

BRUHAT INVERSIONS IN WEYL GROUPS AND TORSION-FREE CLASSES OVER PREPROJECTIVE ALGEBRAS

HARUHISA ENOMOTO

ABSTRACT. For an element w of the simply-laced Weyl group, Buan-Iyama-Reiten-Scott defined a subcategory $\mathcal{F}(w)$ of a module category over a preprojective algebra of Dynkin type. This paper aims at studying categorical properties of $\mathcal{F}(w)$ via its connection with the root system. We show that by taking dimension vectors, simple objects in $\mathcal{F}(w)$ bijectively correspond to Bruhat inversion roots of w . As an application, we obtain a combinatorial criterion for $\mathcal{F}(w)$ to satisfy the Jordan-Hölder property (JHP). To achieve this, we develop a method to find simple objects in a general torsion-free class by using a brick sequence associated to a maximal green sequence of it. For type A case, we give a diagrammatic construction of simple objects, and show that (JHP) can be characterized via a forest-like permutation, introduced by Bousquet-Mélou and Butler in the study of Schubert varieties.

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1. INTRODUCTION

This paper focuses on the interplay between the preprojective algebras of Dynkin type and the root system. More precisely, we study a certain subcategory $\mathcal{F}(w)$ of the module category of the preprojective algebra via an inversion set in the root system.

1.1. Background. Let Φ be the simply-laced root system of type X and W the corresponding Weyl group. Let Q be a quiver of type X , that is, the underlying graph of Q is the Dynkin diagram X . Then the celebrated Gabriel's theorem gives a bijection between indecomposable representations of Q and positive roots in Φ , by taking dimension vectors.

A *preprojective algebra* Π of Φ is a finite-dimensional algebra which unifies the representation theory of all quivers of type X , and has a lot of symmetry compared to path algebras. This algebra has been one of the most important objects in the representation theory of algebras, for example, [AIRT, BIRS, GLS, IRR, Miz], and also plays an important role in the theory of crystal bases of quantum groups, for example, [Lus, KS].

In this paper, we focus on a certain subcategory $\mathcal{F}(w)$ of $\text{mod } \Pi$ associated to an element w of W introduced by Buan-Iyama-Reiten-Scott [BIRS] (under the name \mathcal{C}_w). This category has a nice structure related to cluster algebras, that is, a stably 2-Calabi-Yau Frobenius category admitting a cluster-tilting object. Indeed, Geiss-Leclerc-Schröer later [GLS] proved that $\mathcal{F}(w)$ gives a categorification of the cluster algebra structure on the coordinate ring of the unipotent cell in the complex simple Lie group of Dynkin type X .

The category $\mathcal{F}(w)$ naturally arises also from the viewpoint of the representation theory of algebras, as well as the lattice theoretical study of the Weyl group. This category is a *torsion-free class* in $\text{mod } \Pi$, that is, closed under submodules and extensions. Mizuno [Miz] proved that the map $w \mapsto \mathcal{F}(w)$ is actually a bijection from W to the set $\text{torf } \Pi$ of all torsion-free classes in $\text{mod } \Pi$. He also proved that this bijection is an isomorphism of lattices, where we endow W with the right weak order and $\text{torf } \Pi$ the inclusion order. Via this isomorphism, lattice theoretical properties of W were investigated in [IRRT, DIRRT].

1.2. Main results. It is natural to expect that categorical properties of $\mathcal{F}(w)$ are related to combinatorial properties of w . In this direction, we prove the two main results: we classify *simple objects* in $\mathcal{F}(w)$ (Theorem A), and give a criterion for the validity of the *Jordan-Hölder type theorem* in $\mathcal{F}(w)$ (Theorem C).

A Π -module M in $\mathcal{F}(w)$ is called a *simple object* in $\mathcal{F}(w)$ if there is no non-trivial submodule L of M satisfying $L, M/L \in \mathcal{F}(w)$. This notion was introduced in the context of Quillen's exact categories, and has been investigated by several papers such as [Eno, BHLR]. In [Eno], the author classified simple objects in a torsion-free class in $\text{mod } kQ$ for type A case, and the original motivation of this paper is to generalize this to other Dynkin types and to preprojective algebras.

Our strategy is to consider $\mathcal{F}(w)$ via the root system Φ . For $M \in \text{mod } \Pi$, we can regard its dimension vector as a vector in the ambient space of Φ naturally. Let $\text{inv}(w)$ be the set of *inversions* of w , positive roots which are sent to negative by w^{-1} . Then any $M \in \mathcal{F}(w)$, its dimension vector $\underline{\dim} M$ is a non-negative integer linear combination of inversions of w (Corollary 4.5). In parallel with simples in $\mathcal{F}(w)$, it is natural to consider an inversion of w which cannot be written as a sum of inversions of w non-trivially. We call such a root a *Bruhat inversion* (Definition 2.11, Theorem 2.17). Then the first main result of this paper is the following:

Theorem A (= Corollary 4.9). *Let Π be the preprojective algebra of Dynkin type and w an element of the corresponding Weyl group W . Then by taking dimension vectors, we have the bijection between the following two sets:*

- (1) *The set of isomorphism classes of simple objects in $\mathcal{F}(w)$.*
- (2) *The set of Bruhat inversions of w .*

This immediately deduces the similar result for the case of the path algebra kQ . A torsion-free class in $\text{mod } kQ$ bijectively corresponds to a c_Q -*sortable element* of W by [IT]. For such an element w of W , we have a torsion-free class $\mathcal{F}_Q(w)$ in $\text{mod } kQ$, which is actually equal to the restriction of $\mathcal{F}(w)$ to $\text{mod } kQ$ (Proposition 5.4). Then we have the same result for $\mathcal{F}_Q(w)$:

Theorem B (= Theorem 5.5). *Let Q be the Dynkin quiver, w a c_Q -sortable element of the Weyl group W and $\mathcal{F}_Q(w)$ the corresponding torsion-free class in $\text{mod } kQ$. Then by taking dimension vectors, we have the bijection between the following two sets:*

- (1) *The set of isomorphism classes of simple objects in $\mathcal{F}_Q(w)$.*
- (2) *The set of Bruhat inversions of w .*

As an application, we can characterize the validity of the *Jordan-Hölder property (JHP)* in $\mathcal{F}(w)$ or $\mathcal{F}_Q(w)$. We say that a torsion-free class \mathcal{F} *satisfies (JHP)* if for any M in \mathcal{F} , any relative composition series of M in \mathcal{F} are equivalent (see Definition 4.10 for the precise definition). By using the characterization of (JHP) obtained in [Eno], we prove the following second main result.

Theorem C (= Theorem 4.15, Proposition A.6). *Let Π be the preprojective algebra of Dynkin type and w an element of the corresponding Weyl group W . Then the following are equivalent:*

- (1) *$\mathcal{F}(w)$ satisfies (JHP).*
- (2) *Bruhat inversions of w are linearly independent.*
- (3) *The number of Bruhat inversions of w is equal to that of supports of w .*

Moreover, for the type A case, the above statements are equivalent to the following:

- (4) *w is forest-like in the sense of [BMB], that is, its Bruhat inversion graph is acyclic.*

Here a *support* of w is a vertex i in the Dynkin diagram X such that the reduced expression of w contains the simple reflection s_i . See Appendix A.2 for the details on forest-like permutations. The same result also holds for the case of path algebras (Corollary 5.6).

To show these results, we develop a method to find simple objects in a given torsion-free class \mathcal{F} by using a *brick sequence* associated to a maximal green sequence of \mathcal{F} . A *maximal green sequence* of \mathcal{F} is just a saturated chain $0 = \mathcal{F}_0 \triangleleft \mathcal{F}_1 \triangleleft \cdots \triangleleft \mathcal{F}_l = \mathcal{F}$ of torsion-free classes. We can associate to it a sequence of bricks (modules B such that $\text{End}_\Lambda(B)$ is a division ring) by using the brick labeling introduced in [DIRRT]. Once we can compute one brick sequence of \mathcal{F} , the following gives a way to determine all simple objects in \mathcal{F} :

Proposition D (= Corollary 3.12). *Let Λ be a finite-dimensional algebra and \mathcal{F} a torsion-free class in $\text{mod } \Lambda$. Suppose that there is a maximal green sequence of \mathcal{F} , and let B_1, \dots, B_l be the associated brick sequence. Then the following hold:*

- (1) *Every simple object in \mathcal{F} is isomorphic to B_i for some i .*
- (2) *For $1 \leq i \leq l$, the following are equivalent:*
 - (a) *B_i is a simple object in \mathcal{F} .*
 - (b) *Every morphism $B_i \rightarrow B_j$ with $i < j$ is either zero or injective.*
 - (c) *There is no surjection $B_i \twoheadrightarrow B_j$ with $i < j$.*

1.3. Organization. This paper is organized as follows. In Section 2, we give root-theoretical preliminaries and results. More precisely, we characterize Bruhat inversions (Theorem 2.17). Next we introduce a root sequence associated to a reduced expression of an element of W , which plays an important role later. In Section 3, we develop a general theory of simple objects in a torsion-free class, and prove Proposition D. In Section 4, we focus on torsion-free classes over preprojective algebras of Dynkin type. We show that a brick sequence of $\mathcal{F}(w)$ categorifies a root sequence of w , and prove Theorem A, B. In Section 5, we deduce several results on path algebras using preprojective algebras, and prove Theorem C. In the appendix, we give two combinatorial interpretation of the results for type A case. In Section A.1, we give a diagrammatic construction of simple objects in $\mathcal{F}(w)$ using arc diagrams, and in Section A.2, we show that $\mathcal{F}(w)$ satisfies (JHP) if and only if w is a forest-like permutation.

1.4. Conventions and notation. Throughout this paper, we assume that all categories are *skeletally small*, that is, the isomorphism classes of objects form a set. In addition, all subcategories are assumed to be full and closed under isomorphisms. For a Krull-Schmidt category \mathcal{E} , we denote by $\text{ind } \mathcal{E}$ the set of isomorphism classes of indecomposable objects in \mathcal{E} . We denote by $|X|$ the number of non-isomorphic indecomposable direct summands of X .

For a poset P and two elements $a, b \in P$ with $a \leq b$, we denote by $[a, b]$ the *interval poset* $[a, b] := \{x \in P \mid a \leq x \leq b\}$ with the obvious partial order. For $x, y \in P$, we say that x *covers* y if $x > y$ holds and there exists no $z \in P$ with $x > z > y$. In this case, we write $x \succ y$ or $y \prec x$.

For a set A , we denote by $\#A$ the cardinality of A .

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2. PRELIMINARIES ON ROOT SYSTEM

In this section, we give some results on root systems which we need later. In particular, we give a definition and characterization of Bruhat inversions, and introduce the notion of a root sequence.

2.1. Basic definitions. First we recall some basic definitions and properties of root systems. We refer the reader to [Hum1, Hum2] for the details.

Let V be the Euclidean space, that is, a finite-dimensional \mathbb{R} -vector space with the positive definite symmetric bilinear form $(-, -): V \times V \rightarrow \mathbb{R}$.

For two vectors $\alpha, \beta \in V$ with $\alpha \neq 0$, we put

$$\langle \beta, \alpha \rangle := 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

Note that $\langle -, \alpha \rangle$ is linear but $\langle \alpha, - \rangle$ is not. For $\alpha \in V$ with $\alpha \neq 0$, we denote by $t_\alpha: V \rightarrow V$ the reflection with respect to α , that is,

$$t_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

Definition 2.1. A subset Φ of the Euclidean space V is called a *root system* if it satisfies the following axioms:

- (R0) Φ is a finite subset of V which spans V as an \mathbb{R} -vector space, and does not contain 0.
- (R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for every $\alpha \in \Phi$.
- (R2) $t_\alpha(\Phi) = \Phi$ for every $\alpha \in \Phi$.
- (R3) $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for every $\alpha, \beta \in \Phi$.

A root system is called *simply-laced* if it satisfies the following condition.

- (R4) $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ for every $\alpha, \beta \in \Phi$.

Obviously (R4) is equivalent to that for every $\alpha, \beta \in \Phi$, if $\langle \alpha, \beta \rangle \neq 0$, then α and β have the same length, that is, $(\alpha, \alpha) = (\beta, \beta)$.

Possible values of integers $\langle \alpha, \beta \rangle$ in (R3) are very limited as follows.

Proposition 2.2 ([Hum1, 9.4]). *Let Φ be a root system, and let α and β be two roots in Φ with $(\alpha, \alpha) \leq (\beta, \beta)$. Then the following hold:*

- (1) $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle \geq 0$ and $(|\langle \alpha, \beta \rangle|, |\langle \beta, \alpha \rangle|) \in \{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2)\}$.
- (2) $\beta = \pm\alpha$ if and only if $(|\langle \alpha, \beta \rangle|, |\langle \beta, \alpha \rangle|) = (2, 2)$.
- (3) Φ is simply-laced if and only if $\langle \alpha, \beta \rangle \in \{-1, 0, 1\}$ for every $\alpha, \beta \in \Phi$ with $\alpha \neq \beta$.

Throughout this section, we will use the following notation:

- Φ is a root system in V .
- We fix a choice of simple roots Δ of Φ .
- Φ^+ (resp. Φ^-) is the set of positive roots (resp. negative roots) in Φ with respect to Δ .
- W is the Weyl group associated with Φ , that is, W is a subgroup of $\text{GL}(V)$ generated by t_α with $\alpha \in \Phi$.
- $T \subset W$ is a set of reflections in W , that is, $T = \{t_\alpha \in W \mid \alpha \in \Phi\}$.
- We often write $s_\alpha = t_\alpha$ if α is a simple root.

It is well-known that W is generated by simple reflections. For $w \in W$, let

$$w = s_1 \cdots s_l$$

be such an expression. Then this expression is called a *reduced expression* if l is minimal among all such expressions. In this case, l is called the *length* of w and we write $\ell(w) := l$.

On the reduced expression, we will need the following *exchange property* later.

Lemma 2.3 ([Hum2, 1.6, 1.7]). *Let $w = s_1 \cdots s_l$ be a reduced expression of $w \in W$, and let α be a simple roots. Then the following are equivalent:*

- (1) $s_1 \cdots s_l s_\alpha$ is not reduced, that is, $\ell(ws_\alpha) < \ell(w)$.
- (2) There exists i such that $s_1 \cdots s_l s_\alpha = s_1 \cdots \widehat{s}_i \cdots s_l$ (s_i omitted).
- (3) $w(\alpha) \in \Phi^-$.

2.2. Inversion sets and root sequences. For $w \in W$, we define $\text{inv}(w)$ by

$$\text{inv}(w) := \Phi^+ \cap w(\Phi^-),$$

that is, $\text{inv}(w)$ is the set of positive roots which are sent to negative roots by w^{-1} . We call an element of $\text{inv}(w)$ an *inversion* of w . This set plays an important role in this paper, since it corresponds to the set of dimension vectors of bricks in a torsion-free class $\mathcal{F}(w)$ over the preprojective algebra (Corollary 4.5).

The following description of inversion sets is well-known.

Proposition 2.4 ([Hum2, 1.7]). *Let w be an element of W . Take a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_l}$ of w with $\alpha_i \in \Delta$. Then we have*

$$\text{inv}(w) = \{\alpha_1, s_{\alpha_1}(\alpha_2), s_{\alpha_1}s_{\alpha_2}(\alpha_3), \dots, s_{\alpha_1} \cdots s_{\alpha_{l-1}}(\alpha_l)\},$$

and all the elements above are distinct. In particular, we have $\#\text{inv}(w) = \ell(w)$.

By using this, we can easily show the following property.

Lemma 2.5. *Let v and w elements of W satisfying $\ell(vw) = \ell(v) + \ell(w)$. Then we have $\text{inv}(vw) = \text{inv}(v) \sqcup v(\text{inv}(w))$.*

If we choose a reduced expression of w , then Proposition 2.4 gives a sequence of positive roots. It turns out that the order of appearance are important for our purpose. This leads to the notion of *root sequences*.

Definition 2.6. Let W be the Weyl group of a root system Φ .

- (1) Let w be an element of W and $w = s_{\alpha_1} \cdots s_{\alpha_l}$ a reduced expression of w . Then a *root sequence of w associated to this expression* is a (ordered) sequence of positive roots

$$\alpha_1, s_{\alpha_1}(\alpha_2), s_{\alpha_1}s_{\alpha_2}(\alpha_3), \dots, s_{\alpha_1} \cdots s_{\alpha_{l-1}}(\alpha_l).$$

- (2) A *root sequence* is a sequence of roots which arises as a root sequence of some reduced expression of some element w in W . We call such a sequence a *root sequence of w* .

The notion of root sequences appeared in several papers: they are called *compatible (convex) orderings* in [Pap], *reflection orderings* in [Dye]. We borrowed the terminology *root sequences* from [GL, FS].

We will use the following characterization of inversion sets and root sequences due to Papi later.

Theorem 2.7 ([Pap]). *A sequence \mathcal{R} of positive roots is a root sequence if and only if the following two conditions are satisfied for any pair of positive roots α, β satisfying $\alpha + \beta \in \Phi^+$:*

- (1) If α and β appear in \mathcal{R} , then $\alpha + \beta$ appears between α and β in \mathcal{R} .
- (2) If $\alpha + \beta$ appears in \mathcal{R} , then one of α and β appears and precedes $\alpha + \beta$ in \mathcal{R} .

To give examples of root sequences (and its connection with *brick sequence* defined in the next section), it is convenient to introduce the *right weak order* \leq_R on W and its *Hasse quiver*. Define a quiver $\text{Hasse}(W, \leq_R)$ as follows:

- A vertex set of $\text{Hasse}(W, \leq_R)$ is W .

- We draw an arrow $v \leftarrow vs$ if $\ell(vs) = \ell(v) + 1$ for $v \in W$ and a simple reflection $s \in W$.

Then define a partial order \leq_R on W by $v \leq_R w$ if and only if there is a path from w to v in $\text{Hasse}(W, \leq_R)$. It is known that (W, \leq_R) is actually a lattice, see e.g. [BB, 3.2], and $\text{Hasse}(W, \leq_R)$ is actually a Hasse quiver of (W, \leq_R) defined later.

By construction, each reduced expression of $w = s_1 s_2 \cdots s_l$ gives a reverse path $e \leftarrow s_1 \leftarrow s_1 s_2 \leftarrow \cdots \leftarrow w$ from e to w in $\text{Hasse}(W, \leq_R)$, and this correspondence is a bijection. We define the *root labeling* of arrows in $\text{Hasse}(W, \leq_R)$ by attaching a positive root $v(\alpha)$ to an arrow $v \leftarrow vs_\alpha$. Then a root sequence associated to a reduced expression $w = s_1 \cdots s_l$ is obtained by reading labels of the corresponding reverse path $e \leftarrow s_1 \leftarrow \cdots \leftarrow w$.

Example 2.8. Let Φ be the root system of type A_3 with its Dynkin graph 1—2—3. We denote by s_i (resp. α_i) the simple reflection (resp. simple root) associated to the vertex i for $1 \leq i \leq 3$. Consider $w = s_1 s_2 s_3 s_1 s_2 \in W$. Figure 1 shows all the reverse paths from e to w and its root labeling. Here we write $s_{1231} := s_1 s_2 s_3 s_1$ and $110 := \alpha_1 + \alpha_2$ for example. Then the left most

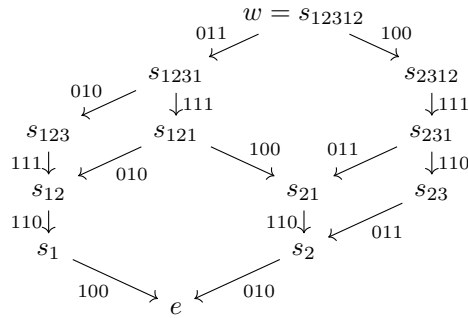


FIGURE 1. Root sequences of $w = s_{12312}$

path corresponds to a reduced expression s_{12312} , and the right most corresponds to s_{23121} . Their associated root sequences are $100, 110, 111, 010, 011$ and $010, 011, 110, 111, 100$ respectively.

Example 2.9. Let Φ be the root system of type D_4 , whose Dynkin diagram is as follows:

$$\begin{array}{c} 2 \\ | \\ 1 - 0 - 3 \end{array}$$

Consider $w = s_{012301230}$ (here we use the same notation as in Example 2.8). Figure 2 shows all the reduced expressions of w and its root sequence. For example, the right most path gives a reduced expression $w = s_{012030210}$ with its associated root sequence ${}^0_{010}, {}^0_{110}, {}^1_{010}, {}^1_{110}, {}^1_{121}, {}^0_{011}, {}^0_{111}, {}^1_{011}, {}^1_{111}$ (here as before, ${}^1_{011}$ denotes $\alpha_0 + \alpha_2 + \alpha_3$ for example).

2.3. Bruhat inversions. A simple root in a root system is *indecomposable* in Φ^+ in the sense that we cannot write it as a positive linear combination of positive roots in a non-trivial way.

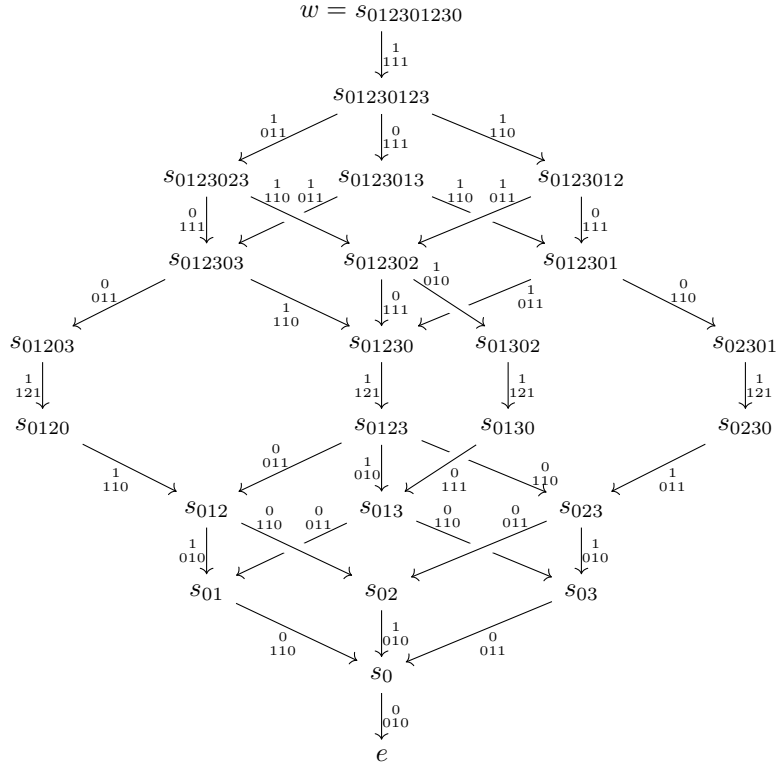
We are interested, not in the whole set of positive roots, but in the inversion set $\text{inv}(w)$ for a fixed $w \in W$. It is natural to ask which elements are *indecomposable* in the above sense. This leads to the notion of *Bruhat inversions*.

Let us begin with the property of reflections with respect to inversion roots of w .

Proposition 2.10 ([Hum2, Theorem 5.8], [BB, Theorem 1.4.3]). *Let w be an element of W and β a positive root. Then the following hold:*

- (1) *Then $\beta \in \text{inv}(w)$ if and only if $\ell(t_\beta w) < \ell(w)$ holds.*
- (2) *Let $s_1 s_2 \cdots s_l = w$ be a reduced expression with $s_i = s_{\alpha_i}$, and let $\beta_1, \beta_2, \dots, \beta_l$ a root sequence associated to it. Then we have*

$$t_{\beta_i} w = s_1 s_2 \cdots \widehat{s_i} \cdots s_l.$$


 FIGURE 2. Root sequences for $w = s_{012301230}$

Next we will define *Bruhat inversions* of elements in W . Recall that the *Bruhat order* on W is the transitive closure of the following relation: for every $t \in T$ and $w \in W$ satisfying $\ell(tw) < \ell(w)$, we have $tw \leq w$.

Definition 2.11. Let w be an element of W and β a positive root. Then β is a *Bruhat inversion* of w if it satisfies $\ell(t_\beta w) = \ell(w) - 1$. We denote by $\text{Binv}(w)$ the set of Bruhat inversions of w . We call an element of $\text{inv}(w) \setminus \text{Binv}(w)$ a *non-Bruhat inversion* of w .

By the description in Proposition 2.10, the number of Bruhat inversions can be computed as follows: fix a reduced expression $w = s_1 \cdots s_l$ of w , then $\#\text{Binv}(w)$ is the number of i 's such that deleting s_i from this expression still yields a reduced expression.

A Bruhat inversion is closely related to the covering relation in the Bruhat order as follows.

Proposition 2.12. For $w \in W$ and $\beta \in \Phi^+$, the following are equivalent:

- (1) $\beta \in \text{Binv}(w)$, that is, β is a Bruhat inversion of w .
- (2) w covers $t_\beta w$ in the Bruhat order of w .

Proof. This easily follows from Proposition 2.10 and the chain property of the Bruhat order, see [BB, Theorem 2.2.6] for example. \square

The following is an example of Bruhat inversions for type A case (we refer the reader to the appendix for the detail).

Example 2.13. Let Φ be the standard root system of type A_n , and $\alpha_1, \dots, \alpha_n$ the simple roots. Then positive roots are of the form $\beta_{(i,j)} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \leq i < j \leq n+1$. We can identify W with the symmetric group S_{n+1} . For w in $W = S_{n+1}$, we have that $\beta_{(i,j)} \in \text{inv}(w)$ if and only if (i,j) is the *classical inversion* of w , that is, $i < j$ and $w^{-1}(i) > w^{-1}(j)$ holds. Moreover, we can easily see that $\beta_{(i,j)} \in \text{Binv}(w)$ if and only if $\beta_{(i,j)} \in \text{inv}(w)$ and there is no $i < k < j$ with $\beta_{(i,k)}, \beta_{(k,j)} \in \text{inv}(w)$. For example, if $w = 42153 \in S_5$, we have $\text{inv}(w) = \{\beta_{(1,2)}, \beta_{(1,4)}, \beta_{(2,4)}, \beta_{(3,4)}, \beta_{(3,5)}\}$ and $\text{Binv}(w) = \text{inv}(w) \setminus \{\beta_{(1,4)}\}$.

For a fixed element $w \in W$, we will give a characterization of Bruhat inversions of w among all inversions of w . To do this, we prepare some lemmas.

The following is known as the *lifting property* of the Bruhat order.

Lemma 2.14 ([BB, Proposition 2.2.7]). *Let v and w be elements in W satisfying $v < w$, and let s be a simple reflection. If $\ell(sv) = \ell(v) + 1$ and $\ell(sw) = \ell(w) - 1$ holds, then we have $sv < w$ and $v < sw$.*

By using the lifting property, we can show the following technical lemma.

Lemma 2.15. *Let w be an element of W and $\beta \in \text{Bin}(w)$, and put $t = t_\beta \in W$. Then either one of the following holds:*

- (1) β is a simple root.
- (2) There exists a simple reflection $s \in W$ which satisfies the following two conditions:
 - (a) $\ell(sw) = \ell(w) + 1$.
 - (b) $\ell(stw) = \ell(tw) + 1$.

Proof. Suppose that (2) does not hold. We will show that β must be simple.

Since (2) does not hold, for every simple reflection $s \in W$, we have that either (a) or (b) (or both) fails to hold. We will show the following claim.

(Claim): *There exists some simple reflection $s \in W$ such that (a) does not hold and (b) holds.*

Proof of (Claim). If this is not the case, then (b) does not hold for every simple reflection $s \in W$. This means that tw is the longest element in W , which contradicts to $\ell(w) = \ell(tw) + 1$. ■

Take such a simple reflection s . Then we have that $\ell(sw) = \ell(w) - 1$ and $\ell(stw) = \ell(tw) + 1$ hold, and that $tw < w$ by the assumption. Then by Lemma 2.14, we have that $tw \leq sw < w$ holds. Since w covers tw in the Bruhat order, we must have $tw = sw$, hence $t = s$. Therefore β is a simple root. □

Now we can show that a Bruhat inversion can be transformed into a simple root:

Proposition 2.16. *Let w be an element of W and β an inversion of w . Then the following are equivalent:*

- (1) β is a Bruhat inversion of w .
- (2) There exists some element $v \in W$ which satisfies the following two conditions:
 - (a) $\ell(vw) = \ell(v) + \ell(w)$ holds.
 - (b) $v(\beta)$ is a simple root.

Proof. (1) \Rightarrow (2): Suppose that $\beta \in \text{Bin}(w)$ holds. If β is a simple root, then $v = e$ satisfies the conditions (2)(a) and (b).

From now on, we assume that β is not simple. Then by Lemma 2.15, there exists a simple reflection s such that $\ell(sw) = \ell(w) + 1$ and $\ell(st_\beta w) = \ell(w)$ hold. Put $w' = sw$ and $\beta' := s(\beta)$, then $\beta' \in \text{inv}(w')$ by Lemma 2.5. Moreover, $\ell(st_\beta w) = \ell(w)$ implies that $\ell(t_{\beta'} w') = \ell(st_\beta s \cdot sw) = \ell(w) = \ell(w') - 1$. Thus β' is a Bruhat inversion of w' . If β' is a simple root, then $v = s$ satisfies the conditions (2)(a) and (b).

If β' is not simple, then we can iterate this process by considering β' and w' instead of β and w . Moreover, this process must stop at some point since otherwise we would have $\ell(w) < \ell(w') < \dots$, which contradicts to the existence of the longest element in W . Therefore, we have that the Bruhat inversion at this point, which can be written as $v(\beta)$ for some v , is a simple root.

(2) \Rightarrow (1): Since $v(\beta)$ is a simple root and $v(\beta) \in \text{inv}(vw)$ by Lemma 2.5, we have $\ell(t_{v(\beta)} vw) = \ell(vw) - 1 = \ell(v) + \ell(w) - 1$. On the other hand, since $t_{v(\beta)} vw = vt_{\beta} v^{-1} vw = vt_{\beta} w$ holds, we have $\ell(t_{v(\beta)} vw) \leq \ell(v) + \ell(t_{\beta} w)$. Thus $\ell(w) - 1 \leq \ell(t_{\beta} w)$. Since β is an inversion of w , we have $\ell(t_{\beta} w) \leq \ell(w) - 1$ by Proposition 2.10. Therefore, we have $\ell(t_{\beta} w) = \ell(w) - 1$, that is, β is a Bruhat inversion of w . □

The following is a main result in this section, which gives a characterization of Bruhat inversions.

Theorem 2.17. *Let w be an element of the Weyl group W of Φ , and $\beta \in \text{inv}(w)$. Then the following are equivalent:*

- (1) β is a non-Bruhat inversion of w .
- (2) There exists $a_\gamma \in \mathbb{R}_{\geq 0}$ for each $\gamma \in \text{inv}(w)$ which satisfies the following conditions:
 - $\beta = \sum_{\gamma \in \text{inv}(w)} a_\gamma \gamma$ holds.
 - This expression is not of the form $\beta = 1 \cdot \beta$, that is, there exists some $\gamma \in \text{inv}(w)$ such that $a_\gamma \neq \delta_{\beta, \gamma}$, where δ is the Kronecker delta.
- (3) There exist $\gamma_1, \gamma_2 \in \text{inv}(w)$ with $\gamma_1 \neq \gamma_2$ and $a_1, a_2 \in \mathbb{R}_{> 0}$ such that $\beta = a_1 \gamma_1 + a_2 \gamma_2$ holds.
- (4) There exist $\gamma_1, \gamma_2 \in \text{inv}(w)$ with $\gamma_1 \neq \gamma_2$ and $n \in \{1, 2, 3\}$ such that $\beta = (\gamma_1 + \gamma_2)/n$ holds.

Moreover, if Φ is simply-laced, then the above statements are equivalent to the following:

- (5) There exist $\gamma_1, \gamma_2 \in \text{inv}(w)$ satisfying $\beta = \gamma_1 + \gamma_2$.

Proof. The implications (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) are clear.

(2) \Rightarrow (1): Suppose that β is a Bruhat inversion. Then by Proposition 2.16, we have an element $v \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$ and that $v(\beta)$ is a simple root. Note that Lemma 2.5 implies $v(\text{inv}(w)) \subset \Phi^+$.

Now take $a_\gamma \in \mathbb{R}_{\geq 0}$ for $\gamma \in \text{inv}(w)$ with $\beta = \sum_{\gamma} a_\gamma \gamma$ as claimed in (2). Then we have

$$v(\beta) = \sum_{\gamma \in \text{inv}(w)} a_\gamma v(\gamma).$$

Since $v(\beta)$ is a simple root and all $v(\gamma)$'s are distinct positive roots, we must have that $a_\gamma = \delta_{\beta, \gamma}$, which contradicts to the condition in (2). Thus β is not a Bruhat inversion.

(1) \Rightarrow (4),(5): Take any reduced expression $w = s_1 s_2 \cdots s_l$ of w , and let $\beta_1, \beta_2, \dots, \beta_l$ be the root sequence associated to it. Then $\beta = \beta_m$ for some m . Since we have $t_\beta w = s_1 \cdots \widehat{s_m} \cdots s_l$ by Proposition 2.10 and β is not a Bruhat inversion, the expression $s_1 \cdots \widehat{s_m} \cdots s_l$ is not reduced.

Take the minimal j such that $s_1 \cdots \widehat{s_m} \cdots s_j$ is not a reduced expression. Then by Lemma 2.3, there exists i such that $s_1 \cdots \widehat{s_m} \cdots s_{j-1} = s_1 \cdots \widehat{s_i} \cdots \widehat{s_m} \cdots s_j$. In this situation, we will prove the following claim.

(Claim): We have $t_\beta(\beta_j) = -\beta_i$.

Proof of (Claim). First we have the following equation:

$$\begin{aligned} t_\beta(\beta_j) &= t_{\beta_m}(\beta_j) \\ &= (s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_1)(\beta_j) \\ &= (s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_1)(s_1 \cdots s_{j-1})(\alpha_j) \\ &= (s_1 \cdots \widehat{s_m} \cdots s_{j-1})(\alpha_j), \end{aligned}$$

where α_j is a simple root with $s_{\alpha_j} = s_j$. On the other hand, since $s_1 \cdots \widehat{s_m} \cdots s_{j-1} s_j$ is not reduced, Lemma 2.3 implies that $(s_1 \cdots \widehat{s_m} \cdots s_{j-1})(\alpha_j)$ is a negative root. Thus $t_\beta(\beta_j)$ is a negative root.

Now consider the reflection with respect to $t_\beta(\beta_j)$:

$$\begin{aligned} t_{t_\beta(\beta_j)} &= t_\beta t_{\beta_j} t_\beta \\ &= (s_1 \cdots s_m \cdots s_1)(s_1 \cdots s_m \cdots s_j \cdots s_m \cdots s_1)(s_1 \cdots s_m \cdots s_1) \\ &= s_1 \cdots \widehat{s_m} \cdots s_{j-1} s_j s_{j-1} \cdots \widehat{s_m} \cdots s_1 \\ &= (s_1 \cdots \widehat{s_i} \cdots \widehat{s_m} \cdots s_j) s_j s_{j-1} \cdots \widehat{s_m} \cdots s_1 \\ &= s_1 \cdots s_i \cdots s_1 \\ &= t_{\beta_i}. \end{aligned}$$

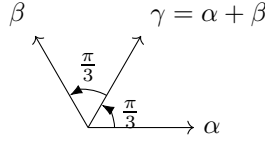
Thus the reflection with respect to $t_\beta(\beta_j)$ coincides with that along β_i . Since $t_\beta(\beta_j)$ is a negative root, we must have $t_\beta(\beta_j) = -\beta_i$. \blacksquare

Now by (Claim), we have $\beta_j - \langle \beta_j, \beta \rangle \beta = t_\beta(\beta_j) = -\beta_i$, thus $\langle \beta_j, \beta \rangle \beta = \beta_i + \beta_j$. Since β, β_i and β_j are positive roots, we must have $\langle \beta_j, \beta \rangle > 0$. We have $\langle \beta_j, \beta \rangle \in \{1, 2, 3\}$ by Proposition 2.2, so (4) holds.

Assume that Φ is simply-laced. Since $m < j$, we have that $\beta_m = \beta$ and β_j are distinct. Thus $\langle \beta_j, \beta \rangle = 1$ holds by Proposition 2.2, so (5) holds. \square

By this theorem, a non-Bruhat inversion γ of w can be written as $\gamma = \alpha + \beta$ with $\alpha, \beta \in \text{inv}(w)$ if Φ is simply-laced. This kind of equation gives a restriction of the relative position of α, β and γ as follows.

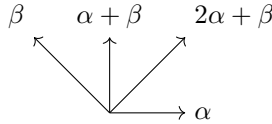
Lemma 2.18. *Suppose that Φ is simply-laced, and that α and β in Φ satisfies $\gamma := \alpha + \beta \in \Phi$. Then $\langle \alpha, \beta \rangle = -1$, $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 1$ hold, thus they look as follows.*



Proof. Clearly α, β, γ are distinct. Thus we have that $\langle \beta, \alpha \rangle, \langle \gamma, \alpha \rangle \in \{-1, 0, 1\}$ by Proposition 2.2. However, we have $\langle \gamma, \alpha \rangle = \langle \alpha + \beta, \alpha \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle = 2 + \langle \beta, \alpha \rangle$. Thus we must have $\langle \beta, \alpha \rangle = -1$ and $\langle \gamma, \alpha \rangle = 1$. The equation $\langle \gamma, \beta \rangle = 1$ can be shown similarly.

Since $(\alpha, \beta), (\alpha, \gamma) \neq 0$ and Φ is simply-laced, α, β, γ has the same length. Then it easily follows from $\langle \beta, \alpha \rangle = -1$ that the angle between α and β is $\frac{2}{3}\pi$. Therefore the situation looks like the figure. \square

Remark 2.19. If Φ is not simply-laced, then a non-Bruhat inversion γ of w may not be written as a sum of other inversions. For example, let Φ be the root system of type B_2 with α a short simple root and β a long simple root:



Then we have four positive roots. Consider $w = s_\beta s_\alpha s_\beta$. Easy computation shows that its root sequence is $\beta, \alpha + \beta, 2\alpha + \beta$, thus we have $\text{inv}(w) = \{\beta, \alpha + \beta, 2\alpha + \beta\}$. By using Theorem 2.17, one can conclude that $\alpha + \beta$ is not a Bruhat inversion of w since $\alpha + \beta = \beta/2 + (2\alpha + \beta)/2$ holds (of course this can be deduced by checking that $s_\beta \widehat{s_\alpha} s_\beta$ is not reduced, which is trivial). Nevertheless, we cannot write $\alpha + \beta$ as a sum of other inversions of w .

3. BRICK SEQUENCE OF A TORSION-FREE CLASS

In this section, we define a brick sequence of a torsion-free class \mathcal{F} , which is associated to a maximal green sequence of \mathcal{F} (a saturated chain of torsion-free classes between 0 and \mathcal{F}). In particular, we focus on the relation between brick sequences and simple objects in \mathcal{F} .

Throughout this section, let Λ be a finite-dimensional k -algebra over a field k . We denote by $\text{mod } \Lambda$ the category of finitely generated right Λ -modules. A module always means a finitely generated right modules.

3.1. Brick labeling. We briefly recall the lattice structure of torsion-free classes in $\text{mod } \Lambda$, following [DIRRT]. Note that in [DIRRT], *torsion classes*, the dual notion of torsion-free classes, were mainly studied, but the same theory works also for torsion-free classes by the standard duality.

A subcategory \mathcal{F} of $\text{mod } \Lambda$ is a *torsion-free class* if it is closed under extensions and submodules in $\text{mod } \Lambda$. We denote by $\text{torf } \Lambda$ the set of all torsion-free classes in $\text{mod } \Lambda$. Then $\text{torf } \Lambda$ is a poset with respect to inclusion. Moreover, since intersection of any torsion-free classes is also a torsion-free class, $\text{torf } \Lambda$ is a complete lattice.

For a poset P , its *Hasse quiver* $\text{Hasse } P$ is a quiver defined as follows:

- A vertex set of $\text{Hasse } P$ is P .
- We draw a unique arrow $x \rightarrow y$ in $\text{Hasse } P$ if $x \succ y$, that is, x covers y in a poset P .

Now $\text{Hasse}(\text{torf } \Lambda)$ has the additional structure called the *brick labeling*, established in [DIRRT]. Let us introduce some terminologies to state it. A Λ -module M in $\text{mod } \Lambda$ is called a *brick* if $\text{End}_\Lambda(M)$ is a division ring. For a collection \mathcal{C} of Λ -modules, we will use the following notations:

- $\text{add } \mathcal{C}$ denotes the category of direct summands of finite direct sums of modules in \mathcal{C} .

- $\text{Sub}\mathcal{C}$ denotes the category of modules X such that there exists an injection from X to a module in $\text{add}\mathcal{C}$.
- $\text{Filt}\mathcal{C}$ denotes the category of modules X such that there exists a filtration $0 = X_0 \leq X_1 \leq \dots \leq X_n = X$ of submodules of X satisfying $X_i/X_{i-1} \in \mathcal{C}$ for each i .
- $\text{F}(\mathcal{C})$ denotes the smallest torsion-free class containing \mathcal{C} , or equivalently, $\text{F}(\mathcal{C}) = \text{Filt}(\text{Sub}\mathcal{C})$.
- ${}^\perp\mathcal{C}$ denotes the category of modules X satisfying $\text{Hom}_\Lambda(X, \mathcal{C}) = 0$.
- \mathcal{C}^\perp denotes the category of modules X satisfying $\text{Hom}_\Lambda(\mathcal{C}, X) = 0$.
- $\text{ind}\mathcal{C}$ denotes the set of isomorphism classes of indecomposable modules in \mathcal{C} .
- $\text{brick}\mathcal{C}$ denotes the set of isomorphism classes of bricks in \mathcal{C} .

For two collections \mathcal{C} and \mathcal{D} of Λ -modules, $\mathcal{C} * \mathcal{D}$ denotes the category of modules X such that there exist an exact sequence

$$0 \longrightarrow C \longrightarrow X \longrightarrow D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. From the classical torsion theory, we have $\text{mod}\Lambda = {}^\perp\mathcal{F} * \mathcal{F}$ for a torsion-free class \mathcal{F} in $\text{mod}\Lambda$, see e.g. [ASS, VI.1]. It can be easily checked that the operation $*$ is associative, that is, we have $(\mathcal{C} * \mathcal{D}) * \mathcal{E} = \mathcal{C} * (\mathcal{D} * \mathcal{E})$ holds for collections \mathcal{C} , \mathcal{D} and \mathcal{E} of Λ -modules. Thus we omit parentheses and just write as $\mathcal{C} * \mathcal{D} * \mathcal{E}$.

In [DIRRT], the following basic observation was established.

Proposition 3.1 ([DIRRT, Theorems 3.3, 3.4]). *Let $\mathcal{G} \subset \mathcal{F}$ be two torsion-free classes in $\text{mod}\Lambda$. Then there exists an arrow $q: \mathcal{F} \rightarrow \mathcal{G}$ in $\text{Hasse}(\text{torf}\Lambda)$ if and only if $\text{brick}({}^\perp\mathcal{G} \cap \mathcal{F})$ contains exactly one element B_q . In this case, we have ${}^\perp\mathcal{G} \cap \mathcal{F} = \text{Filt}(B_q)$, $\mathcal{F} = \text{Filt}(B_q) * \mathcal{G}$ and $\mathcal{G} = \mathcal{F} \cap B_q^\perp$.*

By this, to each arrow q in $\text{Hasse}(\text{torf}\Lambda)$ we can associate a brick B_q , which we call the *label* of q . This labeling is called the *brick labeling* of $\text{torf}\Lambda$.

3.2. Brick sequence associated to a maximal green sequence. To study the structure of a fixed torsion-free class \mathcal{F} , a *brick sequence* associated to a *maximal green sequence* of \mathcal{F} plays an important role later. Let us introduce these notions.

Definition 3.2. Let \mathcal{F} be a torsion-free class in $\text{mod}\Lambda$. Then a *maximal green sequence* of \mathcal{F} is a finite path in $\text{Hasse}(\text{torf}\Lambda)$ which starts at \mathcal{F} and ends at 0 . Or equivalently, a maximal green sequence of \mathcal{F} is a saturated chain in $\text{torf}\Lambda$ of the form $0 = \mathcal{F}_0 \triangleleft \mathcal{F}_1 \triangleleft \dots \triangleleft \mathcal{F}_l = \mathcal{F}$.

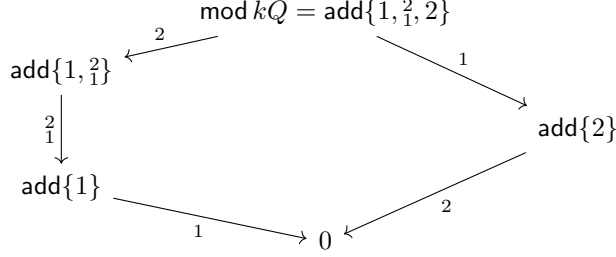
Maximal green sequences were introduced by Keller in the context of quiver mutations in cluster algebras, and have been investigated from various viewpoints. We refer the reader to the recent article [DK] and the reference therein for the details on this notion. The above definition is a straightforward generalization of maximal green sequences of abelian categories, which was introduced in [BST].

To a maximal green sequence of \mathcal{F} , we can associate a sequence of bricks in \mathcal{F} as follows. This is analogous to the root sequence associated to a reduced expression in the Weyl group (and actually gives a categorification as we shall see in Proposition 4.3).

Definition 3.3. Let $0 = \mathcal{F}_0 \triangleleft \mathcal{F}_1 \triangleleft \dots \triangleleft \mathcal{F}_l = \mathcal{F}$ be a maximal green sequence of a torsion-free class \mathcal{F} in $\text{mod}\Lambda$. Then a *brick sequence associated to it* is a sequence B_1, B_2, \dots, B_l of bricks where each B_i is the label of the arrow $\mathcal{F}_i \triangleleft \mathcal{F}_{i-1}$ in $\text{Hasse}(\text{torf}\Lambda)$. We simply call a brick sequence associated to some maximal green sequence of \mathcal{F} a *brick sequence of \mathcal{F}* .

As in the case of root sequences, we take the appearance order of bricks into account. For a fixed \mathcal{F} , brick sequences of \mathcal{F} heavily depend on the choice of maximal green sequences. In general, the lengths of brick sequences may differ, as the following example shows.

Example 3.4. Let Q be a quiver $1 \leftarrow 2$. Then $\text{Hasse}(\text{torf } kQ)$ and its brick labeling are as follows:



Here we write composition series to indicate kQ -modules. Thus there are exactly two brick sequences of $\text{mod } kQ$, namely, $1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 2$ and $2, 1$.

The fundamental property of a brick sequence is the following.

Theorem 3.5. Let B_1, \dots, B_l be a brick sequence of a torsion-free class \mathcal{F} in $\text{mod } \Lambda$. Then the following hold.

- (1) $\text{Hom}_\Lambda(B_j, B_i) = 0$ for $j > i$.
- (2) B_1, \dots, B_l are pairwise non-isomorphic.
- (3) $\mathcal{F} = \text{Filt}(B_l) * \text{Filt}(B_{l-1}) * \dots * \text{Filt}(B_2) * \text{Filt}(B_1)$ holds. In particular, we have $\mathcal{F} = \text{Filt}(B_1, \dots, B_l)$.

Proof. Let $0 = \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \dots \leftarrow \mathcal{F}_l = \mathcal{F}$ be a maximal green sequence of \mathcal{F} which gives a brick sequence B_1, \dots, B_l .

(1) By Proposition 3.1 and the definition of the brick labeling, each B_i is contained in ${}^\perp \mathcal{F}_{i-1} \cap \mathcal{F}_i$. Thus $B_j \in {}^\perp \mathcal{F}_{j-1} \subset {}^\perp \mathcal{F}_i$ holds since $\mathcal{F}_{j-1} \supset \mathcal{F}_i$ for $j > i$. Therefore $\text{Hom}_\Lambda(B_j, B_i) = 0$ by $B_i \in \mathcal{F}_i$.

(2) This is clear from (1).

(3) We will prove by backward induction on i that ${}^\perp \mathcal{F}_{i-1} \cap \mathcal{F} = \text{Filt}(B_l) * \dots * \text{Filt}(B_{i+1}) * \text{Filt}(B_i)$ holds for $1 \leq i \leq l$. For $i = l$, this holds by Proposition 3.1.

Suppose this holds for $i = j+1$ with $1 \leq j < l$, and take $M \in {}^\perp \mathcal{F}_{j-1} \cap \mathcal{F}$. Since ${}^\perp \mathcal{F}_j * \mathcal{F}_j = \text{mod } \Lambda$ holds, we have an exact sequence in $\text{mod } \Lambda$

$$0 \longrightarrow T \longrightarrow M \longrightarrow F \longrightarrow 0$$

with $T \in {}^\perp \mathcal{F}_j$ and $F \in \mathcal{F}_j$. Since \mathcal{F} is closed under submodules and $M \in \mathcal{F}$, we have $T \in {}^\perp \mathcal{F}_j \cap \mathcal{F} = \text{Filt}(B_l) * \dots * \text{Filt}(B_{j+1})$ by the induction hypothesis. On the other hand, since ${}^\perp \mathcal{F}_{j-1}$ is closed under quotients, we have $F \in {}^\perp \mathcal{F}_{j-1} \cap \mathcal{F}_j = \text{Filt}(B_j)$. Thus we have $M \in \text{Filt}(B_l) * \dots * \text{Filt}(B_{j+1}) * \text{Filt}(B_j)$. \square

We remark that this statement appeared in [Tre, Corollary 6.5] for the case $\mathcal{F} = \text{mod } \Lambda$, and the filtration given in (3) above is called the *Harder-Narasimhan filtration* associated to a maximal green sequence.

3.3. Simple objects in a torsion-free class. We introduce the notion of *simple objects* in a torsion-free class.

Definition 3.6. Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$.

- (1) $M \in \mathcal{F}$ is a *simple object* in \mathcal{F} if $M \neq 0$ and for any short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of Λ -modules with $L, N \in \mathcal{F}$, we have $L = 0$ or $M = 0$.

- (2) We denote by $\text{sim } \mathcal{F}$ the set of isomorphism classes of simple objects in \mathcal{F} .

Note that whether a given module M is simple object or not depends on the torsion-free class which contains M . Originally, the notion of simple objects was introduced and studied in the context of exact categories by several papers such as [Eno, BHLR]. Since we are only interested in torsion-free classes, we will not work in full generality.

For a torsion-free class, a simple object can be described by the following property.

Lemma 3.7. *Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$ and M a non-zero object in \mathcal{F} . Then the following are equivalent:*

- (1) M is simple in \mathcal{F} .
- (2) Every morphism $M \rightarrow F$ with $F \in \mathcal{F}$ is either zero or injective.

Proof. (1) \Rightarrow (2): Take a non-zero morphism $\varphi: M \rightarrow F$ with $F \in \mathcal{F}$. Then we have an exact sequence

$$0 \longrightarrow \text{Ker } \varphi \longrightarrow M \longrightarrow \text{Im } \varphi \longrightarrow 0$$

in $\text{mod } \Lambda$. Since $\text{Im } \varphi$ is a submodule of $F \in \mathcal{F}$, we have $\text{Im } \varphi \in \mathcal{F}$. Similarly $\text{Ker } \varphi \in \mathcal{F}$ holds since so is M . On the other hand, $\text{Im } \varphi \neq 0$ by assumption. Therefore, $\text{Ker } \varphi = 0$ holds by the simplicity of M .

(2) \Rightarrow (1): Take a short exact sequence

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$

with $L, N \in \mathcal{F}$. Then by assumption, π should be zero or injection. We have $N = 0$ in the former case, and $L = 0$ in the latter. Thus M is a simple object in \mathcal{F} . \square

We will investigate the relation between simple objects in \mathcal{F} and brick sequences of \mathcal{F} . One of the remarkable property is that simples always appear in every brick sequence of \mathcal{F} :

Proposition 3.8. *Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$ and S a simple object in \mathcal{F} . Then S appears exactly once in every brick sequence of it (if exists).*

Proof. Let B_1, \dots, B_l be a brick sequence of \mathcal{F} . Then we have $\mathcal{F} = \text{Filt}(B_1, \dots, B_l)$ holds by Theorem 3.5. This means that S has a filtration consisting of B_1, \dots, B_l , but since S is a simple object in \mathcal{F} , clearly $S \cong B_i$ holds for some i . By Theorem 3.5, all the bricks in this brick sequence are pairwise non-isomorphic, thus S must appear exactly once. \square

By this, to find a simple object in \mathcal{F} , we only have to work in a fixed brick sequence of \mathcal{F} . In order to give a criterion for a brick in a brick sequence to be simple, we will introduce some technical condition and lemmas.

Definition 3.9. Let M be a Λ -module and \mathcal{C} a collection of Λ -modules. Then we say that \mathcal{C} has the *zero-mono property for M* if every morphism $M \rightarrow C$ with $C \in \mathcal{C}$ is either zero or injective in $\text{mod } \Lambda$.

Then Lemma 3.7 amounts to that M is simple in a torsion-free class \mathcal{F} if and only if \mathcal{F} has the zero-mono property for M .

The important advantage of the zero-mono property is that this property is closed under extensions in the following sense:

Lemma 3.10. *Let M be a Λ -module and \mathcal{C}, \mathcal{D} collections of Λ -modules. If \mathcal{C} and \mathcal{D} have the zero-mono property for M , then so does $\mathcal{C} * \mathcal{D}$.*

Proof. Consider any short exact sequence

$$0 \longrightarrow C \xrightarrow{\iota} X \xrightarrow{\pi} D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Take any morphism $\varphi: M \rightarrow X$, and it suffices to show that φ is either zero or injective.

Consider the following diagram.

$$\begin{array}{ccccccc} & & & M & & & \\ & & & \downarrow \varphi & & & \\ & & & X & \xrightarrow{\pi} & D & \longrightarrow 0 \\ & \swarrow \overline{\varphi} & \searrow \iota & & & & \\ 0 & \longrightarrow & C & \xrightarrow{\iota} & X & \longrightarrow & 0 \end{array}$$

Since \mathcal{D} has the zero-mono property for M , either $\pi\varphi$ is an injection or $\pi\varphi = 0$. In the former case, φ is an injection, so suppose the latter. Then there exists a morphism $\overline{\varphi}: M \rightarrow C$ with $\varphi = \overline{\varphi}\iota$. Since \mathcal{C} has the zero-mono property for M , we have $\overline{\varphi}$ is either zero or injective. Thus φ is either zero or injective respectively. \square

As a first application of this corollary, we can show the following.

Corollary 3.11. *Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$. Then taking labels gives an injection*

$$\{\text{Arrows in Hasse}(\text{torf } \Lambda) \text{ starting at } \mathcal{F}\} \hookrightarrow \text{sim } \mathcal{F}.$$

Proof. Let $\mathcal{F} \rightarrow \mathcal{F}'$ be an arrow in $\text{Hasse}(\text{torf } \Lambda)$ with B its label. We will show that B is simple in \mathcal{F} , or equivalently, \mathcal{F} has the zero-mono property for B by Lemma 3.7.

By Proposition 3.1, we have $\mathcal{F} = \text{Filt}(B) * \mathcal{F}'$ and $\mathcal{F}' = \mathcal{F} \cap B^\perp$. Therefore, according to Lemma 3.10, it suffices to show that B and \mathcal{F}' have the zero-mono property for B . Clearly B has the zero-mono property for B since B is a brick. On the other hand, since $\mathcal{F}' \subset B^\perp$, every morphism from B to $F \in \mathcal{F}'$ should be zero. Thus \mathcal{F}' has the zero-mono property. \square

Another application of Lemma 3.10 is the following complete description of a brick in a given brick sequence to be simple.

Corollary 3.12. *Let B_1, \dots, B_l be a brick sequence of a torsion-free class \mathcal{F} in $\text{mod } \Lambda$. Then any simple objects in \mathcal{F} are contained in $\{B_1, \dots, B_l\}$, and the following are equivalent for $1 \leq i \leq l$.*

- (1) B_i is simple in \mathcal{F} .
- (2) Every morphism $B_i \rightarrow B_j$ is either zero or injective for each $j \neq i$, or equivalently, $j > i$.
- (3) $\{B_1, \dots, B_l\}$ has the zero-mono property for B_i .
- (4) There is no surjection $B_i \twoheadrightarrow B_j$ with $i < j$.

Proof. Since we have $\text{Hom}_\Lambda(B_i, B_j) = 0$ for $j < i$ by Theorem 3.5, the two conditions in (2) are equivalent.

(1) \Rightarrow (2): This is clear by Lemma 3.7 since $B_j \in \mathcal{F}$ for every j .

(2) \Rightarrow (3): It suffices to recall that B_i has the zero-mono property for B_i since B_i is a brick.

(3) \Rightarrow (1): Lemma 3.10 shows that $\text{Filt}(B_1, \dots, B_l)$ has the zero-mono property for B_i . Since $\mathcal{F} = \text{Filt}(B_1, \dots, B_l)$ holds by Theorem 3.5, we have that B_i is simple in \mathcal{F} by Lemma 3.7.

(2) \Rightarrow (4): This is clear since B_i and B_j with $j > i$ are non-isomorphic by Theorem 3.5.

(4) \Rightarrow (2): Suppose that there are some $j > i$ and a map $\varphi: B_i \rightarrow B_j$ which is neither zero nor injective. Take minimal j with this property. We claim that φ is actually a surjection.

Let $0 = \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \dots \leftarrow \mathcal{F}_l = \mathcal{F}$ be a maximal green sequence of \mathcal{F} corresponding to the brick sequence B_1, \dots, B_l . Then we have $\mathcal{F}_j = \text{Filt}(B_j) * \mathcal{F}_{j-1}$ by Proposition 3.1. Moreover, since (1)-(3) are equivalent and B_1, \dots, B_{j-1} is a brick sequence of \mathcal{F}_{j-1} , the minimality of j implies that B_i is a simple object in \mathcal{F}_{j-1} , that is, \mathcal{F}_{j-1} has the zero-mono property for B_i .

We have $\text{Im } \varphi \in \mathcal{F}_j$ since it is a submodule of $B_j \in \mathcal{F}_j$. Thus by $\mathcal{F}_j = \text{Filt}(B_j) * \mathcal{F}_{j-1}$, we have the following exact sequence

$$\begin{array}{ccccccc} & & & & B_i & \xrightarrow{\varphi} & B_j \\ & & & & \downarrow \pi & \nearrow & \\ 0 & \longrightarrow & X & \longrightarrow & \text{Im } \varphi & \longrightarrow & F \longrightarrow 0 \end{array}$$

with $X \in \text{Filt}(B_j)$ and $F \in \mathcal{F}_{j-1}$. Suppose that $X = 0$, then $\text{Im } \varphi \cong F \in \mathcal{F}_{j-1}$ holds. Thus π is either zero or injective. Since $\varphi \neq 0$, we have that π is injective, hence so is φ , which is a contradiction. Thus $X \neq 0$. It follows from $X \in \text{Filt}(B_i)$ and $X \neq 0$ that $\dim B_i \leq \dim X$ holds. On the other hand, since we have injections $X \hookrightarrow \text{Im } \varphi \hookrightarrow B_i$, we have $\dim X \leq \dim(\text{Im } \varphi) \leq \dim B_i$. It follows that $\dim(\text{Im } \varphi) = \dim B_i$, hence the inclusion map $\text{Im } \varphi \hookrightarrow B_i$ is an isomorphism. Therefore, φ is a surjection. \square

Thus, the problem to determine simple objects in a torsion-free class \mathcal{F} can be solved by the following process in principle.

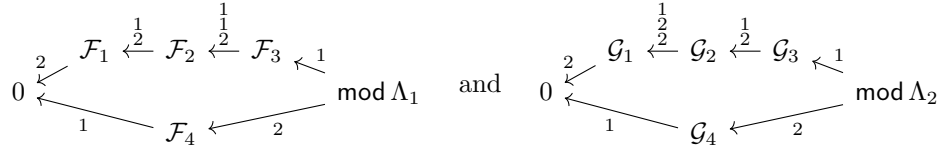
- (1) Find and fix a maximal green sequence of \mathcal{F} (if exists).
- (2) Compute a brick sequence B_1, \dots, B_l associated to it.
- (3) For each i , check whether every morphism $B_i \rightarrow B_j$ with $i < j$ is either zero or injective (or equivalently, check whether there is no surjection $B_i \twoheadrightarrow B_j$ with $i < j$).
- (4) $\text{sim } \mathcal{F}$ consists of B_i with such a property.

Probably the most non-trivial part of the above computation is (1) and (2). If \mathcal{F} is functorially finite, then (1) can be computed by using mutations of support τ -tilting modules in [AIR], and (2) can be computed by using the description of brick labels associated to mutations due to [Asa1].

Example 3.13. We will give two examples of computation of simple objects by considering algebras borrowed from [DIRRT, Example 1.14]. Let Λ_1 and Λ_2 be k -algebras defined by

$$\Lambda_1 = k \left(u \curvearrowright 1 \longrightarrow 2 \right) / (u^2) \quad \text{and} \quad \Lambda_2 = k \left(1 \longrightarrow 2 \curvearrowright v \right) / (v^2).$$

Then $\text{torf } \Lambda_i$ and their brick labeling for $i = 1, 2$ is as follows:



We omit the description of each torsion-free class. By using Corollary 3.12, one can obtain the following table of simple objects. Note that $\text{torf } \Lambda_1$ and $\text{torf } \Lambda_2$ are isomorphic as posets, and \mathcal{F}_2

torsion-free class in $\text{mod } \Lambda_1$	simple objects	torsion-free class in $\text{mod } \Lambda_1$	simple objects
0	\emptyset	0	\emptyset
\mathcal{F}_1	2	\mathcal{G}_1	2
\mathcal{F}_2	$2, \frac{1}{2}$	\mathcal{G}_2	$2, \frac{1}{2}$
\mathcal{F}_3	$2, \frac{1}{2}, \frac{1}{2}$	\mathcal{G}_3	$2, \frac{1}{2}$
\mathcal{F}_4	1	\mathcal{G}_4	1
$\text{mod } \Lambda_1$	1, 2	$\text{mod } \Lambda_1$	1, 2

TABLE 1. Simple objects in torsion-free classes in $\text{mod } \Lambda_1$ and $\text{mod } \Lambda_2$

corresponds to \mathcal{G}_2 . However, the number of simple objects in \mathcal{F}_2 differs from that in \mathcal{G}_2 . This shows that to determine (the number of) simple objects, the poset structure of $\text{torf } \Lambda$ is not enough, and we indeed have to compute brick labels and homomorphisms between labels in general.

Fortunately, in the case of preprojective algebras (or path algebras) of Dynkin type, we can make use of root-theoretical properties developed in the previous section to determine simples, as we shall see in the next section.

We end this section by the following small lemma, which we need later.

Lemma 3.14. *Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$ such that the interval $[0, \mathcal{F}]$ in $\text{torf } \Lambda$ is finite. Then every brick B in \mathcal{F} appears at least one brick sequence of \mathcal{F} .*

Proof. Consider $F(B)$, the smallest torsion-free class containing \mathcal{F} . Then $F(B) \subset \mathcal{F}$ holds, that is, $F(B) \in [0, \mathcal{F}]$.

Since $[0, \mathcal{F}]$ is a finite lattice, clearly there exists a maximal green sequence of \mathcal{F} which is of the form $0 \leftarrow \dots \leftarrow F(B) \leftarrow \dots \leftarrow \mathcal{F}$. In [DIRRT, Theorem 3.4], it was shown that there is only one arrow which starts at $F(B)$ in $\text{Hasse}(\text{torf } \Lambda)$, and that its label is B . Therefore, this maximal sequence gives the desired brick sequence. \square

4. TORSION-FREE CLASSES OVER PREPROJECTIVE ALGEBRAS OF DYNKIN TYPE

In this section, we will classify simples in torsion-free classes in preprojective algebras of Dynkin type, by using root systems and brick sequences.

4.1. Notation and preliminary results. Let us briefly recall the definition of preprojective algebras of Dynkin type, with emphasis on their relation to root systems.

Let Φ be the simply-laced root system of the Dynkin type X , namely, $X \in \{A_n, D_n, E_6, E_7, E_8\}$. We denote by W its Weyl group, and we fix a choice of simple roots of Φ .

Let $Q = (Q_0, Q_1)$ be a Dynkin quiver of the same type X , that is, Q is a quiver whose underlying graph is the Dynkin diagram of type X . Then we may identify Q_0 with the index set of simple roots of Φ . For $i \in Q_0$, we denote by α_i the simple root of Φ corresponding to the vertex i in the Dynkin diagram, and by $s_i = s_{\alpha_i} \in W$ the simple reflection with respect to α_i .

Let \overline{Q} be a *double quiver* of Q , which is obtained from Q by adding an arrow $a^*: j \rightarrow i$ for each arrow $a: i \rightarrow j$ in Q . The *preprojective algebra* of Q is defined by

$$\Pi = \Pi_\Phi := k\overline{Q} / \left(\sum_{a \in Q_1} aa^* - a^*a \right).$$

It is known that Π is a finite-dimensional k -algebra, and it only depends on Φ and does not depend on the choice of Q (the choice of orientations of the Dynkin diagram) up to isomorphism. Thus we call it a *preprojective algebra of Φ* .

It is convenient to consider the dimension vector of Π -modules inside the ambient space of Φ . Let V be the ambient space of Φ , then V has a basis $\{\alpha_i \mid i \in Q_0\}$ as a \mathbb{R} -vector space. Define a group homomorphism $\underline{\dim}: K_0(\text{mod } \Pi) \rightarrow V$ by $[S_i] \mapsto \alpha_i$, where $K_0(\text{mod } \Pi)$ denotes the Grothendieck group of $\text{mod } \Pi$ and S_i denotes the simple Π -module corresponding to the vertex $i \in Q_0$. We simply write $\underline{\dim} M := \underline{\dim}[M]$ for $M \in \text{mod } \Pi$.

To sum up, *throughout this section, we keep the following notation:*

- Φ is a simply-laced root system of Dynkin type X in the ambient space V .
- Q is a Dynkin quiver of the same type X .
- W is the Weyl group of Φ .
- Π is the preprojective algebra of Φ (or Q).
- S_i is the simple Π -module corresponding to the vertex $i \in Q_0$.
- $\underline{\dim}: K_0(\text{mod } \Pi) \rightarrow V$ is a map defined by $[S_i] \mapsto \alpha_i$ for $i \in Q_0$.

Torsion-free classes in $\text{mod } \Pi$ was completely classified by Mizuno in [Miz], and they are in bijection with elements in W . Let us briefly explain his result in our context.

For a vertex $i \in Q_0$, we denote by e_i the corresponding idempotent of Π . We denote by I_i the two-sided ideal of Π generated by $1 - e_i$. For an element $w \in W$, we can define a two-sided ideal I_w of Π as follows: Take any reduced expression $w = s_{u_1} s_{u_2} \cdots s_{u_l}$ of w in W . Then $I(w)$ is defined by

$$I(w) = I_{u_1 \dots u_2 u_1} := I_{u_1} I_{u_{l-1}} \cdots I_{u_2} I_{u_1}.$$

This construction does not depend on the choice of reduced expressions of w by [BIRS, Theorem III.1.9].

The following result of Mizuno gives the key connection between the representation theory of Π and its root system Φ .

Theorem 4.1 ([Miz, Theorem 2.30]). *Let Φ be a simply-laced root system of Dynkin type, W its Weyl group and Π its preprojective algebra.*

- (1) $\mathcal{F}(w) := \text{Sub}(\Pi/I(w))$ is a torsion-free class in $\text{mod } \Pi$.
- (2) The map $w \mapsto \mathcal{F}(w)$ gives a bijection

$$W \xrightarrow{\sim} \text{torf } \Pi.$$

Moreover, this bijection is actually an isomorphism of finite lattices, where we endow W with the right weak order.

4.2. Brick sequences in preprojective algebras. We begin with studying the relation between brick sequences of $\mathcal{F}(w)$ and root sequences of w .

By Theorem 4.1, we can identify a maximal green sequence of $\mathcal{F}(w)$ with a saturated chain in the interval $[e, w]$ in (W, \leq_R) , which in turn is identified with a particular choice of reduced expression of w (see e.g. [BB, Proposition 3.1.2]). In this way, we can talk about a *brick sequence of $\mathcal{F}(w)$*

associated to a reduced expression of w . More precisely, let $w = s_{u_1} \cdots s_{u_l}$ be a reduced expression of $w \in W$. Then we have a maximal green sequence $0 = \mathcal{F}(e) \leftarrow \mathcal{F}(s_{u_1}) \leftarrow \mathcal{F}(s_{u_1}s_{u_2}) \leftarrow \cdots \leftarrow \mathcal{F}(s_{u_1} \cdots s_{u_l}) = \mathcal{F}(w)$ of $\mathcal{F}(w)$, thus we obtain the corresponding brick sequence of $\mathcal{F}(w)$. We remark that this sequence coincides with a sequence of *layer modules* considered in [AIRT].

By a lattice isomorphism in Theorem 4.1, an arrow in $\text{Hasse}(\text{torf } \Pi)$ is of the form $\mathcal{F}(w) \leftarrow \mathcal{F}(ws_i)$ for some $w \in W$ and $i \in Q_0$ satisfying $\ell(w) < \ell(ws_i)$. In this case, we have $I(ws_i) \subset I(w)$, and the following holds.

Proposition 4.2 ([IRRT, Theorem 4.1]). *The brick label of $\mathcal{F}(w) \leftarrow \mathcal{F}(ws_i)$ is given by $I(w)/I(ws_i)$.*

Next we will compute dimension vectors of brick labels following [AIRT].

Proposition 4.3. *Let $w = s_{u_1} \cdots s_{u_l}$ be a reduced expression of $w \in W$ and B_1, B_2, \dots, B_l its associated brick sequence of $\mathcal{F}(w)$. Then the following equality holds in V for $1 \leq m \leq l$:*

$$\underline{\dim} B_m = s_{u_1} \cdots s_{u_{m-1}}(\alpha_{u_m}).$$

In particular, $\{\underline{\dim} B_1, \dots, \underline{\dim} B_l\} = \text{inv}(w) \subset \Phi^+$ holds, and $\underline{\dim} B_1, \underline{\dim} B_2, \dots, \underline{\dim} B_l$ is a root sequence associated to the above reduced expression of w , defined in Definition 2.6.

Proof. In [AIRT, Theorem 2.7], it was shown that

$$[B_m] = [I(s_{u_{m-1}} \cdots s_{u_1})/I(s_{u_m} \cdots s_{u_1})] = R_{u_1} \cdots R_{u_{m-1}}[S_{u_j}]$$

holds in $K_0(\text{mod } \Pi)$, where $R_i: K_0(\text{mod } \Pi) \rightarrow K_0(\text{mod } \Pi)$ for $i \in Q_0$ is a group homomorphism defined by

$$R_i([S_j]) := [S_j] - (2\delta_{ij} - m_{ij})[S_i]$$

on the free basis $\{[S_j] \mid j \in Q_0\}$ of $K_0(\text{mod } \Pi)$. Here δ is the Kronecker delta, and m_{ij} is the number of edges in the Dynkin diagram of Φ which connect i and j .

Therefore, it suffices to check that the following diagram commutes:

$$\begin{array}{ccc} K_0(\text{mod } \Pi) & \xrightarrow{R_i} & K_0(\text{mod } \Pi) \\ \downarrow \underline{\dim} & & \downarrow \underline{\dim} \\ V & \xrightarrow{s_i} & V \end{array}$$

We only have to check on the basis $[S_j]$ for $j \in Q_0$. This follows from the following equality:

$$2\delta_{ij} - m_{ij} = \langle \alpha_j, \alpha_i \rangle = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \text{ and } j \text{ are not connected by an arrow in } Q, \\ -1 & \text{if } i \text{ and } j \text{ are connected by an arrow in } Q, \end{cases} \quad (4.1)$$

which can be checked directly. \square

Therefore, a brick sequence of $\mathcal{F}(w)$ serves as a categorification of a root sequence of w . Note that different bricks may have the same dimension vector.

Example 4.4. Consider an element w in Example 2.8. Then Figure 3 is the Hasse quiver of the interval $[0, \mathcal{F}(w)]$ with its brick labels. For example, two bricks $\frac{1}{2}$ and $\frac{2}{1}$ are non-isomorphic but have the same dimension vector $\alpha_1 + \alpha_2$.

The same computation can be done for Example 2.9, and actually Figure 3 can be seen as a Hasse quiver of $[0, \mathcal{F}(w)]$ with dimension vectors of its brick labels.

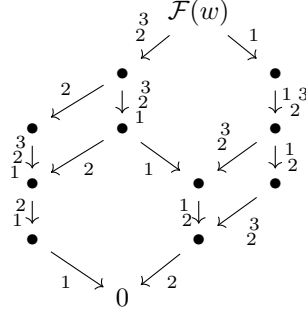
Corollary 4.5. *Let w be an element of W , then the following hold.*

(1) *For any $M \in \mathcal{F}(w)$, we have*

$$\underline{\dim} M = \sum_{\beta \in \text{inv}(w)} n_\beta \beta,$$

where n_β is some non-negative integer for each $\beta \in \text{inv}(w)$.

(2) *For any $B \in \text{brick } \mathcal{F}(w)$, we have $\underline{\dim} B \in \text{inv}(w)$. Thus $\underline{\dim}$ induces a surjection $\underline{\dim}: \text{brick } \mathcal{F}(w) \rightarrow \text{inv}(w)$.*

FIGURE 3. Brick sequences of $w = s_{12312}$

Proof. (1) Let B_1, \dots, B_l be a brick sequence of $\mathcal{F}(w)$. Then by Proposition 4.3, we have $\text{inv}(w) = \{\underline{\dim} B_1, \dots, \underline{\dim} B_l\}$.

Let $M \in \mathcal{F}(w)$. Then Theorem 3.5 implies that $M \in \text{Filt}(B_1, \dots, B_l)$. Thus by taking the dimension vector, the assertion immediately follows.

(2) Let B be a brick in $\mathcal{F}(w)$. Then Lemma 3.14 implies that there is a brick sequence of $\mathcal{F}(w)$ which contains B , since $[0, \mathcal{F}(w)] \subset \text{torf } \Pi \cong W$ is a finite lattice. Now Proposition 4.3 implies that $\underline{\dim} B \in \text{inv}(w)$. Thus the map $\underline{\dim}: \text{brick } \mathcal{F}(w) \rightarrow \text{inv}(w)$ is well-defined. Moreover, it is clearly surjective by Proposition 4.3. \square

4.3. Simple objects versus Bruhat inversions. By Proposition 3.8, to find simple objects in $\mathcal{F}(w)$, it suffices to check whether B_i is simple or not for a fixed brick sequence B_1, \dots, B_l of $\mathcal{F}(w)$. For this, the *homological lemma* due to Crawley-Boevey is useful. To state this, let us introduce the symmetric bilinear form $\langle -, - \rangle_\Pi$ on $\mathbf{K}_0(\text{mod } \Pi)$ defined by

$$\left\langle \sum_i a_i [S_i], \sum_j b_j [S_j] \right\rangle_\Pi = 2 \sum_{i \in Q_0} a_i b_i - \sum_{i \rightarrow j \in Q_1} (a_i b_j + a_j b_i).$$

This coincides with the standard homological symmetric bilinear form associated to the quiver Q (or twice of it, see e.g. [ASS, VII.4]). It can also be interpreted as the restriction of the Euler form of the preprojective algebra $\widehat{\Pi}$ of the extended Dynkin type corresponding to Π , see e.g. [IRRT, Section 3]. Then we have the following formula due to Crawley-Boevey.

Lemma 4.6 ([CB, Lemma 1]). *Let $M, N \in \text{mod } \Pi$. Then we have the following equation:*

$$\langle [M], [N] \rangle_\Pi = \dim \text{Hom}_\Pi(M, N) + \dim \text{Hom}_\Pi(N, M) - \dim \text{Ext}_\Pi^1(M, N).$$

We remark that this formula can be shown easily by using the 2-Calabi-Yau property of the preprojective algebra $\widehat{\Pi}$ of the extended Dynkin type.

We can check the compatibility of the bilinear form on $\mathbf{K}_0(\text{mod } \Pi)$ and the value $\langle \alpha, \beta \rangle$ in $\alpha, \beta \in V$ defined in Section 2 as follows.

Lemma 4.7. *Suppose that $M, N \in \text{mod } \Pi$ satisfy $\underline{\dim} M, \underline{\dim} N \in \Phi^+$. Then we have*

$$\langle [M], [N] \rangle_\Pi = \langle \underline{\dim} M, \underline{\dim} N \rangle.$$

Proof. Since $\underline{\dim} M$ and $\underline{\dim} N$ are positive roots, we can write as $\underline{\dim} M = \sum_i m_i \alpha_i$ and $\underline{\dim} N = \sum_j n_j \alpha_j$ for non-negative integers m_i, n_j . Then we have

$$\begin{aligned} \langle \underline{\dim} M, \underline{\dim} N \rangle &= \left\langle \sum_i m_i \alpha_i, \sum_j n_j \alpha_j \right\rangle \\ &= \sum_i \left(m_i \langle \alpha_i, \sum_j n_j \alpha_j \rangle \right) \\ &= \sum_i \left(m_i \langle \sum_j n_j \alpha_j, \alpha_i \rangle \right) \\ &= \sum_i \sum_j m_i n_j \langle \alpha_j, \alpha_i \rangle. \end{aligned}$$

The first equality follows from definition, the second and the last follow since $\langle -, - \rangle$ is linear with respect to the first variable, and the third follows since Φ is simply-laced and both vectors inside $\langle -, - \rangle$ are roots. By using the equation (4.1), we can compute this as follows:

$$\sum_i \sum_j m_i n_j \langle \alpha_j, \alpha_i \rangle = 2 \sum_{i \in Q_0} m_i n_i - \sum_{i \rightarrow j \in Q} (m_i n_j + m_i n_j)$$

Thus we have the assertion. \square

By using this, we can show the following main result of this paper.

Theorem 4.8. *Let w be an element of W and B_1, \dots, B_l a brick sequence of $\mathcal{F}(w)$. Then the following are equivalent for $1 \leq m \leq l$*

- (1) $\underline{\dim} B_m \in \text{Binv}(w)$ holds.
- (2) B_m is a simple object in $\mathcal{F}(w)$.

Proof. (1) \Rightarrow (2) Suppose that B_m is not a simple object in $\mathcal{F}(w)$. Then there exists an exact sequence

$$0 \longrightarrow L \longrightarrow B_m \longrightarrow N \longrightarrow 0$$

with $L, N \neq 0$. By applying $\underline{\dim}$, we obtain

$$\underline{\dim} B_m = \underline{\dim} L + \underline{\dim} N$$

with $\underline{\dim} L, \underline{\dim} N \neq 0$. By Corollary 4.5, both $\underline{\dim} L$ and $\underline{\dim} N$ are non-negative integer linear combinations of inversions of w such that at least one of the coefficients should be strictly positive. Thus Theorem 2.17 implies that $\underline{\dim} B_m$ is a non-Bruhat inversion.

(2) \Rightarrow (1): Suppose that $\underline{\dim} B_m$ is a non-Bruhat inversion of w . We will use Lemma 3.7 to show that B_m is not a simple object in $\mathcal{F}(w)$.

By Theorem 2.17, there exists some $\alpha, \beta \in \text{inv}(w)$ such that $\alpha + \beta = \underline{\dim} B_m$ since Φ is simply-laced. Moreover, these satisfy $\langle \underline{\dim} B_m, \alpha \rangle = \langle \underline{\dim} B_m, \beta \rangle = 1$ by Lemma 2.18.

On the other hand, since $\{\underline{\dim} B_1, \dots, \underline{\dim} B_l\} = \text{inv}(w)$ by Proposition 4.3, there are i and j such that $\underline{\dim} B_i = \alpha$ and $\underline{\dim} B_j = \beta$. By exchanging α and β if necessary, we may assume that $i < j$. Moreover, since $\underline{\dim} B_1, \dots, \underline{\dim} B_l$ is a root sequence and $\underline{\dim} B_m = \underline{\dim} B_i + \underline{\dim} B_j$ holds, Theorem 2.7 implies $i < m < j$.

To summarize, we have found i and j with $i < m < j$ such that $\underline{\dim} B_m = \underline{\dim} B_i + \underline{\dim} B_j$ and $\langle \underline{\dim} B_m, \underline{\dim} B_j \rangle = 1$ hold. Then Lemma 4.7 implies

$$\langle [B_m], [B_j] \rangle_{\Pi} = \langle \underline{\dim} B_m, \underline{\dim} B_j \rangle = 1.$$

By combining this with Lemma 4.6, we have

$$\dim \text{Hom}_{\Pi}(B_m, B_j) + \dim \text{Hom}_{\Pi}(B_j, B_m) = 1 + \dim \text{Ext}_{\Pi}^1(B_j, B_m) \geq 1.$$

On the other hand, since $m < j$, we must have $\text{Hom}_{\Pi}(B_j, B_m) = 0$ by Theorem 3.5. Therefore, we have $\dim \text{Hom}_{\Pi}(B_m, B_j) \geq 1$, that is, $\text{Hom}_{\Pi}(B_m, B_j) \neq 0$.

Take any non-zero morphism $\varphi: B_m \rightarrow B_j$. Since $\underline{\dim} B_m = \underline{\dim} B_k + \underline{\dim} B_m$, we have $\dim B_m > \dim B_k$, hence φ cannot be an injection. Therefore Lemma 3.7 implies that B_m is not a simple object in $\mathcal{F}(w)$. \square

In conclusion, we have the following classification of simple objects.

Corollary 4.9. *Let w be an element of W . Then the following hold.*

- (1) *A brick B in $\mathcal{F}(w)$ is a simple object in $\mathcal{F}(w)$ if and only if $\underline{\dim} B \in \text{Binv}(w)$ holds.*
- (2) *The map $\underline{\dim}: \text{brick } \mathcal{F}(w) \rightarrow \text{inv}(w)$ in Corollary 4.5 restricts to a bijection*

$$\underline{\dim}: \text{sim } \mathcal{F}(w) \xrightarrow{\sim} \text{Binv}(w).$$

In other words, simple objects in $\mathcal{F}(w)$ bijectively correspond to Bruhat inversions of w by taking dimension vectors.

Proof. (1) Lemma 3.14 implies that B appears in some brick sequence of $\mathcal{F}(w)$. Then (1) is obvious by Theorem 4.8.

(2) Fix a brick sequence B_1, \dots, B_l of $\mathcal{F}(w)$. If S is a simple object in $\mathcal{F}(w)$, then $S \cong B_j$ for some j by Proposition 3.8. Thus $\underline{\dim} S \in \text{Binv}(w)$ holds by Theorem 4.8. Thus we obtain a map $\underline{\dim}: \text{sim } \mathcal{F} \rightarrow \text{Binv}(w)$.

We claim that this map is a bijection. Let $\gamma \in \text{Binv}(w)$. Then since $\{\underline{\dim} B_1, \dots, \underline{\dim} B_l\} = \text{inv}(w) \supset \text{Binv}(w)$ by Proposition 4.3, there exists j with $\underline{\dim} B_j = \gamma$. This B_j is simple in $\mathcal{F}(w)$ by Theorem 4.8. Thus $\underline{\dim}: \text{sim } \mathcal{F} \rightarrow \text{Binv}(w)$ is surjective.

On the other hand, let S and S' be simple objects in $\mathcal{F}(w)$ satisfying $\underline{\dim} S = \underline{\dim} S'$. Then by the above argument, we have $S \cong B_i$ and $S' \cong B_j$ for some i and j , hence $\underline{\dim} B_i = \underline{\dim} B_j$ holds. Since elements in $\{\underline{\dim} B_1, \dots, \underline{\dim} B_l\} = \text{inv}(w)$ are pairwise distinct by Proposition 2.4, we have $i = j$, which shows $S \cong S'$. Thus $\underline{\dim}: \text{sim } \mathcal{F} \rightarrow \text{Binv}(w)$ is injective. \square

4.4. Characterization of the Jordan-Hölder property. Next we will characterize when $\mathcal{F}(w)$ satisfies the Jordan-Hölder property in the sense of [Eno] in terms of the combinatorics of w . Let us recall some related definitions and results from [Eno].

Definition 4.10. Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$ for a finite-dimensional k -algebra Λ .

- (1) For M in \mathcal{F} , a *composition series of M in \mathcal{F}* is a series of submodules of M

$$0 = M_0 \subset M_1 \subset \dots \subset M_m$$

such that M_i/M_{i-1} is a simple object in \mathcal{F} for each i .

- (2) For M in \mathcal{F} , let $0 = M_0 \subset \dots \subset M_m = M$ and $0 = M'_1 \subset \dots \subset M'_n = M$ be two composition series of M in \mathcal{F} . We say that these are *equivalent* if $m = n$ holds and there exists a permutation σ of the set $\{1, 2, \dots, n\}$ such that $M_i/M_{i-1} \cong M'_{\sigma(i)}/M'_{\sigma(i)-1}$ holds for each i .
- (3) We say that \mathcal{F} satisfies the *Jordan-Hölder property*, abbreviated by (JHP), if any composition series of M are equivalent for every object M in \mathcal{F} .

In [Eno, Theorem 5.10], the author gives a numerical criterion for (JHP). To rephrase his result in our context, we introduce the *support* of modules or torsion-free classes.

Definition 4.11. Let Λ be a finite-dimensional k -algebra and $\text{sim}(\text{mod } \Lambda)$ the set of isomorphism classes of simple Λ -modules.

- (1) For a module M , the *support of M* is a set of simple Λ -modules defined by

$$\text{supp } M := \{S \in \text{sim}(\text{mod } \Lambda) \mid S \text{ is a composition factor of } M\}.$$

- (2) For a collection \mathcal{C} of modules, the *support of \mathcal{C}* is a set of simple Λ -modules defined by

$$\text{supp } \mathcal{C} := \bigcup_{M \in \mathcal{C}} \text{supp } M.$$

Then the following gives a numerical criterion for (JHP). Here for a set A , we denote by $\mathbb{Z}^{(A)}$ the free abelian group with basis A . We simply write $\mathbb{Z}^A := \mathbb{Z}^{(A)}$ if A is a finite set.

Theorem 4.12. *Let \mathcal{F} be a torsion-free class in $\text{mod } \Lambda$ for a finite-dimensional k -algebra Λ . Suppose that $\mathcal{F} = \text{Sub } M$ holds for some $M \in \mathcal{F}$. Then the following are equivalent:*

- (1) \mathcal{F} satisfies (JHP).
- (2) The natural map $\mathbb{Z}^{(\text{sim } \mathcal{F})} \rightarrow \mathbf{K}_0(\text{mod } \Lambda)$ which sends $M \in \text{sim } \mathcal{F}$ to $[M]$ is an injection.
- (2') The natural map $\mathbb{Z}^{(\text{sim } \mathcal{F})} \rightarrow \mathbb{Z}^{(\text{supp } \mathcal{F})}$ is an isomorphism, where we identify $\mathbb{Z}^{(\text{supp } \mathcal{F})}$ with a subgroup of $\mathbf{K}_0(\text{mod } \Lambda)$ generated by $[S]$ with $S \in \text{supp } \mathcal{F}$.
- (3) $\# \text{sim } \mathcal{F} = \# \text{supp } \mathcal{F}$ holds.

Moreover, the map in (2') is always surjective.

Proof. We give a proof using τ -tilting theory and the Grothendieck group $\mathbf{K}_0(\mathcal{F})$ of the exact category \mathcal{F} , for which we refer to [AIR] and [Eno] respectively. It is shown in [Eno, Theorem 4.12, Corollary 5.14] that the following are equivalent:

- (i) \mathcal{F} satisfies (JHP).
- (ii) The natural map $\mathbb{Z}^{(\text{sim } \mathcal{F})} \rightarrow \mathbf{K}_0(\mathcal{F})$, which is always surjective, is an isomorphism.
- (iii) $\# \text{sim } \mathcal{F} = |U|$ holds, where U is a support τ^- -tilting module with $\mathcal{F} = \text{Sub } U$ and $|U|$ is a number of non-isomorphic indecomposable direct summands of U .

On the other hand, [AIR, Proposition 2.2] implies $|U| = \# \text{supp } U$. Since $\mathcal{F} = \text{Sub } U$, clearly $\text{supp } \mathcal{F} = \text{supp } U$ holds, hence we have $|U| = \# \text{supp } \mathcal{F}$. Therefore, (1) and (3) are equivalent.

To see that (2) and (2') are also equivalent, let us consider $\mathbf{K}_0(\mathcal{F})$. By using [Eno, Lemma 5.7], one can show that the natural map $\mathbf{K}_0(\mathcal{F}) \rightarrow \mathbf{K}_0(\text{mod } \Lambda)$ is an injection, and that its image is precisely $\mathbb{Z}^{(\text{supp } \mathcal{F})}$. Thus all the conditions are equivalent. \square

To describe a characterization of (JHP) for $\mathcal{F}(w)$, we introduce the *support of $w \in W$* . Recall that simple roots and simple reflections are parametrized by Q_0 in our setting.

Definition 4.13. Let w be an element of W . Then *its support* is a subset $\text{supp}(w)$ of Q_0 defined as follows:

$$\text{supp}(w) = \{i \in Q_0 \mid \text{there is a reduced expression of } w \text{ which contains } s_i\}$$

Then the support of w coincides with the support of $\mathcal{F}(w)$ in the following sense:

Proposition 4.14. *Let w be an element in W . Then a natural bijection $Q_0 \xrightarrow{\sim} \text{sim}(\text{mod } \Pi)$, which sends i to the simple Π -module corresponding to i , restricts to a bijection*

$$\text{supp}(w) \xrightarrow{\sim} \text{supp } \mathcal{F}(w).$$

Proof. For a positive root $\beta \in \Phi^+$, we can write $\beta = \sum_{i \in Q_0} n_i \alpha_i$ with $n_i \geq 0$ in a unique way. Denote by $\text{supp}(\beta)$ the set of $i \in Q_0$ with $n_i > 0$. Then Corollary 4.5 clearly implies that the bijection $Q_0 \rightarrow \text{sim}(\text{mod } \Pi)$ restricts to a bijection

$$\bigcup_{\beta \in \text{inv}(w)} \text{supp}(\beta) \xrightarrow{\sim} \text{supp } \mathcal{F}(w).$$

Thus it suffices to show $\bigcup_{\beta \in \text{inv}(w)} \text{supp}(\beta) = \text{supp}(w)$.

Let $w = s_{u_1} \cdots s_{u_l}$ be a reduced expression of w , and β_1, \dots, β_l its associated root sequence. Then we have $\text{inv}(w) = \{\beta_1, \dots, \beta_l\}$. First suppose that i belongs to $\bigcup_{\beta \in \text{inv}(w)} \text{supp}(\beta)$, then $i \in \text{supp}(\beta_m)$ for some $1 \leq m \leq l$. Recall that $\beta_m = s_{u_1} \cdots s_{u_{m-1}}(\alpha_{u_m})$, and that $\{\alpha_u \mid u \in Q_0\}$ is a basis of V . Since each $s_u: V \rightarrow V$ changes only the α_u -component of roots with respect to this basis, i should appear in $\{u_1, u_2, \dots, u_m\}$. Thus $i \in \text{supp}(w)$ holds.

Conversely, suppose $i \in \text{supp}(w)$. Take the minimal m such that $u_m = i$ holds. We claim $i \in \text{supp}(\beta_m)$. Indeed, we have $\beta_m = s_{u_1} s_{u_2} \cdots s_{u_{m-1}}(\alpha_i)$, and i does not appear in u_1, \dots, u_{m-1} by the minimality of m . Since s_u changes only the α_u -component, the α_i -component of β_m is 1, hence $i \in \text{supp}(\beta_m)$ holds. \square

Now the following immediately follows from these observations.

Theorem 4.15. *Let w be an element of W . Then the following are equivalent:*

- (1) $\mathcal{F}(w)$ satisfies the Jordan-Hölder property.

- (2) A map $\varphi_w: \mathbb{Z}^{\text{Binv}(w)} \rightarrow \mathbb{Z}^{\text{supp}(w)}$ defined by $\varphi_w(\sum_i n_i \alpha_i) = \sum_i n_i e_i$ for $\sum_i n_i \alpha_i \in \text{Binv}(w)$ is a bijection, where e_i denotes the basis of $\mathbb{Z}^{\text{supp}(w)}$ corresponding to $i \in \text{supp}(w)$.
- (3) Elements in $\text{Binv}(w)$ are linearly independent in V .
- (4) $\# \text{Binv}(w) = \# \text{supp}(w)$ holds.

Moreover, the map in (2) is always surjective.

Proof. Since $\mathcal{F}(w) = \text{Sub}(\Pi/I_w)$ by definition, we can apply Theorem 4.12 to $\mathcal{F}(w)$. Hence the following are equivalent:

- (i) $\mathcal{F}(w)$ satisfies (JHP).
- (ii) The map $\mathbb{Z}^{(\text{sim } \mathcal{F}(w))} \rightarrow \mathbb{Z}^{\text{supp } \mathcal{F}(w)}$, which is always a surjection, is an isomorphism.
- (iii) $\# \text{sim } \mathcal{F}(w) = \# \text{supp } \mathcal{F}(w)$ holds.

By identifying Q_0 with simple Π -modules, $\text{supp } \mathcal{F}(w)$ bijectively corresponds to $\text{supp}(w)$ by Proposition 4.14. Moreover, $\text{sim } \mathcal{F}(w)$ bijectively corresponds to $\text{Binv}(w)$ by taking dimension vectors by Corollary 4.9. Thus the map in (ii) are exactly same as φ_w in the assertion under the identification $\mathbb{Z}^{(\text{sim } \mathcal{F}(w))} \cong \mathbb{Z}^{\text{Binv}(w)}$ and $\mathbb{Z}^{\text{supp } \mathcal{F}(w)} \cong \sum_{i \in \text{supp}(w)} \mathbb{Z} \alpha_i$.

The left hand side in (iii) is equal to $\# \text{Binv}(w)$ by Corollary 4.9, and the right hand side is equal to $\# \text{supp}(w)$ by Proposition 4.14, thus (1), (2) and (4) are equivalent. Moreover, it is clear that (2) is equivalent to (3) since φ_w is always surjective. \square

We will use the map φ above later in the appendix to relate our characterization of (JHP) to *forest-like permutations* defined in [BMB] and the Schubert variety X_w for type A case.

Remark 4.16. The equality $\# \text{Binv}(w) = \# \text{supp}(w)$ naturally arises when one consider the Bruhat interval in W and its Poincaré polynomial. Let w be an element of W and $[e, w]$ the interval with respect to the Bruhat order. A *Poincaré polynomial* $P_w(q)$ of w is defined by

$$P_w(q) := \sum_{v \in [e, w]} q^{\ell(v)}.$$

Let us write $P_w(q) = \sum_{i=0}^{\ell(w)} a_i q^i$. Then we have $\# \text{supp}(w) = a_1$ and $\# \text{Binv}(w) = a_{l-1}$, since supports of w are precisely simple reflections which are below w in the Bruhat order, and Bruhat inversions of w are in bijection with elements which are covered by w in the Bruhat order by Proposition 2.12. Thus our criterion is equivalent to $a_1 = a_{l-1}$.

Example 4.17. Consider an element $w = s_{12312}$ in Example 2.8. Then there are three Bruhat inversions of w , namely, 100, 010 and 011. This number is equal to the number of $\text{supp}(w) = \{1, 2, 3\}$, thus $\mathcal{F}(w)$ satisfies (JHP).

On the other hand, consider an element $w = s_{012301230}$ in Example 2.9. Then we have $\text{supp}(w) = \{0, 1, 2, 3\}$, but a computation shows $\text{Binv}(w) = \text{inv}(w) \setminus \{\frac{1}{121}\}$ (for example, this follows from the fact that deleting any letter from $s_{012301230}$ yields a reduced expression except for the middle 0). so there are eight Bruhat inversions of w . Thus $\mathcal{F}(w)$ does not satisfy (JHP).

4.5. Conjectures. In this subsection, we give some natural conjectures on the existence of the particular kind of short exact sequences related to Theorem 4.8.

The most non-trivial part of the proof of Theorem 4.8 is to show that B_m is non-simple if $\underline{\dim} B_m$ is non-Bruhat. If $\underline{\dim} B_m$ is non-Bruhat, then as in the proof, there is a brick sequence $B_1, \dots, B_i, \dots, B_j, \dots, B_l$ of $\mathcal{F}(w)$ such that $\underline{\dim} B_i + \underline{\dim} B_j = \underline{\dim} B$. Then the following conjecture naturally occurs.

Conjecture 4.18. Let w be an element of W . Take a brick sequence B_1, \dots, B_l of $\mathcal{F}(w)$, and suppose $\underline{\dim} B_m = \underline{\dim} B_i + \underline{\dim} B_j$ for $1 \leq i < m < j \leq l$. Then there is an exact sequence

$$0 \longrightarrow B_i \longrightarrow B_m \longrightarrow B_j \longrightarrow 0.$$

In fact, in the proof of Theorem 4.8, we only construct a non-zero non-injection $\varphi: B_m \rightarrow B_j$, which is enough for our purpose. This conjecture can be seen as a natural generalization of the result of Proposition 5.7, where the path algebra case was shown over an algebraically closed field.

We have another conjecture on non-simple objects. A *semibrick* \mathcal{S} in $\text{mod } \Pi$ is a set of bricks in $\text{mod } \Pi$ such that $\text{Hom}_\Pi(S, T) = 0$ holds for every two distinct elements $S, T \in \mathcal{S}$.

Conjecture 4.19. *Let w be an element of W and B a non-simple object in $\mathcal{F}(w)$. Then there is a semibrick $\{S, T\}$ in $\mathcal{F}(w)$ and an exact sequence*

$$0 \longrightarrow S \longrightarrow B \longrightarrow T \longrightarrow 0.$$

This conjecture is closely related to the lattice property (forcing order) of the interval $[e, w]$ or $[0, \mathcal{F}(w)]$, and the root-theoretical combinatorial property of the inversion set (contractibility of inversion triples defined in [GL]). The author has obtained the proof of Conjecture 4.19 for type A_n, D_n using combinatorics of (signed) permutations and E_6 using computer.

Conjecture 4.19 can be shown to be equivalent to the following conjecture. Recall that a simple object in a torsion-free class \mathcal{F} appears in every brick sequence of \mathcal{F} by Proposition 3.8. Then it is natural to ask whether the converse holds:

Conjecture 4.20. *Let w be an element of W . If a brick B appears in every brick sequence of $\mathcal{F}(w)$, then B is a simple object in $\mathcal{F}(w)$.*

This conjecture makes sense for any torsion-free classes over any finite-dimensional algebras, but this fails in general. For example, consider \mathcal{G}_3 in Example 3.13. Then there is only one brick sequence of \mathcal{G}_3 , namely, $2, \frac{1}{2}, \frac{1}{2}$. However, $\frac{1}{2}$ is non-simple in \mathcal{G}_3 .

5. TORSION-FREE CLASSES OVER PATH ALGEBRAS OF DYNKIN TYPE

In this section, we use the results in the previous section to study torsion-free classes over path algebras of Dynkin type. *Throughout this section, let Q be a Dynkin quiver, and we use the same notation as in Section 4.1.* We have a natural surjection of algebras $\Pi \twoheadrightarrow kQ$, defined by annihilating all arrows in \overline{Q} which do not appear in Q . Thereby we have an embedding $\text{mod } kQ \hookrightarrow \text{mod } \Pi$, and we often identify $\text{mod } kQ$ with a subcategory of $\text{mod } \Pi$.

Let us recall the celebrated theorem of Gabriel:

Theorem 5.1. *The assignment $M \mapsto \underline{\dim} M$ for $M \in \text{mod } kQ$ induces a bijection*

$$\underline{\dim}: \text{ind}(\text{mod } kQ) \xrightarrow{\sim} \Phi^+.$$

In other words, indecomposable kQ -modules bijectively correspond to positive roots by taking dimension vectors.

5.1. Coxeter-sortable elements and torsion-free classes. We begin with introducing some terminology which we need to give a description of $\text{torf } kQ$. Put $n := \#Q_0$. Then a *Coxeter element* c_Q of Q is an element $c_Q = s_{u_1} \cdots s_{u_n}$ of W with $Q_0 = \{u_1, \dots, u_n\}$ which satisfies the following condition: if there is an arrow $i \leftarrow j$ in Q , then s_i appears before s_j in this expression of c_Q .

Let $c = c_Q$ be a Coxeter element of Q , and w an element of W . We say that w is *c -sortable* if there exists a reduced expression of the form $w = c^{(0)}c^{(1)} \cdots c^{(m)}$ such that each $c^{(i)}$ is a subword of c satisfying $\text{supp}(c^{(0)}) \supset \text{supp}(c^{(1)}) \supset \cdots \supset \text{supp}(c^{(m)})$. We call such an expression a *c -sorting word* of w .

Now we can state the classification of torsion-free classes in $\text{mod } kQ$, which was first established by [IT], and then generalized to any acyclic quiver by [AIRT] and [Tho].

Theorem 5.2 ([IT, Theorem 4.3]). *Let Q be a Dynkin quiver and W its Weyl group. For $w \in W$, define a subcategory $\mathcal{F}_Q(w)$ of $\text{mod } kQ$ by*

$$\mathcal{F}_Q(w) := \text{add}\{M \in \text{ind}(\text{mod } kQ) \mid \underline{\dim} M \in \text{inv}(w)\}.$$

Then the assignment $w \mapsto \mathcal{F}_Q(w)$ gives a bijection

$$\{w \mid w \text{ is } c_Q\text{-sortable}\} \xrightarrow{\sim} \text{torf } kQ.$$

5.2. Simple objects versus Bruhat inversions. Let w be a c_Q -sortable element, then we have a torsion-free class $\mathcal{F}_Q(w)$ in $\text{mod } kQ$ and a torsion-free class $\mathcal{F}(w)$ in $\text{mod } \Pi$. The relation between these two was stated implicitly in [AIRT] and the proof was involved, thus we present a brief explanation of it.

We begin with the following observation.

Proposition 5.3 ([AIRT, Theorem 3.3]). *Let w be a c_Q -sortable element in W and B_1, \dots, B_l a brick sequence of $\mathcal{F}(w)$ associated with a c_Q -sorting word of w . Then we have $B_i \in \text{mod } kQ$ for each i .*

Using this, we obtain the following description of $\mathcal{F}_Q(w)$ via a brick sequence (c.f. [AIRT, Theorem 3.11]).

Proposition 5.4. *Let w be a c_Q -sortable element in W , and let B_1, \dots, B_l be a brick sequence of $\mathcal{F}(w)$ associated with a c_Q -sorting word of w . Then we have $\mathcal{F}_Q(w) = \text{add}\{B_1, \dots, B_l\}$. Moreover, $\mathcal{F}_Q(w) = \mathcal{F}(w) \cap \text{mod } kQ$ holds.*

Proof. By Proposition 5.3, we have $B_1, \dots, B_l \in \text{mod } kQ$, and $\{\underline{\dim} B_1, \dots, \underline{\dim} B_l\} = \text{inv}(w)$ holds by Proposition 4.3. Since Theorem 5.1 implies that there exists exactly one indecomposable kQ -module which has a fixed dimension vector, every indecomposable kQ -module M with $\underline{\dim} M \in \text{inv}(w)$ should appear in $\{B_1, \dots, B_l\}$. Therefore, by the definition of $\mathcal{F}_Q(w)$, we have $\mathcal{F}_Q(w) = \text{add}\{B_1, \dots, B_l\}$.

We will prove $\mathcal{F}_Q(w) = \mathcal{F}(w) \cap \text{mod } kQ$. Since each B_i belongs to $\mathcal{F}(w) \cap \text{mod } kQ$, we have $\mathcal{F}_Q(w) \subset \mathcal{F}(w) \cap \text{mod } kQ$. Conversely, let $M \in \mathcal{F}(w) \cap \text{mod } kQ$. Then we have $M \in \text{Filt}(B_1, \dots, B_l)$ by Theorem 3.5, where Filt is considered inside $\text{mod } \Pi$. On the other hand, since $\text{mod } kQ \subset \text{mod } \Pi$ is closed under subquotients, clearly we have $M \in \text{Filt}_{kQ}(B_1, \dots, B_l)$, where Filt_{kQ} means we consider it inside $\text{mod } kQ$. However, $\text{add}\{B_1, \dots, B_l\} = \mathcal{F}_Q(w)$ is known to be closed under extensions in $\text{mod } kQ$ since it is a torsion-free class by Theorem 5.2. Thus $M \in \mathcal{F}_Q(w)$ holds. \square

Now we can state our classification of simple objects in $\mathcal{F}_Q(w)$.

Theorem 5.5. *Let w be a c_Q -sortable element of W . Then a bijection $\underline{\dim}: \text{ind } \mathcal{F}_Q(w) \rightarrow \text{inv}(w)$ restricts to a bijection*

$$\underline{\dim}: \text{sim } \mathcal{F}_Q(w) \xrightarrow{\sim} \text{Binv}(w).$$

In other words, simple objects in $\mathcal{F}_Q(w)$ bijectively correspond to Bruhat inversions of w by taking dimension vectors.

Proof. Let B_1, \dots, B_l be a brick sequence of $\mathcal{F}(w)$ (not $\mathcal{F}_Q(w)$!) associated to a c_Q -sorting word of w . Then Proposition 5.4 says that $\mathcal{F}_Q(w) = \{B_1, \dots, B_l\}$. Thus it suffices to show the following:

(Claim): *The following are equivalent for $1 \leq m \leq l$:*

- (1) $\underline{\dim} B_m \in \text{Binv}(w)$ holds.
- (2) B_m is a simple object in $\mathcal{F}(w)$
- (3) B_m is a simple object in $\mathcal{F}_Q(w)$.

The equivalence of (1) and (2) is nothing but Theorem 4.8, thus it suffices to show that (2) and (3) are equivalent. This immediately follows from the fact that $\mathcal{F}_Q(w) = \mathcal{F}(w) \cap \text{mod } kQ$ holds by Proposition 5.4 and that $\text{mod } kQ$ is closed under subquotients in $\text{mod } \Pi$. \square

As in the case of $\mathcal{F}(w)$, we can characterize the validity of the Jordan-Hölder property as follows.

Corollary 5.6. *Let w be an element of W . Then $\mathcal{F}_Q(w)$ satisfies (JHP) if and only if $\#\text{Binv}(w) = \#\text{supp}(w)$ holds.*

Proof. Immediate from Theorems 4.12 and 5.5, once we observe that $\text{supp } \mathcal{F}(w)$ are in bijection with $\text{supp}(w)$, which can be proved similarly to Proposition 4.14. \square

These results generalize the results in [Eno], in which the type A case was proved by direct computation of representations of Q .

If we assume that the base field k is algebraically closed, then we can use the following result of [DR] to give a more quick proof of Theorem 5.5 without using preprojective algebras.

Proposition 5.7 ([DR]). *Let k be an algebraically closed field and Q a Dynkin quiver. Take indecomposable kQ -modules L, M, N such that $\underline{\dim} L + \underline{\dim} N = \underline{\dim} M$ holds in Φ^+ . Then by interchanging L and N if necessary, there is an exact sequence in $\text{mod } kQ$ of the following form:*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

Note that the proof of this given in [DR] is algebro-geometric. The author do not know whether the same method can be used to study preprojective algebras, and whether this kind of exact sequence always exists in the preprojective algebra case (see Conjecture 4.18).

Now we can give another proof of Theorem 5.5 provided that k is algebraically closed.

Proof of Theorem 5.5. Recall that we have a bijection $\underline{\dim}: \text{ind } \mathcal{F}_Q(w) \rightarrow \text{inv}(w)$. We will show that $M \in \text{ind } \mathcal{F}_Q(w)$ is simple in $\mathcal{F}_Q(w)$ if and only if $\underline{\dim} M \in \text{Bin}(w)$. If M is not simple, then Theorem 2.17 clearly implies that $\underline{\dim} M$ is a non-Bruhat inversion of w by taking dimension vectors. Thus it suffices to show that if $\underline{\dim} M$ is a non-Bruhat inversion of w , then M is not a simple object in $\mathcal{F}_Q(w)$.

By Theorem 2.17, there are $\alpha, \beta \in \text{inv}(w)$ such that $\underline{\dim} M = \alpha + \beta$ holds since Φ is simply-laced. Take indecomposable kQ -modules M_α and M_β with $\underline{\dim} M_\alpha = \alpha$ and $\underline{\dim} M_\beta = \beta$, which exist by Theorem 5.1. By definition we have $M_\alpha, M_\beta \in \mathcal{F}_Q(w)$ holds. Then Proposition 5.7 implies that there exists a short exact sequence

$$0 \longrightarrow M_\alpha \longrightarrow M \longrightarrow M_\beta \longrightarrow 0$$

by interchanging α and β if necessary. Clearly this implies that M is not a simple object in $\mathcal{F}_Q(w)$. \square

APPENDIX A. DESCRIPTION AND ENUMERATION FOR TYPE A

In this appendix, we focus on type A case and give more explicit and combinatorial description of results in this paper. First, we give an explicit diagrammatic construction of simple objects in $\mathcal{F}(w)$ by using a *Bruhat inversion graph* G_w . Next, we characterize elements w such that $\mathcal{F}(w)$ satisfies (JHP) in terms of G_w , and deduce some numerical consequences.

Throughout this appendix, we will use the following notation:

- Q is a quiver whose underlying graph is the following Dynkin diagram of type A_n :

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n.$$

- Φ is a standard root system of type A_n in $V := \mathbb{R}^{n+1}$, that is, $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n+1\}$, where ε_i denotes the standard basis of V .
- We fix simple roots by $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n$.
- W is the Weyl group of Φ , and we often identify $W = S_{n+1}$ with the symmetric group S_{n+1} acting on the set $[n+1] := \{1, 2, \dots, n, n+1\}$ so that $w(\varepsilon_i) = \varepsilon_{w(i)}$ holds.
- For $i, j \in [n+1]$, we denote by $(i \ j) \in S_{n+1}$ the transposition of the letter i and j , and put $\beta_{(i,j)} := \varepsilon_i - \varepsilon_j$. Then $(i \ j)$ is identified with the reflection with respect to $\beta_{(i,j)}$.
- For $w \in S_{n+1}$, we often use the *one-line notation for w* , that is, we write as $w = w(1)w(2) \cdots w(n+1)$.
- Π is the preprojective algebra of Φ .
- $\mathcal{F}(w) \in \text{torf } \Pi$ is the torsion-free class in $\text{mod } \Pi$ defined in Theorem 4.1.

First, let us introduce the combinatorial variants of (Bruhat) inversion sets.

Definition A.1. Let w be an element of S_{n+1} .

- (1) $\text{Inv}(w)$ consists of a pair (i, j) with $1 \leq i < j \leq n+1$ such that the letter j appears left to i in the one-line notation for w .
- (2) $\text{BInv}(w)$ consists of a pair (i, j) with $1 \leq i < j \leq n+1$ such that the letter j appears left to i and there is no k with $i < k < j$ such that the letter k appears between j and i .

This notation is justified by the following, which can be proved by direct calculation.

Proposition A.2. *Let w be an element of $W = S_{n+1}$ and $i, j \in [n+1]$. Then the following hold.*

- (1) $\beta_{(i,j)} \in \text{inv}(w)$ if and only if $(i, j) \in \text{Inv}(w)$.
- (2) $\beta_{(i,j)} \in \text{Binv}(w)$ if and only if $(i, j) \in \text{BInv}(w)$.

Let us introduce a way to visualize Bruhat inversions, a *Bruhat inversion graph*. Let w be an element of $W = S_{n+1}$. Consider a square array of boxes with $(n+1)$ rows and $(n+1)$ columns. We name (i, j) to the box in the i -th row and j -th column, and put a dot in $(i, w(i))$ for each $1 \leq i \leq n+1$. We call it a *diagram of w* . A *Bruhat inversion graph G_w* is obtained by connecting every two dots in the diagram of w which correspond to the Bruhat inversion of w , that is, we connect $(w^{-1}(i), i)$ and $(w^{-1}(j), j)$ if $(i, j) \in \text{BInv}(w)$ holds. Figure 4 is examples for G_w for

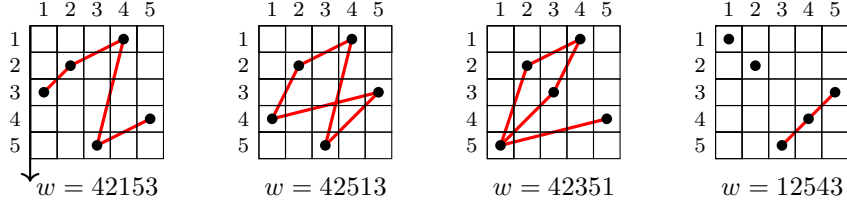
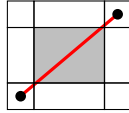


FIGURE 4. Bruhat inversion graphs G_w

elements w in S_5 . It is clear from definition that G_w is obtained by connecting every two dots in the diagram of w which look as follows,



such that there are no dots in the gray region. It is also clear from definition that an edge in G_w bijectively corresponds to a Bruhat inversion of w .

A.1. Diagrammatic description of simples in $\mathcal{F}(w)$. In this subsection, we will give a combinatorial description of simple objects in $\mathcal{F}(w)$ using *arc diagrams* introduced in [Rea] and the description of bricks given in [Asa2]. The author would like to thank Y. Mizuno for explaining to him the interpretation of the description in [Asa2] in terms of arc diagrams.

Let w be an element of $S_{n+1} = W$. We will construct a Π -module B_e for each edge e in G_w in the following way:

- (1) Remove all the edges in G_w except e .
- (2) Move down all the dots into a single horizontal line, allowing e to curve, but not to pass through any dots. We call this diagram an *arc diagram* of e .
- (3) Draw n vertical dashed lines between adjacent dots in the arc diagram, and name these lines as $1, 2, \dots, n$ from left to right.
- (4) Define a (not necessarily full) subquiver $Q(e)$ of \overline{Q} by the following rule:
 - The vertex set of $Q(e)$ consists of $i \in \overline{Q}_0$ such that e and the line i intersect in the arc diagram of e .
 - Suppose that we have $i, i+1 \in Q(e)_0$. If the segment of e cut by the lines i and $i+1$ is above the unique dot between these lines, then we put an arrow $i \rightarrow i+1$, and put $i \leftarrow i+1$ if the segment is below the dot.

We call $Q(e)$ the *defining quiver* of B_e .

- (5) Define a Π -module B_e as follows, where we construct B_e as a representation of \overline{Q} .
 - To each $i \in \overline{Q}_0$, we assign k if $i \in Q(e)_0$, and 0 otherwise.
 - To each arrow $i \rightarrow j \in \overline{Q}$, we assign the identity map if $i \rightarrow j \in Q(e)_1$, and 0 otherwise.

Since two cycles in $k\overline{Q}$ annihilates B_e by construction, we can regard B_e as a Π -module.

Example A.3. Figure 5 is an example of this construction for $w = 42513$ and all the edges in G_w . The middle part is an arc diagram of three edges, and the right part shows defining quivers of B_e corresponding to magenta, green, red, gray and blue edges from top to bottom.

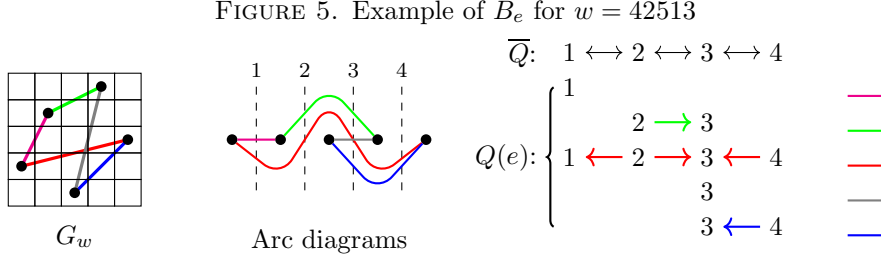
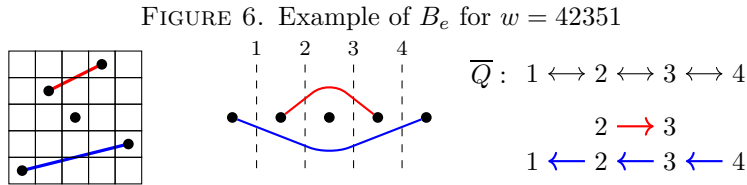


Figure 6 is an example for $w = 42351$ and two particular edges in G_w . Note that the orientations of edges between 2 and 3 in $Q(e)$ may differ as in this example.

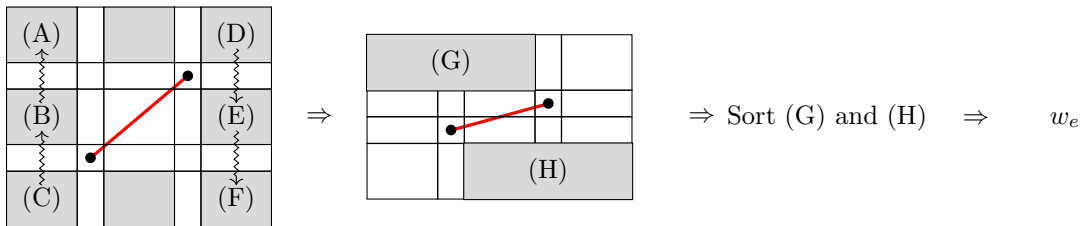


The above construction of arc diagrams is due to [Rea]. More precisely, in [Rea], arcs were assigned only to *descents* of w , which are inversions $(i, j) \in \text{Inv}(w)$ such that w is of the form $\dots ji \dots$. Similarly, our construction of Π -modules B_e is a generalization of the one given in [Asa2, Theorem 3.1], where B_e was given (without using arc diagrams) for elements w with unique descent.

We will confirm that B_e is the simple object in $\mathcal{F}(w)$ associated with the Bruhat inversion corresponding to e .

Proposition A.4. *Let w be an element of $W = S_{n+1}$. Take $(i, j) \in \text{BlInv}(w)$, and let e denotes the edge in G_w corresponding to it. Then B_e is the unique simple object in $\mathcal{F}(w)$ with $\dim B_e = \beta_{(i,j)}$.*

Proof. First, we will construct another element w_e of W with a unique descent. The following picture illustrates the construction, where all the dots lie in the gray regions.



The leftmost diagram is the diagram of w , and the red edge indicates e . Then perform the following procedure, requiring that all the diagrams in each step are diagrams of some elements in W :

- (1) Move all the dots in (B) and (C) to (A), and those in (D) and (E) to (F).
- (2) Sort all the dots in (G) and (H) so that the column number increases from top to bottom.

Denote by w_e the resulting element. Then it is clear from the construction that w_e has the unique descent. See Figure 7 for the example of this process, where $w = 56723814$.

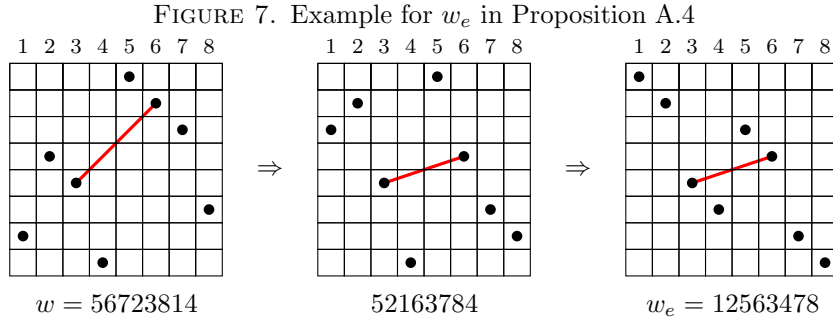
Alternatively, in terms of the one-line notation, we can describe w_e as follow. Let $w = \dots(a) \dots j \dots (b) \dots i \dots (c) \dots$ be the one-line notation for w . First, move all the numbers in

(b) and (c) which are smaller than i to (a), and all the numbers in (a) and (b) which are larger than j to (c). Then we obtain an element of the form $\cdots (a) \cdots \underline{ji} \cdots (c) \cdots$, since (i, j) is a Bruhat inversion of w . Next, sort the part (a) and (c) in ascending order, and denote by w_e the resulting element. Then $w_e = \cdots \underline{ji} \cdots$ has the unique descent at the underlined part.

It is straightforward to see that the module B_e is the same as the module $B_{w_e} := B_{e'}$, where e' is the (unique) edge in the Bruhat inversion graph of w_e . Moreover, it is easily checked that B_{w_e} is nothing but the brick constructed in [Asa2, Theorem 3.1] associated to w_e , which has the unique descent.

It is shown in [Asa2, Theorem 3.1] that B_{w_e} is the label of the unique arrow starting at $\mathcal{F}(w_e)$. In particular, we have $B_{w_e} \in \text{brick } \mathcal{F}(w_e)$. On the other hand, by construction, it is straightforward to check that $\text{Inv}(w_e) \subset \text{Inv}(w)$ holds, which implies $w_e \leq w$ in the right weak order in W , see e.g. [BB, Proposition 3.1.3]. Thus we have $\mathcal{F}(w_e) \subset \mathcal{F}(w)$ in torf II by Theorem 4.1. Hence we have $B_{w_e} \in \text{brick } \mathcal{F}(w)$. Therefore, Corollary 4.9 implies that B_{w_e} is the unique simple object in $\mathcal{F}(w)$ with its dimension vector $\beta_{(i,j)}$, since $\beta_{(i,j)} \in \text{BInv}(w)$. \square

By this, we can obtain all the simple objects in $\mathcal{F}(w)$ by computing B_e for each edge e in G_w , as we have done in Figure 5.



A.2. Forest-like permutations and the Jordan-Hölder property. Next, we will investigate elements $w \in W = S_{n+1}$ such that $\mathcal{F}(w)$ satisfy the Jordan-Hölder property. By Theorem 4.15, this is equivalent to that $\# \text{supp}(w) = \# \text{BInv}(w)$. By using [Sage], we calculated the number of such elements in S_{n+1} and obtained a sequence 2, 6, 22, 89, 379, 1661, \dots . This coincides with [OEIS, A111053], a sequence of the number of *forest-like permutations* defined in [BMB]. These conditions are equivalent (up to a multiplication by the longest element), as we shall see later.

The following is the result of [BMB] in our context.

Theorem A.5 ([BMB, Theorem 1.1]). *Let w be an element of $W = S_{n+1}$. Then the following are equivalent:*

- (1) *The Bruhat inversion graph G_w is a forest, that is, it does not contain any cycles as an undirected graph.*
- (2) *Define a map $L_w: \mathbb{Z}^n \rightarrow \mathbb{Z}^{\text{BInv}(w)}$ by $L_w(e_i) = \sum \{\beta \mid \beta \in \text{BInv}(w), i \in \text{supp}(\beta)\}$, where e_i denotes the i -th standard basis of \mathbb{Z}^n . Then this map is surjective.*
- (3) *w avoids the patterns 4231 and $\overline{34}12$ with Bruhat restriction $4 \leftrightarrow 1$, that is, there exist no $1 \leq a < b < c < d \leq n+1$ such that the one-line notation for w is of the form $w = \cdots d \cdots b \cdots c \cdots a \cdots$ or $w = \cdots c \cdots da \cdots b \cdots$.*

We call w *forest-like* if G_w is a forest. We remark that w is forest-like in our sense if and only if $w_0 w$ is forest-like in the sense of [BMB], where $w_0 = (n+1)n \cdots 21$ is the longest element.

This turns out to be equivalent to our characterization of w such that $\mathcal{F}(w)$ satisfies (JHP):

Proposition A.6. *Let w be an element of $W = S_{n+1}$. Then the following are equivalent:*

- (1) *$\mathcal{F}(w)$ satisfies (JHP), that is, $\# \text{supp}(w) = \# \text{BInv}(w)$ holds.*
- (2) *w is forest-like.*

Proof. We will show that the surjectivity of $L_w: \mathbb{Z}^n \rightarrow \mathbb{Z}^{\text{Binv}(w)}$ is equivalent to the bijectivity of the map $\varphi_w: \mathbb{Z}^{\text{Binv}(w)} \rightarrow \mathbb{Z}^{\text{supp}(w)}$ in Theorem 4.15.

Since $\text{supp}(w)$ is a subset of $Q_0 = \{1, 2, \dots, n\}$, we have the natural inclusion $\iota: \mathbb{Z}^{\text{supp}(w)} \hookrightarrow \mathbb{Z}^n$. Then it is clear from the definition of L_w that L_w is surjective if and only if so is $L'_w := L_w \circ \iota: \mathbb{Z}^{\text{supp}(w)} \rightarrow \mathbb{Z}^{\text{Binv}(w)}$.

Now it is straightforward to check that L'_w is nothing but the \mathbb{Z} -dual of φ_w by calculating matrix representations of L'_w and φ_w : the transpose of the matrix representing L'_w coincides with the matrix representing φ_w . In particular, since φ_w is always surjective, L'_w should be injective. Therefore, the surjectivity of L_w is equivalent to the bijectivity of L'_w , which is in turn equivalent to the bijectivity of φ_w , since \mathbb{Z} -dual $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is a duality of the category of finitely generated free \mathbb{Z} -modules. \square

The motivation of forest-like permutations in [BMB] comes from the study of Schubert varieties in the flag variety. Consider the flag variety $\text{SL}_{n+1}(\mathbb{C})/B$, where B is the subgroup of upper triangular matrices. For $w \in S_{n+1}$, let e_w denote the permutation matrix for w . Then the *Schubert variety* X_w is the Zariski closure of the B -orbit of e_w in the flag variety.

It is known that X_w is a projective variety, which is not necessarily smooth. There are various studies on the relation between algebro-geometric properties of X_w and combinatorial properties of w . For example, X_w is smooth if and only if w avoids the patterns 4231 and 3412. Since there are many excellent papers and books on Schubert varieties, we only refer the reader to the recent survey article [AB] for the details.

We say that a variety is *factorial* if the local ring at every point is a unique factorization domain. In [WY, Proposition 2], it was proved that X_w is factorial if and only if the map L_w in Theorem A.5 is surjective. Thus Theorem A.5 characterizes the factoriality of X_w in a combinatorial way.

By combining this result to Theorems 4.15 and A.5, Proposition A.6 and [BMB, Theorem 3.1], we immediately obtain the following summary:

Corollary A.7. *Let w be an element of $W = S_{n+1}$. Then the following are equivalent:*

- (1) G_w is forest.
- (2) $\#\text{supp}(w) = \#\text{Binv}(w)$ holds.
- (3) Elements in $\text{Binv}(w)$ are linearly independent.
- (4) $\mathcal{F}(w)$ satisfies (JHP).
- (5) X_w is factorial.
- (6) w avoids the pattern 4231 and $\overline{341}2$ with Bruhat restriction $4 \leftrightarrow 1$.

We remark that there is an explicit formula for the generation function of the number of forest-like permutations in [BMB, (2)], and the number grows exponentially, which says that although there are lots of forest-like permutations, the number is relatively small compared to $\#S_{n+1} = (n+1)!$.

Theorem 4.15 shows that (2)-(4) above are equivalent for all Dynkin types. It seems that they are also equivalent to (5) for other Dynkin types, by the same argument as in Proposition A.6 using the maps φ_w in Theorem 4.15 and L_w in Theorem A.5 and the Monk-Chevalley formula. Since the author does not have appropriate knowledge on algebraic geometries and Schubert varieties, we state it in the following conjecture:

Conjecture A.8. *Let G be a simple algebraic group over \mathbb{C} , B a Borel subgroup, Φ the associated root system and W the Weyl group. Fix $w \in W$, and denote by X_w the Schubert variety in G/B . Then the following are equivalent:*

- (1) X_w is factorial.
- (2) $\#\text{supp}(w) = \#\text{Binv}(w)$ holds, or equivalently, Bruhat inversions of w are linearly independent, or equivalently, the torsion-free class $\mathcal{F}(w)$ in $\text{mod } \Pi_{\Phi}$ satisfies (JHP).

For type D case, we can use the realization of the Weyl group as a group of signed permutations with even number of negatives. In particular, it seems to be an interesting problem to find an analogue of Corollary A.7 for type D case.

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