ON TRIANGULAR SIMILARITY OF NILPOTENT TRIANGULAR MATRICES

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Abstract. Let B_n (resp. U_n, N_n) be the set of $n \times n$ nonsingular (resp. unit, nilpotent) upper triangular matrices. We use a novel approach to explore the B_n-similarity orbits in N_n. The Belitskiĭ's canonical form of $A \in N_n$ under B_n-similarity is in QU_n where Q is the subpermutation such that $A \in B_nQB_n$. Using graph representations and U_n-similarity actions stabilzing QU_n, we obtain new properties of the Belitskiĭ's canonical forms and present an efficient algorithm to find the Belitskiĭ's canonical forms in N_n. As consequences, we construct new Belitskiĭ's canonical forms in all N_n's, list all Belitskiĭ's canonical forms for $n = 7, 8$, and show examples of 3-nilpotent Belitskiĭ's canonical forms in N_n with arbitrary numbers of parameters up to $\mathrm{O}(n^2)$.

1. Introduction

Let $\mathbb{F}$ be a fixed field. Let $M_{m,n}$ (resp. M_n, GL_n) be the set of $m \times n$ (resp. $n \times n$, $n \times n$ nonsingular) matrices over $\mathbb{F}$. Let B_n (resp. U_n, N_n) be the set of $n \times n$ nonsingular (resp. unit, nilpotent) upper triangular matrices, and D_n the set of $n \times n$ nonsingular diagonal matrices, over $\mathbb{F}$.

The main goal of this paper is to describe the B_n-similarity orbits in N_n through the Belitskiĭ's canonical forms. We link a B_n-similarity orbit to the corresponding (B_n, B_n) double coset. Given $A \in N_n$, let Q be the unique subpermutation such that $A \in B_nQB_n$. The Belitskiĭ's canonical form of A under B_n-similarity is in QU_n. We improve the Belitskiĭ's algorithm to efficiently search for the Belitskiĭ's canonical forms using graph representations and graph operations on matrices in QU_n. As a consequence, all indecomposable Belitskiĭ's canonical forms for $n = 7$ and $n = 8$ are given, which extends the works of D. Kobal [10] and Y. Chen et al [5]. Moreover, we discover a way to obtain new indecomposable Belitskiĭ's canonical forms of any order n; we also present examples of 3-nilpotent Belitskiĭ's canonical forms in N_n with arbitrary number of parameters up to $\mathrm{O}(n^2)$, which improves the $\mathrm{O}(n)$ result in [5].

The B_n-similarity orbits in N_n is a special case of the Λ-similarity matrix problem explored by V. Sergeichuk in [15]. Sergeichuk showed how the Λ-similarity

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can be used to formulate the representations of quivers and matrix problems [15, Examples 1.1, 1.2], and presented the Belitskiĭ's algorithm to obtain so called the Belitskiĭ's canonical form under Λ-similarity. The strengthen Tame-Wild theorem for matrix problem ([15, Theorem 3.1]) and the existing classification on the Belitskiĭ's canonical forms with two parameters [5] indicate that the B_n-similarity problem on N_n is of wild type.

In 1978, M. Roitman discovered that if $\mathbb{F}$ is infinite, the number of B_n-similarity orbits in N_n is infinite for $n \geq 12$ [14]. D. Djoković and J. Malzan improved the result to $n \geq 6$ in 1980 [7]. D. Kobal in 2005 listed all Belitskiĭ's canonical forms of the B_n-similarity orbits in N_n for $n \leq 5$ [10]. P. Thijsse showed in 1997 that every upper triangular matrix is B_n-similar to a generalized direct sum of irreducible blocks, and gave a classification of indecomposable (non-Belitskiĭ's) canonical forms for $n \leq 6$ [16]. Besides, Thijsse showed that if an upper triangular matrix A is non-derogatory or A has Jordan block sizes no more than 2, then A is B_n-similar to a generalized Jordan canonical form. In 2016, Y. Chen et al classified the indecomposable Belitskiĭ's canonical forms for $n = 6$ and for $n = 7$ which admits a parameter, and showed that there exists an indecomposable Belitskiĭ's canonical form which admits at least $\lfloor \frac{n}{2} \rfloor - 2$ parameters [5].

When $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, the conjugacy orbits on nilpotent matrices or Lie algebra elements were also intensively investigated by Lie theorists and representation theorists. In the book [6] of D. Collingwood and W. McGovern, nilpotent G-orbits in semisimple Lie algebras $\mathfrak{g}$ are bijectively corresponding to the G-orbits of the standard $\mathfrak{sl}_2$-triples, and are parameterized by weighted Dynkin diagrams. L. Fresse gave sufficient and necessary conditions for the intersection of a nilpotent GL_n-orbit with N_n to be a union of finitely many B_n-orbits [8]. A. Melnikov described the B_n-orbits and their geometry on upper triangular 2-nilpotent matrices by link patterns in [11, 12, 13]. M. Boos and M. Reineke described the B_n-orbits and their closure relations of all 2-nilpotent matrices [4]. N. Barnea and A. Melnikov described the Borel orbits of 2-nilpotent elements in nilradicals for the symplectic algebra in 2017 [1]. M. Boos et al described the parabolic orbits of 2-nilpotent elements for classical groups [2, 3].

The structure of this paper is as follows.

In Section 2, we review the classification and invariants of (B_n, B_n) double cosets and the Belitskiĭ's algorithm for the B_n-similarity. We show that the Belitskiĭ's canonical form of $A \in N_n$ is necessarily in QU_n in which Q is the subpermutation such that $A \in B_nQB_n$ (Theorem 2.5). As a by product, we can construct new Belitskiĭ's canonical forms $\begin{bmatrix} A_1 & Q_{12} \\ & A_2 \end{bmatrix}$ when $A_1 \in Q_1U_p$ and $A_2 \in Q_2U_q$ are Belitskiĭ's canonical forms and $\begin{bmatrix} Q_1 & Q_{12} \\ & Q_2 \end{bmatrix}$ is a subpermutation in N_{p+q} (Theorem 2.7). The criteria for D_n-similarity is given in Theorem 2.9. Finally, every matrix in B_nQB_n for a subpermutation $Q \in N_n$ can be transformed via B_n-similarity to a matrix in

QU_n, and this matrix can be transformed via elementary U_n-similarity operations (ESOs) stablizing QU_n to a matrix which is D_n-similar to the Belitskiĭ's canonical form (Theorem 2.16).

Section 3 introduces the graph representations of matrices, and the graph operations corresponding to ESOs stabilizing QU_n. The graph operations visualize the U_n-similarity reduction process on QU_n and help obtain the Belitskiĭ's canonical forms efficiently.

Section 4 is devoted to explore the properties of the Belitskiĭ's canonical form through its graph. The graph of a Belitskiĭ's canonical form in N_n with m connected components and N arcs has exactly m indecomposable components and $N-n+m$ parameters (Theorem 4.1). Theorem 4.4 determines the places of parameters in a Belitskiĭ's canonical form. Theorems 4.6 and 4.9 prove that some entries in a Belitskiĭ's canonical form must be zero, and Theorem 4.8 describes the possible places of nonzero entries. Finally, Theorem 4.11 constructs indecomposable 3-nilpotent Belitskiĭ's canonical forms with r parameters for all $r \leq \frac{1}{2}\lfloor\frac{n-2}{3}\rfloor(\lfloor\frac{n-2}{3}\rfloor-1)$ if $n \equiv 0, 2 \mod 3$, and $r \leq \frac{1}{2}\lfloor\frac{n-2}{3}\rfloor(\lfloor\frac{n-2}{3}\rfloor-1)-1$ if $n \equiv 1 \mod 3$.

In Section 5, we give an efficient graphical algorithm to search for the Belitskiĭ's canonical forms based on Theorems 4.8 and 4.9. The algorithm significantly improves the Belitskiĭ's algorithm. The indecomposable Belitskiĭ's canonical forms for $n = 7$ is given in Theorem 5.4, and those for $n = 8$ is given in Theorem 5.5 and the Appendix. Examples of the algorithm, graph illustrations of Theorem 2.7, and connections to the B_n-similarity orbits of upper triangular matrices are also included in this section.

2. Preliminary

2.1. **$B_n \times B_n$ action on N_n.** Given a subgroup G of GL_n, two matrices $A, C \in M_n$ are called *G-similar*, denoted by $A \overset{G}{\sim} C$, if there exists $B \in G$ such that $C = BAB^{-1}$. The A and C are in the same (B_n, B_n) double coset if there exist $B, B' \in B_n$ such that $C = BAB'$. The B_n-similarity orbit of $A \in M_n$ is contained in the (B_n, B_n) double coset of A:

$$\{BAB^{-1} \mid B \in B_n\} \subseteq B_nAB_n := \{BAB' \mid B, B' \in B_n\}.$$

The (B_n, B_n) double cosets on M_n are well classified as an extension of both the Bruhat decomposition in semisimple Lie groups and Gelfand-Naimark decomposition in matrix theory. We review the results here.

Definition 2.1. A matrix $Q \in M_{m,n}$ is called a *subpermutation* if each of the rows and columns of Q has at most one nonzero entry, which equals 1.

Let $[n] := \{1, 2, \ldots, n\}$. Given $i, j \in [n]$, let $E^{(n)}_{i,j} \in M_n$ (or $E^{(n)}_{ij}$ for simplicity) be the matrix that has 1 on the (i, j) entry and 0's elsewhere, and let $e^{(n)}_i \in \mathbb{F}^n$ be the vector that has 1 on the ith entry and 0's elsewhere. They are abbreviated as $E_{i,j}$ (or E_{ij} for simplicity) and e_i, respectively, if the size n is clear. Every subpermutation $Q \in M_n$ can be determined by a bijective map $\sigma : I \to \sigma(I)$ between two subsets I and $\sigma(I)$ of $[n]$ of the same cardinality:

$$Q = \sum_{i\in I} E_{i,\sigma(i)}; \qquad Q := 0 \ \text{ if } \ I = \emptyset. \tag{2.1}$$

Given $A \in M_n$ and $I, J \subseteq [n]$, let $A[I, J]$ denote the submatrix of A with rows indexed by I and columns indexed by J. Moreover, given $i, j \in [n]$, let

$$r^{i,j}(A) := \text{rank } A[[n] \setminus [n-i], [j]] = \text{rank } A[\{n-i+1, \dots, n\}, \{1, \dots, j\}] \tag{2.2}$$

be the rank of the lower left $i \times j$ submatrix of A; define $r^{0,j}(A) = r^{i,0}(A) := 0$.

The following characterization of (B_n, B_n) double cosets on M_n is classical. Analogic double coset results on GL_n can be found in [9, Theorem 3.5.14].

Lemma 2.2. *The (B_n, B_n) double coset of $A \in M_n$ is completely determined by the set of invariants:*

$$\{r^{i,j}(A) : i, j \in [n]\}. \tag{2.3}$$

There is a unique subpermutation $Q \in M_n$ such that $A \in B_nQB_n$. The entries of $Q = [q_{ij}]$ are determined by:

$$q_{n-i+1,j} = r^{i,j}(A) - r^{i-1,j}(A) - r^{i,j-1}(A) + r^{i-1,j-1}(A), \quad i, j \in [n]. \tag{2.4}$$

Proof. Given arbitrary $B, B' \in B_n$ and $i, j \in [n]$, we look at BAB' from the following partitions:

$$\begin{aligned} BAB' &= \begin{matrix} & \begin{matrix} n-i & i \end{matrix} \\ \begin{matrix} n-i \\ i \end{matrix} & \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \end{matrix} \begin{matrix} & \begin{matrix} j & n-j \end{matrix} \\ \begin{matrix} n-i \\ i \end{matrix} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{matrix} \begin{matrix} & \begin{matrix} j & n-j \end{matrix} \\ \begin{matrix} j \\ n-j \end{matrix} & \begin{bmatrix} B'_{11} & B'_{12} \\ 0 & B'_{22} \end{bmatrix} \end{matrix} \\ &= \begin{matrix} & \begin{matrix} j & n-j \end{matrix} \\ \begin{matrix} n-i \\ i \end{matrix} & \begin{bmatrix} \star & \star \\ B_{22}A_{21}B'_{11} & \star \end{bmatrix} \end{matrix}. \end{aligned}$$

Both $B_{22} \in M_i$ and $B'_{11} \in M_j$ are nonsingular. Therefore, $r^{i,j}(BAB') = r^{i,j}(A)$.

Next, we illustrate how to transform $A = [a_{ij}] \in M_n$ to a subpermutation Q through elemantary row and column operations associated with muliplications of matrices in B_n.

(1) Start from the last row of A. If it is a zero row, we are done for the row. Otherwise, let $\sigma(n) \in [n]$ such that $a_{n\sigma(n)}$ is the first nonzero entry of the row. For each $j \in [n] \setminus [\sigma(n)]$, add a multiple $(-a_{nj}/a_{n\sigma(n)})$ of the $\sigma(n)$th column of A to the jth column of A. These elementary column operations result in multiplying A from the right by a matrix $B'_{(1)} \in B_n$. Denote $A'_1 = AB'_{(1)}$. Then for each $i \in [n-1]$, add a multiple $(-a_{i\sigma(n)}/a_{n\sigma(n)})$ of the nth row of A'_1 to the ith row of A'_1. These elementary row operations

result in multiplying A_1' from the left by a matrix $B_{(1)} \in B_n$. Denote a new matrix $A_1 = [a_{ij}^{(1)}] = B_{(1)} A_1' = B_{(1)} A B_{(1)}'$. Then $a_{n\sigma(n)}^{(1)} = a_{n\sigma(n)}$ is the only nonzero entry of its row and column in A_1.

(2) Repeat the same strategy on the other rows of the new matrix in the reversing row order until all rows are done.

The above process produces a matrix $Q' = B_* A B_*''$ in which each of the rows and columns has at most one nonzero entry. By multiplying an appropriate nonsingular diagonal matrix D' from the right, we get a subpermutation $Q = B_* A B_*'' D' = B_* A B_*'$ for some $B_*, B_*' \in B_n$.

Clearly, $r^{i,j}(Q) = r^{i,j}(A)$ for $i, j \in [n] \cup \{0\}$. Moveover, given $i, j \in [n]$, $Q[[n] \setminus [n-i], [j]]$ has exactly one of the following forms ($k \in [j-1]$, $l \in [i-1]$):

$$\begin{bmatrix} 0 & 1 \\ Q[[n] \setminus [n-i+1], [j-1]] & 0 \end{bmatrix}, \qquad \begin{bmatrix} (e_k^{(j-1)})^T & 0 \\ Q[[n] \setminus [n-i+1], [j-1]] & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 \\ Q[[n] \setminus [n-i+1], [j-1]] & e_l^{(i-1)} \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ Q[[n] \setminus [n-i+1], [j-1]] & 0 \end{bmatrix}.$$

In all cases, the entries of subpermutation $Q = [q_{ij}]$ can be obtained by:

$$q_{n-i+1,j} = r^{i,j}(Q) - r^{i-1,j}(Q) - r^{i,j-1}(Q) + r^{i-1,j-1}(Q), \quad i, j \in [n]. \tag{2.5}$$

Therefore, the set of invariants $\{r^{i,j}(A) : i, j \in [n]\}$ completely determines the unique subpermutation Q and the corresponding (B_n, B_n) double coset of A. □

If two matrices are similar and in the same (B_n, B_n) double coset, are they necessarily B_n-similar? The answer is no.

Example 2.3. Let $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 0 & 1 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 0 & 1 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$. Both A and B have the only eigenvalue 0, and rank (A^m) = rank (B^m) for all $m \in \mathbb{Z}^+$. So A and B are similar. They are also in the same (B_n, B_n) double coset represented by the subpermutation B. However, A and B are not B_n-similar [10, Theorem 2].

The (B_n, B_n) double coset provides a good direction to explore the B_n-similarity orbits it includes. Suppose $A = BQB' \in N_n$ where $B, B' \in B_n$ and $Q \in N_n$ is a subpermutation. Then $A \overset{B_n}{\sim} QB'B$. Write $B'B = DU$ for $D \in D_n$ and $U \in U_n$. Since $Q \in N_n$, there exists $D' \in D_n$ such that $D'QD(D')^{-1} = Q$. Then

$$A \overset{B_n}{\sim} QB'B = QDU \overset{D_n}{\sim} D'QDU(D')^{-1} = QD'U(D')^{-1} \in QU_n.$$

The coset QU_n takes the following form.

Lemma 2.4. *Suppose $Q = \sum_{i \in I} E_{i,\sigma(i)} \in M_n$ is a subpermutation. Then $A = [a_{ij}] \in QU_n$ if and only if A meets the following conditions:*

(1) *A and Q have the same places of nonzero rows indexed by I;*

(2) *A and Q have the same places and values of the first nonzero entry in each nonzero row; precisely, for each $i \in I$, $a_{i,\sigma(i)} = 1$ is the first nonzero entry of the ith row of A.*

The proof can be done by direct computation.

2.2. **The Belitskiĭ's algorithm for the B_n-similarity in N_n.** V. Sergeichuk presented *the Belitskiĭ's algorithm* to find a canonical form, *the Belitskiĭ's canonical form*, for the Λ-similarity matrix problem [15]. On the B_n-similarity of $A \in N_n$, the algorithm can be described as follows.

(1) List the matrix entry positions above the diagonal in a reversal row lexicographical order "$\prec$" called *the Belitskiĭ's order*:

$$(n-1,n) \prec (n-2,n-1) \prec (n-2,n) \prec (n-3,n-2) \prec \cdots \prec (1,n). \tag{2.6}$$

The strictly upper triangular entries will be normalized through B_n-similarity one-by-one in this order.

(2) (Normalizing the first entry) Let $(A^{(0)}, B^{(0)}) := (A, B_n)$. Find $A^{(1)}$ in the $B^{(0)}$-similarity orbit of $A^{(0)} = [a_{ij}]$ such that the $(n-1,n)$ entry of $A^{(1)}$ is either 0 or 1. For example,

$$A^{(1)} := \begin{cases} A & \text{if } a_{n-1,n} = 0, \\ (I_{n-1} \oplus [a_{n-1,n}])A(I_{n-1} \oplus [a_{n-1,n}])^{-1} & \text{if } a_{n-1,n} \neq 0. \end{cases}$$

Denote the group

$$B^{(1)} := \{g \in B^{(0)} \mid gA^{(1)}g^{-1} \text{ fixes the value of the } (n-1,n) \text{ entry of } A^{(1)}\}.$$

(3) (Normalizing the consequent entries) Suppose $(A^{(k)}, B^{(k)})$ has been determined, and the group $B^{(k)}$ fixes the first k entries of $A^{(k)} = [a'_{ij}]$ in the Belitskiĭ's order. Let (p,q) be the $(k+1)$th entry position. There are three situations for the (p,q) entry of matrices $C = [c_{i,j}]$ in the $B^{(k)}$-similarity orbit of $A^{(k)}$:

   (a) $c_{p,q}$ is always 0, or $c_{p,q}$ could take any value of $\mathbb{F}$: we find $A^{(k+1)} = [a''_{i,j}] \overset{B^{(k)}}{\sim} A^{(k)}$ such that $a''_{p,q} = 0$;

   (b) $c_{p,q}$ could take any value of $\mathbb{F} \setminus \{0\}$: we find $A^{(k+1)} = [a''_{i,j}] \overset{B^{(k)}}{\sim} A^{(k)}$ such that $a''_{p,q} = 1$;

   (c) otherwise, $c_{p,q} \equiv \lambda$ for a fixed $\lambda \in \mathbb{F} \setminus \{0\}$: we choose $A^{(k+1)} = A^{(k)}$ with $a''_{p,q} = \lambda$.

   Let $B^{(k+1)}$ denote the subgroup of $B^{(k)}$ that fixes the $(k+1)$th entry value as well as the first k entry values of $A^{(k+1)}$.

(4) Repeat the preceding step until the last position in the Belitskiĭ's order is reached. Denote the last pair (A^∞, B^∞). The matrix A^∞ is called *the Belitskiĭ's canonical form of A under the B_n-similarity.*

The above algorithm shows that each upper triangular entry of the Belitskiĭ's canonical form A^∞ is 0 or 1 or a *parameter* λ in which different λ values correspond to different B_n-similarity orbits. This property is similar to that of a Jordan canonical form. Moreover, the Belitskiĭ's canonical form A^∞ has the following connection to the subpermutation Q in the (B_n, B_n) double coset of A and A^∞.

Theorem 2.5. *Given a Belitskiĭ's canonical form $A \in N_n$, if $A \in B_n Q B_n$ in which Q is a subpermutation, then $A \in QU_n$.*

Proof. The proof is done by induction on n. $n = 1$ is obviously true. Suppose the statement holds for all $n < m$. Given $A \in B_m Q B_m$ where $Q \in N_m$ is a subpermutation, write $A = \begin{bmatrix} 0 & a^T \\ & A_1 \end{bmatrix}$ for $A_1 \in N_{m-1}$ and $a \in \mathbb{F}^{m-1}$. By the Belitskiĭ's algorithm, A_1 is a Belitskiĭ's canonical form in N_{m-1}. Write $Q = \begin{bmatrix} 0 & b^T \\ & Q_1 \end{bmatrix}$ in which Q_1 is a subpermutation in N_{m-1} and $b \in \mathbb{F}^{m-1}$. Then $A_1 \in B_{m-1} Q_1 B_{m-1}$. So by induction hypothesis $A_1 = Q_1 \hat{U}$ for $\hat{U} \in U_{m-1}$.

(1) If $a = 0$, then Lemma 2.2 implies that $Q = \begin{bmatrix} 0 & 0 \\ & Q_1 \end{bmatrix}$. Hence

$$A = \begin{bmatrix} 0 & 0 \\ & Q_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ & \hat{U} \end{bmatrix} \in QU_m.$$

(2) If $a \neq 0$, let $A = [a_{ij}]$ and let a_{1q} $(q \in \{2, \ldots, m\})$ be the leading nonzero entry in the first row of A. Then

$$A \overset{B_m}{\sim} \begin{bmatrix} a_{1q} & 0 \\ & I_{m-1} \end{bmatrix}^{-1} A \begin{bmatrix} a_{1q} & 0 \\ & I_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & a_{1q}^{-1} a^T \\ & A_1 \end{bmatrix}$$

in which the last matrix has the leading entry 1 on the $(1, q)$ position. By the Belitskiĭ's algorithm $a_{1q} = 1$. We claim that there is no $p \in \{2, \ldots, m\}$ such that a_{pq} is the leading nonzero entry of the pth row of A (i.e. the $(p-1)$th row of A_1). Otherwise,

$$\begin{aligned} A \quad &\overset{B_m}{\sim} \quad \begin{bmatrix} 1 & \frac{1}{a_{pq}} (e_{p-1}^{(m-1)})^T \\ & I_{m-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 & a^T \\ & A_1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{a_{pq}} (e_{p-1}^{(m-1)})^T \\ & I_{m-1} \end{bmatrix} \\ &= \quad \begin{bmatrix} 0 & a^T - \frac{1}{a_{pq}} (e_{p-1}^{(m-1)})^T A_1 \\ & A_1 \end{bmatrix} \end{aligned}$$

where the first row of the last matrix has at least q leading zeros; contradicting the Belitskiĭ's algorithm. By Lemma 2.4, the $(q-1)$th column of Q_1 is zero. Using (2.4), we have $Q = \begin{bmatrix} 0 & (e_{q-1}^{(m-1)})^T \\ & Q_1 \end{bmatrix}$. Let $\hat{U}(a^T)$ denote the matrix obtained by replacing the $(q-1)$th row of $\hat{U}$ by a^T. Then $\hat{U}(a^T) \in U_{m-1}$ and

$$A = \begin{bmatrix} 0 & (e_{q-1}^{(m-1)})^T \\ & Q_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ & \hat{U}(a^T) \end{bmatrix} \in QU_m.$$

Overall, the statement holds for $n = m$ and the induction process is completed. □

Remark 2.6. A Belitskiĭ's canonical form needs not be in U_nQ or B_nQ. See the examples in Theorems 5.2, 5.4, and 5.5.

The direct sums of Belitskiĭ's canonical forms are obviously Belitskiĭ's canonical forms. Moreover, Theorem 2.5 implies a way to combine Belitskiĭ's canonical forms together through certain subpermutations to form a new Belitskiĭ's canonical form, as shown below.

Theorem 2.7. *Suppose $A_1 \in N_p$ and $A_2 \in N_q$ are Belitskiĭ's canonical forms, in which $A_1 \in Q_1U_p$ and $A_2 \in Q_2U_q$ for subpermutations $Q_1 \in N_p$ and $Q_2 \in N_q$. If $Q_{12} \in M_{p,q}$ such that $\begin{bmatrix} Q_1 & Q_{12} \\ 0 & Q_2 \end{bmatrix}$ is a subpermutation, then $\begin{bmatrix} A_1 & Q_{12} \\ 0 & A_2 \end{bmatrix}$ is a Belitskiĭ's canonical form in N_{p+q}.*

Proof. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ ($A_{11} \in N_p$) be the Belitskiĭ's canonical form of $A' := \begin{bmatrix} A_1 & Q_{12} \\ 0 & A_2 \end{bmatrix}$. Then $A_{22} = A_2$ by the Belitskiĭ's algorithm.

Let $Q := \begin{bmatrix} Q_1 & Q_{12} \\ 0 & Q_2 \end{bmatrix}$. Write $A_1 = Q_1U'$ and $A_2 = Q_2U''$ for $U' \in U_p$ and $U'' \in U_q$. Then the nonzero rows of $A' = \begin{bmatrix} Q_1U' & Q_{12} \\ 0 & Q_2U'' \end{bmatrix}$ have the same places and values (i.e., 1) of leading nonzero entries as the nonzero rows of Q do. Therefore, $A' \in QU_{p+q}$ by Lemma 2.4, and $A \in QU_{p+q}$ by Theorem 2.5.

Now consider A_{11} and A_{12}. One one hand, each nonzero entry of the subpermutation Q_{12} equals the corresponding row leading nonzero entry of A_{12}. On the other hand, $A' \overset{B_{p+q}}{\sim} A$ implies that $A_1 \overset{B_p}{\sim} A_{11}$; A_{11} cannot be further reduced from the Belitskiĭ's canonical form A_1 in the Belitskiĭ's algorithm. Therefore, $A_{11} = A_1$ and $A_{12} = Q_{12}$ by the Belitskiĭ's algorithm, so that $A = \begin{bmatrix} A_1 & Q_{12} \\ 0 & A_2 \end{bmatrix}$ is a Belitskiĭ's canonical form. □

Remark 2.8. In Theorem 2.7, the form of the Belitskiĭ's canonical form $\begin{bmatrix} A_1 & Q_{12} \\ 0 & A_2 \end{bmatrix}$ could have more parameters in nonzero entries of A_1 and A_2 than in the original Belitskiĭ's canonical forms A_1 and A_2. For an example, see the case $A_1 = A_2 = \left[\begin{smallmatrix} 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{smallmatrix}\right]$ and $Q_{12} = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right]$ in Example 5.6.

2.3. **B_n, D_n, and U_n similarities.** On the group level, $B_n = D_n \ltimes U_n$. Two matrices $A \overset{B_n}{\sim} C$ if and only if $C = BAB^{-1}$ for $B \in B_n$ and $B = UD$ such that $D \in D_n$ and $U \in U_n$, so that $A \overset{D_n}{\sim} DAD^{-1} \overset{U_n}{\sim} C$. The D_n-similarity on M_n is easy to classify.

In this paper, $A \in M_n$ is called *indecomposable* if no permutation matrix $P \in M_n$ satisfies that PAP^T can be written as a direct sum of two proper principal submatrices. The notation is different from that in [5], but they are identical when referring to an indecomposable Belitskiĭ's canonical form.

Given $A \in M_n$ and $i, j \in [n]$, let us define

$$f_{ij}(A) := \begin{cases} a_{ij} & \text{if } a_{ij} \neq 0, \\ \frac{1}{a_{ji}} & \text{if } a_{ij} = 0 \text{ but } a_{ji} \neq 0, \\ 0 & \text{if } a_{ij} = a_{ji} = 0. \end{cases} \tag{2.7}$$

Theorem 2.9. *Two matrices $A = [a_{ij}], C = [c_{ij}] \in M_n$ have $A \overset{D_n}{\sim} C$ if and only if the following two conditions hold:*

1. *A and C have the same places of nonzero entries, namely, $a_{ij} \neq 0$ if and only if $c_{ij} \neq 0$; and*
2. *for every sequence $(i_1, \ldots, i_p)$ of distinct elements in $[n]$ such that at least one of $a_{i_k i_{k+1}}$ and $a_{i_{k+1} i_k}$ is nonzero for each $k \in [p]$ (let $i_{p+1} := i_1$), we have the identity*

$$f_{i_1 i_2}(A) \cdots f_{i_{p-1} i_p}(A) f_{i_p i_1}(A) = f_{i_1 i_2}(C) \cdots f_{i_{p-1} i_p}(C) f_{i_p i_1}(C). \tag{2.8}$$

Proof. Suppose $C = DAD^{-1}$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is nonsingular. Then $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in [n]$. Conditions (1) and (2) in the theorem obviously hold.

Conversely, we use induction on n to prove that (1) and (2) imply $A \overset{D_n}{\sim} C$. $n = 1$ is true. Suppose the claim holds for all cases of $n < m$. Now for $n = m$, let $A, C \in M_n$ satisfy (1) and (2). If A is not indecomposable, then there is a permutation matrix P such that PAP^T and PCP^T are direct sums of respective proper principal submatrices. So by induction hypothesis $PAP^T \overset{D_n}{\sim} PCP^T$ and $A \overset{D_n}{\sim} C$. Otherwise, A is indecomposable. We find $d_1, \ldots, d_n \in \mathbb{F} \setminus \{0\}$ as follows such that $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in [n]$. Let $S_0 := \{1\}$ and $d_1 := 1$.

(1) Since A is indecomposable, there are $j \in [n] \setminus S_0$ such that $a_{1j} \neq 0$ or $a_{j1} \neq 0$, in which we define

$$d_j := \begin{cases} d_1 \frac{a_{1j}}{c_{1j}} & \text{if } a_{1j} \neq 0, \\ d_1 \frac{c_{j1}}{a_{j1}} & \text{if } a_{1j} = 0,\ a_{j1} \neq 0. \end{cases} \tag{2.9}$$

In the case $a_{1j} \neq 0$ and $a_{j1} \neq 0$, (2.8) gives $a_{1j} a_{j1} = c_{1j} c_{j1}$ so that the d_j defined by (2.9) satisfies both $c_{1j} = \frac{d_1}{d_j} a_{1j}$ and $c_{j1} = \frac{d_j}{d_1} a_{j1}$. Let

$$S_1 := S_0 \cup \{j \in [n] \setminus S_0 : a_{1j} \neq 0 \text{ or } a_{j1} \neq 0\}.$$

Then $S_1 \supsetneq S_0$ and $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in S_1$.

(2) If $S_1 \neq [n]$, then A being indecomposable implies that $a_{ij} \neq 0$ or $a_{ji} \neq 0$ for some $(i,j) \in S_1 \times ([n] \setminus S_1)$, in which we define

$$d_j := \begin{cases} d_i \frac{a_{ij}}{c_{ij}} & \text{if } \ a_{ij} \neq 0, \\ d_i \frac{c_{ji}}{a_{ji}} & \text{if } \ a_{ij} = 0, \ a_{ji} \neq 0. \end{cases} \tag{2.10}$$

Let

$$S_2 := S_1 \cup \{ j \in [n] \setminus S_1 : a_{ij} \neq 0 \text{ or } a_{ji} \neq 0 \text{ for some } i \in S_1 \}.$$

Then $S_2 \supsetneq S_1$ and $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in S_2$ by (2.8).

(3) Repeat the process until we reach $S_m = [n]$, where all d_j for $j \in [n]$ are well-defined. Let $D := \operatorname{diag}(d_1, \ldots, d_n)$ then $C = DAD^{-1}$ as desired. □

Theorem 2.9 shows that: if $A \in M_n$ is transformed via a B_n-similarity action to $C \in M_n$, the zero places of C are determined by the associated U_n-similarity transformation. The identities (2.8) in Theorem 2.9 will also be used to determine the places of parameters in a Belitskiĭ's canonical form.

The matrix group U_n is generated by

$$\{ I_n + \lambda E_{pq} : \lambda \in \mathbb{F}, \ p, q \in [n], \ p < q \}. \tag{2.11}$$

Definition 2.10. Given $\lambda \in \mathbb{F}$, $(p,q) \in [n] \times [n]$ and $p < q$, we define an *elementary U_n-similarity operation (ESO)* to be the function $\mathrm{O}^{\lambda}_{p,q} : N_n \to N_n$ such that for $A = [a_{ij}] \in N_n$:

$$\begin{aligned} \mathrm{O}^{\lambda}_{p,q}(A) \;&:=\; (I_n + \lambda E_{pq}) A (I_n + \lambda E_{pq})^{-1} \\ &=\; (I_n + \lambda E_{pq}) \Big(\sum_{i,j=1}^{n} a_{ij} E_{ij} \Big) (I_n - \lambda E_{pq}) \\ &=\; A + \sum_{\substack{j \in [n] \\ a_{qj} \neq 0}} \lambda a_{qj} E_{pj} - \sum_{\substack{i \in [n] \\ a_{ip} \neq 0}} \lambda a_{ip} E_{iq}. \end{aligned} \tag{2.12}$$

Each $\mathrm{O}^{\lambda}_{p,q}$ is also called an $\mathrm{O}_{p,q}$*-operation.*

The ESOs will be described by graph operations in Section 3.

Lemma 2.11. *Given $U \in U_n$, write $U = I_n + \sum_{k=1}^{m} u_{i_k j_k} E_{i_k j_k}$ where $(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_m, j_m)$ in the Belitskiĭ's order (2.6). Then*

$$U = (I_n + u_{i_1 j_1} E_{i_1 j_1}) \cdots (I_n + u_{i_m j_m} E_{i_m j_m}). \tag{2.13}$$

Proof. Left multiply $(I_n + u_{i_1 j_1} E_{i_1 j_1})^{-1}$ onto U. The matrix $(I_n + u_{i_1 j_1} E_{i_1 j_1})^{-1} U = (I_n - u_{i_1 j_1} E_{i_1 j_1}) U$ is the one that eliminates the (i_1, j_1) entry of U. Keep left multiplying $(I_n + u_{i_2 j_2} E_{i_2 j_2})^{-1}, \ldots, (I_n + u_{i_m j_m} E_{i_m j_m})^{-1}$ in order. We will have

$$(I_n + u_{i_m j_m} E_{i_m j_m})^{-1} \cdots (I_n + u_{i_1 j_1} E_{i_1 j_1})^{-1} U = I_n.$$

So (2.13) holds. □

Remark 2.12. Given $U \in U_n$, if we write $U^{-1} = I_n - \sum_{k=1}^{m} u'_{i_k j_k} E_{i_k j_k}$ where $(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_m, j_m)$, then (2.13) implies that

$$U^{-1} = (I_n - u'_{i_1 j_1} E_{i_1 j_1}) \cdots (I_n - u'_{i_m j_m} E_{i_m j_m})$$

so that

$$U = (I_n + u'_{i_m j_m} E_{i_m j_m}) \cdots (I_n + u'_{i_1 j_1} E_{i_1 j_1}). \tag{2.14}$$

Lemma 2.13. *Let $S \subseteq \{(i, j) \in [n] \times [n] : i < j\}$ such that*

$$U_S := \{I_n + \sum_{(i,j) \in S} a_{ij} E_{ij} : a_{ij} \in \mathbb{F}\} \tag{2.15}$$

is a subgroup of U_n. Then U_S is generated by $\{I_n + \lambda E_{ij} : (i, j) \in S, \lambda \in \mathbb{F}\}$, and each element of U_S can be written as a product of no more than $|S|$ elements in $\{I_n + \lambda E_{ij} : (i, j) \in S, \lambda \in \mathbb{F}\}$.

Proof. It is a direct consequence of Lemma 2.11. □

Given a subpermutation Q, the coset QU_n is not closed under the U_n-similarity. However, the following result indicates that U_n-similar matrices in QU_n can be transformed to each other via finitely many ESOs stabilizing QU_n.

Theorem 2.14. *Let $Q \in N_n$ be a subpermutation. Let $A, C \in QU_n$ such that $A \overset{U_n}{\sim} C$. Then there exist a sequence of ESOs $\{\mathrm{O}^{\lambda_1}_{i_1,j_1}, \ldots, \mathrm{O}^{\lambda_m}_{i_m,j_m}\}$, $\lambda_k \in \mathbb{F}$ and $1 \le i_k < j_k \le n$ for $k \in [m]$, such that the followings conditions hold:*

1. *$(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_m, j_m)$ in the Belitskiĭ's order (2.6).*
2. *Let $A_0 := A$ and for $k \in [m]$:*

$$A_k := \mathrm{O}^{\lambda_k}_{i_k,j_k}(A_{k-1}) = (I_n + \lambda_k E_{i_k j_k}) A_{k-1} (I_n + \lambda_k E_{i_k j_k})^{-1}. \tag{2.16}$$

Then $A_0, A_1, \ldots, A_m \in QU_n$ and $A_m = C$.

Proof. Let $Q = \sum_{i \in I} E_{i,\sigma(i)}$ as in (2.1). Let $A = QU'$ and $C = UAU^{-1} = UQU'U^{-1}$ for $U, U' \in U_n$. Write $U = I_n + [u_{ij}]$ where $[u_{ij}] \in N_n$. By direct computation, $C \in QU_n$ if and only if $UQ \in QU_n$, if and only if the nonzero u_{ij} entries have the pairs (i, j) in the set

$$S_Q := \{(i, j) \in I \times I : i < j, \sigma(i) < \sigma(j)\} \cup \{(i, j) \in [n] \times ([n] \setminus I) : i < j\}. \tag{2.17}$$

Therefore, the group $\{T \in U_n : TAT^{-1} \in QU_n\} = U_{S_Q}$ which is generated by $\{I_n + \lambda E_{ij} : (i, j) \in S_Q,\ \lambda \in \mathbb{F}\}$ according to Lemma 2.13. Moreover, $U^{-1} \in U_{S_Q}$. If we write

$$U^{-1} = I_n - \sum_{k=1}^{m} \lambda_k E_{i_k j_k},$$

in which $\lambda_k \in \mathbb{F} \setminus \{0\}$, $(i_k, j_k) \in S_Q$ and $(i_1, j_1) \prec \cdots \prec (i_m, j_m)$ in the Belitskiĭ's order, then by Lemma 2.11, $U^{-1} = (I_n - \lambda_1 E_{i_1 j_1}) \cdots (I_n - \lambda_m E_{i_m j_m})$ and

$$C = (I_n + \lambda_m E_{i_m j_m}) \cdots (I_n + \lambda_1 E_{i_1 j_1}) A (I_n + \lambda_1 E_{i_1 j_1})^{-1} \cdots (I_n + \lambda_m E_{i_m j_m})^{-1}.$$

So Theorem 2.14 (1) and (2) are proved. □

Remark 2.15. Theorem 2.14 also holds if we replace condition (1) by the condition: $(i_1, j_1) \succ (i_2, j_2) \succ \cdots \succ (i_m, j_m)$ in the Belitskiĭ's order (2.6).

Theorem 2.16. *If $A \in QU_n$ and $Q \in N_n$ is a subpermutation, then A can be transformed via a finite number of ESOs stabilizing QU_n to a matrix $\widetilde{A^\infty} \in QU_n$ which is D_n-similar to the Belitskiĭ's canonical form $A^\infty \in QU_n$.*

Proof. Since $B_n = D_n \ltimes U_n$, there exist $D \in D_n$ and $U \in U_n$ such that $A^\infty = (DU)A(DU)^{-1} = D(UAU^{-1})D^{-1}$. Let $\widetilde{A^\infty} := UAU^{-1}$. We first prove that $\widetilde{A^\infty} \in QU_n$. Notice that $A^\infty \in QU_n$ by Theorem 2.5, and $\widetilde{A^\infty}$ and $A^\infty = D\widetilde{A^\infty}D^{-1}$ have the same places of nonzero entries by Theorem 2.9. Using Lemma 2.4, it suffices to show that the leading nonzero entry of each nonzero row of $\widetilde{A^\infty}$ equals 1. Let $R_i(C)$ denote the ith row of a matrix C. By $A \in QU_n$, we have $AU^{-1} \in QU_n$ so that all nonzero rows $R_i(AU^{-1})$ have distinct places of leading nonzero entries 1. Let $U := [u_{i,j}]$. Then $\widetilde{A^\infty} = U(AU^{-1})$ implies that for $i \in [n]$:

$$R_i(\widetilde{A^\infty}) = R_i(AU^{-1}) + u_{i,i+1}R_{i+1}(AU^{-1}) + \cdots + u_{i,n}R_n(AU^{-1}).$$

Suppose $R_i(\widetilde{A^\infty})$ is a nonzero row for a given i. Then $R_i(\widetilde{A^\infty})$, $R_i(A^\infty)$, and $R_i(AU^{-1})$ have the same places of leading nonzero entries as Q does. Moreover, every $u_{i,j} \neq 0$ for $i < j \leq n$ implies that either $R_j(AU^{-1})$ is zero or the place of leading nonzero entry of $R_j(AU^{-1})$ is after that of $R_i(AU^{-1})$. Therefore, the leading nonzero entry of $R_i(\widetilde{A^\infty})$ equals that of $R_i(AU^{-1})$, namely 1. We get $\widetilde{A^\infty} \in QU_n$.

Finally, Theorem 2.14 shows that A can be transformed via a finite number of ESOs stabilizing QU_n to $\widetilde{A^\infty}$, and $\widetilde{A^\infty}$ is D_n-similar to A^∞. □

In summary, here is a simplification process to get the Belitskiĭ's canonical form A^∞ of a given $A \in N_n$ under the B_n-similarity:

(1) Use elementary row and column operations (cf. the proof of Lemma 2.2) to factorize $A = BQB'$ for $B, B' \in B_n$ and $Q \in N_n$ is the subpermutation determined by $\{r^{i,j}(A) : i, j \in [n]\}$. Then $A \overset{B_n}{\sim} QB'B$.

(2) Write $B'B = DU$ for $D \in D_n$ and $U \in U_n$. Find $D' \in D_n$ such that $D'QD(D')^{-1} = Q$. Then

$$QB'B = QDU \overset{D_n}{\sim} D'QDU(D')^{-1} = QD'U(D')^{-1} \in QU_n.$$

(3) Use a sequence of ESOs stabilizing QU_n to simplify $QD'U(D')^{-1}$ to a matrix $\widetilde{A^\infty}$ which is D_n-similar to A^∞ (cf. Theorem 2.14 and Theorem 2.16). Then determine A^∞ (cf. Theorem 2.9).

We will explore the details of step (3) above in the coming sections.

3. Graph representations and graph operations

In this section, given a subpermutation $Q \in N_n$, we use graph representations to visualize matrices in QU_n, then use graph operations to visualize ESOs on matrices in QU_n.

3.1. **Graph representation of matrices in QU_n.** Every $A = [a_{ij}] \in M_n$ is the adjacency matrix of a directed graph $G_A = (V_A, E_A)$ with a weight function $w_A : [n] \times [n] \to \mathbb{F} \setminus \{0\}$ whose support is E_A. Precisely,

$$V_A = [n]; \qquad E_A = \{(i,j) \in [n] \times [n] : a_{ij} \neq 0\}; \qquad w_A(i,j) = a_{ij}. \tag{3.1}$$

Each element of V_A (resp. E_A) is called a *vertex* (resp. an *arc*) of the graph G_A. Each arc $(i,j) \in E_A$ is visualized as $i \to j$, in which i (resp. j) is called the *tail* (resp. the *head*) of the arc (i,j), and $w_A(i,j)$ is called *the weight* of the arc (i,j). Call $G_A = (V_A, E_A)$ *the graph of* A, and $\widetilde{G}_A = (V_A, E_A, w_A)$ *the weighted graph of* A, respectively.

When $A \in N_n$, the graph of A is simple and it consists of some arcs $(i,j) \in [n] \times [n]$ with $i < j$.

A partition of $[n]$ has the form $[n] = S_1 \cup \cdots \cup S_m$ where each partition subset $S_i \neq \emptyset$. For the uniqueness of expression, we assume that the minimal elements of $S_1, \ldots, S_m$ are in ascending order, and write the partition as $\widetilde{S}_1 | \cdots | \widetilde{S}_m$ where $\widetilde{S}_i$ is the list of elements of S_i in ascending order. For example, the parition $\{5,6\} \cup \{7,3\} \cup \{2,4,1\}$ of $[7]$ will be expressed as $124|37|56$ (for $n > 9$, we will add spaces between neighboring numbers).

Lemma 3.1. *Given a subpermutation $Q \in N_n$, the graph G_Q of Q consists of finite connected components, each of which is a directed path of the form:*

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_p, \qquad i_1 < \cdots < i_p, \quad p \in \mathbb{Z}^+. \tag{3.2}$$

There is a bijective correspondence between the set of all subpermutations in N_n and the set of all paritions of $[n]$, in which Q corresponds to the partition $\mathcal{P}_Q$ of the union of the sets $\{i_1, \ldots, i_p\}$, namely, the (i,j) entry of Q is nonzero if and only if $i < j$ are sequential elements in a partition subset of $\mathcal{P}_Q$.

Proof. Since $Q \in N_n$, the graph G_Q only contains arcs (i,j) with $i < j$. Since Q is a subpermutation, each row and column of Q has at most one nonzero entry, so that each vertex i of G_Q is the head (resp. the tail) of at most one arc. Therefore, each connected component of G_Q must have the form (3.2). The rest is obvious. □

We call each connected component subgraph (3.2) of G_Q a *chain of* G_Q. So the graph G_Q is a union of finite disconnected chains. G_Q is connected if and only if Q is indecomposable. When Q is fixed, in the chain (3.2):

- i_1 (resp. i_p) is called *the chain tail* (resp. *the chain head*) of the chain (3.2);

- for each $k \in [p-1]$, i_{k+1} is called *the chain successor* of i_k, denoted by ${i_k}^+ = i_{k+1}$; and i_k is called *the chain predecessor* of i_{k+1}, denoted by ${i_{k+1}}^- = i_k$.

We call the partition $\mathcal{P}_Q$ in Lemma 3.1 *the partition of* Q. $\mathcal{P}_Q$ also determines the permutation matrices P in which each PQP^T is a direct sum of indecomposable submatrices.

Lemma 2.4 for a subpermutation $Q \in N_n$ can be rephrased in graphs as follows.

Lemma 3.2. *$A \in M_n$ is in QU_n for a subpermutation $Q \in N_n$ if and only if the weighted graph of A satisfies the following conditions:*

(1) *$E_A \supseteq E_Q$ and the weights $w_A(i,j) = w_Q(i,j) = 1$ for all $(i,j) \in E_Q$;*
(2) *each $(i,j) \in E_A \setminus E_Q$ satisfies that i is not the chain head of any chain of G_Q, and $i < i^+ < j$ where $(i, i^+) \in E_Q$.*

Proof. Suppose $A \in QU_n$ in which $Q \in N_n$ is a subpermutation. Write $Q = \sum_{i \in I} E_{i,\sigma(i)}$ for $I \subseteq [n]$. Then $i^+ = \sigma(i)$ for all $i \in I$. Moreover, i is a chain head of G_Q if and only if $i \in [n] \setminus I$. Lemma 2.4 (2) shows that $\widetilde{G}_A$ contains $\widetilde{G}_Q$ as a weighted subgraph. Given $(i,j) \in E_A \setminus E_Q$, Lemma 2.4 (1) shows that i is not a chain head of G_Q, and Lemma 2.4 (2) and the assumption $Q \in N_n$ show that $i < \sigma(i) = i^+ < j$.

The converse statement also holds by Lemma 2.4. □

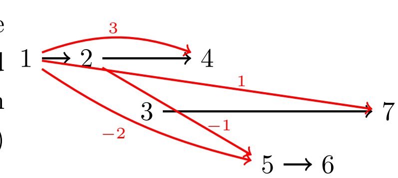
In Lemma 3.2, G_A contains G_Q as a subgraph. When $A \in QU_n$ for a subpermutation Q, we call each element of $E_A \setminus E_Q$ an *extra arc of* G_A. We denote the *graph type* of A as $\mathcal{P}_Q : i_1 j_1 | \cdots | i_t j_t$ where $\mathcal{P}_Q$ is the partition corresponding to Q and $(i_1, j_1), \ldots, (i_t, j_t)$ are the extra arcs of G_A listed in ascending Belitskiĭ's order (2.6). If A has no extra arc (i.e. $A = Q$), its graph type is denoted as $\mathcal{P}_Q : \emptyset$. The graph type of A is a concise expression of the graph G_A.

Example 3.3. Let $n = 7$. Let

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix} \in N_7, \qquad A = Q \begin{bmatrix} 1 & * & * & * & * & * & * \\ & 1 & 0 & 3 & -2 & 0 & 1 \\ & & 1 & * & * & * & * \\ & & & 1 & -1 & 0 & 0 \\ & & & & 1 & * & * \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 3 & -2 & 0 & 1 \\ & 0 & 0 & 1 & -1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}.$$

The subpermutation Q corresponds to the partition $\mathcal{P}_Q = 124|37|56$ of $[7]$. The graph of Q is $G_Q = ([7], E_Q)$ in which $E_Q = \{(1,2),(2,4),(3,7),(5,6)\}$. The graph of A is $G_A = ([7], E_A)$ in which $E_A = E_Q \cup \{(1,4),(1,5),(1,7),(2,5)\}$. So the graph type of A is $124|37|56 : 25|14|15|17$.

In the graph on the right, $\widetilde{G}_Q$ consists of three chains formed by black arcs with weights 1, and $\widetilde{G}_A$ has the extra arcs with weights marked in red. By Lemma 3.2, an arc like $(4,7)$ or $(3,5)$ cannot be an extra arc of G_A.



3.2. **The elementary U_n-similarity graph operations on QU_n.** The ESO in Definition 2.10 can be rephrased using graph operations. Let $\lambda \in \mathbb{F}$ and $(p, q) \in [n] \times [n]$ with $p < q$. For $A = [a_{ij}] \in N_n$, (2.12) shows that

$$\mathrm{O}_{p,q}^{\lambda}(A) = A + \sum_{\substack{j \in [n] \\ a_{qj} \neq 0}} \lambda a_{qj} E_{pj} - \sum_{\substack{i \in [n] \\ a_{ip} \neq 0}} \lambda a_{ip} E_{iq}.$$

Let $A' = [a'_{ij}] := \mathrm{O}_{p,q}^{\lambda}(A)$. The changes made by $\mathrm{O}_{p,q}^{\lambda}$ from $\widetilde{G}_A$ to $\widetilde{G}_{A'}$ are below:

(1) whenever $(i, p) \in E_A$ (i.e. $a_{ip} \neq 0$),

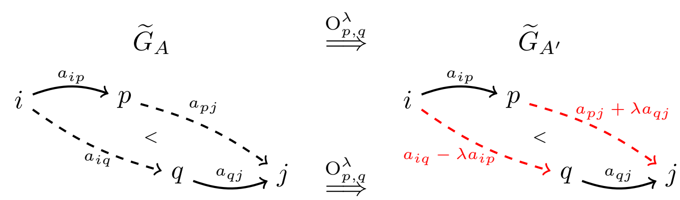
$$w_{A'}(i, q) = a'_{iq} = a_{iq} - \lambda a_{ip} = w_A(i, q) - \lambda w_A(i, p);$$

(2) whenever $(q, j) \in E_A$ (i.e. $a_{qj} \neq 0$),

$$w_{A'}(p, j) = a'_{pj} = a_{pj} + \lambda a_{qj} = w_A(p, j) + \lambda w_A(q, j).$$

These changes are visualized as follows, in which a red arc indicates a change of the weight, and a dashed arc indicates that the weight may be zero:

(3.3)



By abuse of language, we also call transformation (3.3) the *elementary U_n-similarity operation (ESO)* $\mathrm{O}_{p,q}^{\lambda}$ on the weighted graph $\widetilde{G}_A$, denoted by $\mathrm{O}_{p,q}^{\lambda}(\widetilde{G}_A) = \widetilde{G}_{A'}$.

Given a matrix $A = [a_{ij}] \in QU_n$ where $Q \in N_n$ is a subpermutation, Theorem 2.16 shows that A could be transformed via a sequence of ESOs stabilizing QU_n to a matrix $\widetilde{A^\infty} \in QU_n$ which is D_n-similar to the Belitskiĭ's canonical form A^∞. By Theorem 2.9, $\widetilde{A^\infty}$ and A^∞ have the same places of nonzero entries, that is, $G_{\widetilde{A^\infty}} = G_{A^\infty}$, and the relation of weight functions (2.8) holds on each undirected cycle of $G_{\widetilde{A^\infty}}$. Therefore, we will use ESOs (3.3) to eliminate "redundant arcs" on $\widetilde{G}_A$ following the Belitskiĭ's order until we reach $\widetilde{G}_{\widetilde{A^\infty}}$; then we adjust the weights on undirected cycles of $\widetilde{G}_{\widetilde{A^\infty}}$ following the Belitskiĭ's order and get $\widetilde{G}_{A^\infty}$.

Lemma 3.4. *Let $A = [a_{ij}] \in N_n$. An arc (i, j) of $\widetilde{G}_A = ([n], E_A, w_A)$ can be eliminated by an ESO only if one of the following two cases happens:*

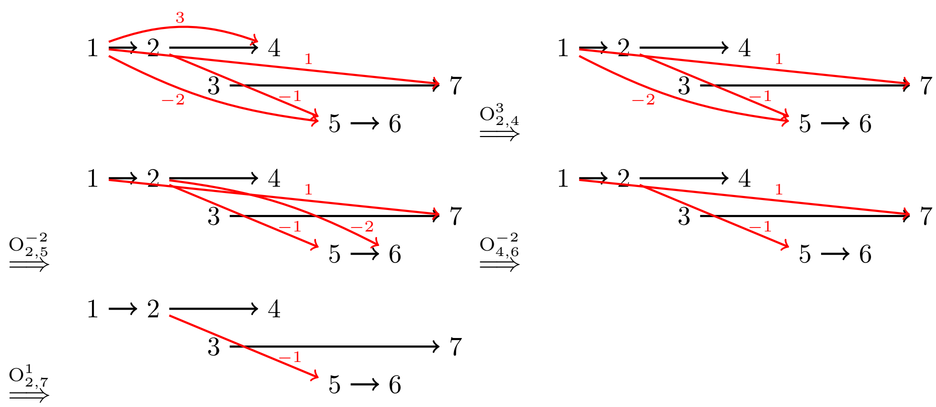
(1) *there is p such that $i < p < j$ and $(i, p) \in E_A$, in which $\mathrm{O}_{p,j}^{\lambda}(\widetilde{G}_A)$ for $\lambda = \frac{a_{ij}}{a_{ip}}$ has no arc (i, j);*

(2) *there is q such that $i < q < j$ and $(q, j) \in E_A$, in which $\mathrm{O}_{i,q}^{\lambda}(\widetilde{G}_A)$ for $\lambda = -\frac{a_{ij}}{a_{qj}}$ has no arc (i, j);*

Proof. The statement is a direct consequence of (3.3). ☐

By Theorem 2.16, when $A \in QU_n$ for a subpermutation $Q \in N_n$, we should try to eliminate the extra arcs of $\widetilde{G}_A$ by ESOs stabilizing QU_n. So in practice, not every ESO satisfying conditions in Lemma 3.4 will be considered.

Example 3.5. Let subpermutation $Q \in N_7$ and matrix $A \in QU_7$ be given in Example 3.3. The extra arcs in $\widetilde{G}_A$ sorted by the Belitskiĭ's order are: $(2,5) \prec (1,4) \prec (1,5) \prec (1,7)$. We test and eliminate them by ESOs stabilizing QU_7 in this order. The first arc $(2,5)$ cannot be eliminated, since the only type of ESOs that can modify the weight of $(2,5)$ is $\mathrm{O}_{4,5}$ which creates the arc $(4,6)$ and does not stabilize QU_7. Then:



Hence $\mathrm{O}^1_{2,7}\mathrm{O}^{-2}_{4,6}\mathrm{O}^{-2}_{2,5}\mathrm{O}^3_{2,4}(A) = \widetilde{A^\infty}$ in which $\widetilde{G}_{\widetilde{A^\infty}}$ is the last weight graph above. There is no undirected cycle on $\widetilde{G}_{\widetilde{A^\infty}}$. So $\widetilde{A^\infty}$ is D_7-similar to the Belitskiĭ's canonical form A^∞ whose weighted graph has weight 1 on the arc $(2,5)$. In other words,

$$A \overset{U_7}{\sim} \widetilde{A^\infty} = Q - E_{2,5} \overset{D_7}{\sim} A^\infty = Q + E_{2,5}.$$

The elimination process in Example 3.5 may be roughly abbreviated as changes on graph types as below, where an appropriate operation on the first row causes the changes of arcs listed on the second row.

$$\begin{array}{rccccc} & \mathrm{O}_{2,4} & \mathrm{O}_{2,5} & \mathrm{O}_{4,6} & \mathrm{O}_{2,7} & \\ 124|37|56 : 25|14|15|17 & -14 & -15+26 & -26 & -17 & = 25 \end{array} \tag{3.4}$$

So the Belitskiĭ's canonical form of A has the type $124|37|56 : 25$. A process like (3.4) only works for generic cases, since the process omits all weight information; for some specific matrix, an ESO on its weighted graph may eliminate several extra arcs simultaneously and lead to a different type. However, we will see that a graphical version of process (3.4) is powerful in classifying the forms of the Belitskiĭ's canonical forms under B_n-similarity.

Another observation about Example 3.5 is that: unlike those ESOs in Theorem 2.14, in $\mathrm{O}^1_{2,7}\mathrm{O}^{-2}_{4,6}\mathrm{O}^{-2}_{2,5}\mathrm{O}^3_{2,4}(A) = \widetilde{A^\infty}$, the pairs $(2,4), (2,5), (4,6), (2,7)$ do not completely follow the Belitskiĭ's order $\prec$. However, we check and (if possible) eliminate the extra arcs of $\widetilde{G}_A$ following the Belitskiĭ's order; the success of this process is guaranteed by the combination of the Belitskiĭ's algorithm and Theorem 2.14.

4. Properties of the Belitskiĭ's canonical forms under B_n-similarity

Given a matrix $A \in B_n Q B_n$ or QU_n where $Q \in N_n$ is a subpermutation, Theorem 2.5 shows that the Belitskiĭ's canonical form $A^\infty \in QU_n$. Here we investigate the nonzero entries in A^∞, or equivalently, what extra arcs and weights could be in $\widetilde{G}_{A^\infty}$. For simplicity, we assume that A is already a Belitskiĭ's canonical form.

4.1. **Characterization of the Belitskiĭ's canonical form.** The (2.8) in Theorem 2.9 indicates that if the graph of a Belitskiĭ's canonical form has an undirected cycle, then at least one arc of this undirected cycle has a parameter weight. It derives the following results.

Theorem 4.1. *Let $A \in N_n$ be a Belitskiĭ's canonical form. If G_A has m connected components and $|E_A| = N$, then A has m indecomposable components and $N-n+m$ parameters.*

Proof. If G_A has m connected components, then the vertex sets of these m connected subgraphs form a partition of $[n]$. For each permutation matrix $P \in M_n$, the graphs G_{PAP^T} and G_A are isomorphic. There is a permutation matrix P such that the vertex set of each connected component of G_{PAP^T} contains sequential integer(s). Then PAP^T is a direct sum of m principal submatrices, each of which is indecomposable. In other words, A has m indecomposable components.

If a connected component of G_A has n_1 vertices and r_1 arcs, then $r_1 \geq n_1 - 1$. When $r_1 = n_1 - 1$, the connected component contains no undirected cycle so that all weights of its arcs are 1 by Theorem 2.9 and the Belitskiĭ's algorithm. When $r_1 > n_1 - 1$, the connected component can be obtained by adding $r_1 - n_1 + 1$ arcs to a connected subgraph with $n_1 - 1$ arcs, and adding each arc creates an undirected cycle on the union of this arc and the subgraph. Therefore, by Theorem 2.9, there are $r_1 - n_1 + 1$ parameter weights on the arcs of this connected component.

Summing over all m connected components of G_A, we see that A has $N - n + m$ parameters. □

Remark 4.2. In matrix way, Theorem 4.1 says that: if a Belitskiĭ's canonical form $A \in N_n$ is permutation similar to a direct sum of m indecomposable squared submatrices, and A has N nonzero entries, then A has $N - n + m$ parameters.

The following two results describe an indecomposable Belitskiĭ's canonical form and its graph. They show that if the graph type or the places of nonzero entries of a Belitskiĭ's canonical form are known, then we can determine the amount and the places of parameters among these nonzero entries.

Corollary 4.3. Let $A \in N_n$ be a Belitskiĭ's canonical form. Then A is indecomposable if and only if the graph $G_A = ([n], E_A)$ is connected. Moreover, if A is indecomposable with N nonzero entries, then A has $N - n + 1$ parameters.

Theorem 4.4. *Let $A \in QU_n$ be an indecomposable Belitskiĭ's canonical form in which $Q \in N_n$ is a subpermutation. List the extra arcs of G_A (i.e. the elements of $E_A \setminus E_Q$) in the Belitskiĭ's order:*

$$(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_t, j_t). \tag{4.1}$$

Then the places of parameters of A (if any) correspond to the marked extra arcs determined by the following steps, starting at the graph $G := G_Q$ in which all arcs in E_Q are unmarked:

(1) *add the extra arcs of G_A one at a time to G according to the Belitskiĭ's order (4.1).*
(2) *when adding an extra arc (i, j) to G creates an undirected cycle in which none of the arcs is marked, mark the extra arc (i, j) and continue;*
(3) *repeat the steps (1) and (2) until all extra arcs of G_A are gone through.*

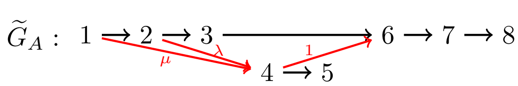
Proof. Since $A \in QU_n$, the parameters of A appear only in the entries corresponding to extra arcs.

In step (2), when adding an extra arc (i, j) results in an undirected cycle in which none of the arcs is marked, we may assume that the undirected cycle has distinct vertices by removing redundant subcycles. By Theorem 2.9 (2), the undirected cycle contains at least one arc with a parameter weight to represent the scalar in (2.8). Moreover, (i, j) is the last arc in the Belitskiĭ's order in this undirected cycle. So by the Belitskiĭ's algorithm, the parameter weight in the undirected cycle should be on (i, j).

After step (3), if we remove all marked arcs from G_A then the remaining subgraph does not have any undirected cycle. By Theorem 2.9, A is D_n-similar (and thus B_n-similar) to a matrix whose unmarked arcs have weights 1 and marked arcs have parameter weights.

The normalization steps (1), (2), (3) allow us to place the parameters of A in accordance with the Belitskiĭ's algorithm. So these steps determine the places of parameters. □

Example 4.5. An analysis similar to Example 3.5 shows that: every matrix $A \in QU_8$ of the graph type 123678|45 : 46|24|14 has no extra arc in G_A that can be eliminated by ESOs stablizing QU_8. So A is a Belitskiĭ's canonical form, which is indecomposable since G_A is connected. Corollary 4.3 shows that A has 2 parameters, and Theorem 4.4 shows that the parameters appear in the $(2, 4)$ and $(1, 4)$ entries. So $\widetilde{G}_A$ and A have the forms $(\lambda, \mu \in \mathbb{F} \setminus \{0\})$:



$$A = \begin{bmatrix} 0 & 1 & 0 & \mu & 0 & 0 & 0 & 0 \\ & 0 & 1 & \lambda & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 1 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{bmatrix}$$

All such Belitskiĭ's canonical forms may be represented by the graph type with additional underlines indicating parameters, namely $123678|45:46|\underline{24}|\underline{14}$.

4.2. **Extra arcs in the graph of the Belitskiĭ's canonical form.** Fix a subpermutation $Q = \sum_{i\in I} E_{i,\sigma(i)} \in N_n$. Using the notations in Section 3.1, the graph G_Q consists of $n - |I|$ chains in which the set S_h of chain heads and the set S_t of chain tails are

$$S_h = [n] \setminus I, \qquad S_t = [n] \setminus \sigma(I). \tag{4.2}$$

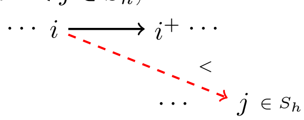
We also denote the maps

$$I \to \sigma(I), \quad i \mapsto i^+ := \sigma(i), \qquad \text{and} \quad \sigma(I) \to I, \quad j \mapsto j^- := \sigma^{-1}(j). \tag{4.3}$$

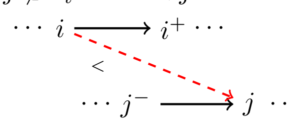
Given a Belitskiĭ's canonical form $A \in QU_n$, Lemma 2.4 and its graph version Lemma 3.2 give a description of the entries of A. In this subsection, we further explore what entries of A should be zero, namely, what extra arcs should not be in G_A.

Theorem 4.6. *Let $A = [a_{ij}] \in QU_n$ be a Belitskiĭ's canonical form in which $Q = \sum_{i\in I} E_{i,\sigma(i)} \in N_n$ is a subpermutation. Then for $i \in I$, $(i,j) \notin E_A$ (i.e. $a_{ij} = 0$) when one of the following situations happen:*

(1) $i^+ < j \in S_h$;



(2) $j \notin S_t$ *and* $i < j^-$.



In particular, if i and j are on the same chain of G_Q but $j \neq i^+$, then $(i,j) \notin E_A$.

Proof. We prove by contradictions that A cannot be a Belitskiĭ's canonical form if an arc $(i,j) \in E_A$ satisfies (1) or (2) of Theorem 4.6. The idea is to find a matrix $A' \in QU_n$ such that $A' \overset{U_n}{\sim} A$ and

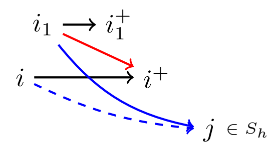
$$E_{A'} \subseteq (E_A \setminus \{(i,j)\}) \cup \{(i',j') \in [n]\times[n] : i' < j',\ (i,j) \prec (i',j')\}, \tag{4.4}$$

which contradicts the Belitskiĭ's algorithm to get the Belitskiĭ's canonical form A.

(1) Suppose $(i,j) \in E_A$ such that $i^+ < j \in S_h$. By Lemma 3.4, there is $\lambda_1 \in \mathbb{F}$ such that the graph of $A_1 := \mathrm{O}^{\lambda_1}_{i^+,j}(A)$ contains no arc (i,j). By Lemma 3.2 (2), $j \in S_h$ is not the tail of any arc of $\widetilde{G}_A$. Thus by (3.3), $E_{A_1} \setminus E_A$ only contains some (i_1, j) in which $(i_1, i^+) \in E_A$ and $i_1 \neq i$ so that $i_1 \in I$

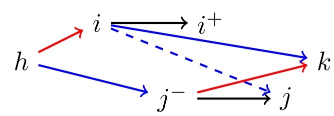
and $i_1^+ < i^+$ by Lemma 3.2 (2).

The changes from G_A to G_{A_1} are illustrated on the right, in which the dashed blue arc is removed and some solid blue arcs are added.



Similarly, for each $(i_1, j) \in E_{A_1} \setminus E_A$, an appropriate $\mathrm{O}_{i_1^+,j}$-operation will remove the arc (i_1, j) from the graph and add the arcs (i_2, j) in which $(i_2, i_1^+) \in E_{A_1}$ and $i_2 \neq i_1$ so that $i_2 \in I$ and $i_2^+ < i_1^+$. Repeating the process results in $i^+ > i_1^+ > i_2^+ > \cdots$. However, the process cannot go on forever. Hence by a finite steps of ESOs we can remove (i, j) from G_A as well as all arcs created by these ESOs. In other words, we get $A' = A - a_{ij}E_{ij}$ such that $A' \overset{U_n}{\sim} A$. This contradicts the assumption that A is a Belitskiĭ's canonical form. Therefore, $(i, j) \notin E_A$.

(2) Suppose $(i, j) \in E_A$ such that $j \notin S_t$ and $i < j^-$. By Lemma 3.4, there is $\lambda \in \mathbb{F}$ such that the graph of $A' := \mathrm{O}^{\lambda}_{i,j^-}(A)$ contains no arc (i, j). By (3.3), $E_{A'} \setminus E_A$ only contains the following possible arcs:



(a) $(h, j^-) \in E_{A'} \setminus E_A$ in which $(h, i) \in E_A$. In such a case, $h < i$ so that $(h, j^-) \succ (i, j)$ in the Belitskiĭ's order.

(b) $(i, k) \in E_{A'} \setminus E_A$ in which $(j^-, k) \in E_A$ and $k \neq j$. In such a case, $j^- < j < k$ by Lemma 3.2 (2) so that $(i, k) \succ (i, j)$ in the Belitskiĭ's order.

Overall, we get $A' \overset{U_n}{\sim} A$ in which $E_{A'}$ satisfies (4.4). It contradicts the assumption that A is a Belitskiĭ's canonical form. Therefore, $(i, j) \notin E_A$. □

The less intuitive matrix version of Theorem 4.6 is as follows.

Theorem 4.7. *Let $A = [a_{ij}] \in QU_n$ be a Belitskiĭ's canonical form where $Q = \sum_{i \in I} E_{i,\sigma(i)} \in N_n$ is a subpermutation. Then $a_{ij} = 0$ whenever:*

(1) *$j \in [n] \setminus (I \cup \{\sigma(i)\})$, or*

(2) *$j \in \sigma(I)$ and $i < \sigma^{-1}(j)$.*

In particular, $a_{ij} = 0$ if $j = \sigma^k(i)$ for some integer $k > 1$.

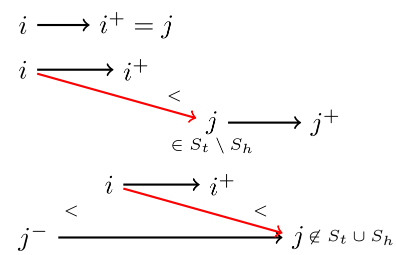
Theorem 4.6 is equivalent to the following result which gives a characterization of the possible arcs in a Belitskiĭ's canonical form.

Theorem 4.8. *Let $A = [a_{ij}] \in QU_n$ be a Belitskiĭ's canonical form in which $Q = \sum_{i \in I} E_{i,\sigma(i)} \in N_n$ is a subpermutation. Then $(i, j) \in E_A$ (i.e. $a_{ij} \neq 0$) implies $i \in I = [n] \setminus S_h$ and one of the following:*

(1) $j = i^+$ *(where $a_{ij} = 1$).*

(2) $j \in S_t \setminus S_h$ *and* $i^+ < j$.

(3) $j \notin S_t \cup S_h$ *and* $j^- < i < i^+ < j$.

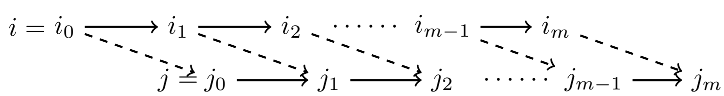


In particular, given $i \in I$, there is at most one vertex j in each chain of G_Q such that $(i, j) \in E_A$.

Proof. The case (1) is $(i, j) \in E_Q$. The cases (2) and (3) cover those extra arcs (i, j) not included in Theorem 4.6 (1) and (2).

It remains to prove the last claim. Suppose $(i, j) \in E_A$ and the vertex j is in a chain G' of G_Q. If the chain G' contains the vertex i, then $j = i^+$. Otherwise, j is the lowest vertex number in the chain G' such that $j > i^+$. $\square$

Theorem 4.9. *Let $A = [a_{ij}] \in QU_n$ be a Belitskiĭ's canonical form in which $Q = \sum_{i \in I} E_{i,\sigma(i)} \in N_n$ is a subpermutation. Given $(i, j) \in [n] \times [n]$ with $i < j$, suppose there exist $m \in \mathbb{N}$ and sequences $i_0 = i, i_1, \ldots, i_m \in [n]$ and $j_0 = j, j_1, \ldots, j_m \in [n]$ such that all of the following conditions hold for $p \in [m]$:*



(1) (i_{p-1}, i_p) *is the only arc in G_A whose head is i_p.*
(2) $(j_{p-1}, j_p) = (j_{p-1}, j_{p-1}^+)$ *is the only arc in G_A whose tail is j_{p-1}.*
(3) $i_p < j_{p-1}$.
(4) $i_m \notin S_h$ *but* $j_m \in S_h$.

Then $(i, j) \notin E_A$.

Proof. Suppose on the contrary $(i, j) = (i_0, j_0) \in E_A$. There exists $\lambda_1 \in \mathbb{F}$ such that the graph of $A_1 := \mathrm{O}_{i_1,j_0}^{\lambda_1}(A)$ does not contain the arc (i, j). Then either $E_{A_1} = E_A \setminus \{(i, j)\}$ or $E_{A_1} = (E_A \setminus \{(i, j)\}) \cup \{(i_1, j_1)\}$. However, $E_{A_1} = E_A \setminus \{(i, j)\}$ is impossible since A is a Belitskiĭ's canonical form. So $E_{A_1} = (E_A \setminus \{(i, j)\}) \cup \{(i_1, j_1)\}$. Similarly, applying a sequence of appropriate $\mathrm{O}_{i_2,j_1}, \ldots, \mathrm{O}_{i_m,j_{m-1}}$ operations to A_1, we will get $A_m \overset{U_n}{\sim} A_1 \overset{U_n}{\sim} A$ such that

$$E_{A_m} = (E_A \setminus \{(i, j), (i_1, j_1), \ldots, (i_{m-1}, j_{m-1})\}) \cup \{(i_m, j_m)\}.$$

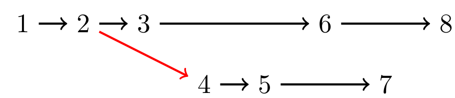
Since $i_m \notin S_h$ and $j_m \in S_h$, the proof of Theorem 4.6 indicates that there is $A_{m+1} \overset{U_n}{\sim} A_m \overset{U_n}{\sim} A$ such that

$$E_{A_{m+1}} = E_{A_m} \setminus \{(i_m, j_m)\} = E_A \setminus \{(i, j), (i_1, j_1), \ldots, (i_m, j_m)\}.$$

It contradicts the assumption that A is a Belitskiĭ's canonical form. $\square$

Example 4.10. Let $Q \in N_8$ be the subpermutation with $\mathcal{P}_Q = 12368|457$. Consider the possible Belitskiĭ's canonical forms $A \in QU_8$. Theorem 4.8 implies that the possible extra arcs in $E_A \setminus E_Q$ are $(1,4)$, $(2,4)$, and $(4,6)$. By Theorem 4.9, neither $(4,6)$ nor $(1,4)$ can be in a Belitskiĭ's canonical form. Therefore, the only indecomposable Belitskiĭ's canonical form in QU_8 is of the graph form $12368|457 : 24$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$



4.3. **Possible numbers of parameters in a Belitskiĭ's canonical form.** In [5, Theorem 2.4], Chen et al showed that for $n \geq 6$, there exists an indecomposable Belitskiĭ's canonical form in N_n which admits at least $\lfloor \frac{n}{2} \rfloor - 2$ parameters. Note that a matrix in N_n has up to $\frac{n(n-1)}{2}$ nonzero entries. We show below the existence of indecomposable 3-nilpotent Belitskiĭ's canonical forms with arbitrary number of parameters up to $\mathrm{O}(n^2)$.

Theorem 4.11. *Let $n, r \in \mathbb{N}$ such that $n \geq 6$ and*

$$r \leq \begin{cases} \frac{1}{2}\lfloor \frac{n-2}{3} \rfloor (\lfloor \frac{n-2}{3} \rfloor - 1) & \text{if } n \equiv 0, 2 \mod 3, \\ \frac{1}{2}\lfloor \frac{n-2}{3} \rfloor (\lfloor \frac{n-2}{3} \rfloor - 1) - 1 & \text{if } n \equiv 1 \mod 3. \end{cases} \tag{4.5}$$

Then there exists an indecomposable Belitskiĭ's canonical form $A \in N_n$ with r parameters, and A has the minimal polynomial x^3.

Proof. We construct the desired Belitskiĭ's canonical forms for $n \geq 6$ according to $n \mod 3$:

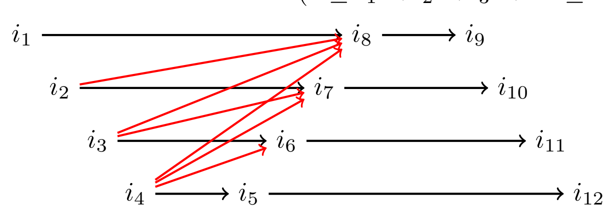
(1) When $n = 3m$, choose the subpermutation $Q \in N_{3m}$ with

$$\mathcal{P}_Q = 1\ 2m\ 2m+1 \mid 2\ 2m-1\ 2m+2 \mid \cdots \mid m\ m+1\ 3m. \tag{4.6}$$

Let $G = ([3m], E)$ be the graph containing G_Q as a subgraph and

$$E \setminus E_Q = \{(i,j) \in [3m] \times [3m] : 2 \leq i \leq m,\ 2m+2-i \leq j \leq 2m\}. \tag{4.7}$$

The graph G is illustrated as below ($1 \leq i_1 < i_2 < i_3 < \ldots \leq n$):



Let $A \in QU_n$ such that $G_A = G$ and the places of parameters in A are given by Theorem 4.4. We claim that A is a Belitskiĭ's canonical form.

By (2.17), the ESOs stabilizing QU_n are those $\mathrm{O}^{\lambda}_{p,q}$ in which either

(a) $(p,q) \in [m] \times ([2m] \setminus [m])$, or

(b) $p < q$ and $q \in [3m] \setminus [2m]$.

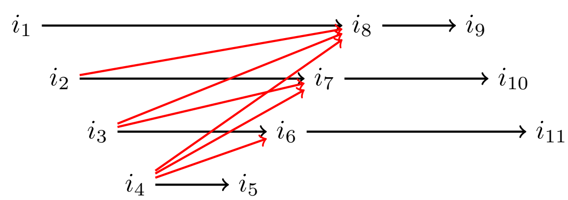
In both cases, these ESOs only changes the weights of (i, j) such that $i < j$ and $j \in [3m] \setminus [2m]$. None of the weights of extra arcs in $E \setminus E_Q$ can be modified by the ESOs stabilizing QU_n. So A is a Belitskiĭ's canonical form.

The number of extra arcs of A is $1 + 2 + \cdots + (m-1) = \frac{1}{2}(m-1)m$. By Theorem 4.1, The number of parameters in A is

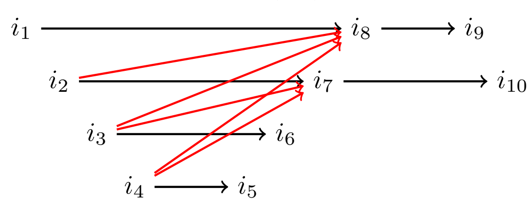
$$\frac{1}{2}(m-1)m + 2m - 3m + 1 = \frac{1}{2}(m-1)(m-2).$$

Given $r \in \{0, 1, \ldots, \frac{1}{2}(m-1)(m-2) - 1\}$, we can remove $\frac{1}{2}(m-1)(m-2) - r$ extra arcs from the graph $G = G_A$ and keep the remaining graph connected. The resulting graph is the graph of an indecomposable Belitskiĭ's canonical form with r parameters. So (4.5) is true for $n \equiv 0 \mod 3$.

(2) When $n = 3m - 1$, let G be the subgraph of the graph in $n = 3m$ case, obtained by removing the vetex $3m$ and the arc $(m + 1, 3m)$. See the illustrated graph below. Similar argument shows that there is a Belitskiĭ's canonical form A with $G_A = G$, and (4.5) is true for $n \equiv 2 \mod 3$.



(3) When $n = 3m - 2$, let G be the subgraph of the graph in $n = 3m - 1$ case, obtained by removing the vetex $3m - 1$ and the arcs $(m + 2, 3m - 1)$ and $(m, m + 2)$. See the illustrated graph below. Similarly, there is a Belitskiĭ's canonical form A with $G_A = G$, and (4.5) is true for $n \equiv 1 \mod 3$.



The graphs of the above Belitskiĭ's canonical forms show that the minimal polynomials of these Belitskiĭ's canonical forms are x^3. □

5. Searches of the Belitskiĭ's canonical forms

5.1. **Algorithms to search for the Belitskiĭ's canonical forms.** We apply the results in Sections 2-4 to get the following efficient algorithm to obtain the Belitskiĭ's canonical forms under the B_n-similarity for a given n.

Algorithm:

(1) List all subpermutations Q in N_n (by the set partitions $\mathcal{P}_Q$ of $[n]$).

(2) For each subpermutation Q, apply Theorems 4.8 and 4.9 to filter out a set S of possible extra arcs of the Belitskiĭ's canonical forms in QU_n and list them in the Belitskiĭ's order, say, $S := \{(i_1, j_1) \prec \cdots \prec (i_m, j_m)\}$.

(3) Explore all possible combinations of the above extra arcs that produce Belitskiĭ's canonical forms. To do this, let $S_0 := \emptyset$ and start at $p = 1$:

  (a) Determine whether there exists an U_n-similarity operation g composed by ESOs stabilizing QU_n such that g changes the graph $([n], E_Q \cup S_{p-1} \cup \{(i_p, j_p)\})$ to a graph $([n], E')$ where
  $$E' \subseteq E_Q \cup S_{p-1} \cup \{(i,j) \in S : (i,j) \succ (i_p, j_p)\}.$$
  If such a g exists, then the arc (i_p, j_p) can be removed by the g operation from the graph of any $A \in QU_n$ whose set of extra arcs no greater than (i_p, j_p) is $S_{p-1} \cup \{(i_p, j_p)\}$; we let $S_p := S_{p-1}$. Otherwise, divide the upcoming process into two cases $S_p := S_{p-1} \cup \{(i_p, j_p)\}$ and $S_p := S_{p-1}$.

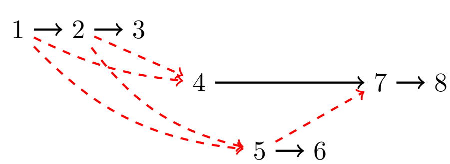
  (b) Increase p by 1. If $p \leq m$, repeat the preceding process.

  (c) The outcoming sets S_m are the sets of extra arcs of all Belitskiĭ's canonical forms in QU_n. Apply Theorem 4.4 to determine the places of parameters for each graph type of the Belitskiĭ's canonical forms.

The above algorithm can be restricted to search for only the indecomposable Belitskiĭ's canonical forms. Moreover, the algorithm can be slightly modified to obtain the Belitskiĭ's canonical form of a given matrix $A \in N_n$ after finding $A' \in QU_n$ such that $A \overset{U_n}{\sim} A'$ by steps (1) and (2) of the simplification process at the end of Section 2. It is much more efficient than the Belitskiĭ's algorithm.

Example 5.1. Let us search the Belitskiĭ's canonical forms in QU_8 in which $\mathcal{P}_Q = 123|478|56$. By Theorems 4.8 and 4.9, the possible extra arcs of a Belitskiĭ's canonical form in QU_8 are listed in the Belitskiĭ's order as follows:

$$(5,7) \prec (2,4) \prec (2,5) \prec (1,4) \prec (1,5).$$



Let S be the set of these arcs. The graph G_Q (in solid arcs) and the set S (in dashed arcs) are shown on the right.

Let $S_0 := \emptyset$. The arc $(5,7)$ cannot be removed by a composition of ESOs stabilizing QU_8, since the only type of ESOs that changes the weight of the arc $(5,7)$ is $\mathrm{O}_{6,7}$ which does not stabilize QU_8. There are two cases $S_1 := \{(5,7)\}$ and $S_1 := \emptyset$. Consider the case $S_1 := \{(5,7)\}$. Similarly, we can reach one of the outcoming cases $S_2 := \{(5,7),(2,4)\}$ and $S_3 := \{(5,7),(2,4),(2,5)\}$. The next arc in consideration is $(1,4)$, which can be removed as illustrated below (cf. (3.4)):

$$\mathcal{P}_Q : 57|24|25|14|\cdots \quad \overset{\mathrm{O}_{12}}{-14+13+15} \quad \overset{\mathrm{O}_{23}}{-13} \quad = 57|24|25|15.$$

Note that $(1,5)$ created in the above process satisfies $(1,5) \succ (1,4)$. Now $S_4 := \{(5,7),(2,4),(2,5)\}$. The next arc $(1,5)$ can be removed as below:

$$\begin{array}{llllllll} & \mathrm{O}_{25} & \mathrm{O}_{36} & \mathrm{O}_{47} & \mathrm{O}_{78} & \mathrm{O}_{68} & \\ \mathcal{P}_Q : 57|24|25|15 & -15+26+27 & -26 & -27+48 & -48+58 & -58 & = 57|24|25. \end{array}$$

So $S_5 := \{(5,7),(2,4),(2,5)\}$. We get a Belitskiĭ's canonical form of the type $\mathcal{P}_Q : 57|24|\underline{25}$ in which the underline indicates the place of a parameter. Explicitly, the 8×8 Belitskiĭ's canonical form is:

$$E_{12}+E_{23}+E_{47}+E_{78}+E_{56}+E_{57}+E_{24}+\lambda E_{25}.$$

The table on the right lists the forms of Belitskiĭ's canonical forms in QU_8 for $\mathcal{P}_Q = 123|478|56$. We use "Y" (resp. "N") to mark the presence (resp. absence) of an extra arc, and "–" to indicate that the arc can be removed given the combination of preceding extra arcs. Totally there are 10 forms of Belitskiĭ's canonical forms in the double coset B_8QB_8, and 5 of them are indecomposable.

The Belitskiĭ's canonical forms for $\mathcal{P}_Q = 123|478|56$

57	24	25	14	15	type	indecomp.
Y	Y	Y	–	–	$57\|24\|\underline{25}$	Yes
Y	Y	N	–	–	57|24	Yes
Y	N	Y	–	–	57|25	Yes
Y	N	N	–	Y	57|15	Yes
Y	N	N	–	N	57	No
N	Y	Y	–	–	24|25	Yes
N	Y	N	–	–	24	No
N	N	Y	–	–	25	No
N	N	N	Y	–	14	No
N	N	N	N	–	∅	No

5.2. **The indecomposable Belitskiĭ's canonical forms for $n \le 8$.** In this subsection, we describe the indecomposable Belitskiĭ's canonical forms in N_n under the B_n-similarity for $n \le 8$ using their graph types together with underlines indicating nonzero parameters (see Example 5.1). The classifications for $n \le 6$ have been done by Kobal [10] and Chen et al [5] (see Theorem 5.2). We apply MAPLE programs to filter out possible extra arcs using the algorithm in the preceding subsection and obtain all classifications for $n \le 8$.

Theorem 5.2 (Kobal [10], Chen et al [5])**.** *The indecomposable Belitskiĭ's canonical forms in N_n under the B_n-similarity for $n \le 6$ are listed by their graph types together with underlines indicating nonzero parameters as follows (29 forms, separated by commas):*

1 : ∅,	12345 : ∅,	123456 : ∅,
12 : ∅,	123|45 : 24,	1234|56 : 35,
123 : ∅,	125|34 : 13,	1236|45 : 24,
1234 : ∅,	145|23 : 24,	1256|34 : $35\|\underline{13}$, 35, 13,
12|34 : 13,	12|345 : 13,	1456|23 : 24,

134|256 : 35,
145|236 : 24,
156|234 : 35,
123|456 : 24, 14,
124|356 : 13,
125|346 : 13,
126|345 : 13,
12|3456 : 13,
12|34|56 : 35|13,
12|36|45 : 13|14,
14|23|56 : 25|15.

Remark 5.3. For $n = 6$, Theorem 5.2 lists 19 forms instead of 18 forms shown in [5, Theorem 2.2], since we use nonzero parameters in our classifications.

The results for $n = 7$ are as follows, including the 8 forms with a parameter discovered in [5, Theorem 2.3].

Theorem 5.4. *The indecomposable Belitskiĭ's canonical forms in N_n under the B_n-similarity for $n = 7$ are of the graph types (85 forms in 58 subpermutations, separated by commas):*

1234567 : ∅,
14567|23 : 24,
12567|34 : 35|$\underline{13}$, 35, 13,
12367|45 : 46|$\underline{24}$, 46, 24,
12347|56 : 35,
12345|67 : 46,
1567|234 : 35, 25,
1467|235 : 24,
1457|236 : 24,
1456|237 : 24,
1367|245 : 46,
1347|256 : 35,
1345|267 : 46,
1267|345 : 46|$\underline{13}$, 46, 13,
1257|346 : 13,
1256|347 : 35|$\underline{13}$, 35, 13,
1247|356 : 13,
1245|367 : 46,
1237|456 : 24, 14,
1236|457 : 24,
1235|467 : 24,
1234|567 : 35, 25,
123|4567 : 24, 14,
124|3567 : 13,
125|3467 : 13,
126|3457 : 13,
127|3456 : 13,
134|2567 : 35,
145|2367 : 46|$\underline{24}$, 46, 24,
156|2347 : 35,
167|2345 : 46,
123|45|67 : 46|24, 46|14,
123|47|56 : 24|25,
124|35|67 : 36|13,
125|34|67 : 36|26|$\underline{13}$, 36|26, 36|13, 26|13,
126|34|57 : 35|13,
127|34|56 : 35|13,
127|36|45 : 13|14,
134|25|67 : 36|26,
145|23|67 : 46|24, 24|16,
146|23|57 : 24|15,
147|23|56 : 24|25|$\underline{15}$, 24|25, 24|15, 25|15,
156|23|47 : 24|25,
167|23|45 : 46|24,
167|25|34 : 36|26,
12|34567 : 13,
15|234|67 : 36|16,
14|237|56 : 25|15,
13|247|56 : 25|15,
13|267|45 : 46|14,
12|345|67 : 46|13,
12|347|56 : 35|13, 35|15,
12|367|45 : 46|13|$\underline{14}$, 46|13, 46|14, 13|14,
12|37|456 : 13|14,
12|36|457 : 13|14,
12|35|467 : 13|14,
12|34|567 : 35|13, 13|15,
14|23|567 : 25|15.

Theorem 5.5. *For $n = 8$, there are 481 forms (in 245 subpermutations) of indecomposable Belitskiĭ's canonical forms in N_n under the B_n-similarity. The graph types of these forms will be listed in the Appendix section.*

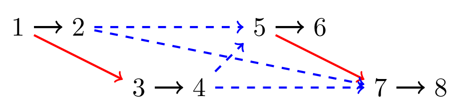
The number of subpermutations in N_n equals the number of partitions of $[n]$, which is called a Bell or exponential number. The first few Bell numbers starting at $n = 1$ are:

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$$

Many properties of the Bell numbers have been studied (cf. http://oeis.org). Both Theorem 2.7 and the Bell numbers imply that the numbers of indecomposable Belitskiĭ's canonical forms in N_n grow in a rate greater than any exponential function of n.

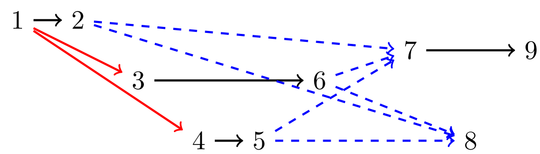
5.3. **Create new Belitskiĭ's canonical forms.** Theorem 2.7 can be used to obtain new Belitskiĭ's canonical forms. Let $A_1 \in Q_1 U_p$ and $A_2 \in Q_2 U_q$ be Belitskiĭ's canonical forms. Theorem 2.7 claims that if $\begin{bmatrix} Q_1 & Q_{12} \\ 0 & Q_2 \end{bmatrix} \in N_{p+q}$ is a subpermutation, then $\begin{bmatrix} A_1 & Q_{12} \\ 0 & A_2 \end{bmatrix}$ is a Belitskiĭ's canonical form in N_{p+q}. Here we rephrase Theorem 2.7 in the language of graphs. Let $G_{A_2} + p$ denote the graph with the vertex set $\{1 + p, \ldots, q + p\}$ and the edge set $\{(i + p, j + p) : (i, j) \in E_{A_2}\}$. If we add some arcs from chain heads of G_{A_1} to chain tails of $G_{A_2} + p$ such that each involving vertex is on at most one such arc, then the resulting graph represents a Belitskiĭ's canonical form in N_{p+q}. The following are two examples.

Example 5.6. Let $A_1 = A_2$ be the Belitskiĭ's canonical forms of the graph type $12|34 : 13$. So $\mathcal{P}_{Q_1} = \mathcal{P}_{Q_2} = 12|34$. By Theorem 2.7, we can obtain Belitskiĭ's canonical forms by adding to the graph $G_{A_1} \cup (G_{A_2} + 4)$ some arcs from the chain heads 2 and 4 of G_{A_1} to the chain tails 5 and 7 of $G_{A_2} + 4$ such that each vertex is on at most one arc. The illustrated graph is as below, in which dashed arcs are possible arcs added to the graph $G_{A_1} \cup (G_{A_2} + 4)$:



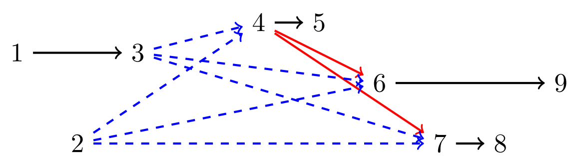
There are 6 indecomposable Belitskiĭ's canonical forms obtained from this way: $1256|34|78 : 57|13$, $1278|34|56 : 57|13$, $12|3456|78 : 57|13$, $12|3478|56 : 57|13$, $1256|3478 : 57|\underline{13}$, $1278|3456 : 57|\underline{13}$. All of them can be found in Theorem 5.5. Note that a parameter is added in each of the last two cases for the general forms. The direct sum $A_1 \oplus A_2$ is the only non-indecomposable Belitskiĭ's canonical form given by Theorem 2.7 here.

Example 5.7. Let $A_1 \in Q_1 U_6$ and $A_2 \in Q_2 U_3$ be the Belitskiĭ's canonical forms of the graph types $12|36|45 : 13|14$ and $13|2 : \emptyset$, respectively. The Belitskiĭ's canonical forms $\begin{bmatrix} A_1 & Q_{12} \\ & A_2 \end{bmatrix}$ constructed in Theorem 2.7 are constructed by adding the following possible dashed arcs to the graph $G_{A_1} \cup (G_{A_2} + 6)$ such that each vertex is on at most one such arc:



There are 6 indecomposable Belitskiĭ's canonical forms constructed in this way: $1279|368|45 : 13|14$, $1279|36|458 : 13|14$, $128|3679|45 : 13|14$, $128|36|4579 : 13|14$, $12|3679|458 : 13|14$, $12|368|4579 : 13|14$.

Similarly, the Belitskiĭ's canonical forms $\begin{bmatrix} A_2 & Q'_{12} \\ & A_1 \end{bmatrix}$ constructed in Theorem 2.7 are constructed by adding the following possible dashed arcs to the graph $G_{A_2} \cup (G_{A_1} + 3)$ such that each vertex is on at most one such arc:



There are also 6 indecomposable Belitskiĭ's canonical forms constructed in this way: $1345|269|78 : 46|47$, $1345|278|69 : 46|47$, $1369|245|78 : 46|47$, $1369|278|45 : 46|47$, $1378|245|69 : 46|47$, $1378|269|45 : 46|47$.

5.4. **The B_n-similarity of upper triangular matrices.** Sylveseter's theorem (cf. [9, Theorem 2.4.4.1]) says that if $M \in M_p$ and $N \in M_q$ have no eigenvalue in common, then the equation $MX - XN = R$ has a unique solution $X \in M_{p,q}$ for each $R \in M_{p,q}$, that is,

$$\begin{bmatrix} M & R \\ 0 & N \end{bmatrix} = \begin{bmatrix} I_p & X \\ 0 & I_q \end{bmatrix}^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} I_p & X \\ 0 & I_q \end{bmatrix} \overset{U_{p+q}}{\sim} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}. \tag{5.1}$$

Similarly, every $n \times n$ upper triangular matrix $A = [a_{ij}]$ is B_n-similar to the matrix $C = [c_{ij}]$ such that $c_{ij} = a_{ij}$ if $a_{ii} = a_{jj}$, and $c_{ij} = 0$ otherwise; C is permutation similar to a direct sum of matrices of the form $\lambda I_k + C'$ for $k \in [n]$ and $C' \in N_k$. See [14] or [16, Section 1] for more details. The B_n-similarity problem of upper triangular A is transformed to the upper triangular similarity problems of nilpotent upper triangular matrices.

In [16, Theorem 1.5], Thijsse showed that if an $n \times n$ upper triangular matrix A satisfies one of the following two conditions:

(1) A is nonderogatory;
(2) $\dim \ker(A - \lambda I)^2 = \dim \ker(A - \lambda I)^3$ for each $\lambda \in \mathbb{C}$.

Then A is B_n-similar to a matrix which is permutation similar to a direct sum of Jordan blocks. We provide a new proof here. By the argument in the preceding paragraph, it suffices to consider the case $A \in N_n$:

- Condition (1) means that $A \in JU_n$ where J is the nilpotent Jordan block of size n. The only Belitskiĭ's canonical form in JU_n is J.
- Condition (2) means that each Jordan block of A has size no more than 2. So $A \overset{B_n}{\sim} A' \in QU_n$ in which $Q \in N_n$ is a subpermutation, $A^2 = (A')^2 = 0$, and each extra arc $(i, j) \in E_{A'} \setminus E_Q$ of A' has j a chain head of a chain of G_Q. So (i, j) is not in the Belitskiĭ's canonical form of A. Therefore, the Belitskiĭ's canonical form of A is exactly Q, which is permutation similar to a direct sum of nilpotent Jordan blocks of sizes one or two.

Another observation is [9, 2.5.P49] which states that: an $n \times n$ upper triangular matrix A is similar to a diagonal matrix if and only if it is B_n-similar to a diagonal matrix. In particular, if A has distinct diagonal entries, then A is B_n-similar to its diagonal. However, the result is not quite useful for nilpotent upper triangular matrices since the only diagonalizable nilpotent upper triangular matrix is the zero matrix.

Appendix

As a complement to Theorem 5.5, the list of indecomposable Belitskiĭ's canonical forms in N_n under the B_n-simlarity for $n = 8$ is as follows (481 forms in 245 subpermutations):

$12345678 : \emptyset,$
$12|345678 : 13,$
$145678|23 : 24,$
$125678|34 : 35|\underline{13}, 35, 13,$
$123678|45 : 46|\underline{24}|\underline{14}, 46|\underline{24}, 46, 24,$
$123478|56 : 57|\underline{35}, 57, 35,$
$123458|67 : 46,$
$123456|78 : 57,$
$123|45678 : 24, 14,$
$124|35678 : 13,$
$125|34678 : 13,$
$126|34578 : 13,$
$127|34568 : 13,$
$128|34567 : 13,$
$134|25678 : 35,$
$145|23678 : 46|\underline{24}, 46, 24,$
$156|23478 : 57|\underline{35}, 57, 35,$
$167|23458 : 46,$
$178|23456 : 57,$
$15678|234 : 35, 25,$
$14678|235 : 24,$
$14578|236 : 24,$
$14568|237 : 24,$
$14567|238 : 24,$
$13678|245 : 46,$
$13478|256 : 57|\underline{35}, 57, 35,$
$13458|267 : 46,$
$13456|278 : 57,$
$12678|345 : 46|\underline{13}, 46, 36|\underline{13}, 36, 13,$
$12578|346 : 35|\underline{13}, 35, 13,$
$12568|347 : 35|\underline{13}, 35, 13,$
$12567|348 : 35|\underline{13}, 35, 13,$
$12478|356 : 57|\underline{13}, 57, 13,$
$12458|367 : 46,$
$12456|378 : 57,$
$12378|456 : 57|\underline{24}, 57|\underline{14}, 57, 24, 14,$
$12368|457 : 24,$
$12367|458 : 46|\underline{24}, 46, 24,$
$12358|467 : 24,$
$12356|478 : 57,$
$12348|567 : 35, 25,$
$12347|568 : 35,$
$12346|578 : 35,$
$12345|678 : 46, 36,$
$1234|5678 : 35, 25, 15,$
$1235|4678 : 24, 14,$
$1236|4578 : 24, 14,$
$1237|4568 : 24, 14,$
$1238|4567 : 24, 14,$
$1245|3678 : 46, 13,$
$1246|3578 : 13,$
$1247|3568 : 13,$
$1248|3567 : 13,$
$1256|3478 : 57|\underline{35}|\underline{13}, 57|\underline{35}, 57|\underline{13}, 57, 35|\underline{13}, 35, 13,$
$1257|3468 : 13,$

$1258|3467 : 13,$
$1267|3458 : 46|\underline{13}, 46, 13,$
$1268|3457 : 13,$
$1278|3456 : 57|\underline{13}, 57, 13,$
$1345|2678 : 46, 36,$
$1346|2578 : 35,$
$1347|2568 : 35,$
$1348|2567 : 35,$
$1356|2478 : 57,$
$1367|2458 : 46,$
$1378|2456 : 57,$
$1456|2378 : 57|\underline{24}, 57, 24,$
$1457|2368 : 24,$
$1458|2367 : 46|\underline{24}, 46, 24,$
$1467|2358 : 24,$
$1478|2356 : 57,$
$1567|2348 : 35, 25,$
$1568|2347 : 35,$
$1578|2346 : 35,$
$1678|2345 : 46, 36,$
$12|34|5678 : 35|13, 13|15,$
$12|35|4678 : 13|14,$
$12|36|4578 : 13|14,$
$12|37|4568 : 13|14,$
$12|38|4567 : 13|14,$
$12|3678|45 : 46|13|\underline{14}, 46|13, 46|14, 13|14,$
$12|3478|56 : 57|\underline{35}|13, 57|\underline{35}|15, 57|13, 57|15, 35|13, 35|15,$
$12|3458|67 : 46|13, 46|16,$
$12|3456|78 : 57|13,$
$12|345|678 : 46|13, 36|13, 36|16,$
$12|346|578 : 35|13, 35|15,$
$12|347|568 : 35|13, 35|15,$
$12|348|567 : 35|13, 35|15, 13|15,$
$12|356|478 : 57|13, 57|14,$
$12|358|467 : 13|14,$
$12|367|458 : 46|13|\underline{14}, 46|13, 46|14, 13|14,$
$12|368|457 : 13|14,$
$12|378|456 : 57|13|\underline{14}, 57|13, 57|14, 13|14,$
$13|2678|45 : 46|14,$
$13|2478|56 : 57|\underline{25}|15, 57|15, 25|15,$
$13|2458|67 : 46|16,$
$13|245|678 : 26|16,$
$13|246|578 : 25|15,$
$13|247|568 : 25|15,$
$13|248|567 : 25|15,$
$13|256|478 : 57|14,$
$13|267|458 : 46|14,$
$13|278|456 : 57|14,$
$14|23|5678 : 25|15,$
$14|2378|56 : 57|\underline{25}|15, 57|15, 25|15,$
$14|2358|67 : 36|16,$
$14|235|678 : 26|16,$
$14|236|578 : 25|15,$
$14|237|568 : 25|15,$
$14|238|567 : 25|15,$
$15|2348|67 : 36|16,$
$15|234|678 : 36|16, 26|16,$
$16|2345|78 : 47|17,$
$1678|23|45 : 46|24, 26|24,$
$1578|23|46 : 24|25,$
$1568|23|47 : 24|25,$
$1567|23|48 : 24|25,$
$1478|23|56 : 57|24|\underline{25}|\underline{15}, 57|24|\underline{25}, 57|24|\underline{15}, 57|24, 57|25|\underline{15}, 57|25, 24|25|\underline{15}, 24|25, 24|15, 25|15,$
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