: Given $n$. Set $n=3 m+i, i \in\{0,1,2\}$. Since we have

$$
\begin{aligned}
& P_{3}=P_{1}+P_{0} \\
& P_{4}=P_{2}+P_{1} \\
& P_{5}=P_{3}+P_{2}=P_{1}+P_{0}+P_{2}
\end{aligned}
$$

we can introduce a matrix

$$
\mathbf{S}=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}\right)
$$

and have

$$
\left(\begin{array}{c}
P_{3 m} \\
P_{3 m+1} \\
P_{3 m+2}
\end{array}\right)=\mathbf{S}^{m}\left(\begin{array}{l}
3 \\
0 \\
2
\end{array}\right)=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}\right)^{m} \cdot\left(\begin{array}{l}
3 \\
0 \\
2
\end{array}\right)
$$

2nd step: The power of $\mathbf{S}$ can be further simplified by using the square $\mathbf{S}^{2}$. Depending on whether $m$ is even or odd, one have

$$
\mathbf{S}^{m}=\left(\mathbf{S}^{\frac{m}{2}}\right)^{2}, \quad 2 \mid m \quad \text { or } \quad \mathbf{S}^{m}=\left(\mathbf{S}^{\frac{m-1}{2}}\right)^{2} \cdot \mathbf{S}, \quad 2 \nmid m
$$

The total power $\mathbf{S}^{m}$ can now be calculated iteratively by using the binary representation of $m$. Let $m=\left(\alpha_{0}, \alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}, \ldots\right), \alpha_{0}=1$ be the dual number representation of $m$. We calculate iteratively matrices $\mathbf{S}_{k}$ in the following way:

$$
\begin{aligned}
\mathbf{S}_{0} & =\mathbf{I} \\
\mathbf{S}_{k+1} & =\left\{\begin{array}{lll}
\mathbf{S}_{k}^{2} & \text { if } & \alpha_{k}=0 \\
\mathbf{S}_{k}^{2} \cdot \mathbf{S} & \text { if } & \alpha_{k}=1
\end{array}\right.
\end{aligned}
$$

Then, $\mathbf{S}^{m}=\mathbf{S}_{k_{0}}$ for some $k_{0}<m$.
For example, we have

$$
\mathbf{S}^{22}=\mathbf{S}^{10110_{2}}=\left(\left(\left(\left(\mathbf{I}^{2} \cdot \mathbf{S}\right)^{2} \cdot \mathbf{S}\right)^{2} \cdot \mathbf{S}\right)^{2}\right)^{2}=\mathbf{S}^{22}
$$

For every 0 (the even digits) one has to square (operation $Q$ ), for every 1 (the odd digits) one has to square and then to multiply (operation $Q M$ ).
3rd step: Observe that

$$
\mathbf{S}^{m}=\left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}\right)^{m}=\left(\begin{array}{ccc}
a & c & b \\
b & a+b & c \\
c & b+c & a+b
\end{array}\right)
$$

Thus, one only has to remember the first column $(a, b, c)$ and to know how this column changes when multiplying $M$ and squaring $Q$.
Operation multiplying $M$ :

$$
\left(\begin{array}{ccc}
a & c & b \\
b & a+b & c \\
c & b+c & a+b
\end{array}\right)\left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}\right)=\left(\begin{array}{ccc}
a+b & a+b+c & b+c \\
b+c & a+2 b+c & a+b+c \\
a+b+c & a+2 b+2 c & a+2 b+c
\end{array}\right)
$$

Thus, $M:(a, b, c) \longrightarrow(a+b, b+c, a+b+c)$.
Operation squaring $Q$ :

$$
\left(\begin{array}{ccc}
a & c & b \\
b & a+b & c \\
c & b+c & a+b
\end{array}\right)^{2}=\left(\begin{array}{ccc}
a^{2}+2 b c & b^{2}+2 a c+2 b c & 2 a b+b^{2}+c^{2} \\
2 a b+b^{2}+c^{2} & a^{2}+2 a b+b^{2}+2 b c+c^{2} & b^{2}+2 a c+2 b c \\
b^{2}+2 a c+2 b c & 2 a b+2 b^{2}+2 a c+2 b c+c^{2} & a^{2}+2 a b+b^{2}+2 b c+c^{2}
\end{array}\right)
$$

Thus, $Q:(a, b, c) \longrightarrow\left(a^{2}+2 b c, b^{2}+c^{2}+2 a b, b^{2}+2 a c+2 b c\right)$.
Furthermore, some numbers $n$ can be excluded from the beginning, because we have

$$
n \equiv 0 \bmod 4 \Longrightarrow f_{n} \not \equiv 0 \bmod 4 \Longrightarrow f_{n} \not \equiv 0 \bmod n
$$

The same happens for $n=9,14, \ldots$. Moreover, we have

$$
n \equiv 0 \bmod 3, \quad n \not \equiv 0,1,3,9 \bmod 13 \Longrightarrow f_{n} \not \equiv 0 \bmod 3 \Longrightarrow f_{n} \not \equiv 0 \bmod n
$$

### 3.4 All steps combined

1. Decompose $n=3 m+i, i \in\{0,1,2\}$
2. Compute the dual representation $D$ of $m$.
3. In $D$, replace every zero with $Q$ and every 1 with $Q M$ and get the word $W$.
4. Set $(a, b, c):=(1,0,0)$ and, following the word $W$ from left to right, perform the following operations modulo $n$ :

$$
\begin{aligned}
M & :(a, b, c):=(a+b, b+c, a+b+c) \\
Q & :(a, b, c):=\left(a^{2}+2 b c, b^{2}+c^{2}+2 a b, b^{2}+2 a c+2 b c\right) .
\end{aligned}
$$

5. Finally, calculate

$$
P_{n} \bmod n=\left\{\begin{array}{ccc}
3 a+2 b & \text { for } & i=0 \\
3 b+2 c & \text { for } & i=1 \\
2 a+2 b+3 c & \text { for } & i=2 .
\end{array}\right.
$$

For Example we test whether 19 divides $P_{19}$ ?

1. $19=3 \cdot 6+1, m=6, i=1$
2. Dual representation of 6 : $D=110$.
3. $W=Q M Q M Q$
4. $(a, b, c)=(1,0,0)$
$\xrightarrow{Q}(a, b, c)=(1,0,0)$
$\xrightarrow{M}(a, b, c)=(1,0,1)$
$\xrightarrow{Q}(a, b, c)=(1,1,2)$
$\xrightarrow{M}(a, b, c)=(2,3,4)$
$\xrightarrow{Q}(a, b, c)=(9,18,11)$
5. $(9,18,11) \xrightarrow{i=1} 3 \cdot 18+2 \cdot 11=76 \equiv 0 \bmod 19$

Thus, we have $19 \mid P_{19}$ and therefore 19 can be a Perrin pseudoprime or a prime.

### 3.5 A mathematica-code for the algorithm

To deal with large integers we used mathematica. Of course, as an interpretive language it is slower than a compiled code. But that saved us the development of an own long integer operation package.
The following mathematica-code was used to check a given number $n$ on whether $n \mid P_{n}$. The code outputs True if $n$ is prime or a Perrin pseudoprime and False otherwise. We used mathematica11.3 at a Intel(R) Core(TM) i5-6500 CPU @ 3.20 GHz . To check the largest known 1436-digit PPP (see page 19) takes 0.18 seconds. Checking the largest Mersenne prime known in $19822^{86243}-1$ takes 4 minutes. Though, at that time the computers were slower. Today, testing $2^{1398269}-1$, the 35 -th Mersenne prime, found in 1996, takes a day.

```
\(\operatorname{PPP}\left[n_{-}\right]\):= (i = Mod[n, 3];
    \(\mathrm{k}=\) Quotient[n, 3];
    lk = IntegerDigits[k, 2];
    b1 = 1; b2 = 0; b3 = 0;
    Do[ If[ lk[[j]] == 0,
                    c 1 = b1 * b1 + 2 * b2 * b3;
                    \(\mathrm{c} 2=\mathrm{b} 2 * \mathrm{~b} 2+\mathrm{b} 3 * \mathrm{~b} 3+2 * \mathrm{~b} 1 * \mathrm{~b} 2\);
                    \(\mathrm{c} 3=\mathrm{b} 2 * \mathrm{~b} 2+2 * \mathrm{~b} 1 * \mathrm{~b} 3+2 * \mathrm{~b} 2 * \mathrm{~b} 3\),
            a1 = b1 * b1 + 2 * b2 * b3;
            \(\mathrm{a} 2 \mathrm{=}\) b2 * b2 + b3 * b3 + 2 * b1 * b2;
            a3 = b2 * b2 + 2 * b1 * b3 + 2 * b2 * b3;
            \(\mathrm{c} 1=\mathrm{a} 1+\mathrm{a} 2\);
            \(\mathrm{c} 2=\mathrm{a} 2+\mathrm{a} 3\);
            c3 = a1 + a2 + a3];
        b1 \(=\operatorname{Mod}[\mathrm{c} 1, \mathrm{n}] ; \mathrm{b} 2=\operatorname{Mod}[\mathrm{c} 2, \mathrm{n}] ; \mathrm{b} 3=\operatorname{Mod}[\mathrm{c} 3, \mathrm{n}]\),
    \{j, 1, Length[lk]\}];
    Which[i \(==0, \mathrm{~b}=3 * \mathrm{~b} 1+2 * \mathrm{~b} 2\),
                i \(==1, b=3 * b 2+2 * b 3\),
        i \(==2, \mathrm{~b}=2\) * b1 + 2 * b2 + 3 * b3];
    \(\operatorname{Mod}[\mathrm{b}, \mathrm{n}]==0)\)
```

The Table on [6] can be tested with

```
ppp = << PPP-new-math;
Do[ If[ Not[ PPP[ ppp[[k1]] ] ] || PrimeQ[ ppp[[k1]] ],
    Print[ ppp[[k1]]," is not a PPP!" ] ], {k1, 1, Length[ppp]}]
```

Do not forget the semicolon, the list ppp is very large. It runs less than two hours.

## 4 How to reduce the number of candidates

It takes many weeks to calculate the 1700 PPP up to $10^{14}$ even with high performance algorithms and computers. One has to check every number (except a few ones like mentioned at page 12 that can be sorted out in advance). Thus, there is no hope, that one could calculate all PPPs, say, up to $10^{20}$ in the next years. Moreover, since they are very rare, if you take a random $n$, you will "never" get a PPP.
So, to calculate more PPPs, one must try to limit the set of potential candidates.
Dana Jacobsen tested other pseudoprimes, hoping that, for example many of the Fermat ${ }_{2}$ - PP are also PPPs. And indeed, she found 101994 PPPs up to $18446724258335155361<10^{20}[5]$. It turns out that 510 of the 1700 PPPs less than $10^{14}$ are Fermat ${ }_{2}$-PP, too.

### 4.1 The structure of most of the PPPs

Let's have a look at the first PPPs and factorize them:


We see that many of them have the structure $P=[k(p-1)+1] \cdot p$, with some $p \in \mathbb{P}$ and $k=1,2,3, \ldots$ is a small number. Clearly, such numbers are never prime. Moreover, to calculate numbers $P$ in the region of $10^{16}$, it is sufficient to consider factors $\sim 10^{8}$. Thus, taking into account that we have 5761455 primes up to $10^{8}$, we get all pseudoprimes of this structure up to $\sim 10^{16}$ for a given $k$ in half an hour.
This was the starting point of a couple of ideas to reduce the amount of candidates to be tested. We list them here in their logical order.

1. Consider numbers $P=[k(p-1)+1] p, \quad p \in \mathbb{P}$

It was amazing that already $k=3$ and $k=2$ gives more than $50 \%$ of the 1700 known PPPs up to $10^{14}$.
2. Next, we considered numbers like $P=\left[k_{1}(p-1)+1\right]\left[k_{2}(p-1)+1\right], p \in \mathbb{P} ; \operatorname{gcd}\left(k_{1}, k_{2}\right)=1$.
3. We saw that some PPPs of this structure were overlooked, because $p$ must not be prime. Thus, we considered numbers like $P=\left[k_{1}(p-1)+1\right]\left[k_{2}(p-1)+1\right], p \notin \mathbb{P}, p$ odd.
4. Clearly, the next step were numbers of the form $P=\left[k_{1}(p-1)+1\right]\left[k_{2}(p-1)+1\right]\left[k_{3}(p-1)+1\right]$
5. and generally $P=\prod_{i=1}^{m}\left[k_{i}(p-1)+1\right]$. For $m>3$ we get only a few new PPP's.

With this method, we calculated all PPP's with 2 factors for given $k_{i}<100$, with 3 factors for given $k_{i}<15$, and with 4 factors for $k_{i}<10$ up to $10^{20}$. More than $95 \%$ of the 1700 known PPPs up to $10^{14}$ have such a structure. Extrapolating this result, we assume that we know now $95 \%$ of the PPPs up to $10^{20}$.
It was not possible to find such a PPP with 5 factors for months.
The largest PPPs have about 40 digits.
To calculate larger PPPs we used two different methods:

- Starting from a PPP with $m$ factors, guess a PPP with $m+1$ factors with the same $p$ and some $k_{m+1}$ resulting form the other $k_{1}, \ldots, k_{m}$. For example, take $k_{m+1}$ as a multiple of the least common multiple of the $k_{1}, \ldots, k_{m}$. In this way we could find some very large PPPs.
- Do we have to test all odd $p$ ? It turns out that only a few remainders of $p$ with respect to 23 occur. In this way we could find millions of new PPPs up to $10^{24}$.


### 4.2 The remainders of $p$

Since 23 is the discriminant of the corresponding polynomial of the Perrin sequence, we look at the remainders of $p$ with respect to 23 in more detail. It turns out that for a given pair $\left(k_{1}, k_{2}\right)$ we have only a few remainders instead of 23 possible ones.
For example:

- Take $\left(k_{1}, k_{2}\right)=(3,1)$, we have the remainders $=(1,2,6,9,18)$
- Take $\left(k_{1}, k_{2}\right)=(2,1)$, we have the remainders $=(1,2,13,16,18)$

The same holds for multiples of 23 . Taking, for example, the number $23 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13=690690$. We have

- For $\left(k_{1}, k_{2}\right)=(3,1)$ only 14853 remainders (a proportion of 0.0215046$)$,
- For $\left(k_{1}, k_{2}\right)=(2,1)$ only 7425 remainders (a proportion of 0.0107501 ).

During our calculation we considered the remainders with respect to $23 \cdot 2 \cdot 3=138$.
Here is a collection of the remainders with respect to 138 for all pairs $\left(k_{1}, k_{2}\right)$ with $k_{1}=5$ and $k_{1}=7$ :

| $k_{1}$ | $k_{2}$ | possible remainders with respect to 138 |
| :---: | :---: | :--- |
| 5 | 1 | $1,25,31,55,73,121$ |
| 5 | 2 | $1,7,15,21,25,43,61,67,93,99,117,135$ |
| 5 | 3 | $1,9,25,43,55,63,75,93,109,117,121,135$ |
| 5 | 4 | $1,7,31,43,67,73$ |
| 7 | 1 | $1,13,25,29,31,35,47,59,71,77,121,127$ |
| 7 | 2 | $1,13,25,67,97$ |
| 7 | 3 | $1,5,11,19,25,29,47,65,71,97,103,121$ |
| 7 | 4 | $1,11,13,19,31,47,59,65,67,77,103,113$ |
| 7 | 5 | $1,25,31,67,121$ |
| 7 | 6 | $1,5,13,29,47,59,67,79,97,113,121,125$ |

These remainders were found experimentally. For a given pair $\left(k_{1}, k_{2}\right)$ we calculated some PPPs for any odd $p$, enough to be sure about the possible remainders. Having obtained these, we test the following $p$ only with these remainders. That resulted in a strong speed-up.
Unfortunately, we have no idea how the remainders can be calculated in advance. We think this is an interesting problem for specialists, for example, in Carmichael numbers.

For PPPs with 3 factors we observed the following interesting experimental result:
Fix a pair $\left(k_{1}, k_{2}\right)$ with $\operatorname{gcd}\left(k_{1}, k_{2}\right)=1$ and let be $R\left(k_{1}, k_{2}\right)$ the set of remainders of $p$. Then, the set of remainders $R\left(k_{1}, k_{2}, k_{3}\right)$ for a PPP with 3 factors is

$$
R\left(k_{1}, k_{2}, k_{3}\right)=R\left(k_{1}, k_{2}\right) \cap R\left(k_{1}, k_{3}\right) \cap R\left(k_{2}, k_{3}\right)
$$

Thus, the number of possible remainders decreases with the number of factors.
A similar result holds for PPPs with more than 3 factors. Again, we do not know how to prove this.

The remainder 1 with respect to multilpes of 23 contains in any set of remainders for any $\left(k_{i}\right)$.

## 5 Numerical results

### 5.1 The state of the art

A current overview can be found in N.J.A. Sloanes famous OEIS (On-Line Encyclopedia of Integer Sequences) [8].
By now, all PPPs - 1700 - up to $10^{14}$ are known. Since we have 3204941750802 primes up to $10^{14}$, using the Perrin prime test, a PPP occurs with probability $W\left(10^{14}\right)=5.3043110^{-10}$. Thus, to check whether a given number less than $10^{14}$ is prime you can use the Perrin test and - if it is true - look at the table whether it is one of the 1700 PPPs. If not, it is prime.

The following table shows the probability $W(n)$ up to $n=10^{14}$. We used [10] for the numbers of primes.

| $n$ | PPPs | primes | probability $W(n)$ |
| :--- | ---: | ---: | :---: |
| $10^{8}$ | 7 | 5761455 | $1.21497 * 10^{-6}$ |
| $10^{9}$ | 17 | 50847534 | $3.34333 * 10^{-7}$ |
| $10^{10}$ | 42 | 455052511 | $9.22970 * 10^{-8}$ |
| $10^{11}$ | 116 | 4118054813 | $2.84115 * 10^{-8}$ |
| $10^{12}$ | 285 | 37607912018 | $7.57819 * 10^{-9}$ |
| $10^{13}$ | 649 | 346065536839 | $1.87537 * 10^{-9}$ |
| $10^{14}$ | 1700 | 3204941750802 | $5.30431 * 10^{-10}$ |

### 5.2 Our results

We calculated 9261931 (by December 2019) PPPs that an be found in the database [6]. (Note, that the database is updated from time to time.)
We tried to find all PPPs up to $10^{20}$ and all with 2 factors and $\left(k_{1}, k_{2}\right)=(3,1)$ and $\left(k_{1}, k_{2}\right)=$ $(2,1)$ up to $10^{22}$. Of course there is a by-catch of many PPPs up to $10^{30}$.
Moreover, we tried to find some very large ones using two methods:
At first, we constructed PPPs with $m+1$ factors starting from a known ones with $m$ factors. Second, knowing that 1 is always a remainder with respect to multilpes of 23 for all $p$, we tested numbers of the form $n=p \cdot(k(p-1)+1)$. with $k=2,3$ and $p=23 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots$ a multiple of 23 and the first primes This yields very large PPPs, for example the one on page 19.

### 5.2.1 Almost all PPPs

Having a look at the table above, we see that $\log W(n)$ behaves largely linearly. We extrapolate this and expect the following numbers of PPPs. The numbers up to $10^{20}$ are "almost all", the numbers up to $10^{22}$ are "more than a half" of all PPPs.


| $n$ | expected PPPs | founded PPPs |
| :--- | ---: | ---: |
| $10^{15}$ | 4360 | 4409 |
| $10^{16}$ | 11236 | 11972 |
| $10^{17}$ | 29076 | 33045 |
| $10^{18}$ | 75520 | 93001 |
| $10^{19}$ | 196790 | 262236 |
| $10^{20}$ | 514287 | 742759 |
| $10^{21}$ | 1347560 | 1502883 |
| $10^{22}$ | 3539332 | 3615622 |
| $10^{23}$ | 9316050 | 7870747 |
| $10^{24}$ | 24569601 | 7874995 |
| $10^{25}$ | 64915566 | 7879187 |
| $10^{26}$ | 171799266 | 7885930 |
| $10^{27}$ | 455365341 | 7898184 |
| $10^{28}$ | 1208691635 | 7920907 |
| $10^{29}$ | 3212505576 | 7964655 |
| $10^{30}$ | 8548808804 | 8049285 |

### 5.2.2 Huge PPPs

## Collected by factors: We found

- 1 PPP with 14 factors.
- 13 PPPs with 13 factors.
- 64 PPPs with 12 factors.
- 113 PPPs with 11 factors.
- 176 PPPs with 10 factors.
- 481 PPPs with 9 factors.
- 1054 PPPs with 8 factors.
- 2591 PPPs with 7 factors.
- 7159 PPPs with 6 factors.
- 29529 PPPs with 5 factors.

Collected by digits: We found

- $\sim 4000$ PPPs with more than 80 decimal digits
- $\sim 1600$ PPPs with more than 100 decimal digits
- 36 PPPs with more than 500 decimal digits
- 6 PPPs with more than 1000 decimal digits
- The largest PPP has 3101 digits. Here it is:

2182001064371918934845924375655593970781204553917566660863280384747887616030277480053172205785183353188400 4126146210865090197070653868880189559625867459754727073713090924616711853613422828119114381617102058517546 8653751496284195684000100419880283999039015488001095163810247785156033211221423472140681188918922518742770 0398996872031544022682029689624783660853880129295123479427747681652039459239579760489615206781614707161883 9138537548347177754556329233097993446947475927879595917904730731452471057039913228447069819231974147528469 7693616171472498459173243671532936165356214403017220481995761095314765972379574827945192124085559691984391 8008661242667729379149221402733564699474653803584334247108722459604844155931040562979301921938928545995807 4207926519074011909871332364749649617141024864366985374867133374038568149858039921667907016960062202008122 9182067899216118132468035588845067378082718617393902077009092862097562284582389695785019716348129717066692 0783325505675383114442119375756418942531432620905077133117297177064802424569877645651274316923030865339422 6661109617675061215430499075868542147459797368102792867066735398199032669816585264700339738266181367925685 9183901438799475057989326512787989244219170992158347364160368593405317157057039942593979747214483064168779 3723363454025576455261406877507795872082604992320378872519383088242811076665512015332176716276340248257164 6729443535184738262902790223792682930259972646770066028255813046639125749771256788743514165965139691554159 3353592560965482315120431456622925845399082336306306234166863238919515156950417488352070194395498058003429 2609689928226091668646468088635185719074533550653987615133601688385577315810376211381436151897390975873498 9194775781036920280653165835092015711042583063595692979056408307560965084104645943087850367750725513620664 9589379996405514942415050679736879467176251813294056719410189773891939434281262409431885675830573414891359 7068260880092249389030829673092944201188379579217564895495418187279934349004962876837044167260718567772046 7521150708667751876125544569499435754902575963129390715770989789849330459963345038762428879760367628428833 7083464467875818139474195085529183097604033933360012552535245232509900842279440109453302234497800743667133 2290093368659872164696682455863309852162786109791145473780233128398296687924256984146263917624053810047106 9132240022024999815261877155099328326233538506570393468310793807821070234336347574184496483617336881484518 9783926914876429525603769119738558257277589955344693025872664161546365759997766592490233729898293133230624 4301770299046097662381531807593304842496115443710755824125123112656492287865978030693101114925766670096297 4043457120990040352730767662860730019992114778921176312285224644592166173374663104973515972020108030670776 0538966132268173354370805800388713443173563909282726774947019900416544732774260586167631835100825092596248 4432038054992189389231847184387110810917603905274409490013362690801082371949435532760468825732391337145460 6507376646884319008228201004154992411941387896249068825523566890040592991334780411481021215235342677940980 162869702039217052132582551

### 5.2.3 Some more information

- Our method found 1647 out of the known 1700 up to $10^{14}$. Thus, 53 or $\sim 3 \%$ left. We call them "sporadic PPPs".
- Dana Jacobsen's list of 101994 PPPs contains 699 that we could not find with our method.
- We found 742759 PPPs up to $10^{20}$. If these compile $97 \%$ of all PPPs, then 22972 sporadic ones are left.
- Among the the first 10000 Carmichael numbers (taken from [9]) there are 16 PPPs:

$$
\begin{aligned}
C_{1353} & =7045248121=821 * 1231 * 6971= \\
& =(2(411-1)+1) *(3(411-1)+1) *(17(411-1)+1) \\
C_{1375} & =7279379941=211 * 3571 * 9661 \\
C_{2142} & =24306384961=19 * 53 * 79 * 89 * 3433 \\
C_{2652} & =43234580143=223 * 5107 * 37963 \\
C_{2837} & =52437986833=23 * 463 * 1453 * 3389 \\
C_{2988} & =60518537641=23 * 89 * 991 * 29833 \\
C_{3336} & =80829302401=89 * 199 * 463 * 9857 \\
C_{3855} & =118805562613=829 * 9109 * 15733 \\
C_{4125} & =144377609419=1319 * 9227 * 11863 \\
C_{4322} & =165321688501=101 * 271 * 691 * 8741 \\
C_{4342} & =167385219121=83 * 6971 * 289297 \\
C_{5046} & =254302215553=307 * 3673 * 225523 \\
C_{5731} & =364573433665=5 * 7 * 23 * 37 * 997 * 12277 \\
C_{6743} & =575687567521=11 * 19 * 79 * 137 * 307 * 829 \\
C_{6810} & =588909469501=1871 * 16831 * 18701= \\
& =1871 *(9(1871-1)+1) *(10(1871-1)+1) \\
C_{7057} & =652270080001=3361 * 9241 * 21001
\end{aligned}
$$

Some of them, namely, $C_{2142}, C_{2837}, C_{3336}, C_{4342}, C_{5731}, C_{6743}$ and $C_{7057}$ we could not find with our method.

Note, that $C_{7057}=(4 *(841-1)+1) *(11 *(841-1)+1) *(25 *(841-1)+1)$ with $841=19^{2}$. We could not find it, since we restrict ourself to $k_{i} \leq 15$ for numbers with 3 factors.

### 5.2.4 Some conjectures

During the calculations, we were led to the following conjectures. We invite everyone to think about the proofs.

- Almost all PPPs have the structure $P=\prod_{i=1}^{m}\left[k_{i}(p-1)+1\right]$
- There are infinitely many of such type.
- The $p$ has few remainders with respect to multiples of 23 . They can be calculated theoretically in advance.
- If $\prod_{i=1}^{m}\left[k_{i}(p-1)+1\right]$ is a PPP, then with "high" probability $\prod_{i=1}^{m+1}\left[k_{i}(p-1)+1\right]$ is a PPP with $k_{m+1}=c k_{m}$. In such a way you can construct large PPPs.
- The set of remainders (with respect to multiples of 23) of $p$ corresponding to given $k_{i}$ with 3 (or more) factors are the intersection of the sets of remainders corresponding to fewer $k_{i}$, requiring $\operatorname{gcd}\left(k_{i}, k_{j}\right)=1$.
- There are a particularly large number of PPPs if the $k_{i}$ are prime, pairwise.
- If for some $p$ the number with $\left\{k_{2} \cdot k_{3}, k_{2}, k_{3}\right\}$ is a PPP then so is the number with $\left\{k_{2}, k_{3}\right\}$.


## 6 Other promising polynomials for pseudoprimes

We tested polynomials of degree 3 and 4 with integer coefficients $a_{i}$ with $\left|a_{i}\right| \leq 20$. Every corresponding sequences we tested for pseudoprimes up to $10^{9}$. For polynomials of third order the Perrin sequence is indeed the rarest.
For polynomials of fourth order we find two polynomials without any pseudoprimes up to $10^{9}$ at all. Here they are:

$$
\begin{aligned}
& Q(x)=-x^{4}+x^{3}-17 x^{2}+0 x+5 \\
& R(x)=-x^{4}+11 x^{3}+x^{2}-12 x+14
\end{aligned}
$$

We have for $Q(x)$ the corresponding sequence

$$
\begin{aligned}
q_{n} & =q_{n-1}-17 q_{n-2}+5 q_{n-4} \\
q_{0} & =4 \\
q_{1} & =1 \\
q_{2} & =-33 \\
q_{3} & =-50
\end{aligned}
$$

and the testing rule $n \in \mathbb{P} \Longrightarrow n \mid\left(q_{n}-1\right)$. For $R(x)$ the sequence is

$$
\begin{aligned}
r_{n} & =11 r_{n-1}+r_{n-2}-12 r_{n-3}+14 r_{n-4} \\
r_{0} & =4 \\
r_{1} & =11 \\
r_{2} & =123 \\
r_{3} & =1328
\end{aligned}
$$

and the testing rule is $n \in \mathbb{P} \Longrightarrow n \mid\left(r_{n}-11^{n}\right)$.
To avoid the term $11^{n}$, it is better to consider

$$
G(x)=Q(x)(x-11)=-x^{5}+22 x^{4}-120 x^{3}-23 x^{2}+146 x-154
$$

instead of $R(x)$. This corresponds to the 5 -th oder sequence

$$
\begin{aligned}
g_{n} & =22 g_{n-1}-120 g_{n-2}-23 g_{n-3}+146 g_{n-4}-154 g_{n-5} \\
g_{0} & =3 \\
g_{1} & =0 \\
g_{2} & =2 \\
g_{3} & =-3 \\
g_{4} & =14
\end{aligned}
$$

with the testing rule $n \in \mathbb{P} \Longrightarrow n \mid g_{n}$.

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