On rearrangement inequalities for multiple sequences

Chai Wah Wu
IBM T. J. Watson Research Center
P. O. Box 218, Yorktown Heights, New York 10598, USA
e-mail: chaiwahwu@ieee.org

February 24, 2020
Latest update: March 21, 2020

Abstract

The classical rearrangement inequality provides bounds for the sum of products of two sequences under permutations of terms and show that similarly ordered sequences provide the largest value whereas opposite ordered sequences provide the smallest value. This has been generalized to multiple sequences to show that similarly ordered sequences provide the largest value. However, the permutations of the sequences that result in the smallest value is in general not known. We show a variant of the rearrangement inequality for which a lower bound can be obtained and conditions for which this bound is achieved for a sequence of permutations. We also study another variation of the rearrangement inequality where the permutations of terms can be across the various sequences.

1 Introduction

The rearrangement inequality [1] states that given two finite sequences of real numbers the sum of the product of pairs of terms is maximal when the sequences are similarly ordered and minimal when oppositely ordered. More precisely, suppose $x_1 \leq x_2 \cdots \leq x_n$ and $y_1 \leq y_2 \cdots \leq y_n$, then for any permutation σ in the symmetric group of permutation on $\{1, \cdots, n\}$,

$$x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots x_n y_n \tag{1}$$

The dual inequality is also true [2], albeit only for nonnegative numbers in general (i.e. $x_1 \geq 0$, $y_1 \geq 0$):

$$(x_1 + y_1) \cdots (x_n + y_n) \leq (x_{\sigma(1)} + y_1) \cdots (x_{\sigma(n)} + y_n) \leq (x_n + y_1) \cdots (x_1 + y_n) \tag{2}$$

Eq. (2) says that similarly ordered terms minimize the product of sums of pairs, while opposite ordered terms maximize the product of sums.

In Ref. [3], these inequalities are generalized to multiple sequences of numbers:

Lemma 1. *Consider a set of nonnegative numbers $\{a_{ij}\}$, $i = 1, \cdots, k$, $j = 1, \cdots, n$. For each i, let $a'_{i1}, a'_{i2}, \cdots, a'_{in}$ be the numbers $a_{i1}, a_{i2}, \cdots, a_{in}$ reordered such that $a'_{i1} \geq a'_{i2} \geq \cdots \geq a'_{in}$. Then*

$$\sum_{j=1}^{n} \prod_{i=1}^{k} a_{ij} \leq \sum_{j=1}^{n} \prod_{i=1}^{k} a'_{ij}$$

$$\prod_{j=1}^{n} \sum_{i=1}^{k} a_{ij} \geq \prod_{j=1}^{n} \sum_{i=1}^{k} a'_{ij}$$

Note that only half of the rearrangement inequality is generalized. In particular, the rightmost inequality (the upper bound) in Eq. (1) and the leftmost inequality (the lower bound) in Eq. (2) are generalized in Lemma 1 by showing that similarly ordered sequences maximizes the sum of products and minimizes the product of sums. No such generalization is known for the other half. This note is an attempt to provide results for the other direction.

Eq. (1) can be used to prove the AM-GM inequality which states that the algebraic mean of nonnegative numbers are larger than or equal to their geometric mean. We will rewrite it in the following equivalent form.

Lemma 2. *For n nonnegative real numbers x_i, $\sum_{i=1}^n x_i \geq n\sqrt[n]{\prod_{i=1}^n x_i}$ and $\prod_{i=1}^n x_i \leq \left(\frac{\sum_{i=1}^n x_i}{n}\right)^n$ with equality if and only if all the x_i are the same.*

This allows us to give the following bounds on the other direction of Lemma 1.

Lemma 3. *Consider a set of nonnegative numbers $\{a_{ij}\}$, $i = 1, \cdots, k$, $j = 1, \cdots, n$. Then*

$$n\sqrt[n]{\prod_{ij} a_{ij}} \leq \sum_{j=1}^n \prod_{i=1}^k a_{ij}$$

$$\left(\frac{\sum_{ij} a_{ij}}{n}\right)^n \geq \prod_{j=1}^n \sum_{i=1}^k a_{ij}$$

In addition, Lemma 2 implies that if there exists k permutations σ_i on $\{1, \cdots, n\}$ such that $\prod_{i=1}^k a_{i\sigma_i(j)} = \prod_{i=1}^k a_{i\sigma_i(1)}$ for all j, then this set of permutations will minimize the sum of products, i.e.

$$\sum_{j=1}^n \prod_{i=1}^k a_{i\sigma_i(j)} \leq \sum_{j=1}^n \prod_{i=1}^k a_{ij}$$

Similarly, if there exists permutations σ_i such that $\sum_{i=1}^k a_{i\sigma_i(j)} = \sum_{i=1}^k a_{i\sigma_i(1)}$ for all j, then this set of permutations will maximize the product of sums, i.e.

$$\prod_{j=1}^n \sum_{i=1}^k a_{i\sigma_i(j)} \geq \prod_{j=1}^n \sum_{i=1}^k a_{ij}$$

In the next section we consider scenarios where these conditions can be satisfied for some sequence of permutations of terms and thus supply the other directions of Lemma 1.

2 Sums of products of permuted sequences

Instead of considering multiple sequences, we restrict ourselves to permutations of the same sequence and look at sum of products of these sequences.

Definition 1. *Let $0 \leq a_1 \leq a_2 \cdots \leq a_n$ be a sequence of nonnegative numbers. Consider k permutations of the integers $\{1, \cdots, n\}$ denoted as $\{\sigma_1, \cdots, \sigma_k\}$ and define the value $v(n, k) = \sum_{i=1}^n \prod_{j=1}^k a_{\sigma_j(i)}$. The maximal and minimal value of v among all k-sets of permutations are denoted as $v_{\max}(n, k)$ and $v_{\min}(n, k)$ respectively.*

An immediate consequence of Lemma 1 is that $v_{\max}(n, k) = \sum_{i=1}^n a_i^k$ and is achieved when all the k permutations σ_i are the same.

$v_{\min}(n, k)$ and $v_{\max}(n, k)$ can be determined explicitly for small value of n or k.

Lemma 4.
- $v(1,k) = a_i^k$,
- $v(n,1) = \sum_{i=1}^n a_i$,
- $v_{\max}(2,k) = a_1^k + a_2^k$.
- $v_{\min}(2,2m) = 2a_1^m a_2^m$
- $v_{\min}(2,2m+1) = (a_1+a_2)a_1^m a_2^m$
- $v_{\max}(n,2) = \sum_{i=1}^n a_i^2$
- $v_{\min}(n,2) = \sum_{i=1}^n a_i a_{n-i+1}$

Proof. For $k = 1$ there is only one sequence and $v(n,1) = \sum_{i=1}^n a_i$. For $n = 1$, the only permutation is (1), so $v(1,k) = a_1^k$. When $n = 2$, there are only two permutations on the integers $\{1,2\}$, and $v_{\max}(2,k) = a_1^k + a_2^k$. If $k = 2m$, $v_{\min}(2,k) = 2a_1^m a_2^m$ is achieved with m of the permutations of one kind and the other half the other kind. If $k = 2m+1$, $v_{\min}(2,k) = (a_1+a_2)a_1^m a_2^m$ is achieved with m of the permutations of one kind and $m+1$ of them the other kind.

The rearrangement inequality (Eq. (1)) implies that for $k = 2$, $v_{\max}(n,2) = \sum_{i=1}^n a_i^2$ and $v_{\min}(n,2) = \sum_{i=1}^n a_i a_{n-i+1}$ by choosing both permutations to be $(1,2,\cdots,n)$ for $v_{\max}(n,2)$ and choosing the two permutations to be $(1,2,\cdots,n)$ and $(n,n-1,\cdots,2,1)$ for $v_{\min}(n,2)$. □

Our next result is a lower bound on $v_{\min}$:

Lemma 5. $v_{\min}(n,k) \geq n\prod_i a_i^{k/n}$.

Proof. The product $\prod_{ij} a_{\sigma_i(j)}$ is equal to $\prod_i a_i^k$. Thus by Lemma 2, $v(n,k) \geq n\sqrt[n]{\prod_i a_i^k} = n\prod_i a_i^{k/n}$. □

Our main result is that this bound is tight when k is a multiple of n.

Theorem 1. *If n divides k, then $v_{min}(n,k) = n\prod_{i=1}^n a_i^{k/n}$ and is achieved by using each cyclic permutation k/n times..*

Proof. By Lemma 5 $v(n,k) \geq n\prod_{i=1}^n a_i^{k/n}$. Consider the n cyclic permutations $r_1 = (1,2,...,n)$, $r_2 = (2,...,n,1)$, ..., $r_n = (n,1,...,n-1)$. It is clear that using k/n copies of each permutation r_i to form k permutations results in $v(n,k) = n\prod_{i=1}^n a_i^{k/n}$. □

3 The dual problem of product of sums

Definition 2. *Let $0 \leq a_1 \leq a_2 \cdots \leq a_n$ be a sequence of nonnegative numbers. Consider k permutations of the integers $\{1,\cdots,n\}$ denoted as $\{\sigma_1,\cdots,\sigma_k\}$ and define the value $w(n,k) = \prod_{i=1}^n \sum_{j=1}^k a_{\sigma_j(i)}$. The maximal and minimal value of v among all k-sets of permutations are denoted as $w_{\max}(n,k)$ and $w_{\min}(n,k)$ respectively.*

Analogous to Section 2 the following result can be derived regarding $w_{\max}$ and $w_{\min}$.

Lemma 6.
- $w_{\min}(n,k) = \prod_{i=1}^n ka_i = k^n\prod_i a_i$
- $w_{\max}(1,k) = ka_1$
- $w_{\max}(n,1) = \prod_i a_i$
- $w_{\min}(2,k) = k^2\prod_i a_i$.
- $w_{\max}(2,2m) = (a_1+a_2)^2 m^2$.

- $w_{\max}(2, 2m+1) = (ma_1 + (m+1)a_2)(ma_2 + (m+1)a_1)$.
- $w_{\min}(n, 2) = 2^n \prod_i a_i$.
- $w_{\max}(n, 2) = \prod_i (a_i + a_{n-i+1})$.
- $w_{\max}(n, k) \leq \left(\frac{k \sum_i a_i}{n}\right)^n$ *with equality if* n *divides* k.

4 The special case where a_i is an arithmetic progression

Consider the special case where the elements a_i form an arithmetic progression, i.e. a_i are equally spaced where $a_{i+1} - a_i$ is constant and does not depend on i. Even though $v_{\min}$ are difficult to compute in general, explicit forms for $w_{\max}$ can be found for many values of n and k.

Theorem 2. *If* $k = 2t + nu$ *for nonnegative integers* t *and* u, *then* $w_{\max}(n, k) = \left(\frac{k(a_1+a_n)}{2}\right)^n$.

Proof. It is easy to see that $\sum_i a_i = n(a_1+a_n)/2$. By Lemma 6 $w_{\max}(n, k) \leq \left(\frac{k(a_1+a_n)}{2}\right)^n$. By using t copies of the permutation $(1, \cdots, n)$ and t copies of the permutation $(n, \cdots, 1)$ followed by u copies each of the cyclic permutations r_i, we see that $\sum_j \sigma_j(i) = t(a_1+a_n)+un(a_1+a_n)/2 = (t+un/2)(a_1+a_n) = k(a_1+a_n)/2$ for all i and thus $w(n, k) = \left(\frac{k(a_1+a_n)}{2}\right)^n$. □

Corollary 1. *If* k *is even, then* $w_{\max}(n, k) = \left(\frac{k(a_1+a_n)}{2}\right)^n$.

Corollary 2. *If* n *is odd and* $k \geq n-1$, *then* $w_{\max}(n, k) = \left(\frac{k(a_1+a_n)}{2}\right)^n$.

The case when k is odd and n is even is more involved. Let $a_i = a_1 + (i-1)d = (a_1 - d) + id$ for $i = 1, \cdots, n$ and $d \geq 0$. Given a k-set of permutations σ_j define w_i as $w_i = \sum_{j=1}^k \sigma_j(i)$. This implies that $\sum_{j=1}^k a_{\sigma_j(i)} = k(a_1 - d) + w_i d$. Next we show there is a sequence of permutations for which $w_i - w_j \leq 1$ for all i, j when $k \geq n-1$.

Lemma 7. *If* n *is even, there exists a sequence* σ_j *of* $n-1$ *permutations of* $\{1, \cdots n\}$ *such that* $w_i = \frac{n^2}{2} - 1$ *for* $i = 1, \cdots \frac{n}{2}$ *and* $w_i = \frac{n^2}{2}$ *for* $i = \frac{n}{2}+1, \cdots, n$.

Proof. Recall the cyclic permutations denoted as r_i. Consider the index set $S = \{i : 2 \leq i \leq n, i \neq n/2+1\}$. Let us compute $\sum_{j\in S} r_j(i)$. Since $r_1(i) = (1, 2, ..., n)$ and $r_{n/2+1} = (n/2+1, n/2+2, ..., n/2)$, $\sum_{j\in S}^{n-1} r_j(i) = n(n+1)/2 - r_1(i) - r_{n/2+1}(i)$ is equal to $n(n+1)/2 - i - (n/2+i) = n^2/2 - 2i$ for $i = 1, \cdots, n/2$ and equal to $n(n+1)/2 - i - (i - n/2) = n^2/2 - (2i - n)$ for $i = n/2+1, \cdots, n$. Let $\tilde{\sigma}$ be the permutation defined as $\tilde{\sigma}(i) = 2i - 1$ for $i = 1 \cdots n/2$ and $\tilde{\sigma}(i) = n - 2i$ for $i = n/2+1 \cdots, n$. Define the $(n-1)$-set of permutations $\{\sigma_i\}$ as $\tilde{\sigma}$ plus the cyclic permutations with index in S, we get $\sum_{j=1}^{n-1} \sigma_j(i) = n^2/2 - 1$ for $i = 1, \cdots, n/2$ and $\sum_j \sigma_j(i) = n^2/2$ for $i = n/2+1, \ldots, n$. □

Corollary 3. *If* n *is even and* k *is odd, there does not exists a* k*-set of permutations such that* $w_i = w_j$ *for all* i, j. *If* $k \geq n-1$, *then there exists* k *permutations such that* $w_i - w_j \leq 1$ *for all* i, j.

Proof. If n is even and k is odd, $\sum_i w_i = kn(n+1)/2$ is not divisible by n as k and $n+1$ are both odd. This means it is not possible for $w_i = w_j$ for all i, j. If n is odd, the case $k = n-1$ can be achieved with $k/2$ permutations $(1, \cdots, n)$ and $k/2$ permutations $(n, n-1, \ldots, 1)$. If n is even, the case $k = n-1$ follows from Lemma 7. If $k > n$, it follows by induction from the $k-2$ case and adding the two permutations $(1, \cdots, n)$ and $(n, n-1, \ldots, 1)$. □

Lemma 8. *If* $w_1 + w_2 = v_1 + v_2$ *and* $|w_2 - w_1| \geq |v_2 - v_1|$, *then* $(x + w_1)(x + w_2) \leq (x + v_1)(x + v_2)$.

Proof. Let $y = w_1 + w_2$. Then $(x+w_1)(x+w_2) = x^2 + yx + w_1(y - w_1)$. Since the function $x(y-x)$ has a maximum at $\frac{y}{2}$, this implies that $(x+w_1)(x+w_2)$ is maximized when $w_1 = w_2$. □

Lemma 9. *If $k \geq n-1$, then for the set permutations σ_j that maximizes $w(n,k)$, the corresponding w_i must satisfy $w_i - w_j \leq 1$ for all i, j. If in addition, n is odd or k is even, then $w_i = w_j$ for all i, j.*

Proof. If $w_i - w_j > 1$ for some pair (w_i, w_j), by Lemma 8 we can reduce w_i and increase w_j by 1 repeatedly until $w_i - w_j \leq 1$ for all i, j without increasing $w_{\max}(n,k) = \prod_{i=1}^n \sum_{j=1}^k a_{\sigma_j(i)} = \prod_{i=1}^n k(a_1 - d) + w_i d$. If n is even and k is odd, $\sum_i w_i$ is not divisible by n and the only set of w_i such that $w_i - w_j \leq 1$ for all i, j is the one described in Lemma 7. If n is odd or k is even, there exists a set of permutations corresponding to $w_{\max}(n,k)$ such that $w_i = w_j$ by Theorem 2. □

Theorem 3. *If n is even and k is odd such that $k \geq n-1$, then*

$$w_{\max}(n,k) = \left(ka_1 + \left(\frac{k(n-1)-1}{2}\right)d\right)^{n/2} \left(ka_1 + \left(\frac{k(n-1)+1}{2}\right)d\right)^{n/2}$$

Proof. Note that k can be written as $k = 2t + (n-1)$. As a consequence of Lemmas 7, 9, the value $w_{\max}(n,k)$ is achieved with t copies of $(1,...,n)$, t copies of $(n,...,1)$, $\tilde{\sigma}$ and the cyclic permutations with index in S. Then $w_i = t(n+1) + n^2/2 - 1 = \frac{k(n+1)-1}{2}$ for $i = 1, \cdots, n/2$, and $w_i = t(n+1) + n^2/2 = \frac{k(n+1)+1}{2}$ for $i = n/2 + 1, \cdots, n$. Thus $w_{\max}(n,k) = \prod_{i=1}^n k(a_1 - d) + w_i d = \left(k(a_1 - d) + \frac{d(k(n+1)-1)}{2}\right)^{n/2} \left(k(a_1 - d) + \frac{d(k(n+1)+1)}{2}\right)^{n/2}$ and the conclusion follows. □

Theorems 2 and 3 show that the value of $w_{\max}(n,k)$ and the corresponding maximizing set of permutations can be explicitly found when $k \geq n-1$ or k is even. It is clear that we get analogous results for $v_{\min}$ if the sequence a_i is a geometric progression, i.e. it is defined as $a_i = c^{b_i}$ for some constant $c \geq 0$ and an arithmetic progression b_i of (not necessarily nonnegative) numbers.

4.1 The special case $a_i = i$

Consider the special case where the sequence a_i is just the first n positive integers, i.e. $v(n,k) = \sum_{i=1}^n \prod_{j=1}^k \sigma_j(i)$ and $w(n,k) = \prod_{i=1}^n \sum_{j=1}^k \sigma_j(i)$. The values of $v_{\min}(n,k)$ and $w_{\max}(n,k)$ can be found in OEIS [4] sequence A260355 (`https://oeis.org/A260355`) and sequence A331988 (`https://oeis.org/A331988`) respectively.

Theorem 4. *If $k = 2t + nu$ for nonnegative integers t and u, then $w_{\max}(n,k) = \left(\frac{k(n+1)}{2}\right)^n$. In particular, if k is even or if n is odd and $k \geq n-1$, then $w_{\max}(n,k) = \left(\frac{k(n+1)}{2}\right)^n$.*

Theorem 5. *If n is even and k is odd such that $k \geq n-1$, then $w_{\max}(n,k) = \left(\frac{k^2(n+1)^2-1}{4}\right)^{n/2}$.*

For example, Theorem 4 shows that $w_{\max}(3,k) = 8k^3$ for $k > 1$. More details about $v_{\min}$ and $w_{\max}$ for this special case, including tables of values, can be found in Ref. [5].

5 Another variation of the rearrangement inequality

In Ref. [6], Eqs (1-2) are generalized as follows:

Theorem 6. *Let f be real valued function of 2 variables defined on $I_a \times I_b$. If*

$$f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1) \geq 0$$

for all $x_1 \leq x_2$ in I_a and $y_1 \leq y_2$ in I_b, then

$$\sum_i f(a_i, b_{n-i+1}) \leq \sum_i f(a_i, b_{\sigma(i)}) \leq \sum_i f(a_i, b_i) \tag{3}$$

for all sequences $a_1 \leq a_2 \cdots \leq a_n$ in I_a, $b_1 \leq b_2 \cdots \leq b_n$ in I_b, and all permutation σ of $\{1, \cdots, n\}$.

Theorem 6 implies Eq. (1) and Eq. (2) by choosing $f(x, y) = xy$ and $f(x, y) = -\log(x + y)$ respectively. In Theorem 6, the sequences a_i and b_i are separate and the permutation σ acts on b_i only. We next introduce a variant of the rearrangement inequality where the permutation acts on the union of a_i and b_i.

Theorem 7. *Let f be real valued function of 2 variables defined on $I \times I$. Let a_i be a set of $2n$ numbers in I and let b_i be the numbers a_i sorted in increasing order, i.e., $b_1 \leq b_2 \cdots \leq b_{2n}$. If*

$$f(x_1, x_2) \leq f(x_2, x_1) \tag{4}$$

and

$$f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1) \geq 0 \tag{5}$$

for all $x_1 \leq x_2$ and $y_1 \leq y_2$ in I, then

$$\sum_{i=1}^{n} f(b_i, b_{2n-i+1}) \leq \sum_{i=1}^{n} f(a_{2i-1}, a_{2i}) \tag{6}$$

If in addition,

$$f(x, y) = f(y, x) \tag{7}$$

for all x, y in I, then

$$\sum_{i=1}^{n} f(a_{2i-1}, a_{2i}) \leq \sum_{i=1}^{n} f(b_{2i-1}, b_{2i}) \tag{8}$$

Proof. Let c_i be a permutation of a_i such that $v = \sum_{i=1}^{n} f(c_i, c_{2n-i+1})$ is minimized. Then by Theorem 6, c_i can be chosen such that $c_i \leq c_{i+1}$ for $1 \leq i \leq n - 1$ and for $n + 1 \leq i \leq 2n - 1$. Suppose $c_{n+1} < c_n$. By Eq. (4) we can swap these two terms without increasing the value of v. Again by Theorem 6, we can reorder c_i for $1 \leq i \leq n$ such that they are nondecreasing and also reorder c_i for $n + 1 \leq i \leq 2n$ such that they are nondecreasing. If $c_{n+1} < c_n$ we repeat the process again. It's clear that this needs to be repeated at most a finite number of times and eventually we have $c_{n+1} \geq c_n$. Thus we have a sequence of c_i such that $c_i \leq c_{i+1}$ for $1 \leq i \leq n - 1$ and for $n + 1 \leq i \leq 2n - 1$, in addition to $c_n \leq c_{n+1}$, i.e., $c_1 \leq c_2 \cdots \leq c_{2n}$.

Next, let d_i be a permutation of a_i such that $v = \sum_{i=1}^{n} f(d_{2i-1}, d_{2i})$ is maximized. Then by Theorem 6, d_i can be chosen such that $d_{2i-1} \leq d_{2i+1}$ and $d_{2i} \leq d_{2i+2}$ for $1 \leq i \leq n - 1$. Furthermore, by repeated use of Theorem 6 and Eq. (7) we can assume $d_{2i-1} \leq d_{2i}$ as well. Suppose $d_{2n-1} < d_{2(n-1)}$. Then $d_{2(n-1)-1} < d_{2(n-1)}$ and by Eq. (7) we can swap $d_{2(n-1)}$ and $d_{2(n-1)-1}$ without decreasing the value of v. Again by repeated application of Theorem 6 and Eq. (7) we can reorder d_{2i} for $1 \leq i \leq n$ such that they are nondecreasing and also reorder d_{2i-1} for $1 \leq i \leq n$ such that they are nondecreasing in addition to ensuring $d_{2i-1} \leq d_{2i}$ without decreasing v. It is easy to see that after this reordering $d_{2n-1} \geq d_{2(n-1)}$. Applying this procedure for $j = n - 1, ..., 3, 2$ sequentially shows that for each $2 \leq j \leq n$, $d_{2j-1} \geq d_{2(j-1)}$. This in addition with the fact that $d_{2i} \geq d_{2i-1}$ shows that $d_1 \leq d_2 \cdots \leq d_{2n}$. □

By choosing $f(x, y) = xy$ or $f(x, y) = -\log(x + y)$, we have the following result.

Corollary 4. *Let a_i be a set of $2n$ numbers and let b_i be the numbers a_i sorted such that $b_1 \leq b_2 \cdots \leq b_{2n}$. Then*

$$\sum_{i=1}^{n} b_i b_{2n-i+1} \leq \sum_{i=1}^{n} a_{2i-1} a_{2i} \leq \sum_{i=1}^{n} b_{2i-1} b_{2i}$$

. *If in addition* $a_i \geq 0$, *then*

$$\prod_{i=1}^{n} (b_{2i-1} + b_{2i}) \leq \prod_{i=1}^{n} (a_{2i-1} + a_{2i}) \leq \prod_{i=1}^{n} (b_i + b_{2n-i+1})$$

.

In Ref. [7] the following result was shown which generalizes the rightmost inequality in Eq. (3).

Theorem 8. *Consider a set of numbers* $\{a_{ij}\} \in I$, $i = 1, \cdots, k$, $j = 1, \cdots, n$. *For each* i, *let* $b_{i1}, b_{i2}, \cdots, b_{in}$ *be the numbers* $a_{i1}, a_{i2}, \cdots, a_{in}$ *sorted such that* $b_{i1} \leq b_{i2} \leq \cdots \leq b_{in}$. *Let* $f(x_1, \cdots, x_k)$ *be a real-value function defined on* I^k *such that Eq. (5) is satisfied for each pair of arguments* x_i *and* x_j. *Then*

$$\sum_{j=1}^{n} f(a_{1j}, a_{2j}, \cdots, a_{kj}) \leq \sum_{j=1}^{n} f(b_{1j}, b_{2j}, \cdots, b_{kj})$$

Similarly, we can generalize Eq. (8) to multiple sequences when the permutation is among all kn numbers $\{a_{ij}\}$.

Theorem 9. *Consider a sequence of* kn *numbers* a_i *in* I. *Let* b_i *be the numbers* a_i *sorted such that* $b_1 \leq b_2 \leq \cdots \leq b_{kn}$. *Let* $f(x_1, \cdots, x_k)$ *be a real valued function defined on* I^k *such that Eq. (7) and Eq. (5) are satisfied for each pair of arguments* x_i *and* x_j. *Then*

$$\sum_{j=1}^{n} f(a_{(j-1)k+1}, a_{(j-1)k+2}, \cdots, a_{jk}) \leq \sum_{j=1}^{n} f(b_{(j-1)k+1}, b_{(j-1)k+2}, \cdots, b_{jk})$$

Proof. The proof is similar to Theorem 7. Let d_i be a permutation of a_i such that

$$v = \sum_{j=1}^{n} f(d_{(j-1)k+1}, d_{(j-1)k+2}, \cdots, d_{jk})$$

is maximized. Then by Theorem 8, d_i can be chosen such that $d_{(j-1)k+i} \leq d_{jk+i}$ for $1 \leq i \leq k$ and $1 \leq j \leq n-1$. Furthermore, by Eq. (7) we can also assume that $d_{(j-1)k+i} \leq d_{(j-1)k+i+1}$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n$. Suppose $d_{k(n-1)+1} < d_{k(n-1)}$. By Eq. (7) we can swap $d_{k(n-2)+1}$ and $d_{k(n-1)}$ without decreasing the value of v. Again by repeated application of Eq. (7) and Theorem 6, we can reorder d_i such that $d_{(j-1)k+i} \leq d_{jk+i}$ for $1 \leq i \leq k$ and $1 \leq j \leq n-1$ without decreasing v while ensuring $d_{(j-1)k+i} \leq d_{(j-1)k+i+1}$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n$. If $d_{k(n-1)+1} < d_{k(n-1)}$ we repeat this process (which terminates after a finite number of times) until $d_{k(n-1)+1} \geq d_{k(n-1)}$. Applying this procedure for j from $n-1, \cdots, 3, 2$ sequentially shows that for each $2 \leq j \leq n$, $d_{(j-1)k+1} \geq d_{k(j-1)}$. This along with $d_{(j-1)k+i} \leq d_{(j-1)k+i+1}$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n$ shows that $d_1 \leq d_2 \cdots \leq d_{kn}$. □

We get the following result when we set $f(x_1, \cdots, x_k) = \prod_{i=1}^{k} x_i$ or $f(x_1, \cdots, x_k) = -\log\left(\sum_{i=1}^{k} x_i\right)$.

Corollary 5. *Let* a_i *be a set of* kn *numbers and let* b_i *be the numbers* a_i *reordered such that* $b_1 \leq b_2 \cdots \leq b_{kn}$. *Then*

$$n \sqrt[n]{\prod_{i=1}^{kn} a_i} \leq \sum_{j=1}^{n} \prod_{i=1}^{k} a_{(j-1)k+i} \leq \sum_{j=1}^{n} \prod_{i=1}^{k} b_{(j-1)k+i}$$

. *Suppose there exists* c_i, *a reordering of the numbers* a_i *such that* $\prod_{i=1}^{k} c_{(j-1)k+i} = \prod_{i=1}^{k} c_{(l-1)k+i}$ *for all* $1 \leq j, l \leq n$. *Then*

$$\sum_{j=1}^{n} \prod_{i=1}^{k} c_{(j-1)k+i} \leq \sum_{j=1}^{n} \prod_{i=1}^{k} a_{(j-1)k+i}$$

If in addition $a_i \geq 0$, then

$$\prod_{j=1}^{n}\sum_{i=1}^{k} b_{(j-1)k+i} \leq \prod_{j=1}^{n}\sum_{i=1}^{k} a_{(j-1)k+i} \leq \left(\frac{\sum_{i=1}^{kn} a_i}{n}\right)^n$$

. Suppose there exists c_i, a reordering of the numbers a_i such that $\sum_{i=1}^{k} c_{(j-1)k+i} = \sum_{i=1}^{k} c_{(l-1)k+i}$ for all $1 \leq j, l \leq n$, then

$$\prod_{j=1}^{n}\sum_{i=1}^{k} a_{(j-1)k+i} \leq \prod_{j=1}^{n}\sum_{i=1}^{k} c_{(j-1)k+i}$$

The bounds $n\sqrt[n]{\prod_{i=1}^{kn} a_i}$ and $\left(\frac{\sum_{i=1}^{kn} a_i}{n}\right)^n$ in Corollary 5 are due to the AM-GM inequality (Lemma 2).

5.1 The special case when a_i is an arithmetic progression

Definition 3. *For a permutation σ of $\{1, \cdots, kn\}$, define $v(n,k) = \sum_{i=1}^{n}\prod_{j=1}^{k} a_{\sigma}((i-1)k+j)$. Let $v_{\min}(n,k)$ and $v_{\max}(n,k)$ be the minimal and maximal values respectively of $v(n,k)$ among all permutations σ of $\{1, \cdots, kn\}$.*

Definition 4. *For a permutation σ of $\{1, \cdots, kn\}$, define $w(n,k) = \prod_{i=1}^{n}\sum_{j=1}^{k} a_{\sigma((i-1)k+j)}$. Let $w_{\min}(n,k)$ and $w_{\max}(n,k)$ be the minimal and maximal values respectively of $w(n,k)$ among all permutations σ of $\{1, \cdots, kn\}$.*

Suppose $a_i \geq 0$ is an arithmetic progression, with $a_i = a_1 + (i-1)d$, for $i = 1, \cdots, kn$, $d \geq 0$. Corollary 5 implies that

Theorem 10.

- $v_{\min}(n,k) \geq nd^k \sqrt[n]{\frac{\Gamma\left(\frac{a_1}{d}+nk\right)}{\Gamma\left(\frac{a_1}{d}\right)}}$.
- $v_{\max}(n,k) = \sum_{i=1}^{n}\prod_{j=1}^{k} a_{(i-1)k+j} = d^k \sum_{i=1}^{n} \frac{\Gamma\left(\frac{a_1}{d}+ik\right)}{\Gamma\left(\frac{a_1}{d}+(i-1)k\right)}$.
- $w_{\max}(n,k) \leq \left(\frac{k(a_1+a_{kn})}{2}\right)^n$.
- $$w_{\min}(n,k) = \prod_{i=1}^{n}\sum_{j=1}^{k} a_{(i-1)k+j} = k^n \prod_{i=1}^{n}\left(a_1 + \left(ik - \frac{k+1}{2}\right)d\right) = k^{2n}d^n \frac{\Gamma\left(n + \frac{2a_1+(k-1)d}{2kd}\right)}{\Gamma\left(\frac{2a_1+(k-1)d}{2kd}\right)}$$

  .

Theorem 11. *If $k = 2t + nu$ for nonnegative integers t and u, then $w_{\max}(n,k) = \left(\frac{k(a_1+a_{kn})}{2}\right)^n$.*

Proof. The proof is similar to the proof of Theorem 2. Instead of using cyclic permutations r_i of $\{1, \cdots, n\}$ and the permutation $(n, n-1, \cdots, 1)$, we apply them to $((j-1)n+1, (j-1)n+2, \cdots, jn)$ and this is equivalent to adding $(j-1)n$ to each term of the j-th permutation. For instance, for $n = k = 3$, $w(n,k)$ is maximized by $(a_1, a_5, a_9, a_2, a_6, a_7, a_3, a_4, a_8)$. □

This implies that if n is odd and $k \geq n-1$ or if k is even, then $w_{\max}(n,k) = \left(\frac{k(a_1+a_{kn})}{2}\right)^n$.

Theorem 12. *If n is even and k is odd such that $k \geq n-1$, then*

$$w_{\max}(n,k) = \left(ka_1 + \left(\frac{k(kn-1)-1}{2}\right)d\right)^{n/2}\left(ka_1 + \left(\frac{k(kn-1)+1}{2}\right)d\right)^{n/2}$$

Proof. The proof is similar to the proof of Theorem 3, except that we add $(j-1)n$ to each term of the j-th permutation in the k-set of permutations of $\{1, \cdots, n\}$. This adds an additional $\sum_{j=1}^{k}(j-1)n = (k-1)kn/2$ to each w_i and thus $w_i = \frac{k(kn+1)-1}{2}$ for $i = 1, \cdots, n/2$, and $w_i = \frac{k(kn+1)+1}{2}$ for $i = n/2+1, \cdots, n$. Thus $w_{\max}(n,k) = \prod_{i=1}^{n} k(a_1-d) + w_i d = \left(k(a_1-d) + \frac{d(k(kn+1)-1)}{2}\right)^{n/2} \left(k(a_1-d) + \frac{d(k(kn+1)+1)}{2}\right)^{n/2}$ and the conclusion follows. □

5.2 The special case $a_i = i$

Definition 5. *For a permutation σ of $\{1, \cdots, kn\}$, define $v(n,k) = \sum_{i=1}^{n} \prod_{j=1}^{k} \sigma((i-1)k+j)$. Let $v_{\min}(n,k)$ and $v_{\max}(n,k)$ be the minimal and maximal values respectively of $v(n,k)$ among all permutations σ of $\{1, \cdots, kn\}$.*

Definition 6. *For a permutation σ of $\{1, \cdots, kn\}$, define $w(n,k) = \prod_{i=1}^{n} \sum_{j=1}^{k} \sigma((i-1)k+j)$. Let $w_{\min}(n,k)$ and $w_{\max}(n,k)$ be the minimal and maximal values respectively of $w(n,k)$ among all permutations σ of $\{1, \cdots, kn\}$.*

We have $v_{\min}(n,1) = w_{\max}(1,n) = n(n+1)/2$, $v_{\min}(1,k) = w_{\max}(k,1) = k!$, $v_{\min}(n,k) \geq n\sqrt[n]{(kn)!}$. Furthermore, $w_{\max}(n,k) \leq \left(\frac{k(nk+1)}{2}\right)^n$ with equality if $k = 2t + nu$ for nonnegative integers t and u.

Theorem 13. *$v_{\min}(n,2) = n(n+1)(2n+1)/3$, $w_{\max}(n,2) = (2n+1)^n$.*

Proof. By Corollary 5, $v_{\min}(n,2) = \sum_{i=1}^{n} i(2n-i+1) = (2n+1)\sum_{i}^{n} i - \sum_{i}^{n} i^2 = n(n+1)(2n+1)/2 - n(n+1)(2n+1)/6 = n(n+1)(2n+1)/3$. Similarly, $w_{\max}(n,2) = \prod_{i=1}^{n}(i + (2n-i+1)) = (2n+1)^n$. □

Theorem 12 implies that

Corollary 6. *If n is even and k is odd such that $k \geq n-1$, then $w_{\max}(n,k) = \left(\frac{k^2(kn+1)^2-1}{4}\right)^{n/2}$.*

The value of $v_{\min}(n,3)$ can be found in OEIS [4] sequence A072368 (`https://oeis.org/A072368`). The values of $v_{\min}(n,k)$ can be found in sequence A331889 (`https://oeis.org/A331889`). The values of $w_{\max}(n,k)$ can be found in sequence A333420 (`https://oeis.org/A333420`). The values of $w_{\min}(n,k)$ can be found in sequence A333445 (`https://oeis.org/A333445`). The values of $v_{\max}(n,k)$ can be found in sequence A333446 (`https://oeis.org/A333446`).

6 Conclusions

We consider several variants of the rearrangement inequality for which we can generalize to multiple sequences and find both the set of permutations that maximizes or minimizes the sum of products or product of sums of terms and where the permutation can be chosen across sequences.

References

[1] G. H. Hardy, J. E. Littlewood, G. Pólya, D. Littlewood, G. Pólya, *et al.*, *Inequalities.* Cambridge university press, 1952.

[2] A. Oppenheim, "Inequalities connected with definite hermitian forms, II," *American Mathematical Monthly*, vol. 61, pp. 463–466, 1954.

[3] H. Ruderman, "Two new inequalities," *American Mathematical Monthly*, vol. 59, no. 1, pp. 29–32, 1952.

[4] The OEIS Foundation Inc., "The on-line encyclopedia of integer sequences," 1996-present. Founded in 1964 by N. J. A. Sloane.

[5] C. W. Wu, “Permutations r_j such that $\sum_{i=1}^{n} \prod_{j=1}^{k} r_j(i)$ is maximized or minimized,” *arXiv e-prints*, p. arXiv:1508.02934, Aug. 2015.

[6] P. W. Day, “Rearrangement inequalities,” *Canadian Journal of Mathematics*, vol. 24, no. 5, pp. 930–943, 1972.

[7] A. Vince, “A rearrangement inequality and the permutahedron,” *The American mathematical monthly*, vol. 97, no. 4, pp. 319–323, 1990.