A bracket polynomial for 2-tangle shadows

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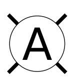
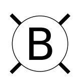
Abstract

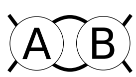
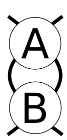
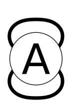
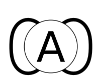
We compute the Kauffman bracket polynomial of the numerator and denominator closures of $A+A+\cdots+A$ (A is repeated n times), where A is a 2-tangle shadow that has at most 4 crossings.

Keywords: tangle shadow, bracket polynomial.

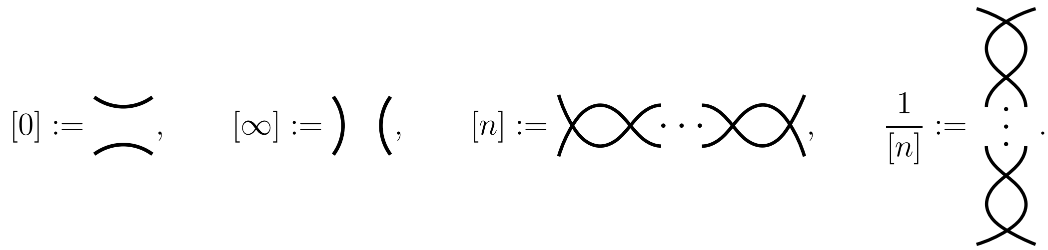
1 Introduction

A 2-tangle shadow is a 2-tangle diagram without under/over crossing information [Kre99, MS19]. The following definition holds for such class of tangles.

Definition 1. Let $A :=$  and $B :=$  be two 2-tangles. Then

- $A+B :=$  denotes the *horizontal sum*;
- $A*B :=$  denotes the *vertical sum*;
- $\frac{1}{A} :=$  denote the *inverse of* A which is obtained by turning the tangle counter-clockwise by 90 degree in the plane;
- $N(A) :=$  and $D(A) :=$  denote the *numerator* and the *denominator closures*, respectively, of A.

The simplest 2-tangles are



Set $A_n := A + A + \cdots + A$, where A is repeated n times (with $A_0 := [0]$). In the present paper, we compute the Kauffman bracket polynomial of the closures $N(A_n)$ and $D(A_n)$. We restrict our computation to 2-tangles of up to 4 crossings. In fact, since $N(A_n)$ and $D(A_n)$ are knot shadows, we extend previous results in which we investigated the generating polynomial of knots of up to 3 crossings [Ram18a].

The rest of the paper is organized as follows. We give in section 2 some properties of the bracket polynomial of 2-tangle shadows. By those properties, we obtain a set of integer polynomials which is presented in section 3. We give as well the tables giving the coefficients in the expansion of each polynomial.

Throughout the paper by "tangle" and "knot" we understand "2-tangle shadow diagram" and "knot shadow diagram", respectively. Also, we study tangles and knots up to planar isotopy.

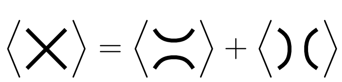
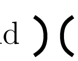
2 Definition and construction

In the present framework, the Kauffman bracket polynomial of a shadow diagram is an integer polynomial in x such that the coefficient of x^k matches the number of state diagrams having exactly k circles [Ram18a, Ram18b]. This definition extends to tangles as follows.

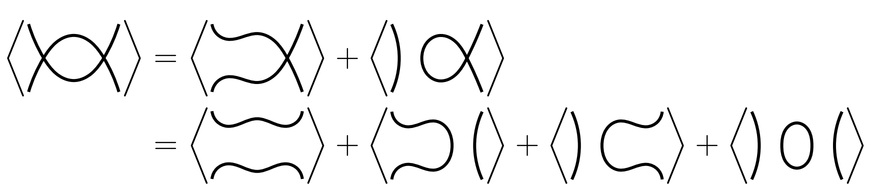
Definition 2. Let A be a tangle. The bracket polynomial of A, denoted $\langle A \rangle$, is defined by

$$\langle A \rangle = a(A) \langle [0] \rangle + b(A) \langle [\infty] \rangle \,, \tag{1}$$

where $a(A)$ and $b(A)$ are integer polynomials. The linear combination of $\langle [0] \rangle$ and $\langle [\infty] \rangle$ in (1) is obtained from the following usual rules for shadow diagrams:

- $\langle \bigcirc \rangle = x$;
- $\langle \bigcirc \sqcup A \rangle = x \langle A \rangle$;
- , where  and  denote the states of the crossing .

For example, $\langle [2] \rangle = \langle [0] \rangle + (x+2) \langle [\infty] \rangle$. Indeed, we have



Lemma 3. *The bracket of the closures $N(A)$ and $D(A)$ are given by*

$$\langle N(A)\rangle = x^2 a(A) + x b(A)$$

and

$$\langle D(A)\rangle = x a(A) + x^2 b(A),$$

respectively.

Proof. The crossings are smoothed without affecting the closures [KL07]. Hence

$$\begin{aligned}\langle N(A)\rangle &= a(A)\,\langle N([0])\rangle + b(A)\,\langle N([\infty])\rangle \\ &= a(A)x^2 + b(A)x\end{aligned}$$

and

$$\begin{aligned}\langle D(A)\rangle &= a(A)\,\langle D([0])\rangle + b(A)\,\langle D([\infty])\rangle \\ &= a(A)x + b(A)x^2.\end{aligned}$$

□

Remark 4. Let $R(A)$ be the operation defined by $N\left(A + [1]\right)$. Then

$$\langle R(A)\rangle = \langle N(A)\rangle + \langle D(A)\rangle = \left(x^2 + x\right)\left(a(A) + b(A)\right).$$

We understand $R(A)$ as an interpretation of the DNA recombination [ES90, KL07]. For simplicity, we call it the R-closure.

Lemma 5. *Given two tangles A and B,*

$$\langle A + B\rangle = a(A)a(B)\,\langle [0]\rangle + \big(a(A)b(B) + b(A)a(B) + xb(A)b(B)\big)\,\langle [\infty]\rangle\,. \tag{2}$$

Proof. First note that for any tangle A,

- $A + [0] = [0] + A = A$,
- $\langle A + [\infty]\rangle = \langle [\infty] + A\rangle = (a(A) + xb(A))\,\langle [\infty]\rangle$.

Then we establish (2) by computing the states of A leaving B intact, and then those of B:

$$\begin{aligned}\langle A + B\rangle &= a(A)\,\langle [0] + B\rangle + b(A)\,\langle [\infty] + B\rangle \\ &= a(A)\,(a(B)\,\langle [0]\rangle + b(B)\,\langle [\infty]\rangle) + b(A)\,(a(B) + xb(B))\,\langle [\infty]\rangle \\ &= a(A)a(B)\,\langle [0]\rangle + \big(a(A)b(B) + b(A)a(B) + xb(A)b(B)\big)\,\langle [\infty]\rangle\,.\end{aligned}$$

□

We now use similar notations to those of Kauffman [Kau83, p. 88]: we set $A^N := \langle N(A)\rangle$, $A^D := \langle D(A)\rangle$ and $A^R := \langle R(A)\rangle$. The following identities hold.

Lemma 6. *If A and B are tangles, then*

$$\begin{aligned} x(A+B)^D &= A^D B^D; \\ x\left(x^2-1\right)(A+B)^N &= \left(A^N B^N + A^D B^D\right)x - \left(A^D B^N + A^N B^D\right); \\ x\left(x^2-1\right)(A+B)^R &= A^D B^D x^2 + \left(A^N B^N + A^D B^D\right)x - \left(A^D B^N + A^N B^D + A^D B^D\right). \end{aligned}$$

Let us now refer to the pair $\begin{pmatrix} a(A) \\ b(A) \end{pmatrix}$, denoted $\operatorname{bp}(A)$, as *bracket pair* of A [EF17], and let $M(A)$ be the matrix defined as $\begin{pmatrix} a(A) & 0 \\ b(A) & a(A)+xb(A) \end{pmatrix}$.

For example $\operatorname{bp}([0]) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $M([0]) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M([\infty]) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $M([\infty]) = \begin{pmatrix} 0 & 0 \\ 1 & x \end{pmatrix}$.

Lemma 7. *With previous notations and usual matrix algebra we have*

- $\operatorname{bp}(A) = M(A)\operatorname{bp}([0])$;
- $\operatorname{bp}(A+B) = M(A)\operatorname{bp}(B) = M(A)M(B)\operatorname{bp}([0])$.

We use Lemma 7 to show by induction that

$$\operatorname{bp}(A_n) = \big(M(A)\big)^n \operatorname{bp}([0]),$$

where

$$\big(M(A)\big)^n = M(A_n) = \begin{pmatrix} \big(a(A)\big)^n & 0 \\ \dfrac{\big(a(A)+xb(A)\big)^n - \big(a(A)\big)^n}{x} & \big(a(A)+xb(A)\big)^n \end{pmatrix}.$$

Theorem 8. *If $\langle A_n \rangle = a(A_n)\langle [0] \rangle + b(A_n)\langle [\infty] \rangle$, then*

$$a(A_n) = \big(a(A)\big)^n$$

and

$$b(A_n) = \frac{\big(a(A)+xb(A)\big)^n - \big(a(A)\big)^n}{x}.$$

Formulas for the closures are then immediate:

Corollary 9. *The closures of A_n verify*

$$\begin{aligned} (A_n)^D &= x\big(a(A)+xb(A)\big)^n; \\ (A_n)^N &= \big(a(A)+xb(A)\big)^n + \left(x^2-1\right)\big(a(A)\big)^n; \\ (A_n)^R &= (x+1)\big(a(A)+xb(A)\big)^n + \left(x^2-1\right)\big(a(A)\big)^n. \end{aligned}$$

For example, knowing that $\langle [1] \rangle = \langle [0] \rangle + \langle [\infty] \rangle$ and $[n] = [1] + [1] + \cdots + [1]$, we have

$$\begin{aligned} [n]^D &= x(x+1)^n; \\ [n]^N &= (x+1)^n + x^2 - 1; \\ [n]^R &= (x+1)^{n+1} + x^2 - 1. \end{aligned}$$

Definition 10 ([KL07])**.** We define the *polynomial fraction*, $F(A)$, of the tangle A as

$$F(A) := \frac{b(A)}{a(A)}. \tag{3}$$

With the previous notations, we have

- $F(A+B) = F(A) + F(B) + xF(A)F(B)$;
- $F\left(\frac{1}{A}\right) = \frac{1}{F(A)}$;
- $F(A_n) = \frac{\big(1 + xF(A)\big)^n - 1}{x}$.

Remark 11. Let A and B be two tangles. We have $\langle A \rangle = \langle B \rangle$

(i) if A is planar isotopic to B, or

(ii) if there exist a tangle T and two knots K and K', with $\langle K \rangle = \langle K' \rangle$, such that $\langle A \rangle = \langle T \# K \rangle$ and $\langle B \rangle = \langle T \# K' \rangle$.

By "tangle # knot" we mean the knot is connected at some segment of the tangle, # denoting the usual connected sum. Recall that for two knots K and K' [Ram18a],

$$\langle K \# K' \rangle = x^{-1} \langle K \rangle \langle K' \rangle . \tag{4}$$

Here, (i) and (ii) suggest that there exists a tangle A for which the identity $\langle A \rangle = \langle T \# K \rangle$ is satisfied only if A is planar isotopic to T, and K is the unknot $\bigcirc$. For such case, we say that A is *prime*, and *locally knotted* otherwise. If $\langle A \rangle = x^{-1} \langle T^* \rangle \langle K \rangle$ and T^* is prime, then we say that T^* is the *skeleton* of A.

The polynomial fraction allows us to "extract" the skeleton of a tangle. Indeed, assume that $\langle A \rangle = x^{-1} \langle T^* \rangle \langle K \rangle$. By (4) we obtain a common factor which affects both brackets $\langle [0] \rangle$ and $\langle [\infty] \rangle$, i.e.,

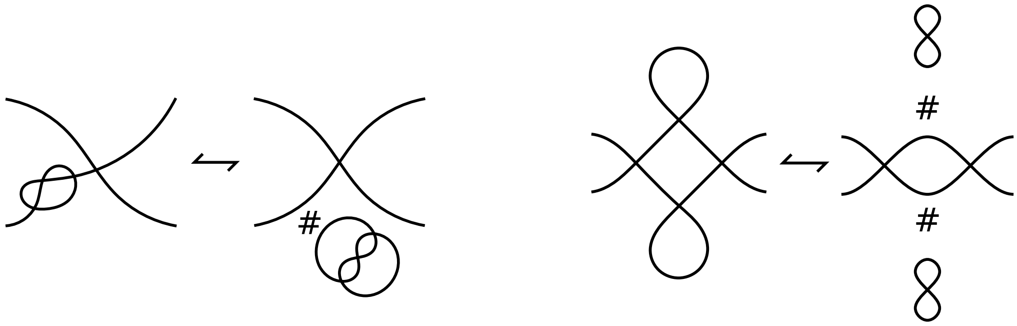
$$\langle A \rangle = \big(x^{-1} \langle K \rangle a(T^*)\big) \langle [0] \rangle + \big(x^{-1} \langle K \rangle b(T^*)\big) \langle [\infty] \rangle ,$$

and consequently

$$F(A) = F(T^*).$$

Last equality implies that the skeleton T verifies $\gcd(a(T^*), b(T^*)) = 1$. For example, consider tangle A as in Figure 1a. We have

$$F(A) = \frac{x^2 + 4x + 3}{x^2 + 4x + 3} = 1 = F([1]).$$



a: A **b:** B

Figure 1: Locally knotted tangles.

Remark 12. Assume that we can disconnect more than one knot in the tangle A, i.e.,

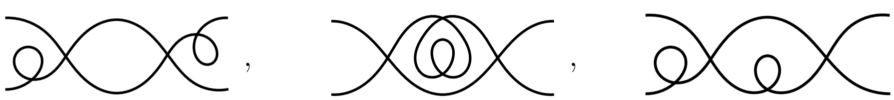
$$A = \left(\cdots\left(\left(T\#K^{(1)}\right)\#K^{(2)}\right)\#\cdots\right)\#K^{(m)},$$

Then by (4) we have

$$\langle A\rangle = x^{-m+1}\langle T\rangle\left\langle K^{(1)}\right\rangle\cdots\left\langle K^{(m)}\right\rangle = x^{-1}\langle T\rangle\langle K\rangle$$

which is also the bracket polynomial of a certain $T\#K$, where $K = K^{(1)}\#\cdots\#K^{(m)}$.

Example 13. Tangle B in Figure 1b has the same bracket polynomial as the tangle belows



and verifies

$$F(B) = \frac{x^3+4x^2+5x+2}{x^2+2x+1} = x+2 = F([2]).$$

3 Results

3.1 Classifying of knots and tangles

Let $c(X)$ denote the number of crossings in the diagram X. Our goal is to find all possible values of $\langle A_n\rangle$ for any tangle A with $c(A)\leq 4$. Since $\langle A\rangle$ can always be written as $x^{-1}\langle T\rangle\langle K\rangle$, with T prime, it suffices to find all values of $\langle T\rangle$ and $\langle K\rangle$ when $c(A)\in\{1,2,3,4\}$ and $c(A) = c(T)+c(K)$. As seen in Remark 11 and 12, two different diagrams can have the same bracket polynomial. Let us then introduce the following equivalence relation which allows us to to gather together tangles and knots that share the same bracket expression.

Definition 14. We say that X is equivalent to X' if $\langle X\rangle = \langle X'\rangle$. We let $\pi(X)$ denote the equivalence class of X.

For each numbered equivalence classes listed below, **Ai** is a representative of a class, **Bi** is the entry for the bracket of **Ai**, and entries **Di**, **Ni** and **Ri** are for the brackets of the denominator, numerator and R closures of $(\mathbf{Ai})_n$, respectively.

We begin with the classes of prime tangles (cf. Figure 2).

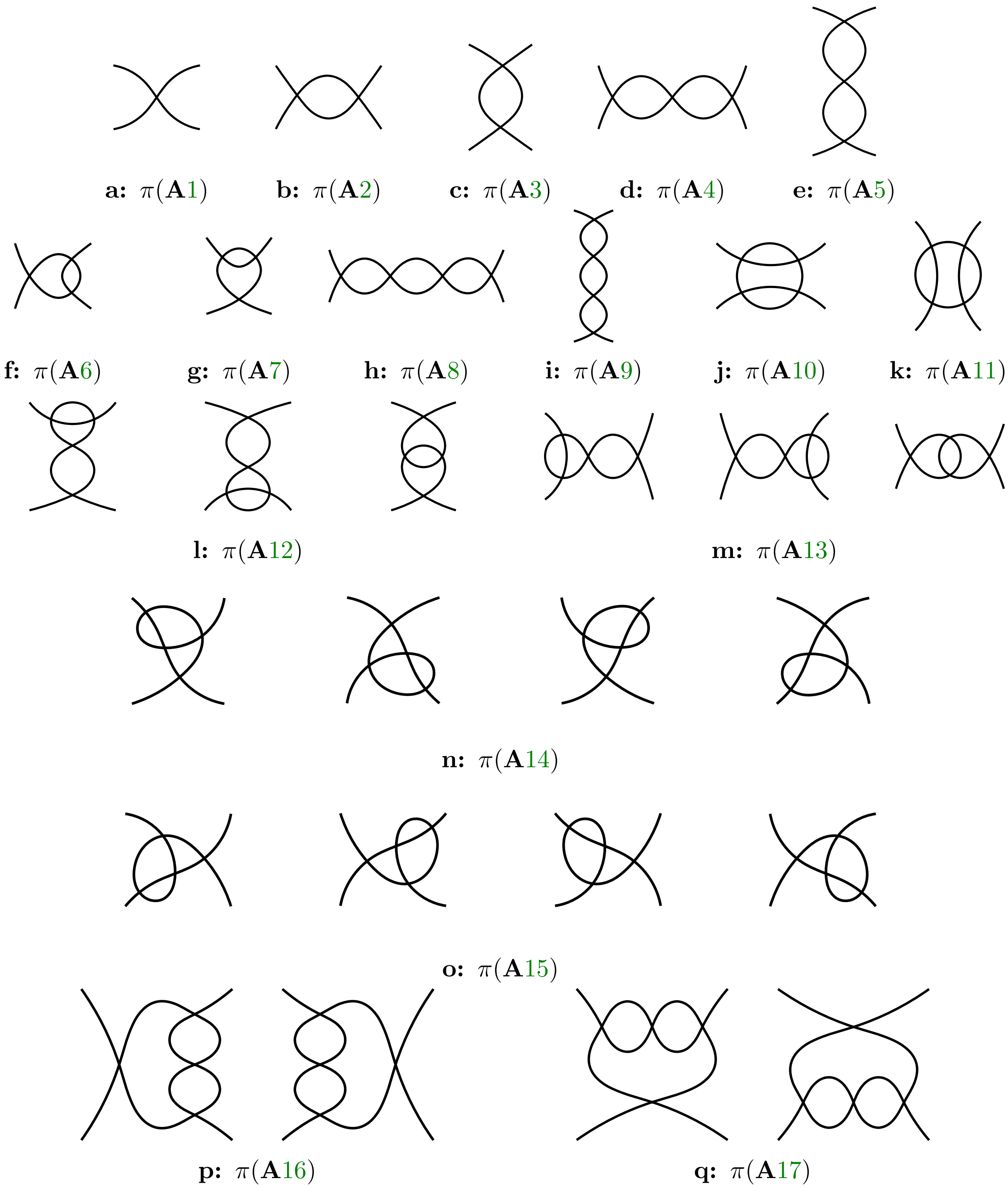


Figure 2: Prime tangles of up to 4 crossings.

1. $\pi\big([1]\big) = \big\{[1]\big\}$.

   B1: $\langle[0]\rangle + \langle[\infty]\rangle$.

   D1: $x(x+1)^n$, Table 1.

   N1: $(x+1)^n + x^2 - 1$, Table 2.

   R1: $(x+1)^{n+1} + x^2 - 1$, Table 3.

2. $\pi\big([2]\big) = \big\{[2]\big\}$.

   B2: $\langle[0]\rangle + (x+2)\,\langle[\infty]\rangle$.

   D2: $x\,(x+1)^{2n}$, Table 4 (cf. **D**18).

   N2: $(x+1)^{2n} + x^2 - 1$, Table 5.

   R2: $(x+1)^{2n+1} + x^2 - 1$, Table 6.

3. $\pi\left(\frac{1}{[2]}\right) = \left\{\frac{1}{[2]}\right\}$.

   B3: $(x+2)\,\langle[0]\rangle + \langle[\infty]\rangle$.

   D3: $x(2x+2)^n$, Table 7.

   N3: $(2x+2)^n + \left(x^2-1\right)(x+2)^n$, Table 8.

   R3: $(x+1)(2x+2)^n + \left(x^2-1\right)(x+2)^n$, Table 9.

4. $\pi\big([3]\big) = \big\{[3]\big\}$.

   B4: $\langle[0]\rangle + \left(x^2+3x+3\right)\langle[\infty]\rangle$.

   D4: $x\,(x+1)^{3n}$, Table 10 (cf. **D**25, **D**19).

   N4: $(x+1)^{3n} + x^2 - 1$, Table 11.

   R1: $(x+1)^{3n+1} + x^2 - 1$, Table 12.

5. $\pi\left(\frac{1}{[3]}\right) = \left\{\frac{1}{[3]}\right\}$.

   B5: $\left(x^3+3x+3\right)\langle[0]\rangle + \langle[\infty]\rangle$.

   D5: $x\left(x^2+4x+3\right)^n$, Table 13 (cf. **D**7).

   N5: $\left(x^2+4x+3\right)^n + \left(x^2-1\right)\left(x^2+3x+3\right)^n$, Table 14.

   R5: $(x+1)\left(x^2+4x+3\right)^n + \left(x^2-1\right)\left(x^2+3x+3\right)^n$, Table 15.

6. $\pi\left([1]+\frac{1}{[2]}\right) = \left\{[1]+\frac{1}{[2]}, \frac{1}{[2]}+[1]\right\}$.

   B6: $(x+2)\,\langle[0]\rangle + (2x+3)\,\langle[\infty]\rangle$.

D6: $x\left(2x^2+4x+2\right)^n$, Table 16 (cf. **D**28, **D**20).

N6: $\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(x+2)^n$, Table 17.

R6: $(x+1)\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(x+2)^n$, Table 18.

7. $\pi\big([1]*[2]\big)=\big\{[1]*[2],[2]*[1]\big\}$.

B7: $(2x+3)\langle[0]\rangle+(x+2)\langle[\infty]\rangle$.

D7: $x\left(x^2+4x+3\right)^n$, Table 13 (cf. **D**5).

N7: $\left(x^2+4x+3\right)^n+\left(x^2-1\right)(2x+3)^n$, Table 19.

R7: $(x+1)\left(x^2+4x+3\right)^n+\left(x^2-1\right)(2x+3)^n$, Table 20.

8. $\pi\big([4]\big)=\big\{[4]\big\}$.

B8: $\langle[0]\rangle+\left(x^3+4x^2+6x+4\right)\langle[\infty]\rangle$.

D8: $x(x+1)^{4n}$, Table 21 (cf. **D**21, **D**26, **D**31).

N8: $(x+1)^{4n}+x^2-1$, Table 22.

R8: $(x+1)^{4n+1}+x^2-1$, Table 23.

9. $\pi\left(\frac{1}{[4]}\right)=\left\{\frac{1}{[4]}\right\}$.

B9: $\left(x^3+4x^2+6x+4\right)\langle[0]\rangle+\langle[\infty]\rangle$.

D9: $x\left(x^3+4x^2+7x+4\right)^n$, Table 24 (cf. **D**10, **D**17).

N9: $\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+4\right)^n$, Table 25.

R9: $(x+1)\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+4\right)^n$, Table 26.

10. $\pi\big([2]*[2]\big)=\big\{[2]*[2]\big\}$.

B10: $(3x+4)\langle[0]\rangle+\left(x^2+4x+4\right)\langle[\infty]\rangle$.

D10: $x\left(x^3+4x^2+7x+4\right)^n$, Table 24 (cf. **D**9, **D**17).

N10: $\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)(3x+4)^n$, Table 27.

R10: $(x+1)\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)(3x+4)^n$, Table 28.

11. $\pi\left(\frac{1}{[2]}+\frac{1}{[2]}\right)=\left\{\frac{1}{[2]}+\frac{1}{[2]}\right\}$.

B11: $\left(x^2+4x+4\right)\langle[0]\rangle+(3x+4)\langle[\infty]\rangle$.

D11: $x(2x+2)^{2n}$, Table 29 (cf. **D**30).

N11: $(2x+2)^{2n}+\left(x^2-1\right)(x+2)^{2n}$, Table 30.

R11: $(x+1)(2x+2)^{2n} + \left(x^2-1\right)(x+2)^{2n}$, Table 31.

12. $\pi\left([2]*\frac{1}{[2]}\right) = \left\{[2]*\frac{1}{[2]}, \frac{1}{[2]}*[2], [1]*[2]*[1]\right\}$.

B12: $\left(2x^2+6x+5\right)\langle[0]\rangle + (x+2)\langle[\infty]\rangle$.

D12: $x\left(3x^2+8x+5\right)^n$, Table 32 (cf. **D**14).

N12: $\left(3x^2+8x+5\right)^n + \left(x^2-1\right)\left(2x^2+6x+5\right)^n$, Table 33.

R12: $(x+1)\left(3x^2+8x+5\right)^n + \left(x^2-1\right)\left(2x^2+6x+5\right)^n$, Table 34.

13. $\pi\left([2]+\frac{1}{[2]}\right) = \left\{[2]+\frac{1}{[2]}, \frac{1}{[2]}+[2], [1]+\frac{1}{[2]}+[1]\right\}$.

B13: $(x+2)\langle[0]\rangle + \left(2x^2+6x+5\right)\langle[\infty]\rangle$.

D13: $x\left(2x^3+6x^2+6x+2\right)^n$, Table 35 (cf. **D**23, **D**27, **D**29, **D**33).

N13: $\left(2x^3+6x^2+6x+2\right)^n + \left(x^2-1\right)(x+2)^n$, Table 36.

R13: $(x+1)\left(2x^3+6x^2+6x+2\right)^n + \left(x^2-1\right)(x+2)^n$, Table 37.

14. $$\pi\left([1]*\left([1]+\frac{1}{[2]}\right)\right) = \left\{[1]*\left([1]+\frac{1}{[2]}\right), \left([1]+\frac{1}{[2]}\right)*[1], [1]*\left(\frac{1}{[2]}+[1]\right), \left(\frac{1}{[2]}+[1]\right)*[1]\right\}.$$

B14: $\left(x^2+5x+5\right)\langle[0]\rangle + (2x+3)\langle[\infty]\rangle$.

D14: $x\left(3x^2+8x+5\right)^n$, Table 32 (cf. **D**12).

N14: $\left(3x^2+8x+5\right)^n + \left(x^2-1\right)\left(x^2+5x+5\right)^n$, Table 38.

R14: $(x+1)\left(3x^2+8x+5\right)^n + \left(x^2-1\right)\left(x^2+5x+5\right)^n$, Table 39.

15. $\pi\big([1]+([2]*[1])\big) = \big\{[1]+([2]*[1]), ([2]*[1])+[1], [1]+([1]*[2]), ([1]*[2])+[1]\big\}$.

B15: $(2x+3)\langle[0]\rangle + \left(x^2+5x+5\right)\langle[\infty]\rangle$.

D15: $x\left(x^3+5x^2+7x+3\right)^n$, Table 40 (cf. **D**16, **D**22, **D**24, **D**32).

N15: $\left(x^3+5x^2+7x+3\right)^n + \left(x^2-1\right)(2x+3)^n$, Table 41.

R15: $(x+1)\left(x^3+5x^2+7x+3\right)^n + \left(x^2-1\right)(2x+3)^n$, Table 42.

16. $\pi\left([1]+\frac{1}{[3]}\right) = \left\{[1]+\frac{1}{[3]}, \frac{1}{[3]}+[1]\right\}$.

B16: $\left(x^2+3x+3\right)\langle[0]\rangle + \left(x^2+4x+4\right)\langle[\infty]\rangle$.

D16: $x\left(x^3+5x^2+7x+3\right)^n$, Table 40 (cf. **D**15, **D**22, **D**24, **D**32).

N16: $\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+3x+3\right)^n$, Table 43.

R16: $(x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+3x+3\right)^n$, Table 44.

17. $\pi\big([1]*[3]\big)=\big\{[1]*[3],[3]*[1]\big\}$.

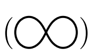
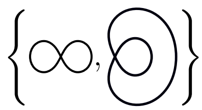
B17: $\left(x^2+4x+4\right)\langle[0]\rangle+\left(x^2+3x+3\right)\langle[\infty]\rangle$.

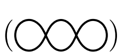
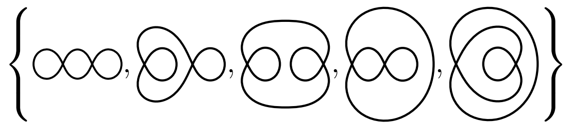
D17: $x\left(x^3+4x^2+7x+4\right)^n$, Table 24 (cf. **D**9, **D**10).

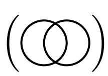
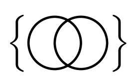
N17: $\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^2+4x+4\right)^n$, Table 45.

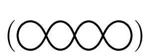
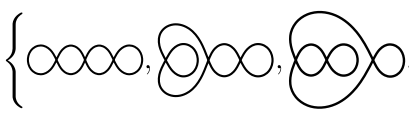
R17: $(x+1)\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^2+4x+4\right)^n$, Table 46.

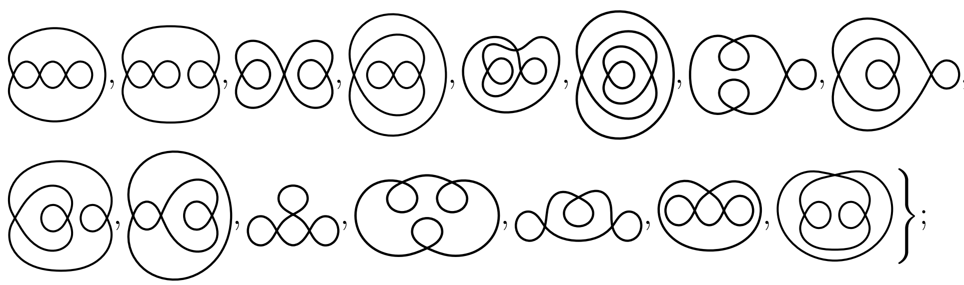
Notation 15. As previously raised in Remark 12, the class $\pi\big(T\#K\big)$ is also equal to the set $\pi\big(T\big)\#\pi\big(K\big):=\big\{T\#K : T\in\pi\big(T\big), K\in\pi\big(K\big)\big\}$. We identify the following classes for $\pi\big(K\big)$, where $c(K)\in\{1,2,3,4\}$ [Arn94, p. 14], [DDTT07].

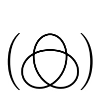
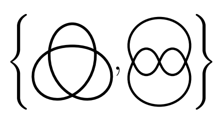
K1: π () $=\big\{K:\langle K\rangle=x^2+x\big\}=$  ;

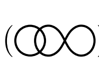
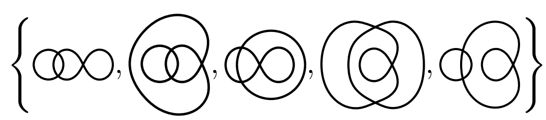
K2: π () $=\big\{K:\langle K\rangle=x^3+2x^2+x\big\}=$  ;

K3: π () $=\big\{K:\langle K\rangle=2x^2+2x\big\}=$  ;

K4: π () $=\big\{K:\langle K\rangle=x^4+3x^3+3x^2+x\big\}=$ 

 ;

K5: π () $=\big\{K:\langle K\rangle=x^3+4x^2+3x\big\}=$  ;

K6: π () $=\big\{K:\langle K\rangle=2x^3+4x^2+2x\big\}=$  .

Note also that

- $\pi\left(\bigcirc\!\!\bigcirc\!\!\bigcirc\right) = \pi\left(\bigcirc\!\!\bigcirc\right) \# \pi\left(\bigcirc\!\!\bigcirc\right)$;
- $\pi\left(\bigcirc\!\!\bigcirc\!\!\bigcirc\!\!\bigcirc\right) = \pi\left(\bigcirc\!\!\bigcirc\!\!\bigcirc\right) \# \pi\left(\bigcirc\!\!\bigcirc\right)$;
- $\pi\left(\bigcirc\!\!\!\!\bigcirc\!\!\bigcirc\right) = \pi\left(\bigcirc\!\!\!\!\bigcirc\right) \# \pi\left(\bigcirc\!\!\bigcirc\right)$.

Now we have the following list for locally knotted tangles.

18. $\pi\big([1]\big) \# \pi\left(\bigcirc\!\!\bigcirc\right)$.

    B18: $(x+1)\,\langle[0]\rangle + (x+1)\,\langle[\infty]\rangle$.

    D18: $x\,(x+1)^{2n}$, Table 4 (cf. **D**2).

    N18: $(x+1)^{2n} + \left(x^2-1\right)(x+1)^n$, Table 47.

    R18: $(x+1)^{2n+1} + \left(x^2-1\right)(x+1)^n$, Table 48.

19. $\pi\big([2]\big) \# \pi\left(\bigcirc\!\!\bigcirc\right)$.

    B19: $(x+1)\,\langle[0]\rangle + (x+3x+2)\,\langle[\infty]\rangle$.

    D19: $x\,(x+1)^{3n}$, Table 10 (cf. **D**4, **D**25).

    N19: $(x+1)^{3n} + \left(x^2-1\right)(x+1)^n$, Table 49.

    R19: $(x+1)^{3n+1} + \left(x^2-1\right)(x+1)^n$, Table 50.

20. $\pi\left(\dfrac{1}{[2]}\right) \# \pi\left(\bigcirc\!\!\bigcirc\right)$.

    B20: $(x+3x+2)\,\langle[0]\rangle + (x+1)\,\langle[\infty]\rangle$.

    D20: $x\left(2x^2+4x+2\right)^n$, Table 16 (cf. **D**6, **D**28).

    N20: $\left(2x^2+4x+2\right)^n + \left(x^2-1\right)\left(x^2+3x+2\right)^n$, Table 51.

    R20: $(x+1)\left(2x^2+4x+2\right)^n + \left(x^2-1\right)\left(x^2+3x+2\right)^n$, Table 52.

21. $\pi\big([3]\big) \# \pi\left(\bigcirc\!\!\bigcirc\right)$.

    B21: $(x+1)\,\langle[0]\rangle + \left(x^3+4x^2+6x+3\right)\langle[\infty]\rangle$

    D21: $x\,(x+1)^{4n}$, Table 21 (cf. **D**8, **D**26, **D**31).

    N21: $(x+1)^{4n} + \left(x^2-1\right)(x+1)^n$, Table 53.

    R21: $(x+1)^{4n+1} + \left(x^2-1\right)(x+1)^n$, Table 54.

22. $\pi\left(\dfrac{1}{[3]}\right) \# \pi\left(\bigcirc\!\!\bigcirc\right)$.

    B22: $\left(x^3+4x^2+6x+3\right)\langle[0]\rangle + (x+1)\,\langle[\infty]\rangle$.

D22: $x\left(x^3+5x^2+7x+3\right)^n$, Table 40 (cf. **D**15, **D**16, **D**24, **D**32).

N22: $\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+3\right)^n$, Table 55.

R22: $(x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+3\right)^n$, Table 56.

23. $\pi\left([1]+\frac{1}{[2]}\right)\#\pi\,(\bigcirc\!\!\bigcirc)$.

B23: $\left(x^2+3x+2\right)\langle[0]\rangle+\left(2x^2+5x+3\right)\langle[\infty]\rangle$.

D23: $x\left(2x^3+6x^2+6x+2\right)^n$, Table 35 (cf. **D**13, **D**27, **D**29, **D**33).

N23: $\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^2+3x+2\right)^n$, Table 57.

R23: $(x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^2+3x+2\right)^n$, Table 58.

24. $\pi\big([1]*[2]\big)\#\pi\,(\bigcirc\!\!\bigcirc)$.

B24: $\left(2x^2+5x+3\right)\langle[0]\rangle+\left(x^2+3x+2\right)\langle[\infty]\rangle$.

D24: $x\left(x^3+5x^2+7x+3\right)^n$, Table 40 (cf. **D**15, **D**16, **D**22, **D**32).

N24: $\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(2x^2+5x+3\right)^n$, Table 59.

R24: $(x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(2x^2+5x+3\right)^n$, Table 60.

25. $\pi\big([1]\big)\#\pi\,(\bigcirc\!\!\bigcirc\!\!\bigcirc)$.

B25: $\left(x^2+2x+1\right)\langle[0]\rangle+\left(x^2+2x+1\right)\langle[\infty]\rangle$.

D25: $x\,(x+1)^{3n}$, Table 10 (cf. **D**4, **D**19).

N25: $(x+1)^{3n}+\left(x^2-1\right)(x+1)^{2n}$, Table 61.

R25: $(x+1)^{3n+1}+\left(x^2-1\right)(x+1)^{2n}$, Table 62.

26. $\pi\big([2]\big)\#\pi\,(\bigcirc\!\!\bigcirc\!\!\bigcirc)$.

B26: $\left(x^2+2x+1\right)\langle[0]\rangle+\left(x^3+4x^2+5x+2\right)\langle[\infty]\rangle$.

D26: $x\,(x+1)^{4n}$, Table 21 (cf. **D**8, **D**21, **D**31).

N26: $(x+1)^{4n}+\left(x^2-1\right)(x+1)^{2n}$, Table 63.

R26: $(x+1)^{4n+1}+\left(x^2-1\right)(x+1)^{2n}$, Table 64.

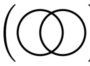
27. $\pi\left(\frac{1}{[2]}\right)\#\pi\,(\bigcirc\!\!\bigcirc\!\!\bigcirc)$.

B27: $\left(x^3+4x^2+5x+2\right)\langle[0]\rangle+\left(x^2+2x+1\right)\langle[\infty]\rangle$.

D27: $x\left(2x^3+6x^2+6x+2\right)^n$, Table 35 (cf. **D**13, **D**23, **D**29, **D**33).

N27: $\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^3+4x^2+5x+2\right)^n$, Table 65.

R27: $(x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^3+4x^2+5x+2\right)^n$, Table 66.

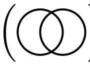
28. $\pi\big([1]\big)\#\pi$ ().

B28: $(2x+2)\langle[0]\rangle+(2x+2)\langle[\infty]\rangle$.

D28: $x\left(2x^2+4x+2\right)^n$, Table 16 (cf. **D**6, **D**20).

N28: $\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(2x+2)^n$, Table 67.

R28: $(x+1)\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(2x+2)^n$, Table 68.

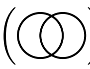
29. $\pi\big([2]\big)\#\pi$ ().

B29: $(2x+2)\langle[0]\rangle+\left(2x^2+6x+4\right)\langle[\infty]\rangle$.

D29: $x\left(2x^3+6x^2+6x+2\right)^n$, Table 35 (cf. **D**13, **D**23, **D**27, **D**33).

N29: $\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)(2x+2)^n$, Table 69.

R29: $(x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)(2x+2)^n$, Table 70.

30. $\pi\left(\frac{1}{[2]}\right)\#\pi$ ().

B30: $\left(2x^2+6x+4\right)\langle[0]\rangle+(2x+2)\langle[\infty]\rangle$.

D30: $x\,(2x+2)^{2n}$, Table 29 (cf. **D**11).

N30: $(2x+2)^{2n}+\left(x^2-1\right)\left(2x^2+6x+4\right)^n$, Table 71.

R30: $(x+1)(2x+2)^{2n}+\left(x^2-1\right)\left(2x^2+6x+4\right)^n$, Table 72.

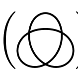
31. $\pi\big([1]\big)\#\pi$ ().

B31: $\left(x^3+3x^2+3x+1\right)\langle[0]\rangle+\left(x^3+3x^2+3x+1\right)\langle[\infty]\rangle$.

D31: $x\,(x+1)^{4n}$, Table 21 (cf. **D**8, **D**21, **D**26).

N31: $(x+1)^{4n}+\left(x^2-1\right)(x+1)^{3n}$, Table 73.

R31: $(x+1)^{4n+1}+\left(x^2-1\right)(x+1)^{3n}$, Table 74.

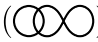
32. $\pi\big([1]\big)\#\pi$ ().

B32: $\left(x^2+4x+3\right)\langle[0]\rangle+\left(x^2+4x+3\right)\langle[\infty]\rangle$.

D32: $x\left(x^3+5x^2+7x+3\right)^n$, Table 40 (cf. **D**15, **D**16, **D**22, **D**24).

N32: $\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+4x+3\right)^n$, Table 75.

R32: $(x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+4x+3\right)^n$, Table 76.

33. $\pi\big([1]\big)\#\pi$ ().

B33: $\left(2x^2+4x+2\right)\langle[0]\rangle+\left(2x^2+4x+2\right)\langle[\infty]\rangle$.

D33: $x\left(2x^3+6x^2+6x+2\right)^n$, Table 35 (cf. **D**13, **D**23, **D**27, **D**29).

N33: $\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(2x^2+4x+2\right)^n$, Table 77.

R33: $(x+2)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(2x^2+4x+2\right)^n$, Table 78.

Remark 16. Except for the classes of $[0]$ and $[\infty]$ below, we do not take into consideration other tangles verifying $a(A).b(A)=0$.

34. $\pi\big([0]\big)=\big\{[0]\big\}$.

B34: $\langle[0]\rangle$.

D34: x.

N34: x^2.

R34: x^2+x.

35. $\pi\big([0]\big)=\big\{[\infty]\big\}$.

B35: $\langle[\infty]\rangle$.

D35: x^{n+1}, Table 79.

N35: x^2 if $n=0$, x^n otherwise, Table 80.

R35: x^2+x if $n=0$, $x^{n+1}+x^n$ otherwise, Table 81.

3.2 Tables of coefficients

Tables that are listed here consist of the coefficients in the expansion of the bracket polynomials seen in subsection 3.1 for small n.

$n\backslash k$	0	1	2	3	4	5	6
0	0	1					
1	0	1	1				
2	0	1	2	1			
3	0	1	3	3	1		
4	0	1	4	6	4	1	
5	0	1	5	10	10	5	1

Table 1: $\left[x^k\right]x(x+1)^n$ (cf. **D**1) [Slo20, A007318].

$n\backslash k$	0	1	2	3	4	5
0	0	0	1			
1	0	1	1			
2	0	2	2			
3	0	3	4	1		
4	0	4	7	4	1	
5	0	5	11	10	5	1

Table 2: $\left[x^k\right]\left((x+1)^n+x^2-1\right)$ (cf. **N**1) [Slo20, A300453].

$n\backslash k$	0	1	2	3	4	5	6
0	0	1	1				
1	0	2	2				
2	0	3	4	1			
3	0	4	7	4	1		
4	0	5	11	10	5	1	
5	0	6	16	20	15	6	1

Table 3: $\left[x^k\right]\left((x+1)^{n+1}+x^2-1\right)$ (cf. **R**1) [Slo20, A300453].

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1										
1	0	1	2	1								
2	0	1	4	6	4	1						
3	0	1	6	15	20	15	6	1				
4	0	1	8	28	56	70	56	28	8	1		
5	0	1	10	45	120	210	252	210	120	45	10	1

Table 4: $\left[x^k\right]\left(x(x+1)^{2n}\right)$ (cf. **D**2, **D**18) [Slo20, A034870].

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	2	2								
2	0	4	7	4	1						
3	0	6	16	20	15	6	1				
4	0	8	29	56	70	56	28	8	1		
5	0	10	46	120	210	252	210	120	45	10	1

Table 5: $\left[x^k\right]\left((x+1)^{2n}+x^2-1\right)$ (cf. **N**2).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	3	4	1								
2	0	5	11	10	5	1						
3	0	7	22	35	35	21	7	1				
4	0	9	37	84	126	126	84	36	9	1		
5	0	11	56	165	330	462	462	330	165	55	11	1

Table 6: $\left[x^k\right]\left((x+1)^{2n+1}+x^2-1\right)$ (cf. **R**2).

$n\backslash k$	0	1	2	3	4	5	6
0	0	1					
1	0	2	2				
2	0	4	8	4			
3	0	8	24	24	8		
4	0	16	64	96	64	16	
4	0	32	160	320	320	160	32

Table 7: $\left[x^k\right](x(2x+2)^n)$ (cf. **D**3) [Slo20, A038208].

$n\backslash k$	0	1	2	3	4	5	6	7
0	0	0	1					
1	0	1	2	1				
2	0	4	7	4	1			
3	0	12	26	19	6	1		
4	0	32	88	88	39	8	1	
5	0	80	272	360	230	71	10	1

Table 8: $\left[x^k\right]\left((2x+2)^n+\left(x^2-1\right)(x+2)^n\right)$ (cf. **N**3) [Slo20, A300184].

$n\backslash k$	0	1	2	3	4	5	6	7
0	0	1	1					
1	0	3	4	1				
2	0	8	15	8	1			
3	0	20	50	43	14	1		
4	0	48	152	184	103	24	1	
5	0	112	432	680	550	231	42	1

Table 9: $\left[x^k\right]\left((x+1)(2x+2)^n+\left(x^2-1\right)(x+2)^n\right)$ (cf. **R**3).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1													
1	0	1	3	3	1										
2	0	1	6	15	20	15	6	1							
3	0	1	9	36	84	126	126	84	36	9	1				
4	0	1	12	66	220	495	792	924	792	495	220	66	12	1	
5	0	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	...

Table 10: $\left[x^k\right]\left(x(x+1)^{3n}\right)$ (cf. **D**4, **D**25, **D**19).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	1											
1	0	3	4	1										
2	0	6	16	20	15	6	1							
3	0	9	37	84	126	126	84	36	9	1				
4	0	12	67	220	495	792	924	792	495	220	66	12	1	
5	0	15	106	455	1365	3003	5005	6435	6435	5005	3003	1365	455	...

Table 11: $\left[x^k\right]\left((x+1)^{3n}+x^2-1\right)$ (cf. **N**4).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	1											
1	0	4	7	4	1									
2	0	7	22	35	35	21	7	1						
3	0	10	46	120	210	252	210	120	45	10	1			
4	0	13	79	286	715	1287	1716	1716	1287	715	286	78	13	...
5	0	16	121	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	...

Table 12: $\left[x^k\right]\left((x+1)^{3n+1}+x^2-1\right)$ (cf. **R**4).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1										
1	0	3	4	1								
2	0	9	24	22	8	1						
3	0	27	108	171	136	57	12	1				
4	0	81	432	972	1200	886	400	108	16	1		
5	0	243	1620	4725	7920	8430	5944	2810	880	175	20	1

Table 13: $\left[x^k\right]\left(x\left(x^2+4x+3\right)^n\right)$ (cf. **D**5, **D**7) [Slo20, A299989].

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	1	3	3	1								
2	0	6	16	20	15	6	1						
3	0	27	90	136	129	84	36	9	1				
4	0	108	459	876	1021	832	501	220	66	12	1		
5	0	405	2133	5085	7350	7321	5420	3103	1375	455	105	15	1

Table 14: $[x^k]\left(\left(x^2+4x+3\right)^n+\left(x^2-1\right)\left(x^2+3x+3\right)^n\right)$ (cf. **N**5).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1										
1	0	4	7	4	1								
2	0	15	40	42	23	7	1						
3	0	54	198	307	265	141	48	10	1				
4	0	189	891	1848	2221	1718	901	328	82	13	1		
5	0	648	3753	9810	15270	15751	11364	5913	2255	630	125	16	1

Table 15: $[x^k]\left((x+1)\left(x^2+4x+3\right)^n+\left(x^2-1\right)\left(x^2+3x+3\right)^n\right)$ (cf. **R**5).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1										
1	0	2	4	2								
2	0	4	16	24	16	4						
3	0	8	48	120	160	120	48	8				
4	0	16	128	448	896	1120	896	448	128	16		
5	0	32	320	1440	3840	6720	8064	6720	3840	1440	320	32

Table 16: $[x^k]\left(x\left(2x^2+4x+2\right)^n\right)$ (cf. **D**6, **D**28, **D**20) [Slo20, A139548].

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	3	4	1							
2	0	12	27	20	5						
3	0	36	122	171	126	49	8				
4	0	96	440	920	1143	904	449	128	16		
5	0	240	1392	3880	6790	8103	6730	3841	1440	320	32

Table 17: $[x^k]\left(\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(x+2)^n\right)$ (cf. **N**6).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	5	8	3								
2	0	16	43	44	21	4						
3	0	44	170	291	286	169	56	8				
4	0	112	568	1368	2039	2024	1345	576	144	16		
5	0	272	1712	5320	10630	14823	14794	10561	5280	1760	352	32

Table 18: $\left[x^k\right]\left((x+1)\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(x+2)^n\right)$ (cf. **R**6).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	2	4	2							
2	0	12	27	20	5						
3	0	54	162	182	93	20	1				
4	0	216	837	1320	1086	496	124	16	1		
5	0	810	3888	8010	9270	6632	3050	912	175	20	1

Table 19: $\left[x^k\right]\left(\left(x^2+4x+3\right)^n+\left(x^2-1\right)(2x+3)^n\right)$ (cf. **N**7).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	5	8	3								
2	0	21	51	42	13	1						
3	0	81	270	353	229	77	13	1				
4	0	297	1269	2292	2286	1382	524	124	17	1		
5	0	1053	5508	12735	17190	15062	8994	3722	1055	195	21	1

Table 20: $\left[x^k\right]\left((x+1)\left(x^2+4x+3\right)^n+\left(x^2-1\right)(2x+3)^n\right)$ (cf. **R**7).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1											
1	0	1	4	6	4	1							
2	0	1	8	28	56	70	56	28	8	1			
3	0	1	12	66	220	495	792	924	792	495	220	66	…
4	0	1	16	120	560	1820	4368	8008	11440	12870	11440	8008	…
5	0	1	20	190	1140	4845	15504	38760	77520	125970	167960	184756	…

Table 21: $\left[x^k\right]\left(x(x+1)^{4n}\right)$ (cf. **D**8, **D**21, **D**26, **D**31).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	4	7	4	1							
2	0	8	29	56	70	56	28	8	1			
3	0	12	67	220	495	792	924	792	495	220	66	...
4	0	16	121	560	1820	4368	8008	11440	12870	11440	8008	...
5	0	20	191	1140	4845	15504	38760	77520	125970	167960	184756	...

Table 22: $\left[x^k\right]\left((x+1)^{4n}+x^2-1\right)$ (cf. **N**8).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	5	11	10	5	1						
2	0	9	37	84	126	126	84	36	9	1		
3	0	13	79	286	715	1287	1716	1716	1287	715	286	...
4	0	17	137	680	2380	6188	12376	19448	24310	24310	19448	...
5	0	21	211	1330	5985	20349	54264	116280	203490	293930	352716	...

Table 23: $\left[x^k\right]\left((x+1)^{4n+1}+x^2-1\right)$ (cf. **R**8).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1										
1	0	4	7	4	1							
2	0	16	56	81	64	30	8	1				
3	0	64	336	780	1063	948	579	244	69	12	1	
4	0	256	1792	5728	11120	14689	13984	9884	5248	2086	608	...
5	0	1024	8960	36480	92000	161300	208967	207300	160805	98600	47910	...

Table 24: $\left[x^k\right]\left(x\left(x^3+4x^2+7x+4\right)^n\right)$ (cf. **D**9, **D**10, **D**17).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	1	4	6	4	1					
2	0	8	29	56	70	56	28	8	1		
3	0	48	220	511	804	927	792	495	220	66	...
4	0	256	1504	4336	8273	11744	13036	11488	8014	4368	...
5	0	1280	9344	33120	76500	130151	174020	189301	170120	126630	...

Table 25: $\left[x^k\right]\left(\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+4\right)^n\right)$ (cf. **N**9).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	5	11	10	5	1					
2	0	24	85	137	134	86	36	9	1		
3	0	112	556	1291	1867	1875	1371	739	289	78	...
4	0	512	3296	10064	19393	26433	27020	21372	13262	6454	...
5	0	2304	18304	69600	168500	291451	382987	396601	330925	225230	...

Table 26: $\left[x^k\right]\left((x+1)\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+4\right)^n\right)$ (cf. **R**9).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	4	8	4							
2	0	32	88	88	39	8	1				
3	0	192	736	1180	1056	606	244	69	12	1	
4	0	1024	5120	11456	15472	14416	9965	5248	2086	608	...
5	0	5120	31744	91520	165440	213044	208920	161048	98600	47910	...

Table 27: $\left[x^k\right]\left(\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)(3x+4)^n\right)$ (cf. **N**10).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	8	15	8	1						
2	0	48	144	169	103	38	9	1			
3	0	256	1072	1960	2119	1554	823	313	81	13	...
4	0	1280	6912	17184	26592	29105	23949	15132	7334	2694	...
5	0	6144	40704	128000	257440	374344	417887	368348	259405	146510	...

Table 28: $\left[x^k\right]\left((x+1)\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)(3x+4)^n\right)$ (cf. **R**10).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1									
1	0	4	8	4							
2	0	16	64	96	64	16					
3	0	64	384	960	1280	960	384	64			
4	0	256	2048	7168	14336	17920	14336	7168	2048	256	
5	0	1024	10240	46080	122880	215040	258048	215040	122880	46080	...

Table 29: $\left[x^k\right]\left(x\,(2x+2)^{2n}\right)$ (cf. **D**11, **D**30).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	4	7	4	1						
2	0	32	88	88	39	8	1				
3	0	192	784	1312	1140	532	123	12	1		
4	0	1024	5632	13568	18592	15680	8176	2480	367	16	...
5	0	5120	35584	112640	213120	265344	225120	129984	49260	11180	...

Table 30: $[x^k]\left((2x+2)^{2n}+\left(x^2-1\right)(x+2)^{2n}\right)$ (cf. **N**11).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	8	15	8	1						
2	0	48	152	184	103	24	1				
3	0	256	1168	2272	2420	1492	507	76	1		
4	0	1280	7680	20736	32928	33600	22512	9648	2415	272	...
5	0	6144	45824	158720	336000	480384	483168	345024	172140	57260	...

Table 31: $[x^k]\left((x+1)(2x+2)^{2n}+\left(x^2-1\right)(x+2)^{2n}\right)$ (cf. **R**11).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1									
1	0	5	8	3							
2	0	25	80	94	48	9					
3	0	125	600	1185	1232	711	216	27			
4	0	625	4000	11100	17440	16966	10464	3996	864	81	
5	0	3125	25000	89375	188000	257650	240368	154590	67680	19305	...

Table 32: $[x^k]\left(x\left(3x^2+8x+5\right)^n\right)$ (cf. **D**12, **D**14).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	2	6	6	2						
2	0	20	63	84	61	24	4				
3	0	150	620	1106	1125	720	295	72	8		
4	0	1000	5325	12520	17150	15216	9188	3840	1089	192	...
5	0	6250	41250	122750	217500	255392	209430	123216	52585	16200	...

Table 33: $[x^k]\left(\left(3x^2+8x+5\right)^n+\left(x^2-1\right)\left(2x^2+6x+5\right)^n\right)$ (cf. **N**12).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	7	14	9	2						
2	0	45	143	178	109	33	4				
3	0	275	1220	2291	2357	1431	511	99	8		
4	0	1625	9325	23620	34590	32182	19652	7836	1953	273	…
5	0	9375	66250	212125	405500	513042	449798	277806	120265	35505	…

Table 34: $[x^k]\left((x+1)\left(3x^2+8x+5\right)^n+\left(x^2-1\right)\left(2x^2+6x+5\right)^n\right)$ (cf. **R**12).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1										
1	0	2	6	6	2							
2	0	4	24	60	80	60	24	4				
3	0	8	72	288	672	1008	1008	672	288	72	8	
4	0	16	192	1056	3520	7920	12672	14784	12672	7920	3520	…
5	0	32	480	3360	14560	43680	96096	160160	205920	205920	160160	…

Table 35: $[x^k]\left(x\left(2x^3+6x^2+6x+2\right)^n\right)$ (cf. **D**13, **D**23, **D**27, **D**29, **D**33).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	5	8	3								
2	0	20	63	84	61	24	4					
3	0	60	290	683	1014	1009	672	288	72	8		
4	0	160	1048	3544	7943	12680	14785	12672	7920	3520	1056	…
5	0	400	3312	14600	43750	96135	160170	205921	205920	160160	96096	…

Table 36: $[x^k]\left(\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)(x+2)^n\right)$ (cf. **N**13).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	7	14	9	2						
2	0	24	87	144	141	84	28	4			
3	0	68	362	971	1686	2017	1680	960	360	80	…
4	0	176	1240	4600	11463	20600	27457	27456	20592	11440	…
5	0	432	3792	17960	58310	139815	256266	366081	411840	366080	…

Table 37: $[x^k]\left((x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)(x+2)^n\right)$ (cf. **R**13).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	3	7	5	1							
2	0	30	84	88	43	10	1					
3	0	225	860	1332	1071	476	116	15	1			
4	0	1500	7475	15940	18941	13664	6101	1644	250	20	1	
5	0	9375	58125	159875	256400	264743	183090	85305	26155	4965	517	...

Table 38: $\left[x^k\right]\left(\left(3x^2+8x+5\right)^n+\left(x^2-1\right)\left(x^2+5x+5\right)^n\right)$ (cf. **N**14).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	8	15	8	1						
2	0	55	164	182	91	19	1				
3	0	350	1460	2517	2303	1187	332	42	1		
4	0	2125	11475	27040	36381	30630	16565	5640	1114	101	...
5	0	12500	83125	249250	444400	522393	423458	239895	93835	24270	...

Table 39: $\left[x^k\right]\left((x+1)\left(3x^2+8x+5\right)^n+\left(x^2-1\right)\left(x^2+5x+5\right)^n\right)$ (cf. **R**14).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1										
1	0	3	7	5	1							
2	0	9	42	79	76	39	10	1				
3	0	27	189	576	1000	1086	762	344	96	15	...	
4	0	81	756	3186	8004	13327	15464	12796	7592	3199	...	
5	0	243	2835	15255	50175	112695	182887	221275	202995	142185	...	

Table 40: $\left[x^k\right]\left(x\left(x^3+5x^2+7x+3\right)^n\right)$ (cf. **D**15, **D**16, **D**22, **D**24, **D**32).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	5	8	3							
2	0	30	84	88	43	10	1				
3	0	135	567	1046	1122	770	344	96	15	1	
4	0	540	3051	8124	13527	15560	12812	7592	3199	932	...
5	0	2025	14418	50265	113535	183575	221515	203027	142185	75945	...

Table 41: $\left[x^k\right]\left(\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)(2x+3)^n\right)$ (cf. **N**15).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	8	15	8	1						
2	0	39	126	167	119	49	11	1			
3	0	162	756	1622	2122	1856	1106	440	111	16	…
4	0	621	3807	11310	21531	28887	28276	20388	10791	4131	…
5	0	2268	17253	65520	163710	296270	404402	424302	345180	218130	…

Table 42: $\left[x^k\right]\left((x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)(2x+3)^n\right)$ (cf. **R**15).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	4	7	4	1						
2	0	24	73	88	53	16	2				
3	0	108	495	1000	1158	834	379	105	16	1	
4	0	432	2673	7680	13462	15896	13189	7796	3264	944	…
5	0	1620	12663	47340	111615	184264	223885	205218	143385	76380	…

Table 43: $\left[x^k\right]\left(\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+3x+3\right)^n\right)$ (cf. **N**16).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	7	14	9	2						
2	0	33	115	167	129	55	12	1			
3	0	135	684	1576	2158	1920	1141	449	112	16	…
4	0	513	3429	10866	21466	29223	28653	20592	10856	4143	…
5	0	1863	15498	62595	161790	296959	406772	426493	346380	218565	…

Table 44: $\left[x^k\right]\left((x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+3x+3\right)^n\right)$ (cf. **R**16).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	3	7	5	1						
2	0	24	73	88	53	16	2				
3	0	144	604	1095	1128	727	303	81	13	1	
4	0	768	4192	10352	15361	15328	10892	5680	2197	624	…
5	0	3840	25984	81760	159380	216263	217380	167909	101780	48850	…

Table 45: $\left[x^k\right]\left(\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^2+4x+4\right)^n\right)$ (cf. **N**17).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	7	14	9	2						
2	0	40	129	169	117	46	10	1			
3	0	208	940	1875	2191	1675	882	325	82	13	...
4	0	1024	5984	16080	26481	30017	24876	15564	7445	2710	...
5	0	4864	34944	118240	251380	377563	426347	375209	262585	147450	...

Table 46: $\left[x^k\right]\left((x+1)\left(x^3+4x^2+7x+4\right)^n+\left(x^2-1\right)\left(x^2+4x+4\right)^n\right)$ (cf. **R**17).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	1	2	1							
2	0	2	6	6	2						
3	0	3	13	22	18	7	1				
4	0	4	23	56	75	60	29	8	1		
5	0	5	36	115	215	261	215	121	45	10	1

Table 47: $\left[x^k\right]\left((x+1)^{2n}+\left(x^2-1\right)(x+1)^n\right)$ (cf. **N**18) [Slo20, A300192].

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	2	4	2								
2	0	3	10	12	6	1						
3	0	4	19	37	38	22	7	1				
4	0	5	31	84	131	130	85	36	9	1		
5	0	6	46	160	335	471	467	331	165	55	11	1

Table 48: $\left[x^k\right]\left(\left((x+1)^{2n+1}+\left(x^2-1\right)(x+1)^n\right)\right)$ (cf. **R**18).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1												
1	0	2	4	2											
2	0	4	15	22	16	6	1								
3	0	6	34	86	129	127	84	36	9	1					
4	0	8	61	220	500	796	925	792	495	220	66	12	1		
5	0	10	96	450	1370	3012	5010	6436	6435	5005	3003	1365	455	105	...

Table 49: $\left[x^k\right]\left((x+1)^{3n}+\left(x^2-1\right)(x+1)^n\right)$ (cf. **N**19).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	1											
1	0	3	7	5	1									
2	0	5	21	37	36	21	7	1						
3	0	7	43	122	213	253	210	120	45	10	1			
4	0	9	73	286	720	1291	1717	1716	1287	715	286	78	13	...
5	0	11	111	555	1825	4377	8013	11441	12870	11440	8008	4368	1820	...

Table 50: $\left[x^k\right]\left((x+1)^{3n+1}+\left(x^2-1\right)(x+1)^n\right)$ (cf. **R**19).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	1	3	3	1								
2	0	4	15	22	16	6	1						
3	0	12	62	133	153	102	40	9	1				
4	0	32	216	632	1047	1076	707	296	77	12	1		
5	0	80	672	2520	5550	7941	7705	5133	2325	695	131	15	1

Table 51: $\left[x^k\right]\left(\left(2x^2+4x+2\right)^n+\left(x^2-1\right)\left(x^2+3x+2\right)^n\right)$ (cf. **N**20).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1										
1	0	3	7	5	1								
2	0	8	31	46	32	10	1						
3	0	20	110	253	313	222	88	17	1				
4	0	48	344	1080	1943	2196	1603	744	205	28	1		
5	0	112	992	3960	9390	14661	15769	11853	6165	2135	451	47	1

Table 52: $\left[x^k\right]\left((x+1)\left(2x^2+4x+2\right)^n+\left(x^2-1\right)\left(x^2+3x+2\right)^n\right)$ (cf. **R**20).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	3	7	5	1							
2	0	6	28	58	71	56	28	8	1			
3	0	9	64	222	498	793	924	792	495	220	66	...
4	0	12	115	560	1825	4372	8009	11440	12870	11440	8008	...
5	0	15	181	1135	4850	15513	38765	77521	125970	167960	184756	...

Table 53: $\left[x^k\right]\left((x+1)^{4n}+\left(x^2-1\right)(x+1)^n\right)$ (cf. **N**21).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	4	11	11	5	1						
2	0	7	36	86	127	126	84	36	9	1		
3	0	10	76	288	718	1288	1716	1716	1287	715	286	...
4	0	13	131	680	2385	6192	12377	19448	24310	24310	19448	...
5	0	16	201	1325	5990	20358	54269	116281	203490	293930	352716	...

Table 54: $[x^k]\left((x+1)^{4n+1}+\left(x^2-1\right)(x+1)^n\right)$ (cf. **R**21).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	1	4	6	4	1					
2	0	6	28	58	71	56	28	8	1		
3	0	27	171	487	834	969	811	498	220	66	...
4	0	108	891	3360	7711	12116	13918	12176	8301	4432	...
5	0	405	4158	19800	58155	118276	177445	204471	186135	136515	...

Table 55: $[x^k]\left(\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+3\right)^n\right)$ (cf. **N**22).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	4	11	11	5	1					
2	0	15	70	137	147	95	38	9	1		
3	0	54	360	1063	1834	2055	1573	842	316	81	...
4	0	189	1647	6546	15715	25443	29382	24972	15893	7631	...
5	0	648	6993	35055	108330	230971	360332	425746	389130	278700	...

Table 56: $[x^k]\left((x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^3+4x^2+6x+3\right)^n\right)$ (cf. **R**22).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	3	7	5	1						
2	0	12	51	86	72	30	5				
3	0	36	230	645	1041	1062	704	297	73	8	
4	0	96	824	3256	7847	12852	15043	12840	7981	3532	...
5	0	240	2592	13240	42510	95973	161145	207213	206805	160535	...

Table 57: $[x^k]\left(\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^2+3x+2\right)^n\right)$ (cf. **N**23).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	5	13	11	3						
2	0	16	75	146	152	90	29	4			
3	0	44	302	933	1713	2070	1712	969	361	80	…
4	0	112	1016	4312	11367	20772	27715	27624	20653	11452	…
5	0	272	3072	16600	57070	139653	257241	367373	412725	366455	…

Table 58: $\left[x^k\right]\left((x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^2+3x+2\right)^n\right)$ (cf. **R**23).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	2	6	6	2							
2	0	12	51	86	72	30	5					
3	0	54	324	830	1179	1007	522	156	23	1		
4	0	216	1701	5964	12252	16324	14741	9152	3879	1092	194	…
5	0	810	7938	35550	96300	176012	229260	219120	155915	82945	32853	…

Table 59: $\left[x^k\right]\left(\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(2x^2+5x+3\right)^n\right)$ (cf. **N**24).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	5	13	11	3						
2	0	21	93	165	148	69	15	1			
3	0	81	513	1406	2179	2093	1284	500	119	16	…
4	0	297	2457	9150	20256	29651	30205	21948	11471	4291	…
5	0	1053	10773	50805	146475	288707	412147	440395	358910	225130	…

Table 60: $\left[x^k\right]\left((x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(2x^2+5x+3\right)^n\right)$ (cf. **R**24).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1												
1	0	1	3	3	1										
2	0	2	10	20	20	10	2								
3	0	3	22	70	126	140	98	42	10	1					
4	0	4	39	172	453	792	966	840	522	228	67	12	1		
5	0	5	61	345	1200	2871	5005	6567	6600	5115	3047	1375	456	105	…

Table 61: $\left[x^k\right]\left((x+1)^{3n}+\left(x^2-1\right)(x+1)^{2n}\right)$ (cf. **N**25).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	1											
1	0	2	6	6	2									
2	0	3	16	35	40	25	8	1						
3	0	4	31	106	210	266	224	126	46	10	1			
4	0	5	51	238	673	1287	1758	1764	1314	723	287	78	13	...
5	0	6	76	450	1655	4236	8008	11572	13035	11550	8052	4378	1821	...

Table 62: $\left[x^k\right]\left((x+1)^{3n+1}+\left(x^2-1\right)(x+1)^{2n}\right)$ (cf. **R**25).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	2	6	6	2							
2	0	4	23	56	75	60	29	8	1			
3	0	6	52	206	495	806	938	798	496	220	66	...
4	0	8	93	512	1778	4368	8050	11488	12897	11448	8009	...
5	0	10	146	1030	4680	15372	38760	77652	126135	168070	184800	...

Table 63: $\left[x^k\right]\left((x+1)^{4n}+\left(x^2-1\right)(x+1)^{2n}\right)$ (cf. **N**26).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	3	10	12	6	1						
2	0	5	31	84	131	130	85	36	9	1		
3	0	7	64	272	715	1301	1730	1722	1288	715	286	...
4	0	9	109	632	2338	6188	12418	19496	24337	24318	19449	...
5	0	11	166	1220	5820	20217	54264	116412	203655	294040	352760	...

Table 64: $\left[x^k\right]\left((x+1)^{4n+1}+\left(x^2-1\right)(x+1)^{2n}\right)$ (cf. **R**26).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	1	4	6	4	1						
2	0	4	23	56	75	60	29	8	1			
3	0	12	98	355	750	1022	938	588	250	70	12	...
4	0	32	344	1688	4999	9952	14037	14400	10854	6000	2402	...
5	0	80	1072	6680	25670	68011	131550	191840	214720	185955	124652	...

Table 65: $\left[x^k\right]\left(\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^3+4x^2+5x+2\right)^n\right)$ (cf. **N**27).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	3	10	12	6	1					
2	0	8	47	116	155	120	53	12	1		
3	0	20	170	643	1422	2030	1946	1260	538	142	...
4	0	48	536	2744	8519	17872	26709	29184	23526	13920	...
5	0	112	1552	10040	40230	111691	227646	352000	420640	391875	...

Table 66: $\left[x^k\right]\left((x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(x^3+4x^2+5x+2\right)^n\right)$ (cf. **R**27).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	2	4	2							
2	0	8	24	24	8						
3	0	24	104	176	144	56	8				
4	0	64	368	896	1200	960	464	128	16		
5	0	160	1152	3680	6880	8352	6880	3872	1440	320	32

Table 67: $\left[x^k\right]\left(\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(2x+2)^n\right)$ (cf. **N**28).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	4	8	4								
2	0	12	40	48	24	4						
3	0	32	152	296	304	176	56	8				
4	0	80	496	1344	2096	2080	1360	576	144	16		
5	0	192	1472	5120	10720	15072	14944	10592	5280	1760	352	32

Table 68: $\left[x^k\right]\left((x+1)\left(2x^2+4x+2\right)^n+\left(x^2-1\right)(2x+2)^n\right)$ (cf. **R**28).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	4	8	4							
2	0	16	60	88	64	24	4				
3	0	48	272	688	1032	1016	672	288	72	8	
4	0	128	976	3520	8000	12736	14800	12672	7920	3520	...
5	0	320	3072	14400	43840	96384	160320	205952	205920	160160	...

Table 69: $\left[x^k\right]\left(\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)(2x+2)^n\right)$ (cf. **N**29).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	6	14	10	2						
2	0	20	84	148	144	84	28	4			
3	0	56	344	976	1704	2024	1680	960	360	80	...
4	0	144	1168	4576	11520	20656	27472	27456	20592	11440	...
5	0	352	3552	17760	58400	140064	256416	366112	411840	366080	...

Table 70: $\left[x^k\right]\left((x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)(2x+2)^n\right)$ (cf. **R**29).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	2	6	6	2							
2	0	16	60	88	64	24	4					
3	0	96	496	1064	1224	816	320	72	8			
4	0	512	3456	10112	16752	17216	11312	4736	1232	192	16	
5	0	2560	21504	80640	177600	254112	246560	164256	74400	22240	4192	...

Table 71: $\left[x^k\right]\left((2x+2)^{2n}+\left(x^2-1\right)\left(2x^2+6x+4\right)^n\right)$ (cf. **N**30).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	6	14	10	2						
2	0	32	124	184	128	40	4				
3	0	160	880	2024	2504	1776	704	136	8		
4	0	768	5504	17280	31088	35136	25648	11904	3280	448	...
5	0	3584	31744	126720	300480	469152	504608	379296	197280	68320	...

Table 72: $\left[x^k\right]\left((x+1)(2x+2)^{2n}+\left(x^2-1\right)\left(2x^2+6x+4\right)^n\right)$ (cf. **R**30).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	1	4	6	4	1						
2	0	2	14	42	70	70	42	14	2			
3	0	3	31	145	405	750	966	882	570	255	75	...
4	0	4	55	352	1391	3796	7579	11440	13299	12012	8437	...
5	0	5	86	700	3585	12956	35120	74088	124540	169390	188188	...

Table 73: $\left[x^k\right]\left((x+1)^{4n}+\left(x^2-1\right)(x+1)^{3n}\right)$ (cf. **N**31).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	1									
1	0	2	8	12	8	2						
2	0	3	22	70	126	140	98	42	10	1		
3	0	4	43	211	625	1245	1758	1806	1362	750	295	...
4	0	5	71	472	1951	5616	11947	19448	24739	24882	19877	...
5	0	6	106	890	4725	17801	50624	112848	202060	295360	356148	...

Table 74: $\left[x^k\right]\left((x+1)^{4n+1}+\left(x^2-1\right)(x+1)^{3n}\right)$ (cf. **R**31).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	3	7	5	1						
2	0	18	66	92	60	18	2				
3	0	81	432	972	1200	886	400	108	16	1	
4	0	324	2295	7236	13413	16264	13574	7976	3306	948	...
5	0	1215	10773	43875	108990	184863	226895	208059	144820	76805	...

Table 75: $\left[x^k\right]\left(\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+4x+3\right)^n\right)$ (cf. **N**32).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	6	14	10	2						
2	0	27	108	171	136	57	12	1			
3	0	108	621	1548	2200	1972	1162	452	112	16	...
4	0	405	3051	10422	21417	29591	29038	20772	10898	4147	...
5	0	1458	13608	59130	159165	297558	409782	429334	347815	218990	...

Table 76: $\left[x^k\right]\left((x+1)\left(x^3+5x^2+7x+3\right)^n+\left(x^2-1\right)\left(x^2+4x+3\right)^n\right)$ (cf. **R**32).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1								
1	0	2	6	6	2						
2	0	8	40	80	80	40	8				
3	0	24	176	560	1008	1120	784	336	80	8	
4	0	64	624	2752	7248	12672	15456	13440	8352	3648	...
5	0	160	1952	11040	38400	91872	160160	210144	211200	163680	...

Table 77: $\left[x^k\right]\left(\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(2x^2+4x+2\right)^n\right)$ (cf. **N**33).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1								
1	0	4	12	12	4						
2	0	12	64	140	160	100	32	4			
3	0	32	248	848	1680	2128	1792	1008	368	80	...
4	0	80	816	3808	10768	20592	28128	28224	21024	11568	...
5	0	192	2432	14400	52960	135552	256256	370304	417120	369600	...

Table 78: $[x^k]\left((x+1)\left(2x^3+6x^2+6x+2\right)^n+\left(x^2-1\right)\left(2x^2+4x+2\right)^n\right)$ (cf. **R**33).

$n\backslash k$	0	1	2	3	4	5	6
0	0	1					
1	0	0	1				
2	0	0	0	1			
3	0	0	0	0	1		
4	0	0	0	0	0	1	
5	0	0	0	0	0	0	1

Table 79: $[x^k]\, x^{n+1}$ (cf. **D**35) [Slo20, A129185].

$n\backslash k$	0	1	2	3	4	5
0	0	0	1			
1	0	1				
2	0	0	1			
3	0	0	0	1		
4	0	0	0	0	1	
5	0	0	0	0	0	1

Table 80: $[x^k]\, p_n(x)$, where $p_0(x)=x^2$ and $p_n(x)=x^n$ for $n\geq 1$ (cf. **N**35).

$n\backslash k$	0	1	2	3	4	5	6
0	0	1	1				
1	0	1	1				
2	0	0	1	1			
3	0	0	0	1	1		
4	0	0	0	0	1	1	
5	0	0	0	0	0	1	1

Table 81: $[x^k]\, p_n(x)$, where $p_0(x)=x^2+x$ and $p_n(x)=x^{n+1}+x^n$ for $n\geq 1$ (cf. **R**35).

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