Remarks on the Erdős Matching Conjecture for Vector Spaces

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Abstract

In 1965, Paul Erdős asked about the largest family Y of k-sets in $\{1, \ldots, n\}$ such that Y does not contain $s+1$ pairwise disjoint sets. This problem is commonly known as the Erdős Matching Conjecture. We investigate the q-analog of this question, that is we want to determine the size of a largest family Y of k-spaces in $\mathbb{F}_q^n$ such that Y does not contain $s+1$ pairwise disjoint k-spaces.

Our main result is, slightly simplified, that if $16s \leq \min\{q^{\frac{n-k-1}{4}}, q^{\frac{n}{2}-k+1}\}$, then Y is either small or a union of intersecting families. Thus showing the Erdős Matching Conjecture for this range. The proof uses a method due to Metsch. We also discuss constructions. In particular, we show that for larger s, there are large examples which are close in size to a union of intersecting families, but structurally different.

As an application, we discuss the close relationship between the Erdős Matching Conjecture for vector spaces and Cameron-Liebler line classes (and their generalization to k-spaces), a popular topic in finite geometry for the last 30 years. More specifically, we propose the Erdős Matching Conjecture (for vector spaces) as an interesting variation of the classical research on Cameron-Liebler line classes.

1 Introduction

In 1961, Erdős, Ko, and Rado famously showed that an *intersecting family* of k-sets in $\{1, \ldots, n\}$ has at most size $\binom{n-1}{k-1}$ and, if $n > 2k$, consists of all k-sets which contain a fixed element in the case of equality [10]. In other words, intersecting families are families of k-sets with no 2 of its elements pairwise disjoint and we know the largest such families. If we replace 2 by a parameter s, then we obtain the setting of the Erdős Matching Conjecture from 1965 [8]. Let us say that a family without $s+1$ pairwise disjoint elements is an s-EM-family. There are two natural choices for s-EM-families of k-sets in $\{1, \ldots, n\}$. The first one, let us call it Y_1, is the family of k-sets which intersect

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$\{1, \ldots, s\}$ non-trivially. The family Y_1 has size $\binom{n}{k} - \binom{n-s}{k}$. The second one, let us call it Y_2, is the family of k-sets which are contained in $\{1, \ldots, k(s+1)-1\}$. The family Y_2 has size $\binom{k(s+1)-1}{k}$. Erdős states in [8] that the following "is not impossible":

Conjecture 1 (The Erdős Matching Conjecture). *Let Y be a largest s-EM-family of k-sets of $\{1, \ldots, n\}$. Then $|Y| = \max\{|Y_1|, |Y_2|\}$.*

The conjecture was proven for $k = 2$ by Erdős and Gallai [9] and for $k = 3$ by Frankl [14]. In particular, Frankl showed the conjecture for $n \geq (2s+1)k - s$ [13] and for $n \leq (s+1)(k+\epsilon)$ where ϵ depends on k [15]. Furthermore, Frankl and Kupavskii showed the conjecture for $n \geq \frac{5}{3}sk - \frac{2}{3}s$ for sufficiently large s. A more complete overview on the history of the problem can be found in [16].

For our purposes, let us rephrase the Erdős Matching Conjecture in a way that makes it more generic and easily transferable between lattices.

Conjecture 2 (The Erdős Matching Conjecture (rephrased)). *Let Y be a largest s-EM-family of k-sets of $\{1, \ldots, n\}$. Then Y is the union of s intersecting families or its complement.*

Note that one can easily deduce Conjecture 1 from Conjecture 2 due to the fact that the structure of large intersecting families of k-sets is well-known. In this paper we consider s-EM-families of k-spaces in $\mathbb{F}_q^n$. The natural conjecture here is as follows.

Conjecture 3. *Let Y be a largest s-EM-family of k-space of $\mathbb{F}_q^n$. Then Y is the union of s intersecting families or its complement.*

One could also write down a more explicit description of the largest examples as in Conjecture 1, but this is far more tedious than in the set case due to the rigid nature of vector spaces.

We consider the setting in vector spaces as particularly interesting: In the set case, we have that if k divides n and Z is a family of k-sets which partitions $\{1, \ldots, n\}$, then Z intersects an s-EM-family Y in at most s elements. It is not hard to see that this implies

$$|Y| \leq s\binom{n-1}{k-1}.$$

One can show that equality in this bound only holds when Y is, in the language of [21], a equitable bipartition of the Johnson graph or, in the language of [12], a Boolean degree 1 function of the Johnson graph. These do not exist except for $s = 0, 1, \frac{n}{k} - 1, \frac{n}{k}$, so the bound above can be instantaneously improved by one.

Write $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for the Gaussian (or q-binomial) coefficient, that is $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-spaces in $\mathbb{F}_q^n$. In the vector space analog, if k divides n and Z is a family of k-spaces which partitions $\mathbb{F}_q^n \setminus \{0\}$, so a *spread* of $\mathbb{F}_q^n$, then the same behavior occurs. In this setting, Boolean degree 1 functions are known as Cameron-Liebler classes of k-spaces

[3, 12]. Here we have the analogous result, that is a s-EM-family Y of k-spaces intersects Z in at most s elements, from which it follows that

$$|Y| \leq s \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

It is easy to find trivial examples for Cameron-Liebler classes which meet this bound for small s, but the general picture is not clear. Throughout the paper, we use projective notation and call 1-spaces *points*, 2-spaces *lines*, 3-spaces *planes*, and $(n-1)$-spaces *hyperplanes*. The trivial examples, up to taking complements and besides the empty set, are all k-spaces through a fixed point, all k in a fixed hyperplane, and the disjoint union of the first two examples. Non-trivial Cameron-Liebler classes appear to exist for $(n,k) = (4,2)$ and any $q > 2$ [5, 6, 7, 11, 19, 31], but not for $n \geq 2k$ when $n > 4$. The latter is at least true for $q \in \{2,3,4,5\}$ [12]. The fact that non-trivial example exist for $(n,k) = (4,2)$ does not imply that the Erdős Matching Conjecture is false as these example might have $s+1$ pairwise disjoint elements which do not extend to a spread of $\mathbb{F}_q^n$. Indeed, all known non-trivial examples investigated by the author are not s-EM families. Nonetheless, it makes one doubt that Conjecture 3 is true.

It is known that there are no non-trivial small examples for Boolean degree 1 functions. Metsch established a proof technique in [28] which essentially shows that small Boolean degree 1 functions are s-EM-families. He used it to show the following.

Theorem 4 (Metsch [28, Theorem 1.4])**.** *All Cameron-Liebler classes Y of k-spaces in $\mathbb{F}_q^{2k}$ with $5 \cdot |Y| \leq q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ are trivial.*

Note that [28] states that q has to be sufficiently large, but this condition can be dropped [25]. Blokhuis, De Boeck and D'haeseleer generalized this to k-spaces in $\mathbb{F}_q^n$ [3, Theorem 4.9], but the proof of their result (and therefore the stated result) contains a minor mistake which we amend with Theorem 7.

Our main result is as follows. Throughout the document ℓ is the smallest integer such that $s \leq \frac{q^\ell - 1}{q-1}$.

Theorem 5. *Let Y be a largest s-EM family of k-spaces in $\mathbb{F}_q^n$. If $16s \leq \min\{ q^{\frac{n-k-\ell+2}{3}}, q^{\frac{n}{2}-k+1}\}$, then Y is the union of s intersecting families.*

Note that we did not optimize the constant 16. Indeed, 16 can be certainly replaced by a constant c_q which is arbitrarily close to 1 for q sufficiently large. Besides this, the argument is optimized to the best knowledge of the author. As $s \geq q^{\ell-1}$, Theorem 5 is satisfied if $16s \leq \min\{q^{\frac{n-k-1}{4}}, q^{\frac{n}{2}-k+1}\}$ as stated in the abstract. For $n \geq 3k-4$, this simplifies further to

Cameron-Liebler classes are completely classified for $q \in \{2,3,4,5\}$ [7, 12, 18, 20], while in general only some limited characterizations are known. For the special case of $(n,k) = (4,2)$ Gavrilyuk and Metsch [20], and Metsch [27] showed highly non-trivial existence conditions. The latter is as follows.

Theorem 6 (Metsch [27])**.** *Let Y be a Cameron-Liebler class of lines in $\mathbb{F}_q^3$ of size $s(q^2+q+1)$. If $s \leq Cq^{4/3}(q^2+q+1)$ for some universal constant C, then Y is the union of s intersecting families.*

From Theorem 5 we deduce the following.

Theorem 7. *Let Y be a Cameron-Liebler class of k-spaces in $\mathbb{F}_q^n$ of size $s\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. If $16s \leq \min\{q^{\frac{n-k-\ell+2}{3}}, q^{\frac{n-2k-r+1}{3}}\}$, where $n = mk - r$ with $0 \leq r < k$, then Y is the union of s intersecting families.*

Our original intend was to improve a result in [3] for certain choices of parameters, but as we discovered a mistake in the argument in [3], this is the only such bound at the time of writing.[1]

2 Preliminaries

2.1 Gaussian Coefficients

For any real number a and $q > 0$, we define $[a]_q := \lim_{r\to q} \frac{r^a-1}{r-1}$ and, for b a non-negative integer, we define the Gaussian coefficient by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b} \frac{[a-i]_q}{[b-i]_q}.$$

We write $[a]$ instead of $[a]_q$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ instead of $\begin{bmatrix} a \\ b \end{bmatrix}_q$ as q is usually fixed. Notice that $\begin{bmatrix} n \\ k \end{bmatrix}$ corresponds to the number of k-spaces in $\mathbb{F}_q^n$. We have by [24, Lemma 2.1] (alternatively, [26, Lemma 34]):

Lemma 8. *Let $a \geq b \geq 0$ and $q \geq 2$. Then*

$$q^{b(a-b)} \leq \begin{bmatrix} a \\ b \end{bmatrix} \leq (1+5q^{-1})q^{b(a-b)} \leq 4q^{b(a-b)}$$

and, if $q \geq 4$,

$$q^{b(a-b)} \leq \begin{bmatrix} a \\ b \end{bmatrix} \leq (1+2q^{-1})q^{b(a-b)} \leq 2q^{b(a-b)}.$$

Let $\rho = 1 + 5q^{-1}$ for $q \in \{2, 3\}$ and $\rho = 1 + 2q^{-1}$ otherwise. We will use that $\begin{bmatrix} a \\ b \end{bmatrix} \leq \rho q^{b(a-b)} \leq 4q^{b(a-b)}$ throughout the document. For $[a]$ we use the better bound of $[a] \leq \frac{q}{q-1}q^{q-1} \leq 2q^{a-1}$.

The Gaussian coefficients satisfies the following generalization of Pascal's identity:

$$\begin{bmatrix} a \\ b \end{bmatrix} = q^b \begin{bmatrix} a-1 \\ b \end{bmatrix} + \begin{bmatrix} a-1 \\ b-1 \end{bmatrix} = q^{a-b} \begin{bmatrix} a-1 \\ b-1 \end{bmatrix} + \begin{bmatrix} a-1 \\ b \end{bmatrix}. \tag{1}$$

This enables us to make the following useful observation.

[1]Our bound is $Cs \leq q^{\frac{n}{2}-k+1}$ for n large enough while the alleged bound in [3] is $Cs \leq q^{\frac{n}{2}-k+\frac{1}{2}}$ and only holds for $n \geq 3k$. We consider the behavior for n close to $2k$ as the most interesting.

Lemma 9. *Let $q \geq 2$, a, x be real numbers with $a \geq x > 1$, and b an integer with $a \geq b \geq 2$. Then*

$$\begin{aligned}\begin{bmatrix} a \\ b \end{bmatrix} - q^{bx} \begin{bmatrix} a-x \\ b \end{bmatrix} &\leq \rho(1 + \tfrac{1}{q-1}) q^{x+(b-1)(a-b)-1} \\ &\leq (1 + 7q^{-1}) q^{x+(b-1)(a-b)-1}.\end{aligned}$$

Proof. Equation (1) together with Lemma 8 implies that

$$\begin{bmatrix} a \\ b \end{bmatrix} = q^b \begin{bmatrix} a-1 \\ b \end{bmatrix} + \begin{bmatrix} a-1 \\ b-1 \end{bmatrix} \leq q^b \begin{bmatrix} a-1 \\ b \end{bmatrix} + \rho q^{(b-1)(a-b)}$$

for some constant ρ. If we repeat this x times, then we obtain (we bound the geometric series by $\frac{q}{q-1}$)

$$\begin{bmatrix} a \\ b \end{bmatrix} \leq q^{bx} \begin{bmatrix} a-x \\ b \end{bmatrix} + \rho(1 + \frac{q}{q-1} \cdot q^{-1}) q^{x+(b-1)(a-b)-1}.$$

The assertion follows. □

We use this bound mostly for $x = b$ and $x = b+1$, so let use restate the bound for these particular cases:

$$\begin{bmatrix} a \\ b \end{bmatrix} - q^{b^2} \begin{bmatrix} a-b \\ b \end{bmatrix} \leq (1 + 7q^{-1}) q^{(b-1)(a-b+1)}.$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} - q^{b^2+b} \begin{bmatrix} a-b-1 \\ b \end{bmatrix} \leq (1 + 7q^{-1}) q^{1+(b-1)(a-b+1)}.$$

Remark 10. The first coefficients of $\begin{bmatrix} a \\ b \end{bmatrix}$ seen as a polynomial in q are the possible ways of partitioning $b-1$, so sequence A000041 in OEIS. This can be seen in a similar way.

2.2 Geometry

We rely on the existing results on intersecting families and partial spreads of k-spaces in $\mathbb{F}_q^n$. If Y is the family of all k-spaces containing a fixed point, then we call Y a *dictator*. If Y is the family of all k-spaces contained in a fixed hyperplane, then we call Y a *dual dictator*.

Extending work by Hsieh [23] and Frankl and Wilson [17], Newman showed the following [29]:

Theorem 11. *If $n \geq 2k$, then the size of an intersecting family Y of k-spaces in $\mathbb{F}_q^n$ is at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Equality holds in one of the following two cases:*

(i) the family Y is a dictator,

(ii) we have $n = 2k$ and the family Y is a dual dictator.

We will use the following corollary later on.

Lemma 12. *Let Y be a dictator, or let Y be a dual dictator with $n = 2k$. A k-space not in Y meets at most $[k]\left[{n-2 \atop k-2}\right]$ elements of Y.*

The following was shown for large q by Blokhuis et al. [2] and for all q by the author [25].

Theorem 13. *Let $n \geq 2k$ and Y is an intersecting family of k-spaces in $\mathbb{F}_q^n$ with $|Y| > 3[k]\left[{n-2 \atop k-2}\right]$. Then Y is contained in a (uniquely determined) dictator or a dual dictator.*

A *partial spread* is a set of pairwise disjoint k-spaces. Beutelspacher showed the following [1].

Theorem 14. *Let $n = mk + r$ with $0 \leq r < k$. Then there exists a partial spread of k-spaces of $\mathbb{F}_q^n$ consisting of size*

$$\frac{q^{k+r}[n-k-r]}{[k]+1}.$$

Let

$$z(n,k,q) := \frac{q^{k+r}[n-k-r]}{[k]} + 1.$$

When n, k, q are clear from the context, we just write z. In particular, $s \leq z$. We denote a partial spread of size z as a *z-spread.*

We will also need the well-known fact that a k-space is disjoint to

$$q^{k\ell} \begin{bmatrix} n-k \\ \ell \end{bmatrix} \tag{2}$$

ℓ-spaces of $\mathbb{F}_q^n$ [22, Theorem 3.3]. It follows that if we fix two disjoint k-spaces, then at least

$$q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} - [k] \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

k-spaces are disjoint to both of them.

Let n_i be the number of z-spreads through i fixed, pairwise disjoint k-spaces. An easy double counting argument show (e.g. see [3, 30]) that

$$\frac{n_1}{n_2} = \frac{q^{k^2}\left[{n-k \atop k}\right]}{z-1} = \frac{q^{k^2}[k]}{q^{k+r}[n-k-r]} \begin{bmatrix} n-k \\ k \end{bmatrix},$$
$$\frac{n_2}{n_3} = \frac{q^{k^2}\left[{n-k \atop k}\right] - [k]\left[{n-1 \atop k-1}\right]}{z-2}.$$

Lemma 15. *Let $n \geq 2k \geq 4$. The number of $(k-1)$-spaces in $\mathbb{F}_q^{n-1}$ which are disjoint to two fixed k-spaces is at most*

$$q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - \frac{1}{4}q^{(k-2)(n-k+1)}.$$

Proof. Without loss of generality we assume that the two fixed k-spaces are disjoint. Let A be one of the fixed k-spaces. For a point $p \subseteq A$ the number of $(k-1)$-spaces B through p with $A \cap B = p$ is, by Equation (2), $q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix}$. The number of $(k-1)$-spaces which meet both fixed k-spaces in a point is at most $[k]^2\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$. Hence, the number of $(k-1)$-spaces disjoint to both fixed k-spaces is at most

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - 2[k]q^{(k-1)(k-2)}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} + [k]^2\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}.$$

By Lemma 9, with $q \geq 7$, the first term in the sum is at most

$$q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} + \frac{3}{2}q^{1+(k-2)(n-k+1)}.$$

Using our bounds on the Gaussian coefficients (see Lemma 8 and the following) with $q \geq 7$, the previous term simplifies to

$$\begin{aligned} q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} &+ \frac{3}{2}q^{1+(k-2)(n-k+1)} - 2q^{1+(k-2)(n-k+1)} + \frac{7}{4}q^{(k-3)(n-k+1)+k+1} \\ &\leq q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} - \frac{1}{4}q^{(k-2)(n-k+1)}. \end{aligned}$$

In the last step we use that $n \geq 2k$ and $q \geq 7$. $\square$

3 The General Case

Let Y be an (n, k, q, s)-EM-family. Choose ℓ such that $s \leq [\ell]$. We assume that Y has size at least

$$y := s\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - [\ell-1]\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}\right).$$

If we take s points in an ℓ-space and let Y be the family of k-spaces which contain at least one of these points, then it is easy to see that $|Y| \geq y$.

Assumption: From now on we assume that $16s \leq \min\{ q^{\frac{n-k-\ell+2}{3}}, q^{\frac{n}{2}-k+1}\}$ till the end of the section. Recall that the first interesting case is $s = 3$. Hence, we assume that that $n \geq 2k+5$ if $q = 2$, $n \geq 2k+3$ if $q \leq 3$, $n \geq 2k+2$ if $q \leq 5$, and $n \geq 2k+1$ if $q \leq 49$ as the theorem does not say anything non-trivial for the excluded cases.

Lemma 16. *Let Z be a z-spread. Then*

$$\mathbb{E}(|Y \cap Z|) > s - 3s\frac{[k-1][\ell-1]}{[n-1]}.$$

Proof. Note that

$$\begin{aligned}\frac{\begin{bmatrix}n\\k\end{bmatrix}}{|Z|} &= \frac{[n]}{[k]}\begin{bmatrix}n-1\\k-1\end{bmatrix} \cdot \frac{[k]}{q^{k+r}[n-k-r]}\\ &= \frac{[n]}{q^{k+r}[n-k-r]}\begin{bmatrix}n-1\\k-1\end{bmatrix} \leq (1+2q^{k+r-n})\begin{bmatrix}n-1\\k-1\end{bmatrix}.\end{aligned}$$

The average size of the intersection is

$$\begin{aligned}\frac{|Y|\cdot|Z|}{\begin{bmatrix}n\\k\end{bmatrix}} &\geq \frac{y}{(1+2q^{k+r-n})\begin{bmatrix}n-1\\k-1\end{bmatrix}}\\ &\geq s(1-2q^{k+r-n})\left(1-\frac{[k-1][\ell-1]}{[n-1]}\right)\\ &\geq s - 2sq^{k+r-n} - s\frac{[k-1][\ell-1]}{[n-1]}.\end{aligned}$$

From here $\ell \leq \frac{n}{2} - k + 2$ implies the claim. ☐

For a k-space S, we let w_S denote $\mathbb{E}(|Y \cap Z| : S \in Z)$ for all z-spreads Z with contain S.

Corollary 17. *There exists a z-spread Z such all elements $S \in Y \cap Z$ satisfy $w_S > s - 3s^2\frac{[k-1][\ell-1]}{[n-1]}$*

Proof. By averaging and Lemma 16, we find a z-spread Z with $\sum_{S\in Y\cap Z} w_S \geq s(s - 3s\frac{[k-1][\ell-1]}{[n-1]})$. We have $w_S \leq s$. The worst case is that $s-1$ elements $S \in Y \cap Z$ have $w_S = s$. Then the remaining element T satisfies

$$w_T \geq \sum_{S\in Y\cap Z} w_S - (s-1)s = s - 3s^2\frac{[k-1][\ell-1]}{[n-1]}.$$

This shows the claim. ☐

Let Y' the set of elements $S \in Y$ such that $\mathbb{E}(|Y \cap Z|) \geq s - 3s^2\frac{[k-1][\ell-1]}{[n-1]}$ for all partial spreads Z of size z with $S \in Z$.

Lemma 18. *(i) An element $S \in Y$ meets at least*

$$\begin{bmatrix}n-1\\k-1\end{bmatrix}\left(1 - 4q^{2\ell+k-n-3} - 2(s-1)q^{k+r-n}\right)$$

elements of Y non-trivially.

(ii) For $S, T \in Y'$, there are at most

$$2q^{(k-2)(n-k+1)+1} + 7sq^{(k-2)(n-k+1)} + 200s^2q^{(k-2)(n-k)+\ell-2}$$

elements of Y which meet S and T non-trivially.

Proof. By double counting (Z, R), where Z is a partial z-spread with $S \in Z$ and $R \in Y$ with R is disjoint to S, we see that R is disjoint to at most $(w_S - 1)\frac{n_1}{n_2}$ elements of Y. Hence, S meets $|Y| - (w_S - 1)\frac{n_1}{n_2}$ elements of Y non-trivially.

Similarly, double counting (Z, R), where Z is a partial spread of size z with $S, T \in Z$ and $R \in Y$ with R is disjoint to S and T, shows that S and T are disjoint to at most $(s-2)\frac{n_2}{n_3}$ elements of Y. Hence, S and T meet at most

$$A := |Y| - (w_S + w_T - 2)\frac{n_1}{n_2} + (s-2)\frac{n_2}{n_3}$$

elements of Y simultaneously non-trivially.

What remains are some tedious calculations. In the case of (i), where we ask for an upper bound, we use $w_S \leq s$. Then

$$\begin{aligned}
|Y| - (w_S - 1)\frac{n_1}{n_2} &\geq y - (s-1)\frac{q^{k^2}\begin{bmatrix}n-k\\k\end{bmatrix}}{z-1}\\
&= y - (s-1)q^{k^2}\frac{[n-k]}{[n-k-r]q^{k+r}}\begin{bmatrix}n-k-1\\k-1\end{bmatrix}\\
&\geq y - (s-1)q^{k^2-k}(1+2q^{k+r-n})\begin{bmatrix}n-k-1\\k-1\end{bmatrix}\\
&\geq \begin{bmatrix}n-1\\k-1\end{bmatrix}\left(1 - \frac{[\ell][\ell-1][k-1]}{[n-1]} - 2(s-1)q^{k+r-n}\right)\\
&\geq \begin{bmatrix}n-1\\k-1\end{bmatrix}\left(1 - 4q^{2\ell+k-n-3} - 2(s-1)q^{k+r-n}\right).
\end{aligned}$$

Set $\delta = 3s^2\frac{[k-1][\ell-1]}{[n-1]}$. For (ii), we use that $w_S, w_T > s - \delta$ For (ii) we have that

$$\begin{aligned}
A &= y - 2(s-1-\delta)\frac{q^{k^2}\begin{bmatrix}n-k\\k\end{bmatrix}}{z-1} + (s-2)\frac{\left(q^{k^2}\begin{bmatrix}n-k\\k\end{bmatrix} - [k]\begin{bmatrix}n-1\\k-1\end{bmatrix}\right)}{z-2}\\
&\geq y - s\frac{q^{k^2}\begin{bmatrix}n-k\\k\end{bmatrix}}{z-1} - (s-2)\frac{[k]\begin{bmatrix}n-1\\k-1\end{bmatrix}}{z-2} + 2\delta\frac{q^{k^2}\begin{bmatrix}n-k\\k\end{bmatrix}}{z-1}\\
&\geq y - sq^{k^2-k}\begin{bmatrix}n-k-1\\k-1\end{bmatrix} - (s-2)q^{(k-2)(n-k+1)+1} + 2\delta q^{k^2}\frac{\begin{bmatrix}n-k\\k\end{bmatrix}}{z-1}.
\end{aligned}$$

Now we apply Lemma 9 together with $y \leq s\begin{bmatrix}n-1\\k-1\end{bmatrix}$ to obtain that

$$A \leq 2q^{(k-2)(n-k+1)+1} + 7sq^{(k-2)(n-k+1)} + 100s^2q^{(k-2)(n-k)+\ell-2}.$$

This shows the assertion. □

Proof of Theorem 5. First we show that Y contains s intersecting families $\mathcal{E}_1, \dots, \mathcal{E}_s$ such that $Y \setminus \bigcup_{i=1}^s \mathcal{E}_i$ is small. From this we then conclude that $Y \setminus \bigcup_{i=1}^s \mathcal{E}_i$ is actually empty.

By Corollary 17, there exists a z-spread Z such that $|Y' \cap Z| = s$. Write $\{S_1, \dots, S_s\} = Y' \cap Z$. Let $\mathcal{E}_i$ denote the set of elements of Y which meet S_i trivially and are disjoint to any S_j with $i \neq j$. By Lemma 18,

$$\begin{aligned} |\mathcal{E}_i| \geq & \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(1 - 4q^{2\ell+k-n-3} - 2(s-1)q^{k+r-n}\right) \\ & - (s-1)\left(2q^{(k-2)(n-k+1)+1} + 7sq^{(k-2)(n-k+1)} + 100s^2 q^{(k-2)(n-k)+\ell-2}\right). \end{aligned}$$

In the following, we will bound the individual terms of the sum.

Recall that $[\ell - 1] \leq s$. If $q \leq 3$, then $\ell \leq \frac{n}{2} - k$. Hence, as $n \geq 2k+3 \geq 7$,

$$4q^{2\ell+k-n-3} \leq 4q^{-k-3} \leq \frac{1}{8}.$$

If $4 \leq q < 16$, then $\ell \leq \frac{n}{2} - k + 1$. Hence,

$$4q^{2\ell+k-n-3} \leq 4q^{-k-1} \leq \frac{1}{8}.$$

If $q \geq 16$, then $\ell \leq \frac{n}{2} - k + 2$. Hence,

$$4q^{2\ell+k-n-3} \leq 4q^{-k+1} \leq \frac{1}{8}.$$

As $16s \leq q^{\frac{n}{2}-k+1}$, we have

$$2(s-1)q^{k+r-n} \leq \frac{1}{8}q^{-\frac{n}{2}+r} \leq \frac{1}{8}q^{-1} \leq \frac{1}{16}.$$

We conclude that

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(1 - 4q^{2\ell+k-n-3} - 2(s-1)q^{k+r-n}\right) \geq \frac{13}{16}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

As $16s \leq q^{\frac{n}{2}-k+1}$, we have that

$$\begin{aligned} 2(s-1)q^{(k-2)(n-k+1)+1} &\leq \frac{1}{8}q^{(k-2)(n-k+1)+\frac{n}{2}-k+2} \\ &\leq \frac{1}{8}q^{(k-1)(n-k)-\frac{n}{2}+k} \\ &\leq \frac{1}{8}(1+5q^{-1})q^{-\frac{n}{2}+k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \leq \frac{3}{16}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \end{aligned}$$

In the last step we use that $\frac{n}{2} - k \geq 3$ for $q \leq 3$, $\frac{n}{2} - k \geq 2$ for $q \leq 5$, $\frac{n}{2} - k \geq 1$ for $q \leq 49$.

As $16s \leq q^{\frac{n}{2}-k+1}$, we have

$$\begin{aligned} 7s(s-1)q^{(k-2)(n-k+1)} &\leq \frac{3}{4} \cdot \frac{1}{16} q^{(k-2)(n-k+1)+n-2k+2} \\ &= \frac{3}{64} q^{(k-1)(n-k)} \leq \frac{3}{16} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \end{aligned}$$

As $16s \leq q^{\frac{n-k-\ell+2}{3}}$, we have

$$100(s-1)s^2 q^{(k-2)(n-k)+\ell-2} \leq \frac{25}{1024} q^{(k-1)(n-k)} \leq \frac{1}{8} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Hence,

$$|\mathcal{E}_i| \geq \frac{5}{16} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} > 3[k] \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$$

for $n \geq 2k+5$, $q \geq 3$ and $n \geq 2k+3$, $q \geq 4$ and $n \geq 2k+1$, and $q \geq 49$ and $n \geq 2k$. Hence, by Theorem 13, $\mathcal{E}_i$ lies in a unique dictator or dual dictator $\mathcal{E}'_i$.

We finish the proof by contradiction. Suppose that there exists a $T \in Y \setminus \bigcup_{i=1}^{s} \mathcal{E}'_i$. By Theorem 13, we do know that that at most $[k]\left[\begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix}\right]$ elements of $\mathcal{E}_i$ meet T. First we consider the case that $n > 2k$. Then, by Theorem 13, $|\mathcal{E}_i \cap \mathcal{E}_j| \leq \left[\begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix}\right]$ for $i \neq j$. Hence, as $16s \leq q^{\frac{n}{2}-k+1}$, we have that

$$|\mathcal{E}_i| - s \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} = \frac{5}{16} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - s \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} > 0.$$

Hence, there exists an element Z_i in each $\mathcal{E}_i \setminus \bigcup_{j \neq i} \mathcal{E}_j$ which is disjoint to T. Thus $\{Z_1, \ldots, Z_s, T\}$ is a subset of $s+1$ pairwise disjoint elements in Y, a contradiction.

For $n = 2k$, we can only guarantee that $|\mathcal{E}_i \cap \mathcal{E}_j| \leq \left[\begin{smallmatrix} n-2 \\ k-1 \end{smallmatrix}\right]$ for $i \neq j$. As $16s \leq q$ and $k \geq 2$, our estimate is

$$|\mathcal{E}_i| - s \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} = \frac{5}{16} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - s \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} > 0.$$

As before, this is a contradiction. □

4 Cameron-Liebler Classes

Cameron-Liebler classes of k-spaces on $\mathbb{F}_q^n$, which the author often refers to as Boolean degree 1 functions of k-spaces on $\mathbb{F}_q^n$ [12], is a well-investigated object [3, 12, 30]. In particular for the case $n = 4$ and $k = 2$ where it is known as Cameron-Liebler line class. When k divides n (so a z-spread is simply a spread), one particular property of Boolean degree 1 functions is that their size is $s\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ for some integer s and that every spread intersects them in exactly s elements [3]. In the following, define s by $|Y| = s\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$, even if k does not divide n. Theorem 4.9 in [3] claims a result similar to Theorem 7. A minor, but sadly consequential sign-error in Lemma 4.6 of [3] makes Theorem 4.9 false in the stated form. Below we provide a fix for Lemma 4.6 of [3] in form of Lemma 19. We use this to show Theorem 7.

Lemma 19. *Let $n \geq 2k+1$. Let Y be a Cameron-Liebler class of k-spaces on $\mathbb{F}_q^n$ of size $s\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. If $s^3 \leq q^{n-2k-r+1}$, where $n = mk - r$ with $0 \leq r < k$, then Y contains at most s pairwise disjoint k-spaces.*

Proof. As shown in [3, Lemma 4.6], this is equivalent to

$$\frac{(1-\lfloor s\rfloor)s\lfloor s\rfloor}{2}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (s-1)(\lfloor s\rfloor^2-1)q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} > \frac{(s-2)(\lfloor s\rfloor+1)\lfloor s\rfloor}{2}W_\Sigma,$$

where W_Σ denotes the number of k-spaces through a point disjoint to two fixed, disjoint k-spaces. Note that [3, Lemma 4.6] requires that $n \geq 2k+1$.

The coefficient of the first term is negative, so (this is the mistake in [3, Lemma 4.6]), we can obtain a sufficient condition by substituting $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ by the upper bound from Lemma 9. We will bound W_Σ by Lemma 15.

Hence, it suffices that

$$\begin{aligned}\frac{(1-\lfloor s\rfloor)s\lfloor s\rfloor}{2}&\left(q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + \frac{3}{2}q^{1+(k-2)(n-k+1)}\right)\\ &+(s-1)(\lfloor s\rfloor^2-1)q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}\\ > \frac{(s-2)(\lfloor s\rfloor+1)\lfloor s\rfloor}{2}&\left(q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - \frac{1}{4}q^{1+(k-2)(n-k+1)}\right).\end{aligned}$$

Rearranging yields

$$8\left(\lfloor s\rfloor - s + 1\right)q^{k^2-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} > \lfloor s\rfloor(7s\lfloor s\rfloor - 2\lfloor s\rfloor - 5s - 2)q^{1+(k-2)(n-k+1)}.$$

Hence, it suffices to guarantee

$$8\left(\lfloor s\rfloor - s + 1\right)q^{n-2k+1} > 7s^3.$$

It is shown in [3, Theorem 2.9.4] that $s[k]$ divisible by $[n]$. Hence, $\lfloor s\rfloor - s + 1$ is at least $(q-1)q^{-r-1}$. The assertion follows using $q \geq 7$. □

Hence, using Theorem 5, we obtain Theorem 7. While $n = 2k$ is technically not included in Lemma 19, this case is implied by Theorem 4. We do not need the condition $16s \leq q^{\frac{n}{2}-k+1}$ in Theorem 7 as it always implied by one of the other two bounds on s.

5 Almost Counterexamples and Future Work

One objective of this project was to find counterexamples to the natural Conjecture 3. Obviously, we did not achieve this goal and it is left to future work. For $(n,k) = (4,2)$, we have $(q^2+1)(q^2+q+1)$ lines. The trivial upper bound is $s(q^2+q+1)$. By combining intersecting families, it is easy to obtain examples of size $s(q^2+q)+2$ for $s \leq 2q$. This number is still very close to the trivial bound, so it seems unreasonable to find counterexamples in this range. If we limit ourselves to $s \leq \frac{q^2+1}{2}$, so we take at most half of all lines, then maybe the first plausible parameter to look at is $q=5$ with $s=11$.

Here we will provide one construction which show that there is not much stability possible in Theorem 5. The examples are limited to $(n,k)=(4,2)$ for the sake of clarity. We take an elliptic quadric $\mathcal{Q}$ in $\mathbb{F}_q^4$. This consists of q^2+1 points, no three of which are collinear. A line which contains two points of $\mathcal{Q}$ is called a secant. Let Y be the family of all secants. Clearly, $|Y| = \binom{q^2+1}{2} = \frac{q^2}{2}(q^2+1)$ and, if q even, then Y contains at most $\frac{q^2}{2}$ pairwise disjoint secants. Hence, $s = \frac{q^2}{2}$. For sufficiently large q, it is not too hard to find a union Y' of $\frac{q^2}{2}$ intersecting families with[2] $|Y'| = \frac{q^2}{2} \cdot q^2 + q^2 + q + 2$. Here $|Y'| - |Y| = \frac{q^2}{2} + q + 2$.

There are several other similar constructions using quadric curves and related objects such as hyperovals, but we could never extend them in a way that it disproves Conjecture 3. We could also not adapt any of the many constructions for non-trivial Cameron-Liebler line classes for $(n,k)=(4,2)$ to obtain such a counterexample. Our search here was surely very incomplete as for instance [19] and [31] show that there are many such examples.

Furthermore, there are other classical geometrical structures for which the Erdős Matching Conjecture might be interesting. For instance, one can easily deduce the following using the same methods as in Theorem 5 for some universal constant C.

Theorem 20. *Let Y be an s-EM family of k-spaces in $AG(n,q)$. If $Cs \leq \min\{ q^{\frac{n-k-\ell+2}{3}}, q^{\frac{n}{2}-k+1}\}$, then Y is the union of s intersecting families.*

Here improvements on this bound might be easier compared to the investigated case as spreads always exist.

Similarly, $k \times (n-k)$-bilinear forms over $\mathbb{F}_q$ can be seen as the set of k-spaces which are disjoint to a fixed $(n-k)$-space [4, §9.5]. Again, a analogous result is easy to show.

Theorem 21. *Let Y be an s-EM family of $k \times (n-k)$-bilinear forms over $\mathbb{F}_q$. If $Cs \leq \min\{ q^{\frac{n-k-\ell+2}{3}}, q^{\frac{n}{2}-k+1}\}$, then Y is the union of s intersecting families.*

The trivial bound here is $s\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}\right)$ (instead of $s\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ for vector spaces) which can be easily obtained for all $s \leq [k]$. It might be easier to find counterexamples to the natural variation of Conjecture 3 in affine spaces or bilinear forms.

Note that the statement of Theorem 7 is empty for $2k < n < \frac{5}{2}k$. We believe that this can be improved by using better estimated in Lemma 19.

[2]Fix a line ℓ and a plane π with ℓ. Let $\mathcal{P}$ a set of $\frac{q^2}{2} - q$ points in $\pi \setminus \ell$. Let Y' be the union of the set of lines in planes through ℓ and the set of all lines which contain a point of $\mathcal{P}$. Then $|Y'| = q(q^2+q)+1+(\frac{q^2}{2}-q)q^2+q+1 = \frac{q^4}{2}+q^2+q+2$.

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