# MULTIPLICATIVE FUNCTIONS WITH 

$$
f\left(p+q-n_{0}\right)=f(p)+f(q)-f\left(n_{0}\right)
$$

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#### Abstract

Let $n_{0}$ be 1 or 3. If a multiplicative function $f$ satisfies $f(p+q-$ $\left.n_{0}\right)=f(p)+f(q)-f\left(n_{0}\right)$ for all primes $p$ and $q$, then $f$ is the identity function $f(n)=n$ or a constant function $f(n)=1$.


## 1. Introduction

In 2016 Chen, Fang, Yuan, and Zheng showed that if a multiplicative function $f$ satisfies $f\left(p+q+n_{0}\right)=f(p)+f(q)+f\left(n_{0}\right)$ with $1 \leq n_{0} \leq 10^{6}$ then $f$ is the identity function provided $f\left(p_{0}\right) \neq 0$ for some prime $p_{0}$ [1]. This is a variation of Spiro's paper in 1992 in which she dealt multiplicative functions satisfying $f(p+$ $q)=f(p)+f(q)$ 7]. She called the set of primes an additive uniqueness set for multiplicative functions $f$ with $f\left(p_{0}\right) \neq 0$ for some prime $p_{0}$.

A natural question follows about $n_{0}$ being negative for Chen et al.'s paper. It is natural to consider the condition $f\left(p+q-n_{0}\right)=f(p)+f(q)-f\left(n_{0}\right)$ with $n_{0}=1,2,3$ because a multiplicative function is defined on positive integers.

The author already studied a multiplicative function satisfying $f(p+q-2)=$ $f(p)+f(q)-f(2)$, which also yields that the set of numbers 1 less than primes is an additive uniqueness set for multiplicative functions [5].

In this article we classify multiplicative functions satisfying $f\left(p+q-n_{0}\right)=$ $f(p)+f(q)-f\left(n_{0}\right)$ with $n_{0}=1,3$. For consistency we state the classification for $n_{0}=2$ as well.

Theorem 1.1. If a multiplicative function $f$ satisfies $f(p+q-1)=f(p)+f(q)-f(1)$ for all primes $p$ and $q$, then $f$ is the identity function $f(n)=n$ or a constant function $f(n)=1$.

Theorem 1.2 ([5]). If a multiplicative function $f$ satisfies $f(p+q-2)=f(p)+$ $f(q)-f(2)$ for all primes $p$ and $q$, then $f$ is the identity function $f(n)=n$, $a$ constant function $f(n)=1$, or $f(n)=0$ for $n \geq 2$ unless $n$ is odd and squareful.

Theorem 1.3. If a multiplicative function $f$ satisfies $f(p+q-3)=f(p)+f(q)-f(3)$ for all primes $p$ and $q$, then $f$ is the identity function $f(n)=n$ or a constant function $f(n)=1$.

Theorem 1.2 for $n_{0}=2$ has one more option. We give a proof for Theorem 1.3 The proof of Theorem 1.1 is similar and the proof of Theorem 1.2 is given in [5, §4].

## 2. Lemmas

Lemma 2.1. Assume a multiplicative function $f$ satisfies $f(p+q-3)=f(p)+$ $f(q)-f(3)$ for all primes $p$ and $q$. Then, $f(n)=1$ or $f(n)=n$ for $n=2,3,5,7$, and 11.

Proof. Note that $f(1)=1$ and the equalities

$$
\begin{aligned}
f(1) & =f(2+2-3)=f(2)+f(2)-f(3), \\
f(7) & =f(5+5-3)=f(5)+f(5)-f(3), \\
f(10) & =f(2) f(5) \\
& =f(11+2-3)=f(11)+f(2)-f(3), \\
f(11) & =f(7+7-3)=f(7)+f(7)-f(3), \\
f(15) & =f(3) f(5) \\
& =f(11+7-3)=f(11)+f(7)-f(3) .
\end{aligned}
$$

Let $a=f(2), b=f(3), c=f(5), d=f(7), e=f(11)$. Then,

$$
\begin{align*}
1 & =2 a-b  \tag{1}\\
d & =2 c-b  \tag{2}\\
a c & =e+a-b  \tag{3}\\
e & =2 d-b  \tag{4}\\
b c & =e+d-b \tag{5}
\end{align*}
$$

The equation (3) becomes

$$
a c=4 c-7 a+4
$$

by the equations (1), (2), and (4). Also, the equation (5) becomes

$$
2 a c=7 c-10 a+5 .
$$

So, $c=4 a-3$ and we obtain an equation $a^{2}-3 a+2=0$.
Thus, $a=1$ or $a=2$ and it follows that

$$
\begin{array}{llll}
f(2)=1, & f(3)=1, & f(5)=1, & f(7)=1,
\end{array} \quad f(11)=1 ; ~ 子=2, \quad f(3)=3, \quad f(7)=7, \quad f(11)=11 .
$$

Lemma 2.2. The results in Lemma 2.1 can be extended up to $n$ odd and $n<10^{10}$.

Proof. We use induction. Let $n$ be odd and $11<n<10^{10}$.
If $n$ is prime, then $n=6 k-1$ or $n=6 k+1$. Suppose $n=6 k-1$. Note that

$$
f(n+4)=f(6 k+3)=f(n+7-3)=f(n)+f(7)-f(3)
$$

Since $6 k+3$ can be factored into the product of two smaller integers, $f(6 k+3)=1$ or $f(6 k+3)=6 k+3$ by induction hypothesis. Thus, $f(n)=1$ or $f(n)=n$ when $n=6 k-1$ is prime .

Similarly, if $n$ is a prime of the form $6 k+1$, then $f(n)=1$ or $f(n)=n$ by

$$
f(n+2)=f(6 k+3)=f(n+5-3)=f(n)+f(5)-f(3)
$$

If $n$ is not a prime, $n$ is either a product of two relatively prime integers or a power of a prime. The first case is easy by the multiplicity of $f$. So the second case remains.

Now, assume that $n$ is a power of a prime with exponent $\geq 2$. Then, $n+3$ is even and can be written as a sum of two primes $p$ and $q$ with $5 \leq p, q<n$ by the numerical verification of the Goldbach Conjecture up to $4 \times 10^{18}$ [4].

Then, since $f(n)=f(n+3-3)=f(p+q-3)=f(p)+f(q)-f(3)$, we obtain that $f(n)=1$ or $f(n)=n$ by the induction hypothesis.

Indeed, those can be extended up to $n \leq 4 \times 10^{18}-3$.
Lemma 2.3. The results in Lemma[2.1] can be extended up to $n$ even and $n<10^{10}$.

Proof. It is enough to investigate $f\left(2^{r}\right)$ with $r \leq 33$. Note that $k \cdot 2^{r}+1$ with $k<2^{r}$ is called Proth number. If a Proth number is prime, it is called a Proth prime. It is verified that there exists an odd integer $k \leq 4141$ such that $k \cdot 2^{r}+1$ is a Proth prime for $1 \leq r \leq 1000$ in The On-Line Encyclopedia of Integer Sequences (OEIS, https://oeis.org/A057778), although the infinitude of Proth primes is not yet proved 6.

Then, $k \cdot 2^{r}+1$ is an odd prime and

$$
f(k) f\left(2^{r}\right)=f\left(\left(k \cdot 2^{r}+1\right)+2-3\right)=f\left(k \cdot 2^{r}+1\right)+f(2)-f(3)
$$

Thus, we are done by Lemmas 2.1 and 2.2 .
If the Goldbach Conjecture and the infinitude of Proth primes for all exponents were proved, Theorem 1.3 could be easily proved. But, neither of them has not yet been proved, so that we need other strategy. In the following lemma, $v_{p}(n)$ means the exponent of $p$ in the prime factorization of $n$ when $p$ is a prime and $n$ is a positive integer. The set $H$ was defined by Spiro and the numerical verification of the Goldbach Conjecture was up to $2 \times 10^{10}$ at that time. We would call the set $H$ in the lemma the Spiro set.

Lemma 2.4. Let

$$
H=\left\{n \mid v_{p}(n) \leq 1 \text { if } p>1000 ; v_{p}(n) \leq\left\lfloor 9 \log _{p} 10\right\rfloor-1 \text { if } p<1000\right\}
$$

$$
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$$

For any integer $m>10^{10}$, there is an odd prime $q \leq m-1$ such that $m+q \in H$.

Proof. This lemma is the consequence of [1, Lemma 2.4] which follows the proof of [7. Lemma 5].

Lemma 2.5 ( $2, ~ 3, ~ 8, ~$. Almost every even positive integer is expressible as the sum of two primes.

Lemma 2.6. The restricted function $\left.f\right|_{H}$ is the identity function or a constant function on $H$.

Proof. Assume $f(n)=n$ for $n=2,3,5,7,11$. If $n<10^{10}$, then $f(n)=n$ from Lemmas 2.2 and 2.3. Let $n \in H$ with $n \geq 10^{10}$ and assume that $f(m)=m$ for all $m \in H$ with $m<n$. If $n$ is not a prime power, then $f(n)=f(a) f(b)$ with $(a, b)=1$ and $a, b>1$. Since $f(a)=a$ and $f(b)=b$ by the induction hypothesis, $f(n)=n$.

Now, if $n$ is a prime power, then $n$ is a prime by the definition of $H$. If $n=6 k-1$, then consider $n+7-3=6 k+3$. Since

$$
f(n+4)=f(6 k+3)=f(n+7-3)=f(n)+f(7)-f(3)
$$

and $6 k+3$ can be factored into the product of two smaller integers, $f(n)=n$.
Similarly, if $n=6 k+1$, then

$$
f(n+2)=f(6 k+3)=f(n+5-3)=f(n)+f(5)-f(3)
$$

yields $f(n)=n$.
By the same reasoning, we can conclude that $f(n)=1$ if $f(2)=f(3)=f(5)=$ $f(7)=f(11)=1$.

Lemma 2.7 ([7, Lemma 7]). For any positive integer n, put

$$
H_{n}= \begin{cases}\{m n: m \in H,(m, n)=1\} & \text { if } 2 \mid n ; \\ \{2 m n: 2 m \in H,(m, n)=1\} & \text { if } 2 \nmid n .\end{cases}
$$

Then $H_{n}$ satisfies the following properties:
(1) Every element of $H_{n}$ is even.
(2) The set $H_{n}$ has positive lower density.

## 3. Proofs of Theorems

Let us start to prove Theorem 1.3. Suppose that there exists $n$ for which $f(n) \neq$ $n$. For $k n \in H_{n}$, we have that

$$
f(k n)=f(k) f(n)=k f(n)
$$

If $f(k n)=k n$, then $f(n)=f(k n) / k=k n / k=n$, which contradicts. So $f(k n) \neq$ $k n$ for every $k n \in H_{n}$.

But, if $k n+3$ with $k$ odd can be represented as a sum of two primes $p$ and $q$, then

$$
f(k n)=f(p+q-3)=f(p)+f(q)-f(3)=p+q-3=k n
$$

Thus, this implies that there exist many counterexamples to the Goldbach Conjecture whose density is positive. But, this contradicts Lemma 2.5. Therefore, $f(n)=n$ for all $n$.

We can prove Theorem 1.1 in the similar way. First, we have that

$$
\begin{aligned}
f(3) & =f(2+2-1)=f(2)+f(2)-f(1) \\
f(5) & =f(3+3-1)=f(3)+f(3)-f(1) \\
f(6) & =f(2) f(3) \\
& =f(5+2-1)=f(5)+f(2)-f(1)
\end{aligned}
$$

Let $a=f(2), b=f(3)$, and $c=f(5)$. Then,

$$
b=2 a-1, \quad c=2 b-1, \quad a b=c+a-1
$$

Thus,

$$
a(2 a-1)=(2(2 a-1)-1)+a-1
$$

and it becomes

$$
a^{2}-3 a+2=0
$$

Hence, $a=1$ or $a=2$.
Next, we should check $f\left(2^{r}\right)$ as in Lemma 2.3. We can use $k \cdot 2^{r}-1$ instead of $k \cdot 2^{r}+1$. The list of prime $k \cdot 2^{r}-1$ with $0 \leq r \leq 10000$ is in OEIS (https://oeis.org/A126717). See also [6].

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