

PRODUCT SUBSET PROBLEM : APPLICATIONS TO NUMBER THEORY AND CRYPTOGRAPHY

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ABSTRACT. We consider applications of Subset Product Problem (SPP) in number theory and cryptography. We obtain a probabilistic algorithm that attack SPP and we analyze it with respect time/space complexity and success probability. In fact we provide an application to the problem of finding Carmichael numbers and an attack to Naccache-Stern knapsack cryptosystem, where we update previous results.

1. INTRODUCTION

In the present paper we study the modular version of subset product problem (MSPP). We consider an application to number theory and cryptography. Furthermore, we shall provide an algorithm for solving MSPP based on birthday paradox attack. Finally we analyze the algorithm with respect to success probability and time/space complexity. Our applications concern the problem of finding Carmichael numbers and as far as the application on cryptography, we update previous results concerning an attack to the Naccache-Stern Knapsack (NSK) public key cryptosystem. We begin with the following definition.

Definition 1.1 (Subset Product Problem). *Given a list of integers L and an integer c , find a subset of L whose product is c .*

This problem is (strong) NP-complete using a transformation from Exact Cover by 3-Sets (X3C) problem [12, p. 224], [39]. Also, see [11, Theorem 3.2], the authors proved that is at least as hard as *Clique* problem (with respect fixed-parameter tractability). In the present paper we consider the following variant.

Definition 1.2 (Modular Subset Product Problem : MSPP $_{\Lambda}$). *Given a positive integer Λ , an integer $c \in \mathbf{Z}_{\Lambda}^*$ and a vector $(u_0, u_1, \dots, u_n) \in (\mathbf{Z}_{\Lambda}^*)^{n+1}$, find a binary vector $m = (m_0, m_1, \dots, m_n)$ such that*

$$(1.1) \quad c \equiv \prod_{i=0}^n u_i^{m_i} \pmod{\Lambda}.$$

The MSPP $_{\Lambda}(\mathcal{P}, c)$ problem can be defined also as follows : Given a finite set $\mathcal{P} \subset \mathbf{Z}_{\Lambda}^*$ and a number $c \in \mathbf{Z}_{\Lambda}^*$, find a subset \mathcal{B} of \mathcal{P} , such that

$$\prod_{x \in \mathcal{B}} x \equiv c \pmod{\Lambda}.$$

2010 *Mathematics Subject Classification.* 11Y16, 11A51, 11T71, 94A60, 68R05.

Key words and phrases. Number Theory, Carmichael Numbers, Product Subset Problem, Public Key Cryptography, Naccache-Stern Knapsack cryptosystem, Birthday attack, Parallel algorithms.

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We can define MSPP for a general abelian finite group G as following. We write G multiplicative.

Definition 1.3 (Modular Subset Product Problem for G , $\text{MSPP}_G(\mathcal{P}, c)$). *Given an element $c \in G$ and a vector $(u_0, u_1, \dots, u_n) \in G^{n+1}$, find a binary vector $m = (m_0, \dots, m_n)$ such that,*

$$(1.2) \quad c = \prod_{i=0}^n u_i^{m_i}.$$

Although in the present work we are interested in $G = \mathbf{Z}_Q^*$ where Q is highly composite number (the case of Carmichael numbers) or prime (the case of NSK cryptosystem).

1.1. Our Contribution. First we provide an algorithm for solving product subset problem based on birthday paradox. This approach is not new, for instance see [29, Section 2.3]. Here we use a variant of [4, Section 3]. We study and implement a parallel version of this algorithm. This result to an improvement of the tables provided in [4]. Further, except the cryptanalysis of NSK cryptosystem, we applied our algorithm to the searching of Carmichael numbers. We used a method of Erdős, to the problem of finding Carmichael numbers with many prime divisors. We managed to generate a Carmichael number with 19589 prime factors¹. Finally, we provide an abstract version of the algorithm in [4, Section 3], to the general product subset problem and further we analyze the algorithm as far as the selection of the parameters (this is provided in Proposition 2.2).

Roadmap. This paper is organized as follows. In section 2 we introduce the attack to MSPP based on birthday paradox. We further provide a detailed analysis. Section 3 is dedicated to the problem of finding Carmichael numbers with many prime factors. We provide the necessary bibliography and known results. In the next section we obtain a second application of MSPP to Naccache-Stern Knapsack cryptosystem. In section 5 we provide experimental results. Finally, the last section contains some concluding remarks.

2. BIRTHDAY ATTACK TO MODULAR SUBSET PRODUCT PROBLEM

We call density of $\text{MSPP}_G(\mathcal{P}, c)$ the positive real number

$$d = \frac{|\mathcal{P}|}{\log_2 |G|}.$$

If $G = \mathbf{Z}_\Lambda^*$, then

$$d = \frac{|\mathcal{P}|}{\log_2 |\mathbf{Z}_\Lambda^*|} = \frac{|\mathcal{P}|}{\log_2 \phi(\Lambda)},$$

where ϕ is the Euler totient function. In a $\text{MSPP}_G(\mathcal{P}, c)$ having a large density, we expect to have many solutions.

A straightforward attack uses birthday paradox paradigm to $\text{MSPP}_\Lambda(\mathcal{P}, c)$. Rewriting equivalence (1.1) as

$$\prod_{i=0}^{\alpha} u_i^{m_i} \equiv c \prod_{i=\alpha+1}^n u_i^{-m_i} \pmod{\Lambda},$$

¹<http://tiny.cc/tm6miz>

for some $\alpha \approx n/2$, we construct two sets, say U_1 and U_2 . The first contains elements of the form $\prod_{i=0}^{\alpha} u_i^{m_i} \pmod{\Lambda}$, and the second $c \prod_{i=\alpha+1}^n u_i^{-m_i} \pmod{\Lambda}$, for all possible (binary) values of $\{m_i\}_i$. So, the problem reduces to finding a common element of sets U_1 and U_2 . Below, we provide the pseudocode of the previous algorithm.

Algorithm 1 : Birthday attack to MSPP $_{\Lambda}(\mathcal{P}, c)$

INPUT : $\mathcal{P} = \{u_i\}_i \subset \mathbf{Z}_{\Lambda}^*$ ($|\mathcal{P}| = n + 1$), $c \in \mathbf{Z}_{\Lambda}^*$ (assume that $\gcd(u_i, \Lambda) = 1$)
 OUTPUT: $\mathcal{B} \subset \mathcal{P}$ such that $\prod_{x \in \mathcal{B}} x \equiv c \pmod{\Lambda}$ or Fail : There is not any solution

- 1: $I_1 \leftarrow \{0, 1, \dots, \lceil n/2 \rceil\}, I_2 \leftarrow \{\lceil n/2 \rceil + 1, \dots, n\}$
- 2: $U_1 \leftarrow \left\{ \prod_{i \in I_1} u_i^{\varepsilon_i} \pmod{\Lambda} : \varepsilon_i \in \{0, 1\} \right\}$
- 3: $U_2 \leftarrow \left\{ c \prod_{i \in I_2} u_i^{-\varepsilon_i} \pmod{\Lambda} : \varepsilon_i \in \{0, 1\} \right\}$
- 4: **If** $U_1 \cap U_2 \neq \emptyset$
- 5: Let y be an element of $U_1 \cap U_2$
- 6: **return** $u_i : \prod u_i \equiv c \pmod{\Lambda}$.
- 7: **else return** Fail : There is not any solution

This algorithm is deterministic, since if there is a solution to MSPP $_{\Lambda}(\mathcal{P}, c)$ the algorithm will find it. To construct the solution in step 6, we use the equation,

$$(2.1) \quad \prod_{i \in I_1} u_i^{\varepsilon_i} = c \prod_{i \in I_2} u_i^{-\varepsilon_i} \quad (\text{in } \mathbf{Z}_{\Lambda}).$$

It turns out $y = \prod_{i=0}^n u_i^{\varepsilon_i} \pmod{\Lambda}$. This algorithm needs $2^{n/2+1}$ elements of \mathbf{Z}_{Λ}^* for storage. To compute the common element (in line 4) we first sort and then we apply binary search. Overall we need $O(2^{n/2} \log_2 n)$ arithmetic operations in the multiplicative group \mathbf{Z}_{Λ}^* .

We can improve the previous algorithm as far as the space complexity. First, we need to define the notion of hamming weight of c . Let $c \in \mathbf{Z}_{\Lambda}^*$. We set

$$\text{sol}(c; \mathcal{P}, \Lambda) = \left\{ I \subset \{0, 1, \dots, n\} : c \equiv \prod_{i \in I, u_i \in \mathcal{P}} u_i \pmod{\Lambda} \right\}.$$

From this set we can easily build the set of all solutions of the MSPP $_{\Lambda}(\mathcal{P}, c)$. Indeed, if J_1, J_2, \dots, J_k are all the sets in $\text{sol}(c; \mathcal{P}, \Lambda)$, then the set of solution is

$$\left\{ \{u_i\} : i \in J_1 \right\} \cup \left\{ \{u_i\} : i \in J_2 \right\} \cup \dots \cup \left\{ \{u_i\} : i \in J_k \right\}.$$

We can simplify if we use the following characteristic function

$$\chi : \text{sol}(c; \mathcal{P}, \Lambda) \rightarrow \{0, 1\}^{n+1},$$

such that $\chi(I) = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$, where $\varepsilon_i = 1$ if $i \in I$ else $\varepsilon_i = 0$. Thus, the set of solutions

$$\text{Sol}(c; \mathcal{P}, \Lambda) = \left\{ (u_0^{\varepsilon_0}, u_1^{\varepsilon_1}, \dots, u_n^{\varepsilon_n}) : \chi(I) = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \text{ for all } I \in \text{sol}(c; \mathcal{P}, \Lambda) \right\}.$$

To each element I of $\text{sol}(c; \mathcal{P}, \Lambda)$ (assuming there exists one) we correspond the natural number $H_I(c) = |I|$ (the cardinality of I). We call this number local

Hamming weight of c at I . We call Hamming weight $H(c)$ of c , the minimum of all these numbers. I.e.

$$H(c) = \min\{H_I(c) : I \in \text{Sol}(c; \mathcal{P}, \Lambda)\}.$$

Now, we consider two positive integers, say h_1, h_2 , such that, $h_1 + h_2 = H_I(c)$, and two disjoint subsets I_1, I_2 of $\{0, 1, \dots, n\}$ with $|I_1| = |I_2| = b$, for some positive integer $b \leq n/2$. Finally, we consider the sets,

$$U_{h_1}(I_1; \mathcal{P}, \Lambda) = \left\{ \prod_{i \in I_1} u_i^{\varepsilon_i} \pmod{\Lambda} : \sum_{i \in I_1} \varepsilon_i = h_1 \right\},$$

$$U_{h_2}(I_2, c; \mathcal{P}, \Lambda) = \left\{ c \prod_{i \in I_2} u_i^{-\varepsilon_i} \pmod{\Lambda} : \sum_{i \in I_2} \varepsilon_i = h_2 \right\}.$$

We usually write them as $U_{h_1}(I_1)$ and $U_{h_2}(I_2, c)$ when \mathcal{P}, Λ are fixed. We have,

$$|U_{h_1}(I_1)| = \binom{|I_1|}{h_1}, \quad |U_{h_2}(I_2, c)| = \binom{|I_2|}{h_2}.$$

Remark 2.1. The set U_1 of Algorithm 1 is written,

$$U_1 = \bigcup_{h_1=0}^{\lceil n/2 \rceil + 1} U_{h_1}(\{0, 1, \dots, \lceil n/2 \rceil\}; \mathcal{P}, \Lambda).$$

Similar for U_2 .

For the following algorithm we assume that we know a local Hamming weight of the target number c .

Algorithm 1(a) $\text{BA_MSPP}_\Lambda(\mathcal{P}, c; b, \ell, Q, \text{Iter})^2$: **Memory efficient attack to** $\text{MSPP}_\Lambda(\mathcal{P}, c)$

INPUT:

- i. A set $\mathcal{P} = \{u_i\}_i \subset \mathbf{Z}_\Lambda^*$ with $|\mathcal{P}| = n + 1$ (assume that $\gcd(u_i, \Lambda) = 1$),
- ii. a number $c \in \mathbf{Z}_\Lambda^*$,
- iii. a local Hamming weight of c , say ℓ ,
- iv. a positive number b : $\ell \leq b \leq n/2$.
- v. a hash function \mathbb{H} , where we assume that the output is given as a series of hex characters.
- vi. Q : is a positive integer. $\mathbb{H}_Q(x)$ is the sequence consists from the first $4Q$ bits (or Q -hex characters) of the hash function \mathbb{H} applied to some number x .
- vii. Iter is a positive integer.

OUTPUT: a set $\mathcal{B} \subset \mathcal{P}$, such that $\prod_{x \in \mathcal{B}} x \equiv c \pmod{\Lambda}$ or Fail

- 1: $(h_1, h_2) \leftarrow (\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil)$
- 2: **For** i in $1, \dots, \text{Iter}$
- 3: $(I_1, I_2) \xleftarrow{\$} \{0, \dots, n\} \times \{0, \dots, n\}$ (with I_1, I_2 disjoint and with b elements.)
- 4: $U_{h_1}^*(I_1) \leftarrow \left\{ \mathbb{H}_Q\left(\prod_{i \in I_1} u_i^{\varepsilon_i} \pmod{\Lambda}\right) : \sum_{i \in I_1} \varepsilon_i = h_1, \varepsilon_i \in \{0, 1\} \right\}$
- 5: **For** each $(\varepsilon_i)_i$ such that $\sum_{i \in I_1} \varepsilon_i = h_2$

²BA : Birthday Attack

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6:      If  $\mathbb{H}_Q(c \prod_{i \in I_2} u_i^{-\varepsilon_i} \pmod{\Lambda}) \in U_{h_1}^*(I_1)$ 
7:          Let  $y$  be an element of  $U_{h_1}^*(I_1) \cap U_{h_2}^*(I_2, c)$ 
8:          return  $\{u_i\}_i$  such that  $\prod u_i \equiv c \pmod{\Lambda}$  and terminate
9: return Fail
    
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This algorithm is memory efficient version of algorithm 1, since we consider subsets of U_1, U_2 . Although, this algorithm may fail, even when $\text{MSPP}_\Lambda(\mathcal{P}, c)$ has a solution. For instance, if we pick I_1, I_2 such that the union $I_1 \cup I_2 \notin \text{sol}(c; \mathcal{P}, \Lambda)$, then the algorithm will fail. This may occur when $b < n/2$, that is $I_1 \cup I_2 \subset \{0, 1, \dots, n\}$. If $I_1 \cup I_2 = \{0, 1, \dots, n\}$, then $I_1 \cup I_2 \in \text{sol}(c; \mathcal{P}, \Lambda)$. In this case, the algorithm remains probabilistic, since we consider a specific choice of (h_1, h_2) and not all the possible (h_1, h_2) , with $h_1 + h_2 = \ell$. If we consider all (h_1, h_2) such that $h_1 + h_2 = \ell$, then the algorithm is deterministic. In this case the analysis below is similar.

We analyze the algorithm line by line.

Line 3: This can be implemented easily in the case where $2b < \sqrt{n}$. Indeed, we can use rejection sampling in the set $\{0, \dots, n\}$ and construct a list of length $2b$. Then, I_1 is the set consisting from the first b elements and I_2 the rest. If $2b \geq \sqrt{n}$ then, we have to sample from the set $\{0, 1, \dots, n\}$, so the memory increases since we have to store the set $\{0, 1, \dots, n\}$. When we apply the algorithm to the searching of Carmichael numbers we indeed have $2b \ll \sqrt{n}$. In the case of NSK cryptosystem we usually have $2b > \sqrt{n}$.

Line 4: The most intensive part (both for memory and time complexity) is the construction of the set $U_{h_1}^*$. Here we can parallelize our algorithm to decrease time complexity. To reduce the space complexity we decrease the two parameters $b \leq n/2$ and Q . If we do not truncate our hash function, then we store all the hex digits of the hash (usually 128-bits). In subsection 2.4 we provide a strategy to choose Q .

Line 5-6: In Line 4 we stored $U_{h_1}^*(I_1)$, in this line we compute *on the fly* the elements of the second set $U_{h_2}^*(I_2, c)$ and check if any is in $U_{h_1}^*(I_1)$. So we do not need to store $U_{h_2}^*$. A suitable data structure for the searching is the hashtable, which we also used in our implementation. Hashtables have the advantage of the fast insert, delete and search operations. Since these operations have $O(1)$ time complexity on the average.

Line 7-8: Having the element found by the previous step (Line 6), say y , we construct the u_i 's such that their product is y . We return Fail if the intersection is empty for all the iterations.

Remark 2.2. *If we do not consider any hash function and $\text{Iter} = 1$, then we write*

$$\text{BA_MSPP}_\Lambda(\mathcal{P}, c; b, \ell).$$

We implemented³ the previous algorithm and in a few seconds we found a solution of $\text{MSPP}_\Lambda(\mathcal{P}, c)$ for $\mathcal{P} = \{2, 3, \dots, 10^7\}$, $\Lambda = 10000019$, $c = 190238$ with local hamming weight 11, $b = 12$ and $\text{iter} = 150$. We got the following solution:

$$\begin{aligned} \mathcal{A} = \{ & 9851537, 303860, 4680021, 9647209, 2006838, 9984877, \\ & 2512434, 2126904, 1942182, 8985302, 2193757\}. \end{aligned}$$

³For a C++/Sagemath implementation see, <https://github.com/drazioti/Carmichael>

Straightforward calculations provide,

$$\prod_{x \in \mathcal{A}} x \equiv 190238 \pmod{\Lambda}.$$

2.1. Space complexity. We assume that there is not any collision in the construction of $U_{h_1}^*(I_1)$ and $U_{h_2}^*(I_2, c)$, or in other words we choose Q to minimize the probability to have a collision. I.e. we choose $2^{4Q} \gg \max\{|U_1|, |U_2|\}$ and we assume that \mathbb{H} is behaving random enough. In practice (or at least in our examples) we always have this constraint.

So we get,

$$|U_{h_1}(I_1)| = |U_{h_1}^*(I_1)| = \binom{|I_1|}{h_1} \quad \text{and} \quad |U_{h_2}(I_2, c)| = |U_{h_2}^*(I_2, c)| = \binom{|I_2|}{h_2},$$

where $(h_1, h_2) = (\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil)$. By choosing $|I_1| = |I_2| = b$, we get

$$|U_{h_1}^*(I_1)| = \binom{b}{h_1} \quad \text{and} \quad |U_{h_2}^*(I_2, c)| = \binom{b}{h_2}.$$

We set

$$S_b = \binom{b}{h_1} = B_b(h_1).$$

In our algorithm, we need to store the S_b hashes of the set $U_{h_1}^*(I_1)$. We need $4Q$ -bits for keeping Q -hex digits in the memory. So, overall we store $4QS_b$ bits. Furthermore, we store the binary exponents $(\varepsilon_i)_i$ that are necessary for the computation of the products, $\prod_{i \in I_1} u_i^{\varepsilon_i}$. This is needed, since we must reconstruct c as a product of $(u_i)_i$. These are S_b , thus we need $b \times S_b$ - bits. Since we also need to store the set \mathcal{P} of length $n + 1$, we conclude that,

$$(2.2) \quad \mathbb{M} < (4Q + b)S_b + (n + 1)B \quad (\text{bits}),$$

where $B = \max\{\log_2(x) : x \in \mathcal{P}\}$. Remark that \mathbb{M} does not depend on the modulus Λ .

ℓ	9	11	13	15	17	19	21
$\mathbb{M} (GB)$	0.029	0.05	0.2	1.16	6.15	28.61	117.2

TABLE 1. For $b = 50$, $Q = 12$ and $\mathcal{P} = \{2, 3, \dots, 10^7\}$.

If we can describe in an efficient way the set \mathcal{P} we do not need to store it. Say, that $\mathcal{P} = \{2, 3, \dots, n\}$. Then, there is no need to store it in the memory, since the sequence $f(x) = x + 1$ describes efficiently the set \mathcal{P} . Also, in other situations B can be stored using $O(|\mathcal{P}| \log_2(|\mathcal{P}|))$ bits. We call such sets *nice* and they can save us enough memory. In fact, for nice sets the inequality (2.2) changes to,

$$(2.3) \quad \mathbb{M} < (4Q + b)S_b + O(n \log_2 n) \quad (\text{bits}).$$

In fact when we apply this algorithm to the problem of finding Carmichael numbers, we shall see that the set \mathcal{P} is *nice*.

Finally, if U_{h_1} is very large we can make chunks of it, to store it in the memory. Note that this can not be done if we directly compute the intersection of $U_{h_1} \cap U_{h_2}$ as in [4]. This simple trick considerable improves the algorithms in [4].

2.2. Time complexity. Time complexity is dominated by the construction of the sets U_{h_1} and U_{h_2} and the calculation of their intersection. We work with U_{h_1} instead of $U_{h_1}^*$ since all the multiplications are between the elements of U_{h_1} , only in the searching phase we move to $U_{h_1}^*$. Let M_Λ be the bit-complexity of the multiplication of two integers $\pmod{\Lambda}$. So

$$M_\Lambda = O((\log_2 \Lambda)^{1+\varepsilon})$$

for some $0 < \varepsilon \leq 1$ (for instance Karatsuba suggests $\varepsilon = \log_2 3 - 1$ [19]). In fact recently was proved $M_\Lambda = O(\log_2 \Lambda \log_2 (\log_2 \Lambda))$ [16]. To construct the sets U_{h_1} and U_{h_2} (ignoring the cost for the inversion $\pmod{\Lambda}$) we need $M_\Lambda h_1 B_b(h_1) = M_\Lambda h_1 S_b$ bit-operations for the set U_{h_1} and $M_\Lambda h_2 B_b(h_2 + 1) \approx M_\Lambda h_2 S_b$ bit-operations for U_{h_2} . So overall,

$$\mathbb{T}_1 = M_\Lambda(h_1 S_b + h_2 S_b) = M_\Lambda S_b(h_1 + h_2) \text{ bits.}$$

Considering the time complexity for finding a collision in the two sets by using a hashtable, we get

$$\mathbb{T} = \mathbb{T}_1 + O(1)S_b \text{ bits on average}$$

and

$$\mathbb{T} = \mathbb{T}_1 + O(1)S_b^2 \text{ in the worst case.}$$

We used that $h_1 \approx h_2$ (they differ at most by 1). In case we have T threads we get about \mathbb{T}/T (bit operations) instead of \mathbb{T} .

2.3. Success Probability. In the following Lemma we compute the probability to get a common element in $U_{h_1}(I_1)$ and $U_{h_2}(I_2, c)$ when $(I_1, I_2) \stackrel{\S}{\leftarrow} \{0, \dots, n\} \times \{0, \dots, n\}$, where I_1, I_2 are disjoint, with b elements. Let $0 \leq y \leq x$. With $B_x(y)$ we denote the binomial coefficient $\binom{x}{y}$.

Lemma 2.1. *Let h_1, h_2 be positive integers and $\ell = h_1 + h_2$. The probability to get $U_{h_1}(I_1) \cap U_{h_2}(I_2, c) \neq \emptyset$ is,*

$$\mathbb{P} = \frac{B_b(h_1)B_b(h_2)}{B_{n+1}(\ell)}.$$

For the proof see [4, section 3].

We can easily provide another and simpler proof in the case $2b = n + 1$ (this occurs very often when attacking Naccache-Stern cryptosystem). Then,

$$\mathbb{P} = \text{hyper}(x; 2b, b, \ell),$$

where *hyper* is the hypergeometric distribution,

$$\text{hyper}(x; N, b, \ell) = Pr(X = x) = \frac{\binom{b}{x} \binom{N-b}{\ell-x}}{\binom{N}{\ell}}.$$

Where,

- $N = n + 1$ is the population size
- ℓ is the number of draws
- b is the number of successes in the population
- x is the number of observed successes

Adapting to our case, we set $N = 2b(= n + 1)$, $\ell(= h_1 + h_2)$ and $x = h_1$. Then,

$$\text{hyper}(x = h_1; n + 1, b, \ell) = Pr(X = h_1) = \frac{\binom{b}{h_1} \binom{b}{\ell-h_1}}{\binom{n+1}{\ell}} = \mathbb{P}.$$

The expected value is $\frac{b\ell}{n+1}$ and since $b = (n+1)/2$ we get $EX = \frac{\ell}{2}$. Since the random variable X counts the successes we expect on average to have $\ell/2$ after considering enough instances (i.e. choices of I_1, I_2). The maximum value of ℓ is $n/2$. So in this case the expected value is maximized. So we expect our algorithm to find faster a solution from another one that uses smaller value for ℓ .

Also, we need about

$$\frac{1}{\mathbb{P}} = \frac{\binom{n+1}{\ell}}{\binom{b}{h_1}\binom{b}{h_2}} \text{ iterations on average to find a solution.}$$

One last remark is that the contribution of b is larger than the contribution of hamming weight, in the probability \mathbb{P} .

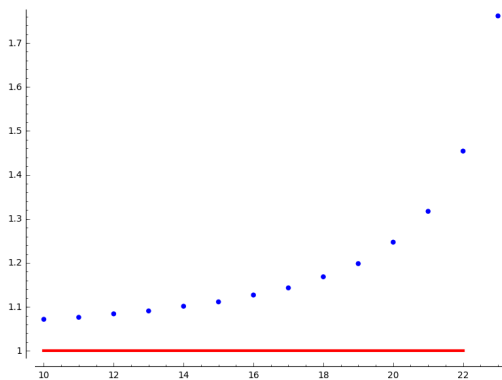


FIGURE 1. We fixed $n = 23000$ and $b = 35$. We consider pairs $(h, f(h))$, where $f(h) = (\mathbb{P}(b+1, h)/\mathbb{P}(b, h+1))$ and h the hamming weight. We finally normalized the values by taking logarithms. So, for a given (b, h) , if we want to increase the probability is better strategy to increase b than h .

2.3.1. *The best choice of h_1 and h_2 .* The choice of h_1, h_2 (in line 1 of algorithm 1(a)) is $h_1 = \lfloor \ell/2 \rfloor$, $h_2 = \lceil \ell/2 \rceil$. This can be explained easily, since these values maximize the probability \mathbb{P} of Lemma 2.1. We set

$$J_\ell = \{(x, y) \in \mathbf{Z}^2 : x + y = \ell, 0 \leq x \leq y\}.$$

Observe that $(\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil) \in J_\ell$.

Proposition 2.2. *Let b and n be fixed positive integers such that $\ell = x + y \leq b \leq n/2$. Then the finite sequence $\mathbb{P} : J_\ell \rightarrow \mathbb{Q}$, defined by*

$$\mathbb{P}(x, y) = \frac{B_b(x)B_b(y)}{B_{n+1}(\ell)},$$

is maximized for $(x, y) = (\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil)$.

We need the following simple lemma.

Lemma 2.3.

$$\binom{b}{x} \binom{b}{\ell-x} \binom{2b}{b} = \binom{\ell}{x} \binom{2b-\ell}{b-x} \binom{2b}{\ell}.$$

Proof. It is straightforward by expressing the binomial coefficients in terms of factorials and rearranging them. \square

of Proposition 2.2. . From the previous lemma we have

$$(2.4) \quad \frac{\binom{b}{x} \binom{b}{\ell-x}}{\binom{2b}{\ell}} = \frac{\binom{\ell}{x} \binom{2b-\ell}{b-x}}{\binom{2b}{b}}.$$

Since b, ℓ are fixed, the maximum of the right hand side is at $x = \lfloor \ell/2 \rfloor$. Indeed, both sequences $\binom{b}{x}$ and $\binom{2b-\ell}{b-x}$ are positive and maximized at $x = \lfloor \ell/2 \rfloor$. So also the product $\binom{\ell}{x} \binom{2b-\ell}{b-x}$ is maximized at $x = \lfloor \ell/2 \rfloor$. The same occurs to the left hand side. Since the denominator of the left hand side in (2.4) is fixed, the numerator $B_b(x)B_b(\ell-x) = \binom{b}{x} \binom{b}{\ell-x}$ is maximized at $x = \lfloor \ell/2 \rfloor$. Since n is also a fixed positive integer, the numerator of \mathbb{P} is maximized at $x = \lfloor \ell/2 \rfloor$ and so \mathbb{P} is maximized at the same point. Finally, $\ell - \lfloor \ell/2 \rfloor = \lceil \ell/2 \rceil = y$. The Proposition follows. \square

2.4. How to choose Q . If we are searching for r -same objects to one set (with cardinality n), when we pick the elements of the set from some largest set (with cardinality m), then we say that we have a r -multicollision. We have the following Lemma.

Lemma 2.4. *If we have a set with m elements and we pick randomly (and independently) n elements from the set, then the expected number of r -multicollisions is approximately*

$$(2.5) \quad \frac{n^r}{r!m^{r-1}}.$$

Proof. [18, section 6.2.1] \square

Say we use md5 hash function. If we use the parameter Q , we have to truncate the output of md5, which has 16-hex strings, to Q -hex strings ($Q < 16$). I.e we only consider the first $\kappa = 4 \cdot Q$ -bits of the output. Our strategy to choose Q uses formula (2.5). In practice, is enough to avoid $r = 3$ -multicollisions in the set $U_{h_1}^*(I_1) \cup U_{h_2}^*(I_2, c)$ of cardinality S_b , where

$$S_b = |U_{h_1}^*(I_1) \cup U_{h_2}^*(I_2, c)| = \binom{|I_1|}{h_1} + \binom{|I_1|}{h_2}.$$

We set the formula (2.5) equal to 1 and we solve with respect to m , which in our case is $m = 2^\kappa$. So, $\frac{S_b^3}{6} = 2^{2\kappa}$. Therefore we get, $\kappa \approx (3 \log_2 S_b)/2$ (since in our examples $\log_2(S_b) \gg \log_2 6$). For instance if we have local Hamming weight ≤ 13 , $b = n/2$, $|I_1| = |I_2| = b$, and $n = 232$, we get $\kappa \approx 52$. So, $Q \approx 13$. In fact we used $Q = 12$ in our attack to Naccache-Stern knapsack cryptosystem.

3. CARMICHAEL NUMBERS

Fermat proved that if p is a prime number, then p divides $a^p - a$ for every integer a . This is known as Fermat's Little Theorem. The question if the converse is true has negative answer. In fact in 1910 Carmichael noticed that 561 provides such a counterexample. A Carmichael number⁴ is a positive composite integer n such that $a^{n-1} \equiv 1 \pmod{n}$ for every integer a , with $1 < a < n$ and $\gcd(a, n) = 1$. They named after Robert Daniel Carmichael (1879-1967). Although, the Carmichael

⁴See also, <http://oeis.org/A002997>

numbers between 561 and 8911 i.e. the first seven, initially they discovered by the Czech mathematician V. Šimerka in 1885 [37]. In 1910 Carmichael conjectured that there is an infinite number of Carmichael numbers. This conjecture was proved in 1994 by Alford, Granville, and Pomerance [2]. Although, the problem if there are infinitely many Carmichael numbers with exactly $R \geq 3$ prime factors, is remained open until today. We have the following criterion.

Proposition 3.1. (*Korselt, 1899, [21]*) *A positive integer n is Carmichael if and only if is composite, square-free and for every prime p with $p|n$, we get $p-1|n-1$.*

For a simple elegant proof see [9]. We define the following function.

$$\lambda(2^a) = \phi(2^a) \text{ if } a = 0, 1, 2$$

and

$$\lambda(2^a) = \frac{1}{2}\phi(2^a), \text{ if } a > 2.$$

If p_i is odd prime and a_i positive integer

$$\lambda(p_i^{a_i}) = \phi(p_i^{a_i}).$$

If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ then

$$\lambda(n) = \text{lcm}(\lambda(p_1^{a_1}), \lambda(p_2^{a_2}), \dots, \lambda(p_k^{a_k})).$$

Korselt's criterion can be written as

Proposition 3.2. (*Carmichael, 1912, [7]*) *n is Carmichael if and only if is composite and $n \equiv 1 \pmod{\lambda(n)}$.*

Using the previous, we can prove that a Carmichael number is odd and have at least three prime factors. Furthermore, we can calculate some Carmichael numbers (the first 16): 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 52633, 62745, 63973, and 75361. In [3] they used an idea of Erdős [10] to find Carmichael numbers with many prime factors. In 1996, Loh and Niebuhr [22] provided a Carmichael with 1,101,518 prime factors using Erdős heuristic algorithm (Algorithm 1). Also, an analysis and some refinements of [22] and an extension to other pseudoprimes was provided by Guillaume and Morain in 1996 in [14]. Further, in the same paper the authors provided a Carmichael number having 5104 prime factors. In 2014 [3, Table 1] the authors provided two large Carmichael numbers with many prime factors. The first one with 1,021,449,117 prime factors and $d = 25,564,327,388$ decimal digits and the second with 10,333,229,505 prime factors, with $d = 29,548,676,178$ decimal digits.

Also, in 1975 J. Swift [36] generated all the Carmichael numbers below 10^9 . In 1979, Yorinaga [40] provided a table for Carmichael numbers up to 10^{10} using the method of Chernick (this method allow us to construct a Carmichael number having already one, see [14, Theorem 2.2]). In 1980, Pomerance, Selfridge and Wagstaff [34] generated Carmichael numbers up to $25 \cdot 10^{10}$. In 1988 Keller [20] calculated the Carmichael numbers up to 10^{13} . In 1990, Jaeschke [17] provided tables for Carmichael numbers up to 10^{12} . Pinch provided a table for all Carmichael numbers up to 10^{18} ([26]). Also the same author in 2006 [27] computed all Carmichael numbers up to 10^{20} and in 2007 [28] a table up to 10^{21} . Furthermore, he found 20138200 Carmichael numbers up to 10^{21} and all of them have at most 12 prime factors.

For an illustration of our algorithm we also generated some tables for Carmichael numbers having many prime factors⁵. For instance we produced Carmichael numbers up to 250 prime factors. Each instance was generated in some seconds. Also Carmichael numbers with 11725 and 19589 prime factors were generated in some hours with our algorithm, in a small home PC (I3/16Gbyte) using a C++/gmp implementation.

The following method is based on Erdős idea [10]. It was used in [3, 22] to produce Carmichael numbers having large number of prime factors.

Algorithm 3 : Generation of Carmichael Numbers

INPUT: A positive integer r and a vector $\mathbf{H} = (h_1, h_2, \dots, h_r) \in \mathbf{Z}^r$, with $h_1 \geq h_2 \geq \dots \geq h_r \geq 1$. Also we consider two positive integers ℓ and b which correspond to the local hamming and the bound, respectively.

OUTPUT: A Carmichael number or Fail

- 1: $Q \leftarrow \{q_1, \dots, q_r\}$ the r -first prime numbers
- 2: $\Lambda \leftarrow q_1^{h_1} \dots q_r^{h_r}$
- 3: $\mathcal{P} \leftarrow \{p : p \text{ prime } p-1 | \Lambda, p \nmid \Lambda\}$
- 4: $S \leftarrow \text{BA_MSPP}_\Lambda(\mathcal{P}, 1; b, \ell)$
- 5: **If** $|S| \geq 2$ **return** $\prod_{p \in S} p$
- 6: $B \leftarrow \prod_{p \in \mathcal{P}} p$
- 7: $T \leftarrow \text{BA_MSPP}_\Lambda(\mathcal{P}, B; b, \ell)$
- 8: **If** $|T| \geq 2$ **return** $B / \prod_{p \in T} p$
- 9: **else return** Fail

In line 8, we return the number $\prod_{p \in \mathcal{P}-T} p$.

Correctness.

It is enough to prove that the numbers returned in steps 5 and 8 are Carmichael. Set $n = \prod_{p \in S} p$. We shall prove it for step 5. The set S contains all the primes of \mathcal{P} such that their product is equivalent to 1 (mod Λ). Since $|S| \geq 2$, n is composite and also is squarefree. Say a prime p is such that $p|n$. Since $n \equiv 1 \pmod{\Lambda}$ i.e. $\Lambda|n-1$ we get $p-1|n-1$. Indeed, this is immediate since $p-1|\Lambda$. From Korselt's criterion we get that n is Carmichael. Similar for the step 8.

We have set $\ell = \text{local_hamming}$. In case of success, the output of the algorithm is a Carmichael number with ℓ or $|\mathcal{P}| - \ell$ prime factors. In fact, if we want to calculate a Carmichael number with many prime factors, we can ignore the lines 4 and 5 and consider a large set \mathcal{P} . An estimation for $|\mathcal{P}|$ was given in [22, formula 4],

$$|\mathcal{P}| \approx g(\Lambda) \prod_{j=1}^r \left(h_j + \frac{q_j - 2}{q_j - 1} \right), \text{ where } g(\Lambda) = \frac{\Lambda}{\phi(\Lambda) \ln \sqrt{2\Lambda}}.$$

In lines 1 and 2 we initialize the algorithm. Since in practice r is not large enough, both these steps are very efficient.

In line 3 we calculate the set \mathcal{P} . One way to construct this set is the following. Say $d|\Lambda$. If $d+1$ is prime with $d+1 \notin Q$ then $d \in \mathcal{P}$. To find the divisors of Λ having their prime divisors is a simple combinatorial problem. We can implement

⁵see, <https://github.com/drazioti/Carmichael>

this without using much memory. Even better, we can use [3, Section 8] where they keep only the exponents of the divisors of Λ . Since the set \mathcal{P} contains integers of the form $2^{a_1}3^{a_2} \cdots p_r^{a_r} + 1$ with $0 \leq a_i \leq h_i$, instead of storing $2^{a_1}3^{a_2} \cdots p_r^{a_r} + 1$ we can store (a_1, \dots, a_r) . Overall $8r|\mathcal{P}|$ bits or $r|\mathcal{P}|$ bytes. So the set \mathcal{P} is *nice*, since the set B in formula (2.2) needs $O(|\mathcal{P}| \log_2(|\mathcal{P}|))$ bits for storage.

In line 4 (and 7), we use algorithm 1(a) with $b = \text{bound}$ and $\ell = \text{local_hamming}$ according to the user choice. We can apply BA_MSPP with the parameters Q and Iter, $\text{BA_MSPP}_\Lambda(\mathcal{P}, c; b, \ell, Q, \text{Iter})$. In [22] they picked T randomly from \mathcal{P} .

Remark 3.1. In [3] they used another algorithm inspired by the quantum algorithm of Kuperberg and they exploit the distribution of the primes in the set \mathcal{P} (which is not uniform).

Remark 3.2. When $|\mathcal{P}|$ is large enough then using $B = 1$ as target number we can easily find a Carmichael number with small number of prime factors (by using small local Hamming weight). If we use $B > 1$ as in line 5 we get a Carmichael number with many prime factors. As we remarked previous the number of prime factors of the Carmichael number is either ℓ or $|\mathcal{P}| - \ell$. One advantage of the algorithm is that we can search for Carmichael numbers near $|\mathcal{P}| - r$. This can be done by considering $\ell = \text{local_hamming}$ close to r . In this way we quickly generated Carmichael numbers up to 250 prime factors in a small PC.

4. NACCACHE-STERN KNAPSACK CRYPTOSYSTEM

In this section we consider a second application of MSPP to cryptography. We shall provide an attack to a public key cryptosystem. Naccache-Stern Knapsack (NSK) cryptosystem is a public key cryptosystem ([29]) based on the Discrete Logarithm Problem (DLP), which is a difficult number theory problem. Furthermore, it is based on another combinatorial problem, the Modular Subset Product Problem. Our attack applies to the latter problem. NSK cryptosystem is defined by the following three algorithms.

i. Key Generation:

Let p be a large safe prime number (that is $p = 2q + 1$, where q is a prime number). Let n denotes the largest positive integer such that:

$$(4.1) \quad p > \prod_{i=0}^n p_i,$$

where p_i is the $(i+1)$ -th prime. The message space of the system is $\mathcal{M} = \{0, 1\}^{n+1}$, this is the set of the binary strings of $(n+1)$ -bits. For instance, if p has 2048 bits, then $n = 232$ and if p has 1024 bits, then $n = 130$.

We randomly pick a positive integer $s < p - 1$, such that $\gcd(s, p - 1) = 1$. This last property guarantees that there exists the (unique) s -th root \pmod{p} of an element in \mathbf{Z}_p^* . Set

$$u_i = \sqrt[s]{p_i} \pmod{p} \in \mathbf{Z}_p^*.$$

The public key is the vector

$$(p, n; u_0, \dots, u_n) \in \mathbf{Z}^2 \times (\mathbf{Z}_p^*)^{n+1}$$

and the secret key is s .

ii. Encryption:

Let m be a message and $\sum_{i=0}^n 2^i m_i$ its binary expansion. The encryption of the $n + 1$ bit message m is $c = \prod_{i=0}^n u_i^{m_i} \pmod p$.

iii. Decryption:

To decrypt the ciphertext c , we compute

$$m = \sum_{i=0}^n \frac{2^i}{p_i - 1} \times (\gcd(p_i, c^s \pmod p) - 1).$$

From the description of the NSK scheme, we see that the security is based on the Discrete Logarithm Problem (DLP). It is sufficient to solve $u_i^x = p_i$ in \mathbf{Z}_p^* , for some i . The best algorithm for computing DLP in prime fields has subexponential bit complexity, [1, 13]. Thus, for large p (at least 2048 bits) the system can not be attacked by using the state of the art algorithms for DLP.

We have also assumed that the prime number p belongs to the special class of safe primes to prevent attacks such as, Pollard rho [33], Pollard $p - 1$ algorithm [32], Pohlig-Hellman algorithm [31] or any similar procedure that exploits properties of $p - 1$.

4.1. **The attack.** Since, $c \equiv \sum_{i=0}^n 2^i m_i \pmod \Lambda$, we get

$$(4.2) \quad \prod_{i \in I_1} u_i^{m_i} = c \prod_{i \in I_2} u_i^{-m_i} \quad (\text{in } \mathbf{Z}_\Lambda).$$

So we can apply BA_MSPP_Λ with input $\mathcal{P} = (u_i)_i$ and c and for some bound b and hamming weight of the message m say H_m i.e. the number of 1's in the binary message m . So in this attack we assume that we know the hamming weight or an upper bound of it. To be more precise, this attack is feasible only for small or large hamming weights. Our parallel version allow us to consider larger hamming weights than in [4].

In the following algorithm we call algorithm 1(a), where we execute steps 4 and 5 in parallel (in function BA_MSPP_Λ)

Algorithm 4 : Attack to NSK cryptosystem

- INPUT: \circ The cryptographic message c
 \circ the Hamming weight H_m of the message m
 \circ a bound $b \leq n/2$
 \circ the public key $pk = (p, n; u_0, \dots, u_n)$ of NSK cryptosystem

OUTPUT: the message m or Fail

- 1: $\mathcal{P} \leftarrow \{u_0, \dots, u_n\}$
- 2: $\mathbb{S} \leftarrow \text{BA_MSPP}_p(\mathcal{P}, c; b, H_m)$
- 3: if $\mathbb{S} \neq \emptyset$ construct m . Else return Fail

4.1.1. *Reduction of the case of large Hamming weight messages.* The case where we have large Hamming weight of a message can be reduced to the case where we have small Hamming weight. Indeed, if the message m has $H_m = n + 1 - \varepsilon$, where ε is a small positive integer, then again we can reduce the problem to one with small Hamming weight. Let $c = \text{Enc}(m)$ and $c' = c^{-1} u_n^2 \prod_{i=0}^{n-1} u_i$. We provide the following Lemma.

Lemma 4.1. *The decryption of c' is $m' = 2^{n+1} + 2^n - m - 1$, where $H_{m'} = \varepsilon + 1$.*

Proof. [4, Lemma 3.4] □

So we can consider $H_m < b$. Indeed, if $H_m \geq b$, we apply the previous Lemma and we get $H_{m'} < b$. So finding m' is equivalent finding m .

4.1.2. *The case of knowing some bits of the message.* If we know the position of some bits of the message m (for instance by applying a fault attack to the system may leak some bits), then we can improve our attack. In this case, we choose I_1 and I_2 in algorithm 4, from the set $\{0, 1, \dots, n\} - K$, where the set K contains the positions of the known bits. Also, in line 10, when we reconstruct the message m (in case of a collision) we need to put the known bits to the right positions.

5. EXPERIMENTAL RESULTS FOR NS

In our implementation⁶ we used C/C++ with GMP library [15] and for parallelization OpenMP [30]. We used 20 threads of an Intel(R) Xeon(R) CPU E5-2630 v4 @2.20GHz, in a Linux platform.

First, in table 2 we present the improvement of the results provided in [4] by using the parallel version. Besides, in table 3 we extend the results of the previous table. In fact, table 3 demonstrates that having a suitable number of threads and considering a suitable bound b we get a practical attack for low Hamming weight messages. In figure 2 we represent some of our data graphically.

len(p)	600			1024			2048	
H_m	8	9	10	8	9	10	7	8
Attack [4]	42s	300s	7.5m	8.1m	1.1h	1.96h	1.17h	2.15h
Parallel Attack	< 1s	6.54s	15.56s	11.44s	70s	284s	70s	11.46m

TABLE 2. We used $b = n/2, Q = 12$, where $n = 84, 130$ and 232 , for len(p) = 600, 1024 and 2048 bits, respectively. For each column, we executed 10 times the attack of [4] and our parallel version (algorithm 4), and we computed the average CPU time.

6. CONCLUSIONS

In the present work we considered a parallel algorithm to attack the modular version of product subset problem. This is a NP-complete problem which have many applications in computer science and mathematics. Here we provide two applications, one in number theory and the other to cryptography.

First we applied our algorithm (providing a C++ implementation) to the the problem of searching Carmichael numbers. We managed to find one with 19589 factors in a small PC in 3 hours.

For the Naccache-Stern knapsack cryptosystem we updated and extended previous experimental cryptanalytic results provided in [4]. The new bounds for H_m concern messages having Hamming weight ≤ 11 or ≥ 223 , for $n = 232$. This is proved by providing experiments. But, our attack is feasible for Hamming weight

⁶The code can be found in <https://goo.gl/t9Fa68>

len(p)	600			1024			2048		
n	84			130			232		
H_m	12	13	14	11	12	13	9	10	11
Parallel (time)	4m	7m	13m	26m	66m	142m	52m	107m	895m
Mem. (GB)	1.43	4.93	7.5	14.28	21.67	67.41	26.9	40.61	127.53
Average Rounds	4.2	2.4	2.6	8.2	2.6	5.6	7.4	2.4	10

TABLE 3. Extension of table 2. For the case $H_m = 13$, $Q = 12$, and $\text{len}(p) = 1024$ we used $b = 60$ instead of $b = \frac{n}{2} = 65$. For the case $H_m = 11$ and $\text{len}(p) = 2048$ we used $b = n/2 - 23 = 93$. For all other cases we used $b = \frac{n}{2}$. The last row, Average Rounds, is the (average) round that eventually our algorithm terminates. Theoretically is approximately $\frac{1}{\mathbb{P}}$ (see Lemma 2.1).

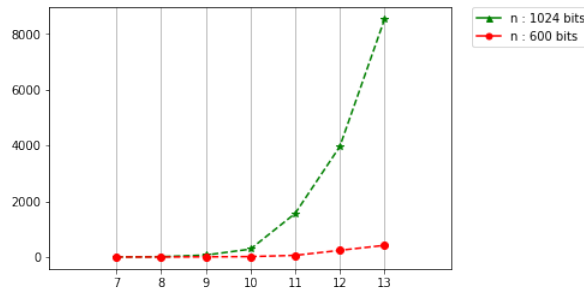


FIGURE 2. The horizontal axis is the hamming weight H_m and the vertical axis is the cpu time in seconds (for the parallel attack). Using `FindFit` of Mathematica [38], we computed the following approximation formulas that best fit to our data, $T_{600}(H_m) = 0.003 \cdot e^{0.71H_m}$, $T_{1024}(H_m) = 0.029 \cdot e^{0.76H_m}$ and $T_{2048}(H_m) = 0.74 \cdot e^{0.9H_m}$ (seconds).

≤ 15 or ≥ 219 . The NSK cryptosystem system could resist to this attack, if we consider Hamming weights in the interval [17, 217].

Acknowledgments. The authors are grateful to Scientific Computing Center (SCC) of the Aristotle’s University of Thessaloniki (Greece), for providing access to their computing facilities and their technical support.

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