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Groups acting on trees with Tits' independence property (P)

Colin D. Reid[1] and Simon M. Smith[2],
with an appendix by Stephan Tornier[1]

[1]The University of Newcastle, School of Mathematical and Physical Sciences, Callaghan, NSW 2308, Australia
[2]Charlotte Scott Research Centre for Algebra, University of Lincoln, Lincoln, U.K.

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Abstract

Let T be a tree (not necessarily locally finite). We give a classification up to conjugacy of the closed subgroups G of $\mathrm{Aut}(T)$ that have Tits' independence property (P) in terms of data called a local action diagram, which is the quotient graph $G\backslash T$ decorated with the 'local actions' of G. We then show how to determine whether the group has certain properties, such as geometric density, compact generation and simplicity, directly from the local action diagram.

Contents

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1 Introduction

Actions on trees have long played an important role in group theory. The most well-established perspective is that of Bass–Serre theory and the theory of ends of groups, in which actions on trees are interpreted as a generalization of the free product and HNN constructions. In addition, a complementary approach has emerged based on local actions, that is, the action of a vertex stabilizer on the neighbouring vertices. In particular, two articles concerning groups acting on trees have been very important for the recent development of the theory of totally disconnected, locally compact (t.d.l.c.) groups: a 1970 article [12] of J. Tits, which introduced property (P) as a condition to produce new examples of simple groups, showing for instance that the automorphism group of a regular tree is virtually simple; and a 2000 article [2] of M. Burger and Sh. Mozes, which rediscovered the approach of Tits and used it to produce an interesting class of (virtually) simple t.d.l.c. groups acting on trees with property (P), which moreover arise naturally in the study of lattices in products of trees. Since then, the majority of new constructions of compactly generated simple t.d.l.c. groups have used the ideas of [12] and [2]. More recently in [11], the second named author generalized the Burger–Mozes construction to obtain a kind of product of permutation groups, often resulting in a permutation group that is both primitive and simple; this was used to show that there are $2^{\aleph_0}$ isomorphism types of nondiscrete compactly generated simple t.d.l.c. groups. Other authors have studied generalizations of property (P), for instance the properties (P_k) introduced in [1], where (P_1) is just (P). One can define the (P_k)-closure of any action on a tree for all $k \geqslant 1$, to obtain a series of approximations to the original action determined by how the original action behaves on balls of radius k, which converge to the closure of the action; thus groups with (P_k) are a general tool to understand all actions on trees.

The goal of the present article is to develop a general method for describing and classifying actions of groups on trees with property (P), or equivalently, to describe all possible (P)-closures of actions on trees, to serve as a 'local action' complement to classical Bass–Serre theory. The basic idea is essentially the same as in [2], namely to describe the group in terms of its local actions; but in contrast to [2] and [11], we make no assumptions about the homogeneity of the local actions or the structure of the orbits of the group on vertices or arcs of the tree. As a result, we need a slightly more complicated way to describe the local actions.

Definition 1.1. A **local action diagram** $\Delta = (\Gamma, (X_a), (G(v)))$ consists of the following information:

- A connected graph Γ. (We define graphs in the sense of Serre, except that a loop may or may not be equal to its own reverse: see Section 2.1.)
- For each arc a of Γ, a nonempty set X_a (called the **colour set** of a).
- For each vertex v of Γ, a group $G(v)$ (called the **local action** at v) with the following properties: write X_v to denote the disjoint union $\bigsqcup_{a \in o^{-1}(v)} X_a$, then the group $G(v)$ is a closed subgroup of $\mathrm{Sym}(X_v)$ and the sets X_a are the orbits of $G(v)$ on X_v.

There is a natural notion of isomorphism of local action diagrams. For actions on trees the natural way to define isomorphism is conjugacy, where we say (T, G) and (T', G') are conjugate if there is a graph isomorphism $\theta : T \to T'$ that intertwines the two actions. The central theorem of this article, which we prove in Section 3, is as follows.

Theorem 1.2. *There is a natural one-to-one correspondence between conjugacy classes of* (P)*-closed actions on trees and isomorphism classes of local action diagrams.*

There are no surprises in how a local action diagram is obtained from a ((P)-closed) action of a group G on a tree T. The graph Γ is the quotient graph $G\backslash T$; $G(v)$ represents the closure of the action of a vertex stabilizer G_{v*} (where v^* lies in the preimage of v) on the arcs $o^{-1}(v^*)$ of T originating at v^*; those arcs are partitioned into G_{v*}-orbits, represented by the colour sets, with the result that there is a natural one-to-one correspondence between $o^{-1}(v)$ and G_{v^*}-orbits on $o^{-1}(v^*)$.

The significance of the correspondence, then, is in the following two observations.

(1) The local action diagram exactly describes the (P)-closure of the original action up to conjugacy. In particular, any 'large-scale' information about the original group can be recovered from the quotient graph Γ together with the local actions.

(2) All possible local action diagrams arise in this manner. In particular, Γ can be any connected graph in our sense, and apart from how Γ limits the number of orbits of the local actions, there are no compatibility conditions for which local actions can be combined.

For comparison, the Burger–Mozes framework corresponds to the case when Γ is a single vertex with a set of loops, each of which is its own reverse; the framework of [11] corresponds to the case that Γ has two vertices and no loops.

As an example of what this means in practice, consider the class $\mathcal{C}(n, d)$ of (P)-closed actions on trees (T, G) such that G has at most n orbits on vertices and no vertex has degree greater than d. Theorem 1.2 immediately shows that for given natural numbers n and d, there are only finitely many conjugacy classes of actions in $\mathcal{C}(n, d)$; but because of all the possible graphs and decorations, the number of conjugacy classes will grow quite rapidly with n and d. Even when $n = 1$, there are more than just the Burger–Mozes groups: see Section 7.1.

The next part of the article is concerned with characterizing various natural properties of interest for (topological) groups acting on trees in terms of the local action diagram.

We recall that Tits' main theorem on property (P), ensuring that the subgroup G^+ generated by arc stabilizers is trivial or simple, only applies to (P)-closed actions that are geometrically dense, meaning that there is no proper invariant subtree or fixed end. Fixed ends and invariant subtrees can be recognized in the local action diagram, as follows.

Definition 1.3. Let $\Delta = (\Gamma, (X_a), (G(v)))$ be a local action diagram. A **strongly confluent partial orientation** (s.c.p.o.) of Δ is a subset O of $A\Gamma$ such that:

(i) If $a \in O$, then $\overline{a} \notin O$ and $|X_a| = 1$;

(ii) For all $v \in V\Gamma$, if O contains an arc a originating at v, then O contains all arcs other than $\overline{a}$ that terminate at v.

Theorem 1.4 (See Section 5.1)**.** *Given a group G acting on a tree T, then the invariant subtrees and fixed ends of the action naturally correspond to s.c.p.o.s of the local action diagram, with the empty s.c.p.o. corresponding to T itself.*

In particular, the action (T, G) is geometrically dense if its local action diagram is **irreducible**, meaning that the only s.c.p.o. is the empty one. Since s.c.p.o.s are quite special, it is easy to write down sufficient conditions for a local action diagram to give rise to an action on the tree that is geometrically dense. Note also that the details of the local actions are not important here, only the structure of the quotient graph Γ and the sizes of the colour sets. In the case that (T, G) is not geometrically dense, we can also describe which of the degenerate cases of actions on trees (if any) it falls into using the local action diagram (see Section 5.2).

Given a group action on a tree (T, G), a **quotient tree** is a surjective graph homomorphism $\theta : T \to T'$ together with a compatible action of G on T'. A natural example of such a quotient tree is the quotient map $\pi_+ : T \to (G^+\backslash T)$ of the action of G^+ on T. We say that θ is **locally surjective** if for each $v \in VT$ and every arc a originating at $\theta(v)$, then a is the image of some arc originating at v. We show a version of Tits' simplicity theorem for locally surjective quotient trees.

Theorem 1.5 (See Proposition 5.15)**.** *Let (T, G) be a geometrically dense* (P)*-closed action on a tree, and let $\theta : T \to T'$ be a quotient tree. Suppose that θ is locally surjective and not injective. Then θ factors through π_+.*

In turn, π_+ is itself locally surjective, although it can be injective on T or an invariant subtree of T, in the case that G^+ acts freely on the arcs of that subtree. The quotient action $(G^+\backslash T, G/G^+)$ is free on arcs (hence is (P)-closed) and has a natural description, either in terms of its local action diagram, or as a fundamental group of a graph of groups in the sense of Bass–Serre theory (see Theorem 5.16). This gives rise to the following simplicity criterion for groups with faithful (P)-closed actions. It is almost an exact characterization of simplicity in this context, except that we need to exclude a few degenerate kinds of action that a simple group could have.

Definition 1.6. Say that $G \leqslant \mathrm{Aut}(T)$ is **strongly closed** if for every G-invariant subtree T' of T, the action of G on T' is closed.

In particular, every closed geometrically dense action is strongly closed.

Theorem 1.7 (See Section 5.3)**.** *Let (T, G) be a faithful* (P)*-closed and strongly closed action on a tree. Then the following are equivalent:*

(i) G is a simple group, G contains a translation, and there is no finite set of vertices whose pointwise stabilizer is trivial.

(ii) There is an invariant subtree T' (possibly equal to T) which is infinite and on which G acts faithfully. Moreover, letting $\Delta = (\Gamma, (X_a), (G(v)))$ be the local action diagram of (T', G), then Δ is irreducible; Γ is a tree; and each of the groups $G(v)$ is closed and generated by point stabilizers, with $G(v) \neq \{1\}$ for some $v \in V\Gamma$.

Note that the condition that there is no finite set of vertices whose pointwise stabilizer is trivial is equivalent to saying that G is nondiscrete in the permutation topology on T.

Next, we describe some topological properties of (P)-closed subgroups of $\mathrm{Aut}(T)$ with the permutation topology; these are already well-understood in the locally finite case, but in the present context we are making no assumptions about the degree of T. We highlight the following special case.

Theorem 1.8 (See Section 6). *Let $\Delta = (\Gamma, (X_a), (G(v)))$ be a local action diagram. Then the following are equivalent:*

(i) $U(\Delta)$ is compactly generated, locally compact and has geometrically dense action on its associated tree;

(ii) Δ is irreducible; Γ is finite; and each of the groups $\overline{G(v)}$ is compactly generated and subdegree-finite.

Moreover, if (i) and (ii) hold, then $U(\Delta)$ is Polish, acting on a countable tree.

Combining Theorems 1.7 and 1.8 yields the following, where $\mathscr{S}$ denotes the class of nondiscrete compactly generated, topologically simple t.d.l.c. groups. This generalizes the criterion used in [11] to construct $2^{\aleph_0}$ nonisomorphic groups in $\mathscr{S}$.

Corollary 1.9. *Let (T, G) be a faithful* (P)*-closed and strongly closed action on a tree. Then the following are equivalent:*

(i) We have $G \in \mathscr{S}$ and the action does not fix any vertex of T.

(ii) There is a unique smallest invariant subtree T' (possibly equal to T) on which G acts faithfully. Moreover, letting $\Delta = (\Gamma, (X_a), (G(v)))$ be the local action diagram of (T', G), then Δ is irreducible; Γ is a finite tree; and each of the groups $G(v)$ is closed, compactly generated, subdegree-finite and generated by point stabilizers, with $G(v) \neq \{1\}$ for some $v \in V\Gamma$.

The final section is devoted to examples. We show how the local action diagram can be used in for classifying group actions on trees by classifying the 70 (P)-closed vertex-transitive actions on trees of degree $0 \leqslant d \leqslant 5$, and give a GAP ([5]) implementation due to S. Tornier that can be used to classify vertex-transitive actions of larger degrees.

Finally, we give an example of how Corollary 1.9 can be used to produce more groups in $\mathscr{S}$, which demonstrates that within $\mathscr{S}$, the groups $U(\Delta)$ have a certain universality property.

Theorem 1.10 (See Section 7.2). *Let n be a positive integer and let $G_1, \dots, G_n$ be nontrivial compactly generated t.d.l.c. groups, such that for each G_i there is a compact open subgroup U_i such that $G_i = \langle gU_ig^{-1} \mid g \in G \rangle$ and $\bigcap_{g \in G} gU_ig^{-1} = \{1\}$. For example, we can take $G_i \in \mathscr{S}$ and U_i to be any compact open subgroup. Then there exists $U(\Delta) \in \mathscr{S}$ acting continuously on a countable tree T, vertex stabilizers $O_1, \dots, O_n$ of $U(\Delta)$ and compact normal subgroups K_i of O_i, such that $O_i \cong K_i \rtimes G_i$ for $1 \leqslant i \leqslant n$.*

2 Preliminaries

2.1 Graphs

A **graph** $\Gamma = (V, A, o, r)$ consists of a vertex set $V = V\Gamma$, a set $A = A\Gamma$ of arcs, a map $o : A \to V$ assigning to each arc an **origin** (or **initial**) **vertex**, and a bijection $r : A \to A$, denoted $a \mapsto \overline{a}$ and called **edge reversal** (or sometimes **edge inversion**), such that $r^2 = \mathrm{id}$. The **terminal vertex** of an edge is $t(a) := o(\overline{a})$. A **loop** is an arc a such that $o(a) = t(a)$. If a is a loop, it is important that we allow both $\overline{a} = a$ and $\overline{a} \neq a$ as possibilities. We call the pair $\{a, \overline{a}\}$ an **edge** between the vertices $o(a)$ and $t(a)$. Two vertices are **adjacent** if there is an edge between them. A **path** (of length n) in Γ is a sequence of vertices $v_1, \ldots, v_{n+1}$ and edges $\{a_1, \overline{a_1}\}, \ldots, \{a_n, \overline{a_n}\}$ such that $\{a_i, \overline{a_i}\}$ is an edge in Γ between v_i and v_{i+1} for all $1 \leqslant i \leqslant n$. We say this is a path **between** v_1 and v_{n+1}. The path is **simple** if all vertices $v_1, \ldots, v_{n+1}$ are unique. If $v_1 = v_{n+1}$ and all vertices $v_1, \ldots, v_n$ are unique then the path is called a **cycle** of length n. If there is a path between two vertices v, w, then there is a shortest path, and the length of this shortest path is the **distance** between v and w; if there is no path then the distance is taken to be infinite.

The graph Γ is **simple** if the map $A \to V \times V$ by $a \mapsto (o(a), t(a))$ is injective and no arc is a loop. In this case, the arc a is sometimes identified with the pair $(o(a), t(a))$. The graph is **connected** if there is a path between any two distinct vertices. In a simple graph Γ a **ray** is a one-way infinite simple path, and a **double-ray** is a two-way infinite simple path. The **ends** of Γ are equivalence classes of rays, in which two rays R_1, R_2 lie in the same end if and only if there is another ray R in Γ that contains infinitely many vertices of R_1 and infinitely many vertices of R_2. A **tree** is a simple, connected graph that contains no cycles.

The **degree** of a vertex $v \in V$ is $\deg(v) := |o^{-1}(v)|$, and the graph is **locally finite** if every vertex has finite degree. The **degree** of the graph is defined to be

$$\deg(\Gamma) := \sup_{v \in V\Gamma} \deg(v).$$

An **automorphism** of a graph Γ is a pair of permutations $\alpha_V : V \to V$ and $\alpha_A : A \to A$ that respect origin vertices and edge reversal: $\alpha_V(o(a)) = o(\alpha_A(a))$ and $\overline{\alpha_A(a)} = \alpha_A(\overline{a})$. The automorphisms of Γ form a group, denoted $\mathrm{Aut}(\Gamma)$. When Γ is a simple graph, the automorphisms of Γ are precisely the permutations of V that respect the edge relation in $V \times V$. In this case we identify $\mathrm{Aut}(\Gamma)$ with the corresponding subgroup of $\mathrm{Sym}(V)$.

If T is a tree, we say that $g \in \mathrm{Aut}(T)$ induces a non-trivial **translation** on a double-ray R of T if and only if $gR = R$ and no vertex in R is fixed by g; we then call R the **axis** of g, and say that g is **hyperbolic**. A group G acting on a tree T is said to act **without inversion** if there is no pair $a \in AT$

and $g \in \mathrm{Aut}(T)$ such that $ga = \overline{a}$. We recall that every element $g \in \mathrm{Aut}(T)$ satisfies precisely one of the following: it fixes a vertex or transposes two adjacent vertices (in which case g is called **elliptic**) or it is a translation, with a unique axis.

For G a group acting on a graph Γ and a vertex, arc or edge e of G, the orbit of e under G is denoted Ge. The action of G gives a **quotient graph** $G\backslash\Gamma$ as follows: the vertex set V_G is the set of G-orbits on V and the edge set A_G is the set of G-orbits on A. The origin map $\tilde{o} : A_G \to A_G$ is defined by $\tilde{o}(Ga) := Go(a)$; this is well-defined since graph automorphisms send origin vertices to origin vertices. The reversal $\tilde{r} : A_G \to A_G$ is given by $Ga \mapsto G\overline{a}$; this map is also well-defined. We will abuse notation and write o and r for $\tilde{o}$ and $\tilde{r}$. We denote the quotient map of the action (Γ, G) by $\pi_{(\Gamma,G)}$.

A subset of the vertices of a connected graph is **bounded**, respectively **unbounded**, if it has finite, respectively infinite diameter in the graph metric. Given a vertex v of a graph Γ, write $B_n(v)$ (the **ball of radius** n) for the induced subgraph formed by all vertices w such that $d_\Gamma(v, w) \leqslant n$, and $S_n(v)$ (the **sphere of radius** n) for the set of vertices w such that $d_\Gamma(v, w) = n$.

A graph homomorphism $\theta : \Gamma \to \Gamma'$ is **locally surjective** if for each $v \in V\Gamma$, we have $o_{\Gamma'}^{-1}(\theta(v)) = \theta(o_\Gamma^{-1}(v))$.

Lemma 2.1. *Let Γ be a graph and let $G \leqslant \mathrm{Aut}(\Gamma)$. Then $\pi_{(\Gamma,G)}$ is locally surjective.*

Proof. Let $\pi = \pi_{(\Gamma,G)}$ and let $\Gamma' = \pi(\Gamma)$. Let $v \in V\Gamma$ and $a' \in o_{\Gamma'}^{-1}(\pi(v))$. Since π is surjective, there exists $a \in A\Gamma$ such that $\pi(a) = a'$, and hence

$$\pi(o(a)) = o(\pi(a)) = o(a') = \pi(v);$$

there is then $g \in G$ such that $go(a) = v$, and hence $ga \in o^{-1}(v)$.

Thus $\pi(ga) = \pi(a) = a'$; in particular, $a' \in \pi(o_\Gamma^{-1}(v))$. Given the choice of v and a, we conclude that π is locally surjective. ☐

2.2 The (P)-closure and property (P)

Given a set X, we equip $\mathrm{Sym}(X)$ with the permutation topology, that is, the coarsest group topology such that the stabilizer of every $x \in X$ is open. Given a tree T, we give $\mathrm{Aut}(T)$ the subspace topology, regarding $\mathrm{Aut}(T)$ as a subgroup of $\mathrm{Sym}(VT)$. Observe that in fact $\mathrm{Aut}(T)$ corresponds to a closed subgroup of $\mathrm{Sym}(VT)$; if VT is countable, this ensures that $\mathrm{Aut}(T)$ is Polish (that is, separable and completely metrizable) and also totally disconnected, but $\mathrm{Aut}(T)$ is not necessarily locally compact. Assuming T has no leaves, one could equivalently define the topology of $\mathrm{Aut}(T)$ with respect to the permutation topology on arcs or undirected edges, or the permutation topology on one part of the natural bipartition of the vertices of T: this can be seen by noting that two undirected edges suffice to specify a vertex, two vertices to specify an arc, and two vertices in one part of the bipartition to specify a vertex in the other part.

Given $G \leqslant \mathrm{Aut}(T)$ and $k \geqslant 1$, the (P_k)**-closure**[1] of G, denoted by $G^{(\mathrm{P}_k)}$, is the set of automorphisms $g \in \mathrm{Aut}(T)$ such that for all $v \in VT$, and every finite set of vertices X all of which are at distance at most k from v, there exists $g_X \in G$ such that $gw = g_X w$ for every vertex $w \in X$. We say G is (P_k)**-closed** if $G = G^{(\mathrm{P}_k)}$.

We will use some basic properties of the (P_k)-closure. We include proofs here because the approach of [1] implicitly assumes that trees are locally finite.

For the rest of this article, we define $G_{(X)} := \{g \in G \mid \forall x \in X : gx = x\}$, where X is a set of vertices of T.

Proposition 2.2 (See [1] Proposition 3.4)**.** *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$ and let $k \in \mathbb{N}$.*

(i) $G^{(\mathrm{P}_k)}$ is a closed subgroup of $\mathrm{Aut}(T)$.

(ii) $G^{(\mathrm{P}_l)} = (G^{(\mathrm{P}_k)})^{(\mathrm{P}_l)}$ whenever $l \leqslant k$. In particular, $(G^{(\mathrm{P}_k)})^{(\mathrm{P}_k)} = G^{(\mathrm{P}_k)}$, so $G^{(\mathrm{P}_k)}$ is (P_k)-closed.

Proof. (i) Write $A := \mathrm{Aut}(T)$. Let $g, h \in G^{(\mathrm{P}_k)}$, let $v \in VT$ and let X be a finite set of vertices all of which are at distance at most k from v. Then there exists $h_X \in G$ such that $h_X w = hw$ for all $w \in X$. In turn, $hX := \{hw \mid w \in X\}$ is a finite set of vertices, all of which are at distance at most k from hv, so there exists $g_{hX} \in G$ such that $g_{hX} w = gw$ for all $w \in hX$. Thus $g_{hX} h_X$ is an element of G such that $g_{hX} h_X w = ghw$ for all $w \in X$. We conclude that $gh \in G^{(\mathrm{P}_k)}$. Similarly, it is clear that $G^{(\mathrm{P}_k)}$ is closed under inverses. Thus $G^{(\mathrm{P}_k)}$ is a subgroup of A.

Let $\mathcal{X}_k$ be the set of all finite sets X of vertices in T, such that there is a vertex v at distance at most k from every vertex in X. Then $A_{(X)}$ is an open subgroup of A for every $X \in \mathcal{X}_k$. Observe that given $g \in A \smallsetminus G^{(\mathrm{P}_k)}$, then there exists $X_g \in \mathcal{X}_k$ such that no element of G agrees with g on X_g, and hence no element of $G^{(\mathrm{P}_k)}$ agrees with g on X_g, that is, $G^{(\mathrm{P}_k)} \cap gA_{(X_g)} = \varnothing$. We can therefore express the complement of $G^{(\mathrm{P}_k)}$ as the following union of open sets:

$$A \smallsetminus G^{(\mathrm{P}_k)} = \bigcup_{g \in A \smallsetminus G^{(\mathrm{P}_k)}} gA_{(X_g)}.$$

Hence $G^{(\mathrm{P}_k)}$ is closed in $\mathrm{Aut}(T)$.

[1] The notion of the (P_k)-closure of a group acting on a tree was introduced in [1] where it was simply called the k-closure; however the term k-closure has a well-established meaning in permutation group theory due to Wielandt, so here we use the term (P_k)-closure.

(ii) Since $G \leqslant G^{(\mathrm{P}_k)}$ then $G^{(\mathrm{P}_l)} \leqslant (G^{(\mathrm{P}_k)})^{(\mathrm{P}_l)}$. Let $g \in (G^{(\mathrm{P}_k)})^{(\mathrm{P}_l)}$ and let X be a finite set of vertices of T, all of which are at distance at most l from some vertex v. Then there exists $g_X \in G^{(\mathrm{P}_k)}$ such that $g_X w = gw$ for all $w \in X$. But then all the vertices in X are at distance at most k from v, so there exists $g'_X \in G$ such that $g'_X w = g_X w = gw$ for all $w \in X$. Hence $G^{(\mathrm{P}_l)} = (G^{(\mathrm{P}_k)})^{(\mathrm{P}_l)}$. The remaining conclusions are clear. □

It is useful to note that the property of being (P_k)-closed is inherited by fixators of vertices and is stable under taking intersections of subgroups.

Lemma 2.3. *Let T be a tree, let k be a positive integer and let $\mathcal{G}$ be a family of (P_k)-closed subgroups of $\mathrm{Aut}(T)$. Then $H = \bigcap_{G \in \mathcal{G}} G$ is (P_k)-closed.*

Proof. Let $h \in H^{(\mathrm{P}_k)}$ and let $G \in \mathcal{G}$. Then for each $v \in VT$ and each finite set X of vertices in $B_k(v)$, there is some $g_X \in H$ such that $g_X w = hw$ for all $w \in X$. In particular, $g_X \in G$. Since G is (P_k)-closed, it follows that $h \in G$; since $G \in \mathcal{G}$ was arbitrary, in fact $h \in H$. Thus $H = H^{(\mathrm{P}_k)}$, so H is (P_k)-closed. □

Lemma 2.4. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$, let $k \geqslant 1$ and let X be a set of vertices of T. If G is (P_k)-closed, then so is $G_{(X)}$.*

Proof. Suppose G is (P_k)-closed, and let $H = (G_{(X)})^{(\mathrm{P}_k)}$. Given $v \in X$ and $g \in H$, we see from the definition of the (P_k)-closure that g must fix v. So in fact H is a subgroup of $(G^{(\mathrm{P}_k)})_{(X)}$, which is just $G_{(X)}$. Hence $H = G_{(X)}$. □

We also recall Tits' property (P), introduced in [12].

Definition 2.5. Let T be a tree and let $\theta : G \to \mathrm{Aut}(T)$ be a group homomorphism. Given a non-empty (finite or infinite) path L in T, let $\pi_L : VT \to VL$ be the closest point projection of the vertices of T onto L; observe that for each $x \in L$, the set $\pi_L^{-1}(x)$ is a non-empty subtree of T. Write $\theta(G)_{(L)}$ for the pointwise stabilizer of L (so $\theta(G)_{(L)}$ preserves setwise each of the fibres $\pi_L^{-1}(x)$ of π_L). Then for each vertex $x \in L$, there is a natural homomorphism $\phi_x : \theta(G)_{(L)} \to \mathrm{Sym}(\pi_L^{-1}(x))$ induced by the action of $\theta(G)_{(L)}$ on $\pi_L^{-1}(x)$. We can combine the homomorphisms ϕ_x in the obvious way to obtain a homomorphism

$$\phi_L : \theta(G)_{(L)} \to \prod_{x \in VL} \phi_x(\theta(G)_{(L)}).$$

In general, ϕ_L is injective but not necessarily surjective. We say G (or more precisely, the action of G on T) has **property** (P) (with respect to a collection $\mathcal{L}$ of paths) if $\theta(G)$ is closed in $\mathrm{Aut}(T)$ and ϕ_L is surjective for every possible choice of L (such that $L \in \mathcal{L}$).

A major motivation of [1] was to generalize Tits' property (P), and indeed property (P) has a natural interpretation in terms of the (P_1)-closure.

Theorem 2.6 (See [1] Theorem 5.4 and Corollary 6.4)**.** *Let T be a tree and let G be a closed subgroup of* $\mathrm{Aut}(T)$*. Then $G = G^{(\mathrm{P}_1)}$ if and only if G satisfies Tits' property* (P)*. Furthermore, if G has property* (P) *with respect to the edges of T, then $G = G^{(\mathrm{P}_1)}$, so G has property* (P) *with respect to all paths.*

Proof. Let L be a non-empty path in T, and let $g \in \mathrm{Aut}(T)_{(L)}$ such that

$$\phi_L(g) \in \prod_{v \in VL} \phi_v(G_{(L)});$$

say $\phi_L(g) = (s_v)_{v \in VL}$. We now claim that $g \in G^{(\mathrm{P}_1)}$ (indeed, $g \in (G_{(L)})^{(\mathrm{P}_1)}$). Let X be a finite set of vertices, all adjacent to some vertex w of T. We will show that there exists $g_X \in G_{(L)}$ such that g agrees with g_X on X. We may assume that $X \cap VL = \varnothing$, since the vertices of L all are all fixed by both g and $G_{(L)}$. Let $x = \pi_L(w)$. We observe that since X is disjoint from VL and the set $X \cup \{w\}$ spans a subtree, any path from X to L must pass through x, in other words $X \subseteq \pi_L^{-1}(x)$. There is then $g_X \in G_{(L)}$ such that $\phi_v(g_X) = s_x$, so that g agrees with g_X on $\pi_L^{-1}(x)$ and in particular on X. Given the freedom of choice of X, we conclude that $g \in G^{(\mathrm{P}_1)}$ as claimed. Thus if $G = G^{(\mathrm{P}_1)}$, then G has property (P).

Conversely, suppose that G is closed and satisfies property (P) with respect to the edges of T. Suppose that $G \neq G^{(\mathrm{P}_1)}$ and let $g \in G^{(\mathrm{P}_1)} \setminus G$. Since G is closed, the set $G^{(\mathrm{P}_1)} \setminus G$ is a neighbourhood of g in $G^{(\mathrm{P}_1)}$, so there is a finite set X of vertices such that $g(G^{(\mathrm{P}_1)})_{(X)} \cap G = \varnothing$. Let S be the smallest subtree of T containing X; note that $\mathrm{Aut}(T)_{(X)} = \mathrm{Aut}(T)_{(S)}$, since every vertex of S lies on the unique path between a pair of vertices in X. Let us suppose that X has been chosen so that $|S|$ is minimized.

By the definition of $G^{(\mathrm{P}_1)}$, we see that S is not a star, so for every $x \in S$, there is a vertex in S at distance 2 from x. Hence there exist adjacent vertices x and y of S such that neither x nor y is a leaf of S. Let L be the path consisting of the single arc (x, y). By the minimality of $|S|$, there is some $h \in G$ such that $gx = hx$ and $gy = hy$, so that $h^{-1}g$ fixes L pointwise. Let

$$S_1 = (S \cap \pi_L^{-1}(x)) \cup \{y\} \text{ and } S_2 = (S \cap \pi_L^{-1}(y)) \cup \{x\}.$$

Note that for $i = 1, 2$, then S_i is the set of vertices of a subtree of S that contains L. The condition that neither x nor y is a leaf of S ensures that there is some neighbour of x in S that is not contained in S_2, and similarly there is some neighbour of y in S that is not contained in S_1. Hence S_1 and S_2 are both proper subtrees of S, so by the minimality of $|S|$, there exists $h_1, h_2 \in G$ such that $h_i w_i = h^{-1} g w_i$ for all $w_i \in S_i$ $(i = 1, 2)$. Indeed, h_1 and h_2 are elements of $G_{(L)}$, since h_1 and h_2 both agree with $h^{-1}g$ on L. In particular, we see that the action of $h^{-1}g$ induces an element of $\phi_x(G_{(L)}) \times \phi_y(G_{(L)})$. But then by (the restricted) property (P), we have $h^{-1}g \in G_{(L)}$ and hence $g \in G$, a contradiction. □

From now on we can refer to the (P)**-closure** of an action, meaning the smallest group with property (P) that contains it. We will use repeatedly without comment the fact that (P) is equivalent to (P_1). We will also describe an action as "(P)-closed" if it has property (P).

Lemma 2.7. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$ and let T' be a G-invariant subtree of T. Suppose that (T, G) has property* (P) *and that the action of G on T' is closed. Then (T', G) also has property* (P).

Proof. Let L be a path in T'. Since (T, G) has property (P), the natural homomorphism

$$\phi_L : G_{(L)} \to \prod_{x \in VL} \phi_x(G_{(L)})$$

is surjective, where ϕ_L and ϕ_x are defined with respect to T. Now consider what happens if we replace ϕ_L and ϕ_x with ϕ'_L and ϕ'_x respectively, which are now defined with respect to T'. We also have closest point projections π_L for L as a path in T and π'_L for L as a path in T', but in fact π_L and π'_L agree on VT'. If we choose $g_x \in \phi'_x(G_{(L)})$ for each $x \in VL$, then there is some $h_x \in G_{(L)}$ such that $\phi'_x(h_x) = g_x$, and then by the surjectivity of ϕ_L, there is $g \in G_{(L)}$ such that $\phi_L(g) = (\phi_x(h_x))_{x \in VL}$. But then since

$$(\pi')^{-1}_L(x) = \pi^{-1}_L(x) \cap VT' \subseteq \pi^{-1}_L(x),$$

we immediately see that

$$\phi'_L(g) = (\phi'_x(h_x))_{x \in VL} = (g_x)_{x \in VL}.$$

Thus ϕ'_L is surjective, so G has property (P) on T'. □

2.3 Bass–Serre theory

Here we recall some standard results in Bass–Serre theory for groups acting on trees. In this article we will not be using Bass–Serre theory to construct the groups but we will use it occasionally to analyse them. Note that conventional Bass–Serre theory considers only actions on trees without reversal, whereas we allow reversal of edges; we keep track of these edge reversals in the quotient graph by allowing a loop to be its own inverse. This added generalization has no deep significance, since an action with reversal can always be converted to an action without reversal by subdividing edges, but it necessitates some adjustments to the statements.

Given a group G acting on a tree T, we define the **reversal-free subdivision** T^i by subdividing in two parts those edges a of T such that $\overline{a} \in Ga$. Analogously, in the quotient graph $\Gamma = G \backslash T$, we define the **reversal-free subdivision** of Γ^i of Γ by taking each loop a such that $a = \overline{a}$ (with $o(a) = t(a) = v$, say), adding a new vertex v_a, and replacing a with the geometric edge $\{a', \overline{a'}\}$ where $o(a') = v$ and $t(a') = v_a$. The action of G on

T^i is then without reversal, and the quotient map from T to Γ naturally gives rise to a quotient map from T^i to Γ^i.

Lemma 2.8 ([9, I.3.1, Proposition 14]). *Let G be a group acting without reversal on a tree T. Then every subtree of $G\backslash T$ lifts to a subtree of T.*

Theorem 2.9 ([9, I.5.4, Theorem 13]). *(Bass–Serre structure theorem for groups acting on trees) Let G be a group acting on a tree T. For the reversal-free subdivision T^i, let $\pi : T^i \to G\backslash T^i$ be the quotient map of (T^i, G). Choose a subtree T' of T^i that is a lift of a maximal subtree of $G\backslash T^i$. Choose a subset $E^+ \subseteq AT^i$ such that $o(a) \in VT'$ for all $a \in E^+$, such that π is injective on E^+ and $\pi(E)$ is an orientation of $G\backslash T^i$ and set $E = E^+ \cup \overline{E^+}$. For each $a \in AT^i$ let τ_a be the inclusion of G_a into $G_{t(a)}$. For each $a \in E^+$ choose $s_a \in G$ so that $s_a^{-1}t(a) \in VT'$, with $s_a = 1$ if $a \in AT'$, and set $s_{\overline{a}} = s_a^{-1}$. Write $F(E)$ for the free group over $\{s_a \mid a \in E\}$. Then G has the form*

$$\frac{F(E) * \ast_{v \in VT'} G_v}{\langle\langle s_a\tau_a(g)s_{\overline{a}}\tau_{\overline{a}}(g)^{-1}\ (a \in E, g \in G_a),\ s_a s_{\overline{a}},\ s_a\ (a \in AT')\rangle\rangle}.$$

Definition 2.10. Retain the hypotheses and notation of the previous theorem. Let c be a directed path in $G\backslash T^i$, that is, a sequence $(v_0, a_1, v_1, \dots, a_{n-1}, v_n)$ consisting alternately of vertices and arcs such that $o(a_i) = v_{i-1}$ and $t(a_i) = v_i$. A **word of type** c is then a word $w = g_0 s_{e_1} g_1 \dots g_{n-1} s_{e_n} g_n$ over $E \sqcup \bigsqcup_{v \in VT'} G_v$ such that $g_i \in G_{v'_i}$ where $\pi(v'_i) = v_i$ and $\pi(e_i) = a_i$. Say that w is **reduced** if it is of type c for some path c in $G\backslash T^i$, and satisfies the following conditions:

If $n = 0$ then $g_0 \neq 1$; if $n \geqslant 1$, then for each index i such that $a_{i+1} = \overline{a_i}$, then $g_i \notin G_{e_i}$.

Theorem 2.11 ([9, I.5.2, Theorem 11]). *(Normal form theorem of Bass–Serre theory) Under the hypotheses of Theorem 2.9, every reduced word evaluates to a nontrivial element of G.*

The following corollary is valid without assuming that G acts without reversal, since on the one hand, every group generated by vertex stabilizers acts without reversal (since it preserves each part of the natural bipartition of vertices of the tree) and on the other, if $\Gamma = G\backslash T$ is a tree then Γ has no loops, so certainly G acts on T without reversal.

Corollary 2.12 ([9, I.5.4, Exercise 2]). *Let G be a group acting on a tree T. Then $G\backslash T$ is a tree if and only if G is generated by vertex stabilizers. Moreover, if G is generated by vertex stabilizers, then $G = \langle G_v \mid v \in VT' \rangle$ where T' is a lift of a maximal subtree in $G\backslash T$.*

Lemma 2.13. *Let T be a tree and let G be a compactly generated locally compact subgroup of $\mathrm{Aut}(T)$, such that every element of G fixes a vertex of T. Then G fixes a vertex of T.*

Proof. Note that the hypotheses imply that G acts without reversal.

In the case that G is finitely generated, the conclusion is [9, I.6.5, Corollary 3].

It is a general fact (see [7, Lemma 2], for example) that given any compactly generated t.d.l.c. group G and compact open subgroup U of G, there is a finitely generated subgroup H of G such that $G = UH$. Now H fixes a vertex v of T, which means that the orbit of v under $G = UH$ is finite. Let S be the smallest subtree of T spanned by Gv. Then S is a finite tree, so it has a canonical centre, which is either a vertex or a pair of adjacent vertices. Thus G either fixes a vertex or preserves a pair of adjacent vertices. Since G acts without reversal, in fact G fixes a vertex. □

3 A parametrization of (P)-closed groups

Definition 3.1. A **local action diagram** $\Delta = (\Gamma, (X_a), (G(v)))$ consists of the following information:

- A connected graph Γ.
- For each arc a of Γ, a nonempty set X_a (called the **colour set** of a).
- For each vertex v of Γ, a group $G(v)$ (called the **local action** at v) with the following properties: write X_v to denote the disjoint union $\bigsqcup_{a \in o^{-1}(v)} X_a$, then the group $G(v)$ is a closed subgroup of $\mathrm{Sym}(X_v)$ and the sets X_a are the orbits of $G(v)$ on X_v.

Definition 3.2. Let $\Delta = (\Gamma, (X_a), (G(v)))$ and $\Delta' = (\Gamma', (X'_a), (G'(v)))$ be local action diagrams.

An **isomorphism** from Δ to Δ' is an isomorphism $\theta : \Gamma \to \Gamma'$ of graphs, together with a bijection $\theta_v : X_v \to X'_{\theta(v)}$ for each $v \in V\Gamma$ that restricts to a bijection from X_a to $X_{\theta(a)}$ for each $a \in o^{-1}(v)$, and such that $\theta_v G(v) \theta_v^{-1} = G'(v')$.

Local action diagrams have the advantage of having a simple description from a combinatorial perspective. In terms of the permutation groups $G(v)$, there are no interactions between them or compatibility conditions to check, except that $G(v)$ should have the specified orbit structure. However, we will see that they provide a parametrization of all (P)-closed groups of tree automorphisms, taken up to isomorphisms of the tree. Our aim in this section is to prove the following:

Theorem 3.3. *There is a natural one-to-one correspondence between isomorphism classes of local action diagrams, and isomorphism classes of pairs (T, G) where T is a tree and G is a* (P)*-closed subgroup of* $\mathrm{Aut}(T)$.

Definition 3.4. Given a local action diagram Δ, a **Δ-tree T** is a tree T together with a surjective graph homomorphism $\pi : T \to \Gamma$ and a **Γ-colouring**, that is, a map $\mathcal{L} : AT \to \bigsqcup_{a \in A\Gamma} X_a$, such that for each vertex $v \in VT$, and each arc a in $o^{-1}(\pi(v))$, the map $\mathcal{L}$ restricts to a bijection $\mathcal{L}_{v,a}$ from $\{b \in o^{-1}(v) \mid \pi(b) = a\}$ to X_a. Given $v \in VT$, write $\mathcal{L}_v$ for the restriction of $\mathcal{L}$ to a bijection from $o^{-1}(v)$ to $X_{\pi(v)}$.

Note that the groups $G(v)$ play no role in the definition of **T**.

Lemma 3.5. *Let Δ be a local action diagram. Then there exists a Δ-tree. Moreover, given any two Δ-trees $(T, \pi, \mathcal{L})$ and $(T', \pi', \mathcal{L}')$, there is a graph isomorphism $\alpha : T \to T'$ such that $\pi' \circ \alpha = \pi$.*

Proof. Choose a base vertex $v_0 \in V\Gamma$. We construct a Δ-tree **T** as follows.

Given $v \in V\Gamma$ and $c \in X_v$, the **type** $p(c)$ of c is the unique $a \in A\Gamma$ such that $c \in X_a$. A **coloured walk (of length n)** in Γ is a finite sequence $(c_1, c_2, \ldots, c_n)$, where for each $1 \leqslant i < n$, we have $o(p(c_{i+1})) = t(p(c_i))$. The **origin** of the coloured walk is $o(p(c_1))$.

Vertices $v \in VT$ will be labelled by coloured walks with origin v_0. For vertex labels $v = (c_1, c_2, \ldots, c_n)$ and $w = (c_1, c_2, \cdots, c_n, c_{n+1}, \cdots, c_m)$ we say that v is a **prefix** of w. For each vertex label $v = (c_1, c_2, \ldots, c_n)$, there will also be a reverse label $\overline{v} = (d_1, d_2, \ldots, d_n)$ of the same length, where d_i is a colour such that $p(d_i) = \overline{p(c_i)}$, and such that if v is a prefix of w, then $\overline{v}$ is the corresponding prefix of $\overline{w}$. We produce the vertices of VT inductively starting at a root vertex $()$.

Suppose we have defined a vertex $v = (c_1, c_2, \ldots, c_n)$ with reverse label $\overline{v} = (d_1, d_2, \ldots, d_n)$. Then we define vertices $v_{+c_{n+1}} = (c_1, \ldots, c_n, c_{n+1})$, for all c_{n+1} such that $o(p(c_{n+1})) = t(p(c_n))$ and $c_{n+1} \neq d_n$. We then set $\overline{v_{+c_{n+1}}} = (d_1, d_2, \ldots, d_{n+1})$, where d_{n+1} is some element of $X_{\overline{p(c_{n+1})}}$ (chosen arbitrarily).

The set AT_+ of forward arcs of T consists of ordered pairs (v, w), where v is a prefix of w of length one less than w; then $AT_- =: \{(w, v) \mid (v, w) \in AT_+\}$ and $AT := AT_- \sqcup AT_+$. Origin and terminal vertices and edge reversal are defined in the obvious way, and it is clear that we obtain a tree. The colouring $\mathcal{L}$ is defined as follows: given $(v, w) \in AT_+$, then $\mathcal{L}(v, w)$ is the last entry of w and $\mathcal{L}(w, v)$ is the last entry of $\overline{w}$.

The graph homomorphism $\pi : T \to \Gamma$ is given by $\pi(()) = v_0$ for the base vertex; $\pi(v) = t(p(c_n))$ for any vertex $v = (c_1, \ldots, c_n)$ in VT; and $\pi(a) = p(\mathcal{L}(a))$ for $a \in AT$. Given the way in which the entries c_i and d_i were chosen and used to define $\mathcal{L}$, one sees that π is a surjective graph homomorphism.

Given a vertex $v = (c_1, \ldots, c_n)$ with reverse label $\overline{v} = (d_1, d_2, \ldots, d_n)$, then v has one parent vertex $(c_1, \ldots, c_{n-1})$ and a set of child vertices of the form $(c_1, \ldots, c_n, c')$, where c' ranges over the set $X_{t(p(c_n))} \setminus \{d_n\} = X_{\pi(v)} \setminus \{d_n\}$. The set $o^{-1}(v)$ is thus in a natural bijection with $X_{\pi(v)}$ in a manner that respects the partition into sets X_a for $a \in o^{-1}(\pi(v))$, and the colouring produces the same bijection. In particular, for each arc $a \in o^{-1}(\pi(v))$, we see that $\mathcal{L}$ restricts to a bijection from $\{b \in o^{-1}(v) \mid \pi(b) = a\}$ to X_a. Thus the object $\mathbf{T}$ we have constructed is a Δ-tree.

Now suppose that we have two Δ-trees $(T, \pi, \mathcal{L})$ and $(T', \pi', \mathcal{L}')$. We construct a graph isomorphism $\alpha : T \to T'$ compatible with (π, π') inductively as follows.

Choose $v_1 \in VT$ and $w_1 \in VT'$ such that $\pi(v_1) = \pi'(w_1) = v_0$, and set $\alpha(v_1) = w_1$. Suppose we have defined α for vertices and arcs in $B_n(v_1)$ $(n \geqslant 0)$, let v be a vertex in T at distance n from v_1 and let $w = \alpha(v)$. Then $\pi(v) = \pi'(w)$ by the induction hypothesis. Given $a \in o^{-1}(\pi(v))$, we have bijections $\mathcal{L}_{v,a} : \{b \in o^{-1}(v) \mid \pi(b) = a\} \to X_a$ and $\mathcal{L}'_{w,a} := \{b \in o^{-1}(w) \mid \pi'(b) = a\} \to X_a$. In particular, the sets $\{b \in o^{-1}(v) \mid \pi(b) = a\}$ and $\{b \in o^{-1}(w) \mid \pi'(b) = a\}$ have the same size, so we can extend α to include $o^{-1}(v)$ in its domain, in such a way that it restricts to a bijection from $\{b \in o^{-1}(v) \mid \pi(b) = a\}$ to $\{b \in o^{-1}(w) \mid \pi'(b) = a\}$. The choice of bijection is unimportant here, except in the case that $\pi^{-1}(a)$ contains an arc a' starting at v in the direction of v_1: in this case, $\alpha(a')$ has already been chosen, so we choose a bijection from $\{b \in o^{-1}(v) \mid \pi(b) = a, b \neq a'\}$ to $\{b \in o^{-1}(w) \mid \pi'(b) = a, b \neq \alpha(a')\}$. For $b \in t^{-1}(v)$ and $v' = o(b)$, we set $\alpha(b) = \overline{\alpha(\overline{b})}$ and $\alpha(v') = o(\alpha(b))$ respectively. This extends the definition of α to a ball of radius $n + 1$ about v_1; notice that α still produces a graph isomorphism from $B_{n+1}(v_1)$ to $B_{n+1}(w_1)$, completing the inductive step. We can thus extend α to a graph isomorphism from T to T' such that $\pi' \circ \alpha = \pi$. □

Note: we do not claim that α can be chosen to map $\mathcal{L}$ to $\mathcal{L}'$.

Definition 3.6. Let G be a group of automorphisms of a tree T. We define an **associated local action diagram** Δ and equip T with the structure of a Δ-tree as follows.

- Γ is the quotient graph $G\backslash T$, and π is the natural quotient map.
- For each $v \in V\Gamma$, choose a vertex $v^* \in \pi^{-1}(v)$; write V^* for the set of vertices so obtained. Given $a \in A\Gamma$ such that $v = o(a)$, let $X_a = \{b \in o^{-1}(v^*) \mid \pi(b) = a\}$. The set $X_v := o^{-1}(v^*)$ is then naturally partitioned as required. Define the group $G(v)$ to be the permutation group induced on X_v by $\overline{G_{v^*}}$.
- For each $w \in VT$, choose $g_w \in G$ such that $g_w w \in V^*$. Then g_w also induces a bijection from $o^{-1}(w)$ to X_v. Given $b \in o^{-1}(w)$, set $\mathcal{L}(b) = g_w b$.

The definition is such that given $v, w \in VT$ such that $\pi(v) = \pi(w)$, the maps $\mathcal{L}_v$ and $\mathcal{L}_w$ form two sides of a commuting triangle: if $r_{v,w}$ is the map from $o^{-1}(v)$ to $o^{-1}(w)$ induced by $g_w^{-1}g_v$, then

$$\mathcal{L}_v = \mathcal{L}_w r_{v,w}.$$

There are many choices for the associated local action diagram, but they are all isomorphic, as we see in the following lemma.

Lemma 3.7. *Let T be a tree and let G be a group of automorphisms of T. Then any two local action diagrams $\Delta = (\Gamma, (X_a), (G(v)))$ and $\Delta' = (\Gamma, (X'_a), (G'(v)))$ associated to G are isomorphic, via an isomorphism θ that is the identity map on the graph Γ.*

Proof. Without loss of generality we can assume G is closed in $\mathrm{Aut}(T)$. From the definition, we see that Δ and Δ' have the same associated graph $\Gamma = G\backslash T$; let θ be the trivial graph automorphism of Γ. Given $v \in V\Gamma$, say the chosen element of $\pi^{-1}(v)$ is v^* in the construction of Δ, and v^{**} in the construction of Δ'. Then $v^{**} = g_v v^*$ for some $g_v \in G$, since $\pi(v^*) = Gv^*$. We can thus define a bijection θ_v from $X_v := o^{-1}(v^*)$ to $X'_v := o^{-1}(v^{**})$ by setting $\theta_v(a) = g_v a$. Given that $g_v G_{v^*} g_v^{-1} = G_{v^{**}}$, and $G(v)$ and $G'(v)$ are determined by the actions of the vertex stabilizers G_{v^*} and $G_{v^{**}}$ respectively, we see that $\theta_v G(v) \theta_v^{-1} = G'(v)$. In particular, θ_v sends orbits of $G(v)$ to orbits of $G'(v)$, so it restricts to a bijection from X_a to X'_a for each $a \in o^{-1}(v)$. Thus (θ, θ_v) is an isomorphism of local action diagrams. ☐

Thus from now on, we can talk about *the* local action diagram $\Delta(T, G)$ associated to (T, G) without ambiguity.

An **automorphism** of the Δ-tree $\mathbf{T}$ is a graph automorphism θ of T such that $\pi \circ \theta = \pi$. Write $\mathrm{Aut}_\pi(T)$ for the group of all such automorphisms. Given $g \in \mathrm{Aut}_\pi(T)$, a vertex $v \in VT$, and $\mathcal{L}$ the colouring associated to $\mathbf{T}$, we define the **$\mathcal{L}$-local action** of g at v as follows:

$$\sigma_{\mathcal{L},v}(g) : X_{\pi(v)} \to X_{\pi(v)} \qquad \sigma_{\mathcal{L},v}(g)(c) := \mathcal{L} g \mathcal{L}|^{-1}_{o^{-1}(v)}(c).$$

We see that $\sigma_{\mathcal{L},v}(g)$ is a permutation of $X_{\pi(v)}$. Finally, we define the **universal group of $\mathbf{T}$ with respect to local actions** $(G(v))_{v \in V\Gamma}$ to be the set $U(\mathbf{T}, (G(v)))$ of all elements g of $\mathrm{Aut}_\pi(T)$ such that for every $v \in VT$, the permutation $\sigma_{\mathcal{L},v}(g)$ belongs to $G(\pi(v))$.

Theorem 3.8. *Let Δ be a local action diagram, let $\mathbf{T}$ be a Δ-tree, and let $H = U(\mathbf{T}, (G(v)))$. Then H is a* (P)*-closed subgroup of* $\mathrm{Aut}(T)$*; Δ is isomorphic to a local action diagram associated to H; and for every vertex $v \in VT$ and $g \in G(\pi(v))$, there is $h \in H_v$ such that $\sigma_{\mathcal{L},v}(h) = g$.*

Proof. Let $g, h \in U(\mathbf{T}, (G(v)))$ and let $v \in VT$. It is clear that $g^{-1}, gh \in \mathrm{Aut}_\pi(T)$, so there is a fixed vertex $w \in V\Gamma$ such that $w = \pi(v) = \pi(hv) =$

$\pi(gv) = \pi(g^{-1}v)$. It is easily seen that $\sigma_{\mathcal{L},v}(g^{-1})$ and $\sigma_{\mathcal{L},v}(gh)$ are given by the following formulae:

$$\sigma_{\mathcal{L},v}(gh) = \sigma_{\mathcal{L},hv}(g)\sigma_{\mathcal{L},v}(h)$$

$$\sigma_{\mathcal{L},v}(g^{-1}) = \left(\sigma_{\mathcal{L},g^{-1}v}(g)\right)^{-1}$$

Since $\sigma_{\mathcal{L},hv}(g)$, $\sigma_{\mathcal{L},v}(h)$ and $\sigma_{\mathcal{L},g^{-1}v}(g)$ are all in the group $G(w)$, we see that $\sigma_{\mathcal{L},v}(gh)$ and $\sigma_{\mathcal{L},v}(g^{-1})$ are also elements of $G(w)$. This proves $U(\mathbf{T}, (G(v)))$ is closed under products and inverses. Since $U(\mathbf{T}, (G(v)))$ clearly also contains the trivial automorphism of T, we conclude that $H := U(\mathbf{T}, (G(v)))$ is a subgroup of $\mathrm{Aut}_\pi(T)$.

Since $H \leqslant \mathrm{Aut}_\pi(T)$, certainly every orbit of H is contained in a fibre of π. We claim that in fact H acts transitively on $\pi^{-1}(r)$ where r is a vertex or arc of Γ. It is enough to show that H is transitive when r is an arc of Γ, as the vertex case will then follow by considering origin vertices of arcs. So fix arcs $a, b \in \pi^{-1}(r)$; we aim to construct $g \in H$ such that $ga = b$. We define g in stages on balls of radius n about $v_0 := o(a)$.

Let $v_0' = o(b)$ and let $w = \pi(v_0)$. Choose an element $h_0 \in G(w)$ such that $h_0\mathcal{L}(a) = \mathcal{L}(b)$; this is possible since by definition, $\mathcal{L}(a)$ and $\mathcal{L}(b)$ must lie in the same $G(w)$-orbit. Then there is a unique graph isomorphism g_1 from $B_1(v_0)$ to $B_1(v_0')$ such that $\mathcal{L}g_1\mathcal{L}|_{o^{-1}(v_0)}(c) = h_0(c)$ for all $c \in X_w$.

Let us also pause to note that by varying b, we can obtain every element of $G(w)$ as a suitable h_0 whilst also fixing v_0: specifically, given $h \in G(w)$, then $h\mathcal{L}(a) = \mathcal{L}(b)$ for some unique $b \in o^{-1}(v_0)$, and hence in this case we can take $h_0 = h$. Thus, provided we can extend g_1 to an element of H, we will have shown that H_{v_0} achieves all possible values of $\sigma_{\mathcal{L},v_0}$ at the vertex v_0. By varying a, the vertex v_0 can also be made an arbitrary vertex of T.

Suppose we have defined a graph isomorphism g_n from $B_n(v_0)$ to $B_n(v_0')$, such that $\mathcal{L}g_n\mathcal{L}|_{o^{-1}(v)}(c) \in G(\pi(v))$ for all $v \in VB_{n-1}(v_0)$. Let $S_n(v_0)$ be the sphere of radius n about v_0 and let $v \in S_n(v_0)$. We have already defined g_nv and also g_nr for the unique arc $r \in o^{-1}(v)$ in the direction of v_0. Similar to before, we see that $\mathcal{L}(r)$ and $\mathcal{L}(g_nr)$ lie in the same $G(\pi(v))$-orbit, so there is $h_n \in G(\pi(v))$ such that $h_n\mathcal{L}(r) = \mathcal{L}(g_nr)$. There is then a unique graph isomorphism $h'(v)$ from $B_1(v)$ to $B_1(g_nv)$ such that $\mathcal{L}h'\mathcal{L}|_{o^{-1}(v)}(c) = h_n(c)$ for all $c \in X_{\pi(v)}$. Note the domains of the maps $\{h'(v) \mid v \in S_n(v_0)\}$ are pairwise disjoint; for each $v \in S_n(v_0)$, the domains of g_n and $h'(v)$ overlap only on a single edge and its endpoints, and for this overlap, g_n and $h'(v)$ agree. We can thus combine g_n with the set of maps $\{h'(v) \mid v \in S_n(v_0)\}$ to produce a graph isomorphism g_{n+1} from $B_{n+1}(v_0)$ to $B_{n+1}(v_0')$. By construction, we see that $\mathcal{L}g_{n+1}\mathcal{L}|_{o^{-1}(v)}(c) \in G(\pi(v))$ for all $v \in VB_n(v_0)$, completing the inductive step.

By combining the sequence (g_n) of graph isomorphisms, we thus obtain $g \in \mathrm{Aut}_\pi(T)$ such that $ga = b$ and such that $\sigma_{\mathcal{L},v}(g) \in G(\pi(v))$ for all $v \in VT$, so $g \in H$.

Now let $\Delta' = (\Gamma', (X'_a), (H(v)))$ be a local action diagram associated to H. We aim to construct an isomorphism of local action diagrams from Δ' to Δ. We have shown that H acts transitively on each fibre of π, so the quotient graph $\Gamma' = H\backslash T$ can be naturally identified with Γ and θ can be taken to be the trivial graph isomorphism on Γ. Given $v \in V\Gamma$, let v^* be the chosen vertex of VT in the construction of Δ'. Then for each $a \in A\Gamma$ such that $v = o(a)$, by definition $X'_a = \{b \in o^{-1}(v^*) \mid \pi(b) = a\}$. The definition of $\mathbf{T}$ then provides a bijection $\mathcal{L}_{v^*,a}$ from X'_a to X_a. We can thus construct a bijection θ_v from X'_v to X_v by setting $\theta_v(c) = \mathcal{L}_{v^*,a}(c)$. By definition, $H(v)$ is just the group of permutations induced by H_{v^*} on $o^{-1}(v^*)$. As previously observed, we obtain in this way the set of all permutations h of $o^{-1}(v^*)$ such that the permutation induced by $\mathcal{L}h\mathcal{L}|_{o^{-1}(v^*)}$ on X_v is an element of $G(v)$. Given $c \in X_v$, we see from the definition of θ_v that

$$\mathcal{L}h\mathcal{L}|^{-1}_{o^{-1}(v^*)}(c) = \theta_v h \theta_v^{-1}(c).$$

Thus $\theta_v H(v) \theta_v^{-1} = G(v)$, completing the proof that $(\theta, (\theta_v))$ is an isomorphism of local action diagrams from Δ' to Δ.

Finally, let g be an element of the (P)-closure of H. Then for every finite subset Y of $o^{-1}(v)$, there is some $h \in H$ such that $gv = hv$ and $gy = hy$ for all $y \in Y$. In other words, h is such that $\sigma_{\mathcal{L},v}(h)(c) = \sigma_{\mathcal{L},v}(g)(c)$ for all $c \in \mathcal{L}(Y)$; note that by definition, $\sigma_{\mathcal{L},v}(h) \in G(\pi(v))$. Since Y can be any finite subset of $o^{-1}(v)$ and $G(\pi(v))$ is closed, it follows that $\sigma_{\mathcal{L},v}(g) \in G(\pi(v))$ for all $v \in VT$, and hence $g \in H$. Thus H is (P)-closed. □

Theorem 3.9. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$, let $\Delta = \Delta(T, G)$ and let $\mathbf{T}$ be an associated Δ-tree structure on T. Then $U(\mathbf{T}, (G(v)))$ is the (P)-closure of G.*

Proof. It is clear that G and $\overline{G}$ have the same associated local action diagram, so we may assume G is closed. By Theorem 3.8, the set $H = U(\mathbf{T}, (G(v)))$ is a (P)-closed subgroup of $\mathrm{Aut}(T)$. Let V^* be the set of vertices of T used to define the Δ-tree structure, and let $v \in VT$. Then there is some $w \in V^*$ such that $\pi(v) = \pi(w)$. In particular, v and w lie in the same G-orbit, and given the way in which $\mathbf{T}$ is constructed, there is some $g_v \in G$ such that $g_v v = w$ and the colouring for $b \in o^{-1}(v)$ is given by $\mathcal{L}(b) = g_v b$.

Let $g \in G$. Then $\pi(v) = \pi(gv) = \pi(w)$, so $g_{gv}(gv) = w$ where g_{gv} is as in the definition of the colouring. Let $b \in o^{-1}(v)$ and let $c = \mathcal{L}(b)$. Then $c = g_v b$ and $\mathcal{L}(gb) = g_{gv} gb = (g_{gv} g g_v^{-1})c$, so

$$\sigma_{\mathcal{L},v}(g)(c) = (g_{gv} g g_v^{-1})c.$$

We see that $(g_{gv} g g_v^{-1})w = w$, so $g_{gv} g g_v^{-1} \in G_w$. Thus by definition, the permutation induced by $g_{gv} g g_v^{-1}$ on w is an element of $G(\pi(w)) = G(\pi(v))$, that is, $\sigma_{\mathcal{L},v}(g) \in G(\pi(v))$. Since this holds for all $v \in VT$, we see that $g \in H$. Thus $G \leqslant H$; since H is (P)-closed, in fact $G^{(\mathrm{P})} \leqslant H$.

Conversely, let $h \in H$. Then by the definition of H,

$$\sigma_{\mathcal{L},v}(h) \in G(\pi(v)).$$

Given the definition of $G(\pi(v))$, there is $g' \in G_w$ such that for all $c \in o^{-1}(w)$,

$$g'c = \sigma_{\mathcal{L},v}(h)(c).$$

Since $h \in \mathrm{Aut}_\pi(T)$ we have $\pi(hv) = \pi(v)$, so we can write $g_{hv}(hv) = w$. Given $b \in o^{-1}(v)$, we have $\mathcal{L}(hb) = \sigma_{\mathcal{L},v}(h)(g_v b)$ and also $\mathcal{L}(hb) = g_{hv}hb$. Thus

$$hb = g_{hv}^{-1}\sigma_{\mathcal{L},v}(h)(g_v b) = g_{hv}^{-1}g'g_v b.$$

Thus on the set $o^{-1}(v)$, we see that h agrees with the element $g_{hv}^{-1}g'g_v$ of G. Since this can be achieved at every vertex $v \in VT$, we conclude that $h \in G^{(\mathrm{P})}$. This proves that $G^{(\mathrm{P})} = H$ as required. □

Since the local action diagram can be recovered from the group $U(\mathbf{T}, (G(v)))$, we have the following corollary.

Corollary 3.10. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$ and let Δ be an associated local action diagram. Then Δ is also an associated local action diagram for the action of $G^{(\mathrm{P})}$ on T.*

We have now shown that the (P)-closed subgroups of $\mathrm{Aut}(T)$ are exactly the groups realizable as a group $U(\mathbf{T}, (G(v)))$ definable from some local action diagram. It remains to show that each local action diagram Δ gives rise to only one group $U(\mathbf{T}, (G(v)))$ up to tree isomorphisms, in other words, the choices made in defining the Δ-tree $\mathbf{T}$ are not significant.

Theorem 3.11. *Let $\Delta = (\Gamma, (X_a), (G(v)))$ be a local action diagram and let $\mathbf{T} = (T, \pi, \mathcal{L})$ and $\mathbf{T}' = (T', \pi', \mathcal{L}')$ be Δ-trees. Then there is a graph isomorphism $\phi : T \to T'$ such that $\phi U(\mathbf{T}, (G(v)))\phi^{-1} = U(\mathbf{T}', (G(v)))$.*

Proof. Let $G = U(\mathbf{T}, (G(v)))$ and $H = U(\mathbf{T}', (G(v)))$. By Lemma 3.5, there is a graph isomorphism $\alpha : T \to T'$ such that $\pi' \circ \alpha = \pi$. Thus by applying α to $\mathbf{T}$ and replacing G with $\alpha G \alpha^{-1}$, we may assume that $T = T'$ and $\pi = \pi'$, in other words, G and H act on the same tree with the same orbits, in such a way that the quotient graph can be naturally identified with Γ.

By Theorem 3.8, for every pair of vertices $v, w \in VT$ such that $\pi(v) = \pi(w)$, we have

$$\sigma_{\mathcal{L},v}(G_v) = G(\pi(v)) = \sigma_{\mathcal{L}',w}(H_w).$$

Let G_v^* and H_w^* be the permutation groups induced by G_v on $o^{-1}(v)$ and H_w on $o^{-1}(w)$ respectively. We see that G_v^* and H_w^* are both isomorphic to $G(\pi(v))$ as permutation groups; moreover, since $\pi = \pi'$, given any element $g \in G$ such that $gv = w$, then the groups $gG_v^*g^{-1}$ and H_v^* also have the same orbits on $o^{-1}(w)$, with the same correspondence between orbits on

$o^{-1}(w)$ and elements of $o^{-1}(\pi(v))$. There is thus a bijection $\rho_{v,w}$ from $o^{-1}(v)$ to $o^{-1}(w)$ such that $\pi(\rho_{v,w}(a)) = \pi(a)$ for all $a \in o^{-1}(v)$ and such that $\rho_{v,w} G^*_v \rho^{-1}_{v,w} = H^*_w$. Since G^*_v is transitive on each π-fibre in $o^{-1}(v)$, for a single given $a \in o^{-1}(v)$, we are free to choose $\rho_{v,w}(a)$ from the set $\{b \in o^{-1}(w) \mid \pi(b) = \pi(a)\}$.

We now aim to construct $\phi \in \mathrm{Aut}_\pi(T)$ such that

$$\phi U(\mathbf{T}, (G(v)))\phi^{-1} = U(\mathbf{T}', (G(v))).$$

We construct ϕ successively on balls of radius n centred on some vertex $v_0 \in VT$, starting with $\phi(v_0) = v_0$, such that at each stage ϕ is a graph automorphism on $B_n(v_0)$ that commutes with π. To define ϕ on $B_1(v_0)$, we let it act as ρ_{v_0,v_0} on $o^{-1}(v_0)$, and then extend to the remaining arcs and vertices in $B_1(v_0)$ via the equations $\phi(\overline{a}) = \overline{\phi(a)}$ and $\phi(t(a)) = t(\phi(a))$. Now suppose we have defined ϕ on $B_n(v_0)$ for some $n \geqslant 1$, and let $v \in S_n(v_0)$; write ϕ_n for the automorphism of $B_n(v_0)$. Let a be the unique arc in $o^{-1}(v)$ in the direction of v_0. Then we have already specified $\phi(a)$, and it has been chosen in such a way that $\pi(\overline{\phi(a)}) = \pi(\overline{a})$; hence it is also the case that $\pi(\phi(a)) = \pi(a)$. We can thus choose $\rho_{v,\phi(v)}$ so that $\rho_{v,\phi(v)}(a) = \phi(a)$. There is then a unique isomorphism ϕ_v from $B_1(v)$ to $B_1(\phi(v))$ that is compatible with both ϕ_n and with ρ_{v,v_0}. We then define $\phi(e)$ for e a vertex or arc of $B_{n+1}(v_0)$ to be $\phi_n(e)$ or $\phi_v(e)$ as applicable, and observe that we have produced a graph automorphism of $B_{n+1}(v_0)$ that commutes with ϕ. By induction, we produce $\phi \in \mathrm{Aut}_\pi(T)$.

Now let $v \in VT$, write $w = \phi(v)$ and consider $\phi G_v \phi^{-1}$. The construction of ϕ was such that the permutation group induced by $\phi G_v \phi^{-1}$ on $o^{-1}(w)$ is $\rho_{v,w} G_v \rho^{-1}_{v,w}$, or in other words, it is the same permutation group as the one induced by H_w on $o^{-1}(w)$. Thus $\sigma_{\mathcal{L}',w}(\phi G_v \phi^{-1}) = G(\pi(w))$. By varying v so that w ranges over VT, we see (from the definition of $H = U(\mathbf{T}', (G(v)))$) that $\phi G \phi^{-1} \leqslant H$. On the other hand, we see by a similar argument that $\phi^{-1} H \phi \leqslant G$, so $\phi G \phi^{-1} \geqslant H$. So in fact $\phi G \phi^{-1} = H$, as required. □

We can thus define *the* universal group of a local action diagram: $U(\Delta)$ is defined as $U(\mathbf{T}, (G(v)))$ where $\mathbf{T}$ is some Δ-tree. Then $U(\Delta)$ is defined up to isomorphisms of the tree on which it acts; we write $U_{\mathbf{T}}(\Delta)$ if we want to impose a specific action on a specific tree.

To conclude the section, we now prove the correspondence theorem.

Proof of Theorem 3.3. Given a local action diagram Δ, we have an associated pair $(T, U(\Delta))$, and by Theorem 3.11 the pair $(T, U(\Delta))$ is specified uniquely up to isomorphisms; on the other hand, it is clear from the construction that if Δ and Δ' are isomorphic as local action diagrams, then they will produce isomorphic pairs $(T, U(\Delta))$ and $(T', U(\Delta'))$. Thus we have a well-defined mapping β from isomorphism classes of local action diagram to isomorphism classes of pairs (T, G) where T is a tree and $G \leqslant \mathrm{Aut}(T)$ is

(P)-closed on T. Theorem 3.9 shows that β is surjective and Theorem 3.8 shows that β is injective. Thus we have a natural one-to-one correspondence as claimed. ☐

4 (P)-closed subgroups of (P)-closed groups

Let G be a (P)-closed group of automorphisms of the tree T. In this section, we identify a natural class of (P)-closed groups related to G, namely: actions of certain subgroups of G on subtrees of T. We reinterpret this class in terms of local action diagrams.

4.1 Sufficient conditions for a subgroup to be (P)-closed

Lemma 4.1. *Let G be a (P)-closed subgroup of $\mathrm{Aut}(T)$, and let $H \leqslant G$. Suppose that H contains an arc stabilizer of G. Then H is (P)-closed.*

Proof. Suppose that H contains the arc stabilizer $G_{(x,y)}$ of G. Note that G must be a closed subgroup of $\mathrm{Aut}(T)$ by Proposition 2.2. Since H is an open subgroup of G, it is also a closed subgroup of $\mathrm{Aut}(T)$. Thus, by Theorem 2.6, it suffices for us to show that H has property (P) with respect to the edges of T.

Let (v, w) be an arc that is not (x, y) or its reverse. Then there is a path that passes through v and w in some order, and then later through x and y in some order; without loss of generality, let us say that the path passes through these points in the order v, w, x, y.

Let $K_{w,v}$ and $K_{v,w}$ be the fixators in G of $\pi^{-1}(v)$ and $\pi^{-1}(w)$ respectively. Since G has property (P), we have $G_{v,w} = K_{v,w}K_{w,v}$. For $h \in H_{v,w}$ we can write $h = k_1k_2$ with $k_1 \in K_{v,w} \leqslant G_{x,y} \leqslant H$ and $k_2 \in K_{w,v}$. Hence $k_2 \in H$ and $H_{v,w} \leqslant K_{v,w}(K_{w,v} \cap H)$. On the other hand, if $k_1 \in K_{v,w}$ and $k_2 \in (K_{w,v} \cap H)$ then of course $k_1k_2 \in H_{v,w}$. The arc stabilizer $H_{(v,w)}$ therefore decomposes as

$$H_{(v,w)} = K_{v,w} \times (K_{w,v} \cap H).$$

Thus H satisfies property (P) with respect to all edges in T. ☐

Every vertex stabilizer contains the stabilizers of the arcs incident with that vertex, so the following is a special case of Lemma 4.1.

Corollary 4.2. *Let G be a (P)-closed subgroup of $\mathrm{Aut}(T)$, and let $H \leqslant G$. Suppose that H contains a vertex stabilizer of G. Then H is (P)-closed.*

Any subgroup H of G containing an arc stabilizer is open. In the other direction, we note in Proposition 4.3 that all 'large' open subgroups of G contain an arc stabilizer, and hence inherit the (P)-closed property.

Proposition 4.3. *Let T be a tree, let G be a non-discrete* (P)*-closed subgroup of* $\mathrm{Aut}(T)$ *and let H be an open subgroup of G. Then exactly one of the following holds:*

(i) Every finitely generated subgroup of H fixes an edge.

(ii) H has a finitely generated subgroup that fixes exactly one vertex v and $|H : H_v| \leqslant 2$.

(iii) H contains an element g that is hyperbolic on T. For any such g, then H contains G_e for every arc e along the axis of g. Consequently, H is (P)*-closed.*

Proof. It is clear that the three given possibilities are mutually exclusive.

Let H^p be the subgroup of H that preserves the parts of the natural bipartition of the vertices of T. Then $|H : H^p| \leqslant 2$, so H^p is open in G; moreover, H has an element with hyperbolic action on T if and only if H does, and H^p acts on T without inversion.

Suppose that H^p does not contain any hyperbolic elements on T. Then by Lemma 2.13, every finitely generated subgroup of H^p fixes a vertex.

Assume that (i) fails, so there is a finitely generated subgroup K of H that does not fix any edge of T. Then there is a vertex v fixed by $K^p = K \cap H^p$, so that $|Kv| \leqslant 2$. If $kv \neq v$ for some $k \in K$, then the midpoint of the shortest path from v to kv is fixed by K. This midpoint is either a vertex or the midpoint of an edge. If it is the midpoint of an edge, then the edge would be fixed by K which is impossible. Therefore this midpoint must itself be a vertex. Thus K fixes a vertex; this ensures $K \leqslant H^p$. If K fixes more than one vertex, then it fixes every edge on the shortest path between any two fixed vertices. Thus in fact K fixes exactly one vertex v. By Lemma 2.13, for any $g \in H^p$, the group $\langle K, g \rangle$ fixes some vertex w; since K only fixes one vertex, we must have $w = v$. Thus H^p fixes v, showing that $|H : H_v| \leqslant 2$, so (ii) holds.

Now suppose instead that there is $g \in H^p$ such that g is hyperbolic on T. Since H is open, there is a finite tuple $\mathbf{v} = (v_1, \dots, v_k)$ of vertices such that the fixator $G_{\mathbf{v}}$ is contained in H. Let L be the axis of g, let ω be the attracting end of L, and let $e = (v, w)$ be an arc along L directed away from the attracting end. Let π be the closest point projection from VT to $\{v, w\}$, and let $K_{w,v}$ and $K_{v,w}$ be the fixators in G of $\pi^{-1}(v)$ and $\pi^{-1}(w)$ respectively. Then there exist $m, n \in \mathbb{Z}$ such that $\pi(g^m v_i) = v$ and $\pi(g^n v_i) = w$ for all $1 \leqslant i \leqslant k$, so $K_{w,v} \leqslant g^m G_{\mathbf{v}} g^{-m}$ and $K_{v,w} \leqslant g^n G_{\mathbf{v}} g^{-n}$. Since $g \in H$ and $G_{\mathbf{v}} \leqslant H$, it follows that $K_{w,v}$ and $K_{v,w}$ are both subgroups of H. Since G is (P)-closed, it has property (P), so $G_e = K_{w,v} \times K_{v,w}$, and hence $G_e \leqslant H$. We conclude by Lemma 4.1 that H is (P)-closed. □

In the special case that G has compact arc stabilizers on $\mathrm{Aut}(T)$, we have the following.

Corollary 4.4. *Let T be a tree and let G be a non-discrete* (P)*-closed subgroup of* $\mathrm{Aut}(T)$*. Suppose that G has compact arc stabilizers. Let H be an open subgroup of G. Then exactly one of the following holds:*

(i) Every compactly generated closed subgroup of H is compact.

(ii) H has a non-compact, compactly generated closed subgroup that fixes exactly one vertex v and $|H : H_v| \leqslant 2$.

(iii) H contains an element g that is hyperbolic on T. For any such g, then H contains G_e for every arc e along the axis of g. Consequently, H is (P)*-closed.*

Proof. It is clear that no hyperbolic element of H can be contained in a compact subgroup; from this fact, it easily follows that the three cases given are mutually exclusive. Case (iii) is the same as in Proposition 4.3, so we may assume it fails and that one of the cases (i) and (ii) in Proposition 4.3 holds. Note also that the condition that G has compact arc stabilizers ensures that G is a t.d.l.c. group.

Suppose case (i) of Proposition 4.3 holds, that is, every finitely generated subgroup of H fixes an edge. Given a compactly generated subgroup K of H, then in fact $K = UF$, where U is a compact open subgroup of H and F is finitely generated. There is thus an arc a such that $|F : F_a| \leqslant 2$; since G_a is compact, it follows that F has compact closure in G, and hence K is compact. Thus (i) holds.

Now suppose case (ii) of Proposition 4.3 holds. There is then a vertex v such that $|H : H_v| \leqslant 2$. Given case (i) of the present corollary, we may assume that there is a compactly generated closed subgroup K of H that is not compact. It follows that K_v is also a compactly generated closed subgroup of H that is not compact. In particular, K_v has infinite orbits on the arcs of T, and hence cannot fix more than one vertex. Thus v is the unique vertex fixed by K_v, so (ii) holds. □

4.2 Vertex stabilizers and (P)-closure

Given a (P)-closed action of a group G on a tree T, then every vertex stabilizer of G itself has (P)-closed action, by Corollary 4.2. In turn, (P)-closed subgroups of $\mathrm{Aut}(T)$ that fix a vertex have a special structure. This generalizes the observation [11, Proposition 15] that the box product of two permutation groups M and N contains isomorphic copies of M and N as subgroups.

Proposition 4.5. *Let G be a group acting on a tree T with property* (P)*, let $\epsilon \in VT$ and let B_r be the closed ball of radius r around ϵ. Let G_r be the pointwise stabilizer of B_r in G. Then there is an increasing sequence $(C_r)_{r \geqslant 1}$ of subgroups of G_0, each closed in* $\mathrm{Aut}(T)$*, such that $G_0 = G_r \rtimes C_r$*

for all $r \geqslant 1$. In particular, there is a permutational isomorphism between the action of C_1 on $o^{-1}(\epsilon)$ and the corresponding vertex group $G(\pi(\epsilon))$ of the local action diagram of G.

Proof. Let $\Gamma = G\backslash T$, let $\pi = \pi_{(T,G)}$ and let $v_0 = \pi(\epsilon)$. Form the local action diagram $\Delta = (\Gamma, (X_a), (G(v)))$ for (T, G). If we replace G with $G_0 = G_\epsilon$, then it will not affect the permutational isomorphism type of $G(v_0)$, nor the structure of G_0, so let us assume that $G = G_0$. Then we see that $|X_a| = 1$ for all $a \in A\Gamma$ such that $d_\Gamma(t(a), v_0)) < d_\Gamma(o(a), v_0)$. In other words, in any colouring $\mathcal{L}$ such that $\mathbf{T} = (T, \pi, \mathcal{L})$ is a Δ-tree, and given $a \in AT$ such that $d(t(a), \epsilon) < d(o(a), \epsilon)$, then $|X_{\pi(a)}| = 1$, so $\mathcal{L}(a)$ must be the unique element of $X_{\pi(a)}$. Fix such a colouring $\mathcal{L}$ and for $r = 1$, set

$$C_r = \{g \in G_0 \mid \forall w \in VT : d(w, \epsilon) \geqslant r \Rightarrow \sigma_{\mathcal{L},w}(g) = 1\}.$$

Given that G_0 preserves distance from ϵ, it is easy to see that C_r is a subgroup of G_0; $C_r \cap G_r = \{1\}$; and $C_r \leqslant C_{r'}$ whenever $r \leqslant r'$. It is also clear that C_r is determined as a subgroup of G_0 by its orbits on arcs, so C_r is closed in G_0; since G_0 is closed in $\mathrm{Aut}(T)$, it follows that C_r is closed in $\mathrm{Aut}(T)$. To see that $G_0 = G_rC_r$, consider some $r \geqslant 1$, an element $h \in G_0$, and a vertex $w \in VT$ with $d(\epsilon, w) = r$. Then h maps the arc in $o^{-1}(w)$ directed towards ϵ to the arc in $o^{-1}(hw)$ directed towards ϵ. This corresponds to a fixed point for the permutation $\sigma_{\mathcal{L},w}(h) \in G(\pi(w))$. Of course there is $g \in \mathrm{Aut}(T)$ that has the same action as h on B_r, but has trivial local action for every vertex $w \in VT$ such that $d(w, \epsilon) \geqslant r$. We then see that in fact $g \in C_r$ and $h \in G_rg$, showing that $h \in G_rC_r$. In particular, it is now clear that $\sigma_{\mathcal{L},\epsilon}$ restricts to a permutational isomorphism from C_1 acting on $o^{-1}(\epsilon)$ to $G(v_0)$ acting on X_{v_0}. □

Definition 4.6. Given $G \leqslant \mathrm{Aut}(T)$, write G^+ for the subgroup of G generated by arc stabilizers in G.

Given $a \in AT$, we define the associated **half-tree** to be the subgraph T_a induced on the vertices v such that $d(t(a), v) < d(o(a), v)$. Write G^{++} for the closure of the subgroup of G generated by the pointwise stabilizers of half-trees in G .

We note that for an arbitrary group action (T, G) on a tree, the operation of taking the (P)-closure behaves well with respect to the subgroup generated by the arc stabilizers.

Proposition 4.7. *Let G be a group acting on a tree T. Then*

$$(G^{(\mathrm{P})})^+ = (G^+)^{(\mathrm{P})}.$$

Proof. We see that $(G^+)^{(\mathrm{P})}$ has the same orbits on arcs as G^+, so $(G^+)^{(\mathrm{P})} = G^+(G^+)^{(\mathrm{P})}_a$ for any $a \in AT$. In turn, it is clear that $G^+ \leqslant (G^{(\mathrm{P})})^+$ and $(G^+)^{(\mathrm{P})}_a \leqslant (G^{(\mathrm{P})})_a \leqslant (G^{(\mathrm{P})})^+$, Thus $(G^{(\mathrm{P})})^+ \geqslant (G^+)^{(\mathrm{P})}$.

It remains to show that $(G^{(\mathrm{P})})^+ \leqslant (G^+)^{(\mathrm{P})}$. In fact it suffices to show that $(G^{(\mathrm{P})})_a \leqslant (G^+)^{(\mathrm{P})}$ for $a \in AT$. In other words we wish to show, for all $a \in AT$, $v \in VT$, F finite subsets of $o^{-1}(v)$ and $g \in (G^{(\mathrm{P})})_a$, there exists $h \in G^+$ such that hg fixes F pointwise. For this discussion we fix a and g; we will proceed by induction on $d = d(v, o(a))$.

In the base case, $d = 0$, in other words $v = o(a)$. Since $g \in G^{(\mathrm{P})}$ there is $h \in G$ such that hg fixes F pointwise; since g fixes a, in fact $h \in G_a \leqslant G^+$.

From now on, we may assume $d > 0$. Let b be the arc in $o^{-1}(v)$ pointing towards $o(a)$. Then by induction there is $h' \in G^+$ such that $h'g$ fixes $\overline{b} \in o^{-1}(t(b))$, and hence fixes b. In turn there is $h \in G$ such that $h(h'g)$ fixes F pointwise. Since $h'g$ fixes b, in fact $h \in G_b \leqslant G^+$, and hence $hh' \in G^+$. This completes the inductive step and hence completes the proof. □

5 Invariant structures

Let G be a group acting on a tree T. In this section, we describe how certain kinds of G-invariant structure in T can be detected from the local action diagram of the action.

Recall that a group action on a tree T is **geometrically dense** if it does not preserve any end or proper subtree of T. Geometrically dense actions often give rise to a simple normal subgroup. We recall two relevant theorems from the literature.

Theorem 5.1 ([12] Théorème 4.5)**.** *Let T be a tree and let G be a geometrically dense subgroup of* $\mathrm{Aut}(T)$ *with property* (P)*. Then every non-trivial subgroup of G normalized by G^+ contains G^+. In particular, G^+ is trivial or abstractly simple.*

Theorem 5.2 ([8] Theorem 6)**.** *Let T be a tree and let G be a closed geometrically dense subgroup of* $\mathrm{Aut}(T)$*. Then every non-trivial closed subgroup of G normalized by G^{++} contains G^{++}. In particular, G^{++} is trivial or topologically simple.*

Remark 5.3. Note that in general $G^{++} \leqslant G^+$; if G is (P)-closed, then equality holds. Since every proper subtree is contained in a half-tree, another way of expressing the condition that G^{++} is nontrivial is the following: there exists $g \in G$, such that the convex hull of $\{v \in VT \mid gv \neq v\}$ is not the whole of T.

5.1 Invariant partial orientations

Definition 5.4. An **orientation** of a graph Γ is a subset O of $A\Gamma$ such that for all $a \in A\Gamma$, either a or $\overline{a}$ is in O, but not both. In particular, an **orientable graph** is a graph in the sense of Serre, in other words, there are no edges such that $a = \overline{a}$. More generally, a **partial orientation** is any (possibly empty) subset O of $A\Gamma$ such that given $a \in A\Gamma$, O does not contain both of a and $\overline{a}$, but it could contain neither of them. We say the (partial) orientation is **G-invariant** if $gO = O$ for all $g \in G$.

Every G-invariant partial orientation of T gives rise to a partial orientation of $G\backslash T$, and conversely. In particular, the local action diagram provides enough information to give a list of the G-invariant partial orientations of T.

Lemma 5.5. *Let Γ be a graph, let $G \leqslant \mathrm{Aut}(\Gamma)$, let $\Gamma' = (G\backslash\Gamma)$ and let $\pi = \pi_{(\Gamma,G)}$. Then a subset O of $A\Gamma$ is a G-invariant partial orientation of Γ if and only if $O = \pi^{-1}(O')$ for some partial orientation O' of Γ'. Moreover, O is a full orientation of Γ if and only if O' is a full orientation of Γ'.*

Proof. Suppose O is a G-invariant partial orientation of Γ. Since O consists of arcs and is G-invariant, we have $O = \pi^{-1}(O')$ for some subset O' of $A\Gamma$. Suppose O' is not a partial orientation, that is, there is $a \in O'$ such that also $\overline{a} \in O'$. Let $b \in \pi^{-1}(a)$. Then

$$\pi(\overline{b}) = \overline{\pi(b)} = \overline{a} \in O',$$

so $\overline{b} \in O$ contradicting the assumption that O is a partial orientation. Thus every G-invariant partial orientation O of Γ arises as $\pi^{-1}(O')$ where O' is a partial orientation of Γ'.

Conversely, suppose O' is a partial orientation of Γ' and let $O = \pi^{-1}(O')$. Then certainly O is a G-invariant set of arcs of Γ; moreover, given $a \in O$, then $\pi(a) \in O'$, and hence

$$\pi(\overline{a}) = \overline{\pi(a)} \notin O',$$

so $\overline{a} \notin O$. Thus O is a partial orientation of Γ.

If O' is an orientation of Γ', then for all $a \in A\Gamma$, either $\pi(a) \in O'$, in which case $a \in O$, or else $\pi(\overline{a}) = \overline{\pi(a)} \in O'$, in which case $\overline{a} \in O$; thus in this case, O is an orientation of Γ. Conversely if O' is not an orientation of Γ', say $O' \cap \{a, \overline{a}\} = \varnothing$ for $a \in A\Gamma'$, then for each $b \in \pi^{-1}(a)$, neither b nor its reverse is contained in O, so O is not an orientation of Γ. ☐

More interesting is to determine, given a partial orientation O of the local action diagram, what kind of invariant structure is being described in the tree. Given Theorem 5.1, partial orientations of T that determine subtrees or ends are of particular interest. Our goal in the rest of this subsection is to use partial orientations to characterize the existence of invariant subtrees or ends in terms of the local action diagram.

Definition 5.6. Say that a partial orientation O of a graph Γ is **confluent** if for every vertex $v \in V\Gamma$, we have $|o^{-1}(v) \cap O| \leqslant 1$. A **strongly confluent**

partial orientation (s.c.p.o.) is a confluent partial orientation such that in addition, for all $v \in V\Gamma$, we have

$$|o^{-1}(v) \cap O| = 1 \Rightarrow \forall a \in o^{-1}(v) : |\{a, \overline{a}\} \cap O| = 1.$$

or in words: if O includes any arc originating at v, then O is a full orientation of the edges incident with v.

Since the quotient map $\pi_{(T,G)}$ is locally surjective and the strongly confluent property is defined using local information, we can easily identify the G-invariant s.c.p.o.s of the tree from the local action diagram.

Lemma 5.7. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$, let $\Delta = (\Gamma, (X_a), (G(v)))$ be the associated local action diagram and let $\pi = \pi_{(T,G)}$. Let O be a partial orientation of Γ. Then the preimage $\pi^{-1}(O)$ is confluent, respectively strongly confluent, on T, if and only if O is confluent, respectively strongly confluent, on Γ and $|X_a| = 1$ for all $a \in O$.*

Proof. Let $v \in VT$. We can calculate the size of $o^{-1}(v) \cap \pi^{-1}(O)$ as follows:

$$|o^{-1}(v) \cap \pi^{-1}(O)| = \sum\{|X_a| \mid a \in o^{-1}(\pi(v)) \cap O\}.$$

In particular, we see that $|o^{-1}(v) \cap \pi^{-1}(O)| \leqslant 1$ if and only if $|o^{-1}(\pi(v)) \cap O| \leqslant 1$ and $|X_a| = 1$ for all $a \in o^{-1}(\pi(v)) \cap O$. This establishes that $\pi^{-1}(O)$ is confluent if and only if O is confluent and $|X_a| = 1$ for all $a \in O$.

Now suppose O and $\pi^{-1}(O)$ are both confluent and that $|X_a| = 1$ for all $a \in O$. We see that

$$o^{-1}(v) \cap \pi^{-1}(O) \neq \varnothing \Leftrightarrow o^{-1}(\pi(v)) \cap O \neq \varnothing.$$

If $o^{-1}(v) \cap \pi^{-1}(O)$ is empty, we do not need to check the strong confluence condition at v or $\pi(v)$, so let us assume that $o^{-1}(v) \cap \pi^{-1}(O)$ is nonempty. Then for O to be strongly confluent, it must induce a full orientation of the edges of Γ incident with $\pi(v)$. In fact, since π is locally surjective, this is equivalent to the condition that $\pi^{-1}(O)$ induces a full orientation of the edges of T incident with v. Thus O is strongly confluent if and only if $\pi^{-1}(O)$ is strongly confluent. □

Accordingly, given a local action diagram $\Delta = (\Gamma, (X_a), (G(v)))$, we define a **(strongly) confluent partial orientation** O of Δ to be a (strongly) confluent partial orientation of Γ such that $|X_a| = 1$ for all $a \in O$.

As we shall see, s.c.p.o.s of a graph only occur in a few special forms, and in the tree case they correspond exactly to subtrees and ends.

Given a graph Γ, we say an induced subgraph Γ' is a **cotree** if for all $v \in V\Gamma \smallsetminus V\Gamma'$, there is a unique path $(v_0, \dots, v_n)$ in Γ such that $v = v_0$, $v_n \in V\Gamma'$, and $v_i \notin V\Gamma'$ for $i < n$, with a unique arc from v_i to v_{i+1} for

$0 \leqslant i < n$. In other words, Γ becomes a tree if we collapse Γ' to a single vertex. Note that a cotree of a connected graph is connected.

A **cycle graph** is a finite connected graph in which all vertices have degree 2. As conventions can differ here, we emphasize that the cycle graph of order 1 consists of a vertex with a loop, but edge-reversal is nontrivial on the loop; the cycle graph of order 2 consists of two vertices with two edges between them. We say a graph is **acyclic** if it has no cycle subgraphs. In particular, trees are precisely the orientable acyclic connected graphs.

We now define three kinds of s.c.p.o.s of the graph Γ.

(a) Given a cotree z of Γ, the associated partial orientation O_z consists of all arcs a such that $o(a) \notin Vz$ and a lies on the directed path from $o(a)$ to z. (In particular, $O_\Gamma = \varnothing$.)

(b) Suppose z is a cycle graph equipped with one of its two cyclic orientations, such that z occurs as a cotree of Γ. Then the associated partial orientation O_z^+ is the union of O_z with the cyclic orientation of z.

(c) If Γ is a tree and we are given an end z of Γ, then for each arc $a \in AT$, exactly one of a and $\overline{a}$ is directed towards ξ, that is, it belongs to a ray in the equivalence class z. The set O_z is then defined to be directed towards z is thus an orientation of T.

Here are some observations on these partial orientations:

(i) The associated partial orientations of type (a), (b) and (c) are all strongly confluent.

(ii) The partial orientations of types (b) and (c) are in fact full orientations of Γ; a partial orientation of type (a) is full if and only if z consists of a single vertex with no edges.

(iii) If z is a cotree or end of Γ and G is a group of automorphisms of Γ, then z is G-invariant if and only if O_z is G-invariant.

Our next goal is to show that the types (a)–(c) actually account for all s.c.p.o.s of graphs, so in particular, in the case of trees they correspond to subtrees and ends.

A confluent partial orientation O of a graph Γ defines a map f_O on $V\Gamma$, as follows: if $o^{-1}(v) \cap O = \{a\}$ we set $f_O(v) = t(a)$, and if $o^{-1}(v) \cap O = \varnothing$ we set $f_O(v) = v$. The **attractor** $K(O)$ of O is then defined to consist of the following:

(i) All vertices of Γ belonging to periodic orbits of f_O;

(ii) All ends of Γ defined by an aperiodic orbit $(v, f_O(v), f_O^2(v), \dots)$.

Thus each $v \in V\Gamma$ defines a nonempty finite subset $z_O(v)$ of the attractor: if $(v, f_O(v), f_O^2(v), \dots)$ is eventually periodic then $z_O(v)$ is the associated periodic orbit, whereas if $(v, f_O(v), f_O^2(v), \dots)$ is aperiodic then $z_O(v)$ is the associated end. We then have $K(O) = \bigcup_{v \in V\Gamma} z_O(v)$.

Attractors of s.c.p.o.s are of a special form, which allows us to recognize the types (a)–(c).

Theorem 5.8. *Let Γ be a connected graph, let O be a s.c.p.o. on Γ and let K be the attractor of O. Then exactly one of the following occurs:*

(a) There is a cotree Γ' of Γ such that $V\Gamma' = K$ and $O = O_{\Gamma'}$;

(b) There is a cotree Γ' of Γ forming a cycle graph such that $V\Gamma' = K$ and $O = O^+_{\Gamma'}$ for one of the cyclic orientations of Γ';

(c) There is an end ξ of Γ such that $K = \{\xi\}$, Γ is a tree and $O = O_\xi$.

Most of the proof will consist of the next two lemmas.

Lemma 5.9. *Let Γ be a graph and let O be a s.c.p.o. of Γ.*

(i) If Γ' is a subgraph of Γ, then $O \cap A\Gamma'$ is a s.c.p.o. of Γ'.

(ii) If Γ is a cycle graph, then O is either empty or it is one of the two cyclic orientations of Γ.

Proof. (i) It is clear that any subset of O is a confluent partial orientation. We also see that

$$o_{\Gamma'}^{-1}(v) \cap O = \{a\} \Rightarrow o_{\Gamma}^{-1}(v) \cap O = \{a\} \Rightarrow t_{\Gamma}^{-1}(v) \subseteq O \cup \{\overline{a}\} \Rightarrow t_{\Gamma'}^{-1}(v) \subseteq O \cup \{\overline{a}\},$$

which ensures that $O \cap A\Gamma'$ is strongly confluent on Γ'.

(ii) It is easy to see that the two cyclic orientations of Γ' are strongly confluent.

Conversely, suppose that O is nonempty, that is, there exists $a \in O$. Then the strong confluence condition means that we must also have $s(a) \in O$, where $s(a)$ is the unique element of $t^{-1}(o(a)) \setminus \{\overline{a}\}$. We then have $s^n(a) \in O$ for all $n \geqslant 0$, and since Γ' is finite, eventually the sequence repeats; without loss of generality, $s^k(a) = a$. The sequence of arcs $a, s^{k-1}(a), s^{k-2}(a), \dots, s(a)$ then defines a directed path from $o(a)$ to $o(a)$ without backtracking; since Γ' is a cycle graph, we conclude that $O = \{a, s^{k-1}(a), s^{k-2}(a), \dots, s(a)\}$ and that O is a cyclic orientation of Γ'. ☐

Lemma 5.10. *Let Γ be a connected graph and let O be a s.c.p.o. on Γ.*

(i) The attractor $K(O)$ contains the vertices of every cycle subgraph of Γ and the endpoint of every non-orientable loop of Γ.

(ii) Suppose there exist $v, w \in V\Gamma$ such that $z_O(v) \neq z_O(w)$. Then $K(O)$ consists exactly of those $v \in V\Gamma$ such that $o^{-1}(v) \cap O = \varnothing$.

Proof. (i) Let v be the endpoint of a loop, that is, there is $a \in A\Gamma$ such that $o(a) = t(a) = v$. If no arc of O originates at v, then $f_O(v) = v$. Otherwise we see that O must contain one of a or $\overline{a}$; we thus end up with an arc in O originating at v that also terminates at v, so $f_O(v) = v$. In either case, we see that $v \in K(O)$.

Let Γ' be a cycle subgraph of Γ of order $\geqslant 2$. By Lemma 5.9, the restriction $O' := O \cap A\Gamma'$ is either empty or one of the two cyclic orientations of Γ'. If O' is a cyclic orientation of Γ', we immediately see that Γ' is a periodic orbit of f_O, so $V\Gamma' \subseteq K(O)$. If instead O' is empty, then for each $v \in V\Gamma'$, O is missing at least two of the arcs of Γ that terminate at v, and hence O is disjoint from $o_\Gamma^{-1}(v)$; this means $f_O(v) = v$, so $v \in K(O)$.

(ii) Let $v_i = f_O^i(v)$ and let $w_i = f_O^i(w)$. Choose $i, j \in \mathbb{N} \times \mathbb{N}$ in such a way that the distance $d(v_i, w_j)$ is minimized; note that $v_i \neq w_j$, so $d(v_i, w_j) > 0$. Let $v'_0 = v_i$ and let $(v'_0, v'_1, \dots, v'_n)$ be a path of minimal length from v_i to w_j. Suppose that there is $v' \in \{v_i, w_j\}$, say $v' = v_i$, such that $o^{-1}(v') \cap O \neq \varnothing$. Then by strong confluence, O must include the arc from v'_1 to v'_0, and then from v'_2 to v'_1 and so on, up to the arc from $v'_n = w_j$ to v'_{n-1}. The definition of f_O then implies that $w_{j+1} = v'_{n-1}$. But this contradicts the choice of (i, j), which was supposed to minimize $d(v_i, w_j)$. Thus

$$o^{-1}(v_i) \cap O = o^{-1}(w_j) \cap O = \varnothing.$$

In particular, v_i and w_j are both fixed by f_O; hence $z_O(v) = \{v_i\}$ and $z_O(w) = \{w_j\}$. We then see by the same argument that given any $w' \in V\Gamma$, then $z_O(w') = \{w''\}$ for some w'' such that $o^{-1}(w'') \cap O = \varnothing$. Conversely, if w'' is any vertex of Γ such that $o^{-1}(w'') \cap O = \varnothing$, then w'' is fixed by f_O, so $w'' \in K(O)$. □

Proof of Theorem 5.8. Suppose that K contains a vertex of Γ. Then by Lemma 5.10(ii), we see that K consists solely of vertices of Γ, and by Lemma 5.10(i), K is the set of vertices of a cotree Γ' of Γ. There are then two possibilities. If K consists of those $v \in V\Gamma$ such that $o^{-1}(v) \cap O = \varnothing$, then we see that case (a) holds. Otherwise, by Lemma 5.10(ii), Γ' is a cycle graph and $O \cap A\Gamma'$ is a cyclic orientation of Γ', and we see that case (b) holds.

The remaining possibility is that K does not contain any vertex of Γ. Then by Lemma 5.10(i), Γ is a tree; by Lemma 5.10(ii), we have $K = \{\xi\}$ for a unique end ξ of Γ. It is then clear that case (c) holds. □

5.2 Tits' theorem revisited

Theorem 5.8 immediately provides a characterization of geometrically dense action in terms of the local action diagram $(\Gamma, (X_a), (G(v)))$: specifically, it should be **irreducible**, meaning that the only s.c.p.o. of Δ is the empty one. (In fact, we only need to know Γ and the colour sets X_a; the additional information provided by the groups $G(v)$ is not needed.)

Corollary 5.11. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$ and let $\Delta = (\Gamma, (X_a), (G(v)))$ be the associated local action diagram. Then G is geometrically dense if and only if Δ is irreducible.*

As stated, Theorem 5.1 leaves open the question of whether or not the group G^+ generated by the arc stabilizers is simple or trivial. In fact, this distinction is easy to detect in the local action diagram.

Say that a local action diagram $\Delta = (\Gamma, (X_v), (G(v)))$ is **free** if for all $v \in V\Gamma$, $G(v)$ acts freely on X_v.

Lemma 5.12. *Let T be a tree, let $G \leqslant \mathrm{Aut}(T)$ and let $\Delta = (\Gamma, (X_v), (G(v)))$ be the associated local action diagram. Then G^+ is trivial if and only if Δ is free.*

Proof. Let $\pi = \pi_{(T,G)}$. Suppose that G^+ is trivial. Then for all $a \in A\Gamma$, G_a is trivial. Let $v \in VT$. Then the action of G_v on $o^{-1}(v)$ is free, since the set of stabilizers of this action is exactly $\{G_a \mid a \in o^{-1}(v)\}$. Thus $G(\pi(v))$ acts freely on $X_{\pi(v)}$.

Conversely, suppose that for all $v \in V\Gamma$, $G(v)$ acts freely on X_v. Let $a \in AT$ and let $g \in G_a$. Suppose $g \neq 1$: then there is some arc $b \notin \{a, \overline{a}\}$ such that b is directed away from a, g fixes $w = o(b)$, but g does not fix b. Then the action of G_w on $o^{-1}(w)$ corresponds as a permutation group to the action of $G(\pi(w))$ on $X_{\pi(w)}$; that is, the action is free modulo kernel. If $w \in \{o(a), t(a)\}$, then clearly g fixes an element of $o^{-1}(w)$; otherwise, g fixes the unique arc in $o^{-1}(w)$ in the direction of $\{o(a), t(a)\}$. Thus g is an element of G_w fixing some element of $o^{-1}(w)$; since the action of G_w on $o^{-1}(w)$ is free modulo kernel, we conclude that $gb = b$, contradicting the choice of b. Thus in fact $g = 1$, showing that G_a is trivial. Since this holds for all $a \in AT$, we conclude that G^+ is trivial. ∎

For a more detailed version of Corollary 5.11, recall that possible structures of invariant subtrees and ends for a group G acting on a tree T fall into a few cases:

(Fixed vertex) G fixes some vertex (not necessarily unique);

(Inversion) G preserves a unique undirected edge and includes a reversal of that edge;

(Lineal) G fixes exactly two ends and includes a translation on the axis between them;

(Horocyclic) G fixes a unique end, does not fix any vertices, and does not include any translations;

(Focal) G fixes a unique end and includes a translation towards this end;

(General type) G preserves a unique minimal subtree T' and acts geometrically densely on T', where T' is the tree spanned by all axes of hyperbolic elements of G.

From a geometric perspective, the fixed vertex and inversion cases can be grouped together as 'bounded' (in other words, every orbit has finite diameter), but from the perspective of local actions it is useful to distinguish them. We can recognize these cases from the local action diagram $\Delta = (\Gamma, (X_a), (G(v)))$ as follows. Given a local action diagram $\Delta = (\Gamma, (X_a), (G(v)))$, we define a **cotree of** Δ to be a cotree Γ' of Γ such that $|X_a| = 1$ for all $a \in O_{\Gamma'}$.

(Fixed vertex) If G fixes a vertex of T, then Γ is a tree. If Γ is a tree, the fixed vertices of G on T (if there are any) correspond to cotrees Γ' of Δ consisting of a single vertex of Γ with no edges.

(Inversion) G preserves a unique undirected edge if and only if there is a cotree Γ' of Δ consisting of a single vertex and a single non-orientable loop a with $|X_a| = 1$.

(Lineal) G fixes exactly two ends and includes a translation on the axis between them if and only if there is a cotree Γ' of Δ that is a cycle graph, such that additionally $|X_a| = 1$ for all $a \in A\Gamma'$.

(Horocyclic) Assuming that G does not fix any vertices, then G fixes a unique end while not including any translations exactly in following situation: Γ is a tree, and there is a unique end ξ of Γ such that $|X_a| = 1$ for every arc a directed towards ξ.

(Focal) The following characterizes the situation where G fixes a unique end and includes a translation towards this end: There is a cotree Γ' of Δ that is a cycle graph, and a cyclic orientation O' of Γ', such that $|X_a| = 1$ for $a \in O'$, but $|X_a| \geqslant 2$ for some $a \in A\Gamma' \setminus O'$.

(General type) In the remaining case, the unique minimal subtree T' on which G acts geometrically densely corresponds to the unique smallest cotree Γ' of Δ, where Γ' is not of the special form indicating a bounded, lineal or focal action, and Δ is not of the form prescribed by a horocyclic action. The action of G on T is geometrically dense if and only if $\Gamma' = \Gamma$.

In all cases except when G fixes a vertex, there are at most two invariant ends, and the invariant ends are easily identified. If G fixes a vertex v of T, then the invariant ends correspond to rays starting at v that are fixed pointwise by G, so they are accounted for by invariant subtrees. We also note that for lineal and focal actions, there is a unique minimal invariant subtree T' spanned by all axes of hyperbolic elements of G; the only distinction from general type is the existence of one or two fixed ends of this subtree.

Note also the following: if Γ is a tree, then the action has a fixed vertex, or is horocyclic or of general type; in particular, by Corollary 2.12, if G is generated by vertex stabilizers then the action must be of one of these types. If Γ is not a tree, then the possibilities are: inversion, lineal, focal and general type.

We also have the following correspondence between types of s.c.p.o. given by $O \mapsto \pi_{(T,G)}(O)$:

(i) Invariant subtrees of T, or equivalently, invariant s.c.p.o.s of T of type (a), correspond to s.c.p.o.s of Δ of type (a).

(ii) Inversion and general type actions do not have invariant ends. Otherwise, there are two kinds of invariant end to consider:

  (1) If Γ is a tree, the action could have a fixed vertex or be horocyclic, with no translations. In this case invariant ends of T correspond to s.c.p.o.s of Δ of type (c).

  (2) If Γ is not a tree, the action could be lineal or focal with a translation towards an invariant end. In this case invariant ends of T correspond to s.c.p.o.s of Δ of type (b), with the order of the associated cycle graph in Γ corresponding to the minimal translation length of a translation towards the fixed end.

5.3 Quotient trees

Given a group G acting on a tree T, a **quotient tree** of the action is a surjective graph homomorphism $\theta : T \to T'$, such that the fibres of θ on vertices and arcs form a system of imprimitivity for G, together with the induced action of G on T'. Our methods are not sufficient to classify quotient trees in general, even if the original action is (P)-closed. However, we can say something in the case that θ is locally surjective.

Given $a \in AT$, say that $\theta : T \to T'$ **backtracks at** a, or a is a **backtracking for** θ, if there exists $b \in AT$ such that

$$o(a) \neq o(b); t(a) = t(b); \theta(a) = \theta(b).$$

It is then easy to see that a graph homomorphism between trees is injective if and only if it has no backtracking. Say that $a \in AT$ is **pre-backtracking** for θ if there is a backtracking arc a' such that $T_{a'} \subseteq T_a$.

The locally surjective property is particularly useful for restricting the possible images of half-trees.

Lemma 5.13. *Let T and T' be trees and let $\theta : T \to T'$ be a locally surjective graph homomorphism.*

(i) *Given* $a' \in AT'$, *then* $T'_{a'} \subseteq \theta(T_a)$, *for all* $a \in \theta^{-1}(a')$.

(ii) *If* $a \in AT$ *is pre-backtracking for* θ *then* $\theta(T_a) = T'$.

Proof. (i) We see that $T'_{a'}$ is the smallest subgraph Γ of T' with the properties that $a' \in A\Gamma$ and for all $v' \in V\Gamma \smallsetminus \{t(a')\}$, then $o_{T'}^{-1}(v') \subseteq A\Gamma$. Consider now the graph $\Gamma = \theta(T_a)$ where $a \in \theta^{-1}(a')$. Clearly $a \in A\Gamma$; given a vertex $v' \in V\Gamma \smallsetminus \{t(a')\}$, then there is $v \in VT_a \smallsetminus \{t(a)\}$ such that $\theta(v) = v'$. We then have $o_T^{-1}(v) \subseteq T_a$ and hence, by local surjectivity, $o_{T'}^{-1}(v') \subseteq A\Gamma$. Thus $T'_{a'} \subseteq \Gamma$.

(ii) Since the half-tree defined by a pre-backtracking arc contains a half-tree defined by a backtracking one, we may assume that a is backtracking. Let $v = t(a)$, let $\Gamma = \theta(T_a)$ and let $b \in AT$ such that

$$o(a) \neq o(b); t(a) = t(b); \theta(a) = \theta(b).$$

Then T_a contains $o_T^{-1}(v) \smallsetminus \{\overline{a}\}$, so in particular $\overline{b} \in A\Gamma$, and we have $\theta(\overline{a}) = \theta(\overline{b})$; thus

$$o_{T'}^{-1}(v) = \theta(o_T^{-1}(v)) = \theta(o_{T_a}^{-1}(v)) \subseteq A\Gamma,$$

since θ is locally surjective. By part (i), we also have $o_{T'}^{-1}(v') \subseteq A\Gamma$ for all other vertices v' of Γ. Since T' is connected we conclude that $\Gamma = T'$. □

Suppose now that we have an action (T, G) on a tree and $\theta : T \to T'$ is a quotient tree that is locally surjective. There are a few ways in which θ can be 'almost injective', which are useful to distinguish.

Lemma 5.14. *Let* (T, G) *be an action on a tree and let* $\theta : T \to T'$ *be a quotient tree. Then one of the following holds.*

(a) *There is a* G*-invariant subtree* T_0 *of* T*, such that the restriction of* θ *to* T_0 *is injective.*

(b) *There is a* G*-invariant end* ξ *of* T*, such that* $a \in AT$ *is pre-backtracking for* θ *if and only if* a *is oriented towards* ξ.

(c) *There is a* G*-invariant subtree* T_0 *of* T *such that every* $a \in AT_0$ *is pre-backtracking for* θ*, and for all arcs* $a \in AT \smallsetminus AT_0$*, then* a *is pre-backtracking for* θ *if and only if* a *is oriented towards* T_0.

Proof. Let B be the set of pre-backtracking arcs for θ, and let C be the set of vertices $v \in VT$ such that $o^{-1}(v) \cap B = \varnothing$. We see that B and C are G-invariant. Moreover, C is convex as a subset of VT, and θ is injective on the subgraph induced by C. Thus as soon as C is nonempty, C spans a subtree T_0 on which (a) holds. We may therefore assume that C is empty. From this, we see that B contains an infinite forwards ray defining some end ξ. It then follows that in fact, every arc pointing towards ξ is pre-backtracking; in particular, B contains an orientation of T.

Let B^* be the set of $a \in AT$ such that $\{a, \overline{a}\} \subseteq B$. There are now two cases. If B^* is empty, then B is exactly the orientation of arcs of T towards some unique end ξ, which is then G-invariant; thus (b) holds. Otherwise we see that B^* spans a G-invariant subtree T_0 satisfying (c). □

By analogy with the usual analysis of actions on trees, we can regard cases (a) and (b) of Lemma 5.14 as degenerate cases of quotient trees; in general there could be many such quotient trees arising from systems of imprimitivity on sets of vertices at a given distance from the invariant tree, or horospheres around the invariant end. In case (c), it makes sense to focus on the restriction of θ to a map from T_0 to $\theta(T_0)$, considered as a quotient tree of the action (T_0, G). So by analogy with the definition of a geometrically dense action, we will say that θ is **densely non-injective** if case (c) of Lemma 5.14 holds with $T = T_0$, that is, every arc is pre-backtracking. Note in particular that if (T, G) is geometrically dense, then it follows from Lemma 5.14 that every non-injective quotient tree is densely non-injective. We then have the following consequence of the previous three lemmas.

Proposition 5.15. *Let (T, G) be an action on a tree and let $\theta : T \to T'$ be a quotient tree. Suppose that θ is locally surjective and densely non-injective. Then G^{++} acts trivially on T'; in particular, θ factors through the locally surjective quotient tree $\pi_+ := \pi_{(T,G^{++})}$.*

Proof. By our hypotheses and Lemma 5.14, every $a \in AT$ is pre-backtracking for θ, so by Lemma 5.13, we have $\theta(T_a) = T'$ for every half-tree T_a of T. In particular, the pointwise stabilizer of T_a must act trivially on T'. Thus the group G^{++} generated by pointwise stabilizers of half-trees acts trivially on T', that is, for all $g \in G^{++}$ and $e \in VT \sqcup AT$, we have $\theta(e) = \theta(ge)$. Thus θ factors through π_+. Since G^{++} is normal in G, it is clear that $G^{++}\backslash T$ is equipped with an action of G such that π_+ is G-equivariant. We see that $(G^{++}\backslash T)$ is a tree by Corollary 2.12, so π_+ is a quotient tree for G. By Lemma 2.1, π_+ is locally surjective. □

Thus if (T, G) is an action on a tree with property (P), then to describe the locally surjective, densely non-injective quotient trees of the action, it is enough to describe the locally surjective quotient trees of the action $(G^+\backslash T, G/G^+)$, starting with $(G^+\backslash T, G/G^+)$ itself. In fact we can describe the action $(G^+\backslash T, G/G^+)$ in general in terms of local action diagrams, without assuming that the original action (T, G) has property (P).

Theorem 5.16. *Let (T, G) be an action on a tree, with local action diagram $\Delta = (\Gamma, (X_a), (G(v)))$. Then $(G^+\backslash T, G/G^+)$ is a (P)-closed action (indeed, a free action on arcs) which depends only on $(\Gamma, (X_a), (\overline{G(v)}))$; in particular it is unchanged if we replace G with $G^{(\mathrm{P})}$. The action $(G^+\backslash T, G/G^+)$ admits the following two equivalent descriptions:*

(i) *It is the* (P)*-closed action admitting the local action diagram* $\Delta^* = (\Gamma^*, (X^*_a), (G^*(v)))$*, where* $\Gamma^* = \Gamma$*; for each* $v \in V\Gamma$*,* $G^*(v) = G(v)/G(v)^+$ *where* $G(v)^+$ *is the subgroup of* $G(v)$ *generated by point stabilizers; and each of the colour sets* X^*_a *is the regular* $G(v)/G(v)^+$*-set obtained as the set of orbits of* $G(v)^+$ *on* X_a*.*

(ii) *It is the fundamental group of a graph of groups over the graph* Γ^i*, arising as the quotient by the action of* G *on the reversal-free subdivision* T^i *of* T*, with the following data. Given* $v \in V\Gamma^i$*, if* v *is the image of a vertex of* T*, then the vertex group is* $G^*(v)$ *as in (i) for the corresponding vertex of* Γ*; if instead* v *is the centre of an edge of* T *that is reversed by* G*, then the vertex group is* C_2 *in its natural action on two points. The edge groups and associated embeddings for the graph of groups are all trivial.*

Proof. We immediately see that in the action $(G^+\backslash T, G/G^+)$, all arc stabilizers are trivial; it is then clear that this action has property (P). In this context, it is easy to see how the notions of local action diagrams and graphs of groups are equivalent in such a manner that (i) and (ii) are equivalent.

Let $L = G^{(\mathrm{P})}$. By Proposition 4.7 we have $L^+ = (G^+)^{(\mathrm{P})}$. In particular, L^+ and G^+ have the same orbits on arcs, so $L^+\backslash T = G^+\backslash T$. Since G and L have the same orbits on arcs, we have $L = GL_a$ and in particular $L = GL^+$, so for every element of L, there is an element of G with the same action on $L^+\backslash T$. We can also take $\overline{\Delta} = (\Gamma, (X_a), (\overline{G(v)}))$ as the local action diagram of L; since point stabilizers in permutation groups are open, we see for all $v \in V\Gamma$ that

$$\overline{G(v)}/\overline{G(v)}^+ = G(v)/G(v)^+$$

as permutation groups acting on $X_v/G(v)^+ = X_v/\overline{G(v)}^+$. So for the rest of the proof, it makes no difference if we replace G with L, so we may assume that G is (P)-closed.

It is now clear that $(G^+\backslash T, G/G^+)$ has local action diagram $(\Gamma, X^*_a, (G^*(v)))$ for some permutation groups $G^*(v) = G(v)/N(v)$ and colour sets $X^*_a = N(o(a))\backslash X_a$. All that remains is to derive the vertex groups $G^*(v)$ and their orbits from Δ.

Let $\pi = \pi_{(T,G)}$. We may suppose that $G = U(\mathbf{T}, (G(v)))$ where $\mathbf{T}$ is T equipped with some Δ-colouring $\mathcal{L}$; we can moreover choose the colouring $\mathcal{L}$ so that for each vertex $v \in VT$, any two elements of $t^{-1}(v)$ of the same type have the same colour. For each $v \in V\Gamma$, let $G(v)^+$ be the subgroup of $G(v)$ generated by the point stabilizers. Let H be the set of $g \in G$ such that $\sigma_{\mathcal{L},v}(g) \in G(\pi(v))^+$ for all $v \in VT$. Applying the usual product and inverse formulae for $\sigma_{\mathcal{L},v}$, we see that H is in fact a subgroup of G; since each of the groups $G(v)^+$ is closed (indeed, open) in $G(v)$, we also see that H is closed.

Our next aim is to show that $G^+ \leqslant H$; since H is a group, it is enough to show $G_a = H_a$ for each $a \in AT$. It is easy to see that H_a is the product of the pointwise stabilizers of the two half-trees defined by a, from which we conclude that H has property (P). Write $v_0 = o(a)$, let B_r be the ball of the radius r around v_0 and let G_r, respectively H_r, be the pointwise stabilizer of B_r in G, respectively H. We see that $\sigma_{\mathcal{L},v_0}(H_a) = \sigma_{\mathcal{L},v_0}(G_a) = G(v)_{\mathcal{L}(a)}$, so $G_a = H_a G_1$. Suppose that $G_a = H_a G_r$ for some $r \geqslant 1$, let $g \in G_r$ and let v be a vertex at distance r from v_0. By the same argument as for v_0, we can realise the action of g on $o^{-1}(v)$ (that is, we obtain an element of the same left coset of the fixator of $o^{-1}(v)$) using some element h_v of $H_{a'}$, where a' is the element of $o^{-1}(v)$ contained in B_r. By property (P) we can take h_v to lie in the pointwise stabilizer of $T_{a'}$; in particular, $h_v \in H_r$. By taking a product of such elements as v ranges over the sphere of radius r around v_0, we in fact obtain $h \in H_r$ such that $g \in hG_{r+1}$. Thus by induction we have $G_a = H_a G_r$ for all $r \geqslant 1$. Since H_a is closed, we conclude that $G_a = H_a$ as desired.

Thus $G^+ \leqslant H$, and hence the local action of G^+ at each vertex is a subgroup of $G(v)^+$ for the appropriate $v \in V\Gamma$. On the other hand, given $v \in VT$, we see that $\sigma_{\mathcal{L},v}(G^+_v)$ contains $G^+(\pi(v))$, by considering the subgroups G_a of G^+_v for $a \in o^{-1}(v)$. Thus in fact the local action of G^+ at each vertex is exactly $G(v)^+$, which is the same as the local action of the vertex stabilizer in G^+ at that vertex. Given that π_+ is locally surjective, we then see that $G^*(v)$ is exactly $G(v)/G(v)^+$ acting on the set of orbits of $G(v)^+$ on X_v; this set decomposes into regular $G(v)/G(v)^+$-sets as described. □

So far, in our effort to describe interesting quotient trees of a (P)-closed action (T, G), we have reduced to an action $(T', G') = (G^+\backslash T, G/G^+)$ on a tree in which arc stabilizers are trivial. In general, such an action can admit further quotient trees, such as those arising from the quotient of the action of some normal subgroup generated by vertex stabilizers. However, from this point onwards, the local action diagram approach effectively reduces to a well-known special case of classical Bass–Serre theory, so it is unlikely that the methods developed in this article will give new insights. We therefore leave any further investigation to the interested reader.

We can now derive Theorem 1.7.

Proof of Theorem 1.7. Suppose that (i) holds. We can rule out a lineal or focal action by the fact that G is a simple group, and a bounded or horocyclic action is ruled out since G contains a translation. Thus the action is of general type, so there is a unique smallest G-invariant subtree T', which has infinite diameter, and the action of G on T' is geometrically dense. In particular, G acts nontrivially, hence faithfully on T'. The action on G on T' is (P)-closed by Lemma 2.7. From now on we focus on the action (T', G) and define subgroups of G relative to this action. Since the action is geometrically dense, Δ is irreducible by Theorem 1.4. Our hypotheses ensure that all arc stabilizers in G are nontrivial. In particular, G^+ is nontrivial; since

G is simple it follows that $G = G^+$. In the terminology of Theorem 5.16, we then have $\{1\} \neq G(v) = G(v)^+$, that is, $G(v)$ is generated by point stabilizers; the group $G(v)$ is closed since G acts as a closed subgroup of $\mathrm{Aut}(T')$. We also see that G is generated by vertex stabilizers, so Γ is a tree by Corollary 2.12. Thus (i) implies (ii).

For the remainder of the proof we suppose that (ii) holds. The action on G on T' is faithful by assumption; it is (P)-closed, in particular, closed, by Lemma 2.7; and it is geometrically dense by Corollary 5.11. The fact that G has a geometrically dense action on an infinite subtree ensures that the original action is not bounded or horocyclic. Let $v \in V\Gamma$ be such that $G(v) \neq \{1\}$. Since we are assuming that $G(v)$ is generated by point stabilizers, there is a nontrivial point stabilizer in $G(v)$, which implies that there is a nontrivial arc stabilizer G_a for some $a \in AT'$. By property (P), the pointwise stabilizer of one of the half-trees of T' defined by a, say T'_a, is nontrivial. Since the action of G on T' is geometrically dense, any half-tree of T' can be mapped inside any other by the action of G, so in fact the pointwise stabilizer of T'_a is nontrivial for every $a \in AT'$. In particular, the pointwise stabilizer of any finite set of vertices of T' is nontrivial. The stabilizer of a vertex of T outside T' is just the stabilizer of the closest vertex in T', so we see that there is no finite set of vertices of T whose pointwise stabilizer is trivial.

Define G^+ with respect to the action (T', G). We have seen that arc stabilizers are nontrivial, so G^+ is nontrivial and hence simple by Theorem 5.1. The fact that $G(v)$ is generated by point stabilizers for every $v \in V\Gamma$ implies, by Theorem 5.16, that G acts freely on vertices of $(G^+ \backslash T')$, in other words, G^+ contains the vertex stabilizers of G. On the other hand, since Γ is a tree, Corollary 2.12 implies that G is generated by vertex stabilizers. Thus $G = G^+$ and hence G is simple. □

6 The group topology

Our definitions ensure that whenever $G \leqslant \mathrm{Aut}(T)$ is such that $G = G^{(\mathrm{P})}$, then G is a closed subgroup of $\mathrm{Aut}(T)$ in the permutation topology. In particular, it follows that G is a non-Archimedean topological group in its own right.

There are natural characterizations of when $G^{(\mathrm{P})}$ is Polish or locally compact as a subgroup of $\mathrm{Aut}(T)$.

Lemma 6.1. *Let T be a tree and let $G \leqslant \mathrm{Aut}(T)$. Let $a \in AT$ and let B be a bounded subtree of VT. Let S be the smallest subtree containing Ga; suppose that S has no leaves. Then there exists $g \in G$ such that*

$$0 < d(t(ga), B) < d(o(ga), B).$$

Proof. Suppose a lies on the axis of some hyperbolic element g of G. Without loss of generality, we identify the vertices of the axis with $\mathbb{Z}$, where $o(a) = 1$,

$t(a) = 0$ and there is $n \in \mathbb{N}$ such that $g(v) = v + n$ for all $v \in \mathbb{Z}$. Let B' be the set of vertices on the axis that realise the minimum distance from the axis to B. Then B' is a finite set, since B is bounded. We then see that the inequalities

$$0 < d(t(g^m a), B') < d(o(g^m a), B').$$

are achieved for all sufficiently large $m > 0$. Taking such an m, it then follows (since all paths from the axis to B must pass through B') that

$$0 < d(t(g^m a), B) < d(o(g^m a), B).$$

In the remaining case, a does not lie on the axis of any hyperbolic element of G. Since S has no leaves, it is unbounded, and hence Ga is unbounded. Thus by replacing a with some G-translate of a, without loss of generality we may assume that neither $o(a)$ nor $t(a)$ is a vertex of B. Let ρ be the projection of T onto the subgraph spanned by a. Then B lies in one of the half-trees defined by a, in other words, either $\rho(B) = \{o(a)\}$ or $\rho(B) = \{t(a)\}$. If $\rho(B) = \{t(a)\}$ then the required inequalities hold for $g = 1$, so we may assume $\rho(B) = \{o(a)\}$. Consider now the tree T' spanned by $\rho^{-1}(o(a)) \cup \{a\}$. If $Ga \subseteq T'$, then also $S \subseteq T'$. But in that case, $t(a)$ would be a leaf of S, which is forbidden by hypothesis. Thus there exists $g \in G$ such that $\rho(ga) = \{t(a)\}$. If $d(t(ga), t(a)) > d(o(ga), t(a))$, then we see from the relative orientations of a and ga that both arcs lie on the axis of g, which contradicts our assumptions. Thus $d(t(ga), t(a)) < d(o(ga), t(a))$. Since all paths from ga to B must pass through $t(a)$, it then follows that

$$0 < d(t(ga), B) < d(o(ga), B). \qquad \square$$

Here is the characterization of local compactness of the (P)-closure.

Proposition 6.2. *Let T be a tree and let $G \leqslant \mathrm{Aut}(T)$. Suppose that there is a unique minimal G-invariant subtree T', such that $|VT'| \geqslant 3$. Then the following are equivalent.*

(i) The (P)*-closure of G is locally compact.*

(ii) For all $a \in AT'$, the stabilizer of a in the (P)*-closure of G is compact.*

(iii) Let $\Delta = (\Gamma, (X_a), (G(v)))$ be the local action diagram for (T, G) and let Γ' be the unique smallest cotree of Δ. Then for all $a \in A\Gamma$ such that $\overline{a} \notin O_{\Gamma'}$, every $G(o(a))$-stabilizer of every point in X_a has finite orbits on X_v.

Proof. The hypotheses ensure that T' has infinite diameter and is leafless. Let $\Gamma = G\backslash T$ and let $\pi : T \to \Gamma$ be the quotient map.

Suppose that (i) holds. Then there is some finite set B of vertices of T, such that the pointwise stabilizer H of B in $G^{(\mathrm{P})}$ is compact. Let $a \in AT'$. Then the hypotheses of Lemma 6.1 hold; thus there is $g \in G$ such that

$$0 < d(t(ga), B) < d(o(ga), B).$$

We now have

$$G^{(\mathrm{P})}_{ga} = R_1 \times R_2$$

where R_1 fixes the half-tree T_{ga} pointwise and R_2 fixes the half-tree $T_{\overline{ga}}$ pointwise. Our choice of g ensures that $B \subseteq T_{ga}$; thus $R_1 \leqslant H$, so R_1 is compact. In particular, $G^{(\mathrm{P})}_{ga}$ has finite orbits on $T_{\overline{ga}}$. After conjugating by g^{-1} we see that $G^{(\mathrm{P})}_{a}$ has finite orbits on the half-tree $T_{\overline{a}}$. A similar argument using $\overline{a}$ in place of a shows that $G^{(\mathrm{P})}_{a} = G^{(\mathrm{P})}_{\overline{a}}$ also has finite orbits on the complementary half-tree T_a, so a satisfies (ii). Thus (i) implies (ii). It is immediately clear that (ii) implies (i), so (i) and (ii) are equivalent.

We observe that any compact subgroup of $\mathrm{Aut}(T)$ has finite orbits, and that G is a subgroup of $G^{(\mathrm{P})}$. Given these observations, it is immediate that (ii) implies (iii).

Suppose (ii) holds; note that the unique smallest cotree Γ' of the local action diagram is the image of T'. Let $v \in VT$ and $a \in o^{-1}(v)$ such that $\overline{\pi(a)} \notin O_{\Gamma'}$. If $\pi(a) \notin O_{\Gamma'}$, then $\pi(a) \in A\Gamma'$ and it is clear from (ii) that G_a has finite orbits on VT. Otherwise, the fact that $\pi(a) \in O_{\Gamma'}$ ensures that a points towards T'. By property (P) the action of $G^{(\mathrm{P})}_a$ on $T_{\overline{a}}$ is then the same as the action of the pointwise fixator of T_a on $T_{\overline{a}}$; since $T' \subseteq T_a$, it follows from (ii) that G_a has finite orbits on $T_{\overline{a}}$, and in particular on $o^{-1}(v)$. It follows that in the local action diagram, the action of $G(\pi(v))$ on X_v is such that every point stabilizer has finite orbits. Thus (ii) implies (iii).

Suppose (iii) holds. We can regard $G^{(\mathrm{P})}$ as the universal group of $\overline{\Delta} = (\Gamma, (X_a), (\overline{G(v)}))$. Then Γ' is the unique smallest cotree of $\overline{\Delta}$ and we see that $\Gamma' = \pi(T')$.

Consider a path $(v_0, \dots, v_n)$ in VT; let a_i be the arc from v_{i-1} to v_i and suppose that $a_1 \in AT'$. We then see that each arc a_i is either contained in T' or points away from it, so for all i we have $\pi(a_i) \notin O_{\Gamma'}$. Thus for each $i \geqslant 0$, the stabilizer in $G(v_i)$ of any point in $X_{\overline{\pi(a_i)}}$ has finite orbits on $X_{\pi(v_i)}$. Translating this information back to the tree: in the action of the stabilizer in $G^{(\mathrm{P})}$ of $\overline{a_i}$, or equivalently of a_i, the orbit of v_{i+1} is finite. We conclude that in the action of $G^{(\mathrm{P})}_{a_1}$, the orbit of v_n is finite. Given the freedom of choice of the path $(v_0, \dots, v_n)$, we conclude that $H = G^{(\mathrm{P})}_{a_1}$ has finite orbits on the half-tree T_{a_1}; by replacing a_1 with $\overline{a_1}$, a similar argument shows that H has finite orbits on the complementary half-tree $T_{\overline{a_1}}$. Since H is also closed in $\mathrm{Aut}(T)$, it is compact. Moreover, a_1 can be chosen to be any arc of AT'; thus every stabilizer in $G^{(\mathrm{P})}$ of an arc of T' is compact.

Thus (iii) implies (ii), completing the proof that all three statements are equivalent. □

Corollary 6.3. *Let T be a tree and let G be a closed subgroup of* $\mathrm{Aut}(T)$ *that does not fix any vertex or preserve any undirected edge. Suppose that $G^{(\mathrm{P})}$ is locally compact, and let T' be a G-invariant subtree of T. Then the kernel of the action of G on T' is compact.*

Proof. Since G does not fix any vertex or preserve any undirected edge, we are in the situation of Proposition 6.2: there is a unique G-invariant subtree T'', such that $|VT''| \geqslant 3$. In particular, $T'' \subseteq T'$.

Let K be the kernel of the action of G on T'. We see by Proposition 6.2(ii) that K has finite orbits on VT. Since G is closed in $\mathrm{Aut}(T)$, we see that K is also closed, so K is compact. □

Here is a characterization of Polish (P)-closed groups. For clarity, we note that the word 'countable' here is understood to allow finite sets as well as countably infinite sets.

Let X be a set, let G act by permutations on X and let $Y \subseteq X$ be G-invariant. We say that the action of G on Y is **strongly faithful (relative to X)** if for all $x \in X$, there exists a finite subset $\{y_1, \dots, y_n\}$ of Y such that $\bigcap_{i=1}^{n} G_{y_i}$ fixes x.

Lemma 6.4. *Let X be a set and let G be a closed subgroup of* $\mathrm{Sym}(X)$*. Then G is Polish if and only if there is a countable subset Y of X on which G acts strongly faithfully. Moreover, if Y is a countable strongly faithful set for G, then the induced homomorphism $\theta : G \rightarrow$* $\mathrm{Sym}(Y)$ *is a closed topological embedding.*

Proof. We note first that $\mathrm{Sym}(X)$ is a non-Archimedean topological group, that is, it has a base of neighbourhoods of the identity consisting of open subgroups. It follows that any subgroup of $\mathrm{Sym}(X)$ with the subspace topology is also non-Archimedean.

Suppose Y is a countable strongly faithful set for G and let $\theta : G \rightarrow \mathrm{Sym}(Y)$ be the natural homomorphism. Then θ is clearly injective and continuous. To show that θ is a closed embedding, we need to show that given a net $(g_i)_{i \in I}$ in G such that $\theta(g_i)$ converges to the identity, then (g_i) converges to the identity. Indeed, since $\theta(g_i) \rightarrow 1$ as $i \rightarrow \infty$, we see that for all $n \geqslant 0$, there exists i_n such that g_i fixes $y_0, \dots, y_n$ for all $i > i_n$, so $g_i \in G_n$; since G_n is a base of neighbourhoods of the identity, it follows that $g_i \rightarrow 1$ as $i \rightarrow \infty$. Thus θ is a closed topological embedding as claimed.

In particular, we see from the previous paragraph that if a countable strongly faithful set Y exists for G, then G is isomorphic to a closed subgroup of the Polish group $\mathrm{Sym}(Y)$; thus G is Polish.

Conversely, suppose that G is Polish. In particular, G is separable, so all open subgroups of G have countable index; thus G has countable orbits on X.

Since G is non-Archimedean and metrizable, there is a countable set of open subgroups of G forming a base of neighbourhoods of the identity; recalling the standard base of topology for $\mathrm{Sym}(X)$, in fact there is a sequence $(y_i)_{i \geqslant 0}$ of points such that $\{G_n \mid n \geqslant 0\}$ is a base of neighbourhoods of the identity, where

$$G_n = \bigcap_{i=0}^{n} G_{y_i}.$$

Given $x \in X$, we then see that $G_n \leqslant G_x$ for some n, showing that the action of G on Y is strongly faithful. □

Proposition 6.5. *Let T be a tree and let $G \leqslant \mathrm{Aut}(T)$ be* (P)*-closed. Then the following are equivalent.*

(i) The permutation topology on G is Polish.

(ii) There is a countable G-invariant subtree T' of T on which G acts strongly faithfully.

(iii) There is a countable G-invariant subtree T' of T on which G acts faithfully and such that for each $v \in VT'$, the action of G_v on $o^{-1}_{T'}(v)$ is strongly faithful relative to $o^{-1}_T(v)$.

(iv) The local action diagram $\Delta = (\Gamma, (X_a), (G(v)))$ for (T, G) is such that all the colour sets X_a are countable and the permutation groups $G(v)$ are Polish, and there exists a countable cotree Γ' of Δ such that $G(v) = \{1\}$ whenever $v \in V\Gamma \setminus V\Gamma'$.

Proof. Suppose (i) holds. Then by Lemma 6.4, there is a countable set Y of vertices on which G acts strongly faithfully relative to VT. We then see that the unique smallest subtree T' of T containing Y is countable; by the construction, T' is G-invariant and G acts strongly faithfully on VT' relative to VT. Thus (i) implies (ii).

Suppose (ii) holds; let $v \in VT'$ and let $a \in o^{-1}(v)$ such that $a \notin AT'$. Since G acts strongly faithfully on T', there are vertices $w_1, \dots, w_n$ in T' such that any element of G that fixes $w_1, \dots, w_n$ also fixes $t(a)$. For each w_i, let a_i be the first arc on the path from v to w_i, and let $g \in G_v$ be such that g fixes a_i for $1 \leqslant i \leqslant n$. Using property (P) and induction on n, we see that there is $g' \in G$ such that g' fixes each of the half-trees T_{a_i} pointwise, but has the same action as g on the vertices outside of $\bigcup_{i=1}^{n} T_{a_i}$. In particular, g' fixes $w_1, \dots, w_n$, so g' also fixes $t(a)$; since $a \notin AT'$, we see that $t(a)$ is not contained in $\bigcup_{i=1}^{n} T_{a_i}$, from which it follows that g also fixes $t(a)$, and hence g fixes a. This proves that relative to the action on $o^{-1}_T(v)$, the action of G_v on $o^{-1}_{T'}(v)$ is strongly faithful. Thus (ii) implies (iii).

Suppose (iii) holds. Then every vertex stabilizer in G also has countable orbits on T, ensuring that the colour sets X_a are countable. The countable

G-invariant subtree T' gives rise to a countable cotree Γ'. Consider $v_0 \in VT \smallsetminus VT'$. Then there is a unique shortest path $(v_0, \dots, v_n)$ from v_0 to T'; in particular, G_{v_0} fixes the arc a from v_0 to v_1. Then T' is contained in the half-tree T_a, so G_a acts faithfully on T_a. By property (P), it follows that the action of G_a on $T_{\overline{a}}$ is trivial; in particular, the action of G_{v_0} on $T_{\overline{a}}$ is trivial. Since G_{v_0} also fixes a, we conclude that G_{v_0} fixes every neighbour of v_0. Thus $G(\pi(v_0)) = \{1\}$ and we see that $|X_a| = 1$ whenever $a \in o^{-1}(\pi(v_0))$. Now consider $v \in VT'$; we see that the condition that the action of G_v on $o^{-1}_{T'}(v)$ is strongly faithful relative to $o^{-1}_T(v)$ translates exactly to the condition that for the action of $G(\pi(v))$ on $X_{\pi(v)}$, we have a strongly faithful action on the subset

$$Y = \bigsqcup_{a \in o^{-1}_{\Gamma'}(\pi(v))} X_a$$

of X. We observe that Y is countable; since G is closed we see that $G(\pi(v))$ is closed, and hence $G(v)$ is Polish by Lemma 6.4; in particular, $G(\pi(v))$ has countable orbits, so $|X_a| \leqslant \aleph_0$ for all $a \in o^{-1}(\pi(v))$. This completes the proof that (iii) implies (iv).

Suppose (iv) holds. Consider the construction of a Δ-tree T starting from a root vertex () mapping to a base vertex $v_0 \in V\Gamma'$. The cotree Γ' then gives rises to a G-invariant subtree T' of T; the fact that both Γ' and the colour sets X_a are countable ensures that T' is countable. For each $v \in VT'$, we add a countable union of G_v-orbits of neighbours of v to T' to produce a new tree T'', such that the action of G_v on $o^{-1}_{T''}(v)$ is strongly faithful relative to $o^{-1}_T(v)$; this is possible by Lemma 6.4 since $G(\pi(v))$ is Polish, and we see that it can be done in such a way that T'' is G-invariant. Now let $v_0 \in VT \smallsetminus VT''$ and let $(v_0, \dots, v_n)$ be the shortest path from v_0 to T'. Let a be the arc from v_n to v_{n-1}. Our choice of T'' ensures that there are arcs $a_1, \dots, a_m \in o^{-1}_{T''}(v_n)$ such that given $g \in G$ that fixes $a_1, \dots, a_m$, or equivalently, given $g \in G$ that fixes $t(a_1), \dots, t(a_m), v_n$, then g fixes a, and hence g fixes v_{n-1}. In turn, using the fact that the local actions for vertices outside of Γ' are trivial, we see that $G_{v_{n-1}}$ fixes each of the vertices $v_0, \dots, v_{n-2}$. We have thus obtained a finite subset $\{t(a_1), \dots, t(a_m), v_n\}$ of VT'' whose pointwise stabilizer also fixes v_0. Since $v_0 \in VT \smallsetminus VT''$ was arbitrary, we conclude that G acts strongly faithfully on T''. It then follows by Lemma 6.4 that G is Polish. Thus (iv) implies (i) and the cycle of implications is complete. □

When G is locally compact, it is natural to ask if G is compactly generated. As in Proposition 6.2, for simplicity we will avoid the case when G fixes a vertex or preserves an undirected edge. Excluding these degenerate cases, compact generation of G is easily seen to be equivalent to compact generation of $G^{(\mathrm{P})}$, so the question of whether or not G is compactly generated can be reduced to the local action diagram.

Proposition 6.6. *Let T be a tree and let $G \leqslant \mathrm{Aut}(T)$ be closed with unbounded action. Let $\Delta = (\Gamma, (X_a), (G(v)))$ be the local action diagram. Suppose that there exists $a \in AT$ for which G_a is compact. Then G and $G^{(\mathrm{P})}$ are locally compact, and the following are equivalent.*

(i) G is compactly generated;

(ii) $G^{(\mathrm{P})}$ is compactly generated;

(iii) there is a unique smallest G-invariant subtree T' such that G has finitely many orbits on $VT' \sqcup AT'$ and G_v is compactly generated for each $v \in VT'$;

(iv) there is a unique smallest cotree Γ' of Δ such that Γ' is finite and $G(v)$ is compactly generated for each $v \in V\Gamma'$.

Proof. Since G_a is open in G, we see that G is locally compact. We also see that G_a has finite orbits on VT; since $G_a^{(\mathrm{P})}$ is closed and has the same orbits it follows that $G_a^{(\mathrm{P})}$ is compact, and hence $G^{(\mathrm{P})}$ is locally compact. Since $G^{(\mathrm{P})}$ has the same orbits on AT as G does, we have $G^{(\mathrm{P})} = GG_a^{(\mathrm{P})}$, so G is cocompact in $G^{(\mathrm{P})}$. Thus G is compactly generated if and only if $G^{(\mathrm{P})}$ is compactly generated, that is, (i) and (ii) are equivalent.

Suppose that the action is horocyclic with unique fixed end ξ. We see that G preserves every horoball around ξ; in particular, there is no minimal G-invariant subtree, so (iii) is false. For the same reason every cotree of Δ is infinite, so (iv) is false. We also see that G acts without reversal and every element of G fixes a vertex, and yet the action is unbounded; thus by Lemma 2.13, G cannot be compactly generated, that is, (i) is false, so (ii) is also false. So if the action is horocyclic then (i)–(iv) are all false, which is consistent with them being equivalent.

For the remainder of the proof we may suppose that the action is not horocyclic. It follows that G contains a translation, and hence there is a unique smallest G-invariant subtree T' of T, which is spanned by the axes of translations in T.

Suppose (i) holds. Choose a compact symmetric generating set S for G and a vertex $v \in VT'$. Then the set $\{sv \mid s \in S\}$ is finite and for each $s \in S$, the path $[v, sv]$ from v to sv is contained in T'. Let T'' be the subtree spanned by the paths $[v, sv]$ as s ranges over S. Then T'' is finite and for each $s \in S$, the graph $T'' \cup sT''$ is connected: specifically, both T'' and sT'' are connected and contain sv. From here, we see that the graph $\bigcup_{g \in G} gT''$ is also connected, and hence equal to T'. This shows that G has finitely many orbits on $VT' \sqcup AT'$.

The rest of the proof that (i) implies (iii) is [4, Proposition 4.1], however for clarity we give a more elementary proof using Bass–Serre theory.

Write Γ^i for the quotient graph of the action of G on the reversal-free subdivision of T'. From the hypotheses, we see that G acts on T' with compact kernel; let G^* be the group defined by this action. Recall the decomposition of G^* given by Theorem 2.9:

$$\frac{F(E) * \ast_{v \in VT^*} G_v^*}{\langle\langle s_a \tau_a(g) s_{\overline{a}} \tau_{\overline{a}}(g)^{-1} \ (a \in E, g \in G_a), \ s_a s_{\overline{a}}, \ s_a \ (a \in AT^*) \rangle\rangle}.$$

where now T^* is a lift in T' of a maximal subtree of Γ^i and E is a lift of the arcs of Γ^i such that every arc in E is incident with T'. We remark that this decomposition is also well-behaved with respect to the topology of G^*, and hence of G, since the factors are amalgamated along open subgroups.

We claim that G_v is compactly generated for all $v \in VT'$; it is enough to show that G_v^* is compactly generated for each $v \in VT^*$. Fix $v \in VT^*$. We have a compactly generated open subgroup H_0 of G_v^* generated by $\tau_a(G_a^*)$ for all $a \in E$ such that $t(\pi_{(T',G)}(a)) = \pi_{(T',G)}(v)$. In particular, in the expression for G^*, every element of G_v^* that is amalgamated with other vertex groups is contained in H_0.

Now write G_v^* as a directed union $\bigcup_{i \in I} H_i$, where each H_i is compactly generated and 0 is the least element of I, and recall the normal form theorem for graphs of groups (Theorem 2.11). Let K_i be the set of elements of G^* expressible as a reduced word (including the empty word), such that all letters taken from G_v^* belong to H_i. Then $G^* = \bigcup_{i \in I} K_i$; since G^* is compactly generated and H_0 is open in G^*, in fact $G' = K_i$ for some i. In particular, every $g \in H_i$ is expressible as a reduced word using $F(E)$, H_i and $G_{v'}^*$ for vertices $v' \in VT^*$ other than v. Given the reduction rules for words in a graph of groups, we conclude that $g \in H_i$. Thus $G^* = H_i$m showing that G^* is compactly generated as required. This completes the proof that (i) implies (iii).

Suppose now that (iii) holds. Then G_v^* is also compactly generated for each vertex v of the reversal-free subdivision of T', and hence we see from the free product decomposition that G^* is compactly generated. Since G acts on T' with compact kernel it follows that G is also compactly generated. Thus (iii) implies (i) and hence (i), (ii) and (iii) are equivalent.

Note that Γ' is finite if and only if G has finitely many orbits on $VT' \sqcup AT'$. We also see that, given $v \in V\Gamma'$ and the chosen representative v^* of $\pi^{-1}_{(T',G)}(v)$, the local action map $\theta : G_{v^*} \to G(v)$ is a quotient homomorphism with compact kernel: the kernel is compact since at least one of the arc stabilizers G_a for $a \in o^{-1}(v^*)$ is compact. It follows that G_{v^*} is compactly generated if and only if $G(v)$ is compactly generated, and in particular, (iii) and (iv) are equivalent. This completes the proof that (i)–(iv) are equivalent. □

We can now deduce Theorem 1.8 and Corollary 1.9 as special cases of the previous propositions.

Proof of Theorem 1.8. Let T be the defining tree of $U(\Delta)$.

Suppose (i) holds. Then by Proposition 6.2, since $U(\Delta)$ is locally compact and the action is geometrically dense, all arc stabilizers of $U(\Delta)$ acting on T are compact. By Proposition 6.6 and the fact that $U(\Delta)$ is locally compact, Γ is finite and $G(v)$ is closed and compactly generated for all $v \in V\Gamma$. The fact that arc stabilizers are compact implies that each of the groups $G(v)$ is subdegree-finite. Finally, Δ is irreducible by Theorem 1.4. Thus (i) implies (ii).

Conversely, suppose (ii) holds. Since Δ is irreducible, $(T, U(\Delta))$ is geometrically dense by Theorem 1.4. Since the local actions are all subdegree finite, $U(\Delta)$ has compact arc stabilizers and hence is locally compact. We see that $U(\Delta)$ is compactly generated by Proposition 6.6. Thus (ii) implies (i).

Suppose (i) and (ii) hold. Taking $v \in V\Gamma$, then as a permutation group, $G(v)$ has finitely many orbits and is a compactly generated locally compact group; it follows that X_v is countable. Since Γ is finite we conclude that T is countable. Thus $\mathrm{Sym}(VT \sqcup AT)$ is Polish, and hence its closed subgroup $U(\Delta)$ is Polish. □

Proof of Corollary 1.9. Suppose (i) holds. By hypothesis the action does not fix a vertex, and the fact that G is nondiscrete and simple rules out actions that are of inversion, lineal or focal type. Proposition 6.6 then rules out a horocyclic action. By process of elimination, the action is of general type, with a unique smallest G-invariant subtree T' on which the action of G is faithful and geometrically dense. In fact the action of G on T' is strongly faithful: the topology of G is already σ-compact, so there is no coarser locally compact group topology on G. Thus (via Lemma 2.7) we can regard G as a (P)-closed, hence closed, subgroup of $\mathrm{Aut}(T')$ as a topological group. Since G is nondiscrete, there is no finite set of vertices whose pointwise stabilizer is trivial. We then conclude by Theorem 1.7 that Δ is irreducible, Γ is a tree, and each of the groups $G(v)$ is closed and generated by point stabilizers, with $G(v) \neq \{1\}$ for some $v \in V\Gamma$. By Theorem 1.8, Γ is finite and each of the groups $G(v)$ is compactly generated and subdegree-finite. This completes the proof that (i) implies (ii).

Conversely, suppose that (ii) holds; as before, we can regard G as a (P)-closed, in particular closed, subgroup of $\mathrm{Aut}(T')$, and thus identify G with $U(\Delta)$ as a topological group. We see immediately by Theorem 1.7 that G is a nondiscrete simple group, and it is clear that G does not fix any vertex of T. By Theorem 1.8, $U(\Delta)$ is compactly generated and locally compact. Thus $G \in \mathscr{S}$, showing that (ii) implies (i). □

7 Examples

7.1 Vertex-transitive actions on trees of small degree

Since there is a one-to-one correspondence, isomorphism types of local action diagrams can be used to classify isomorphism types of (P)-closed groups acting on trees. This classification is most useful for families of groups acting on trees where the associated local action diagram is 'small'.

Let us consider the special case of a vertex-transitive (P)-closed group G acting on a locally finite tree T. The tree is necessarily regular, of some degree d; let us write $T = T_d$, to indicate that T is a regular tree of degree d. In the local action diagram $\Delta = (\Gamma, (X_a), G(v))$ for (G, T), the graph Γ has a single vertex v; the set X_v has size d and there is a single permutation group $G(v)$, which is defined on X_v. The set $\{X_a \mid a \in A\Gamma\}$ is the partition of X_v into $G(v)$-orbits. The only remaining piece of information in the local action diagram is the edge-reversal map r on Γ; since there is only one vertex, this can be any permutation of $A\Gamma$ whose square is the identity.

Thus, up to conjugacy in $\mathrm{Aut}(T_d)$, there are only finitely many vertex-transitive (P)-closed subgroups of $\mathrm{Aut}(T_d)$. The relevant conjugacy classes are in one-to-one correspondence with the set $\mathcal{V}_d$ of equivalence classes of pairs (H, r), where H is a subgroup of $\mathrm{Sym}(d)$ and r is an **orbit pairing** for H, meaning a permutation of the set $H\backslash[d]$ of H-orbits whose square is the identity. Here we say two pairs (H_1, r_1) and (H_2, r_2) are equivalent if there is $g \in \mathrm{Sym}(d)$ such that $gH_1g^{-1} = H_2$ and the map $g' : H_1 \setminus [d] \rightarrow H_2\backslash[d]$ induced by g' satisfies $g'r_1 = r_2g'$. Write $U(H, r)$ for the subgroup of $\mathrm{Aut}(T_d)$ associated to (H, r) (here $U(H, r)$ should be understood as specified up to conjugacy in $\mathrm{Aut}(T_d)$). The orbit pairing captures the difference between the number of arc-orbits of $U(H, r)$ and the number of edge-orbits: the arc-orbits of $U(H, r)$ correspond to orbits of H, whereas the edge-orbits correspond to orbits of r on $H\backslash[d]$. The orbits of H fixed by r correspond to those arc-orbits of $U(H, r)$ that are closed under the reverse map on T_d, in other words, those arcs that are reversed by some element of $U(H, r)$. We see that $U(H, r)$ fixes an end exactly in the following situation: H has a fixed point and acts transitively on the remaining points, and the orbit pairing is nontrivial.

Let us first deal with the special case that H is a free permutation group, that is, point stabilizers of H are trivial. In this case, $U(H, r)$ acts freely on the arcs of T_d, and it follows by Theorem 2.9 that it can be expressed as a free product of copies of H, $\mathbb{Z}$ and C_2 with no amalgamation. Specifically, writing K^{*n} to mean a free product of n copies of K, we have

$$U(H, r) \cong H * C_2^{*a} * \mathbb{Z}^{*b}$$

where a is the number of fixed points of r (in other words, the number of reversible arc orbits of $U(H, r)$ on the tree) and b is the number of nontrivial orbits of r. In fact, in this case we see that every group acting on T_d with the same local action diagram as $U(H,r)$ is $\mathrm{Aut}(T_d)$-conjugate

For small values of d, it is feasible to list the conjugacy classes of vertex-transitive 1-tree-closed subgroups of $\mathrm{Aut}(T_d)$; we will describe the list for $d \leqslant 5$. The pairs (H,r) given below should be understood as being taken up to equivalence. Where it is unambiguous we will indicate the orbit pairing simply by the size of the paired orbits, so for instance $[12,22]$ indicates an orbit pairing where an orbit of size 1 is paired with an orbit of size 2, another orbit of size 2 is paired with a third orbit of size 2, and all other orbits are fixed by r. This notation is especially convenient when $d \leqslant 5$, as in this case, for any $H \leqslant \mathrm{Sym}(d)$, all the orbits of H of the same size lie in a single orbit of the normalizer of H. For brevity we will write (H,id) as (H). Write $S_n := \mathrm{Sym}(n)$, $A_n = \mathrm{Alt}(n)$, C_n for a cyclic group of order n and D_n for a dihedral group of order n. We first recall the conjugacy classes of subgroup of S_d.

In S_0 and S_1 there is only the trivial group 1.

In S_2 there are two subgroups, namely 1 and S_2 itself. (Below, S_2 is the group of order 2 acting as a local action at a vertex of degree 2, whereas C_2 without further decoration will represent an edge-reversing involution.)

In S_3 there are four conjugacy classes of subgroup, namely: 1; one class C_{2+1} of subgroup of order 2; the alternating group $A_3 = C_3$; and S_3 itself.

In S_4 there are 11 conjugacy classes of subgroup, namely: the trivial group 1; two classes C_2^- and C_2^+ of subgroup of order 2 (acting with two and zero fixed points respectively); one class each of cyclic subgroups C_{3+1} and C_4 of orders 3 and 4; two classes V^- and V^+ of the Klein 4-group (the plus sign denoting the regular action, and the minus sign the faithful intransitive action); one class of point stabilizers S_{3+1}; one class of the dihedral group D_8 of order 8; the alternating group A_4; and the symmetric group itself S_4.

In S_5 there are 19 conjugacy classes of subgroup of S_5. These are:

11 classes of subgroup that fix a point (corresponding to conjugacy classes of subgroup of S_4);

Three classes of subgroup with orbit partition $(3,2)$, as follows:

cyclic group C_{3+2} of order 6; 'twisted S_3', viz. $S_3^* = \langle(1,2,3),(1,2)(4,5)\rangle$; direct product $S_{3+2} = S_3 \times S_2$.

Five classes of transitive subgroup, as follows:

cyclic group C_5 of order 5; dihedral group D_{10} of order 10; general affine group $GA(1,5)$, a group of order 20; alternating group A_5; S_5 itself.

The number of conjugacy classes of Burger–Mozes subgroup in $\mathrm{Aut}(T_d)$ is thus $1,1,2,4,11,19$ for $d = 0,1,2,3,4,5$ respectively. However, due to nontrivial orbit pairings, the total number of conjugacy classes of (P)-closed subgroup in $\mathrm{Aut}(T_d)$ is larger, starting from $d = 2$: for $d = 2,3,4,5$ there are a total of $3,6,19,40$ conjugacy classes.

Given $G = U(H,r)$, we can determine the quotient G/G^+ as the fundamental group of a graph of groups with trivial edge groups by passing to the reversal-free subdivision of in Theorem 5.16. In particular, G/G^+ can easily be written as a free product with no amalgamation.

In the tables below, a blank entry means a repeat of the previous entry. Note that the same group may appear several times, but with different actions on the tree. 'l.p.c.' stands for 'local prime content', in other words, the primes p such that the p-Sylow subgroup of a compact open subgroup of $U(H,r)$ is infinite; in the present context, the local prime content is empty if and only if $U(H,r)$ is discrete.

d	Local action	orbit pairing	l.p.c.	fixed end	G/G^+	G^+ local action
0	1	id	$\varnothing$	N/A	1	1
1	1	id	$\varnothing$	N/A	C_2	1
2	1	id	$\varnothing$	No	C_2^{*2}	1
		[11]		Yes	$\mathbb{Z}$	1
	S_2	id	$\varnothing$	No	$S_2 * C_2$	1
3	1	id	$\varnothing$	No	C_2^{*3}	1
		[11]		No	$C_2 * \mathbb{Z}$	
	S_2	id	$\{2\}$	No	C_2^{*2}	S_2
		[12]		Yes	$\mathbb{Z}$	
	C_3	id	$\varnothing$	No	$C_3 * C_2$	1
	S_3	id	$\{2\}$	No	C_2	S_3
4	1	id	$\varnothing$	No	C_2^{*4}	1
		[11]		No	$C_2^{*2} * \mathbb{Z}$	
		$[11, 11]$		No	$\mathbb{Z}^{*2}$	
	C_2^-	id	$\{2\}$	No	C_2^{*3}	C_2^-
		[11]		No	$C_2 * \mathbb{Z}$	
		[12]		No	$C_2 * \mathbb{Z}$	
	C_2^+	id	$\varnothing$	No	C_2^{*2}	1
		[22]		No	$\mathbb{Z}$	
	C_3	id	$\{3\}$	No	C_2^{*2}	C_3
		[13]		Yes	$\mathbb{Z}$	
	C_4	id	$\varnothing$	No	$C_4 * C_2$	1
	V^-	id	$\{2\}$	No	C_2^{*2}	V^-
		[22]		No	$\mathbb{Z}$	
	V^+	id	$\varnothing$	No	$V^+ * C_2$	1
	S_3	id	$\{2,3\}$	No	C_2^{*2}	S_3
		[13]		Yes	$\mathbb{Z}$	
	D_8	id	$\{2\}$	No	$S_2 * \mathbb{Z}$	V^-
	A_4	id	$\{3\}$	No	C_2	A_4
	S_4	id	$\{2,3\}$	No	C_2	S_4

Table 1: The 30 vertex-transitive (P)-closed actions on trees of degree $d \leqslant 4$

Local action	orbit pairing	l.p.c.	fixed end	G/G^+	G^+ local action
1	id	$\varnothing$	No	C_2^{*5}	1
	[11]		No	$C_2^{*3} * \mathbb{Z}$	
	[11, 11]		No	$C_2 * \mathbb{Z}^{*2}$	
C_2^-	id	$\{2\}$	No	C_2^{*4}	C_2^-
	[11]		No	$C_2^{*2} * \mathbb{Z}$	
	[12]		No	$C_2^{*2} * \mathbb{Z}$	
	[11, 12]		No	$C_2 * \mathbb{Z}^{*2}$	
C_2^+	id	$\{2\}$	No	C_2^{*3}	C_2^+
	[12]		No	$C_2 * \mathbb{Z}$	
	[22]		No	$C_2 * \mathbb{Z}$	
C_3	id	$\{3\}$	No	C_2^{*3}	C_3
	[11]		No	$C_2 * \mathbb{Z}$	
	[13]		No	$C_2 * \mathbb{Z}$	
C_4	id	$\{2\}$	No	C_2^{*2}	C_4
	[14]		Yes	$\mathbb{Z}$	
V^-	id	$\{2\}$	No	C_2^{*3}	V^-
	[12]		No	$C_2 * \mathbb{Z}$	
	[22]		No	$C_2 * \mathbb{Z}$	
V^+	id	$\{2\}$	No	C_2^{*2}	V^+
	[14]		Yes	$\mathbb{Z}$	
S_3	id	$\{2, 3\}$	No	C_2^{*3}	S_3
	[11]		No	$C_2 * \mathbb{Z}$	
	[13]		No	$C_2 * \mathbb{Z}$	
D_8	id	$\{2\}$	No	C_2^{*2}	D_8
	[14]		Yes	$\mathbb{Z}$	
A_4	id	$\{2, 3\}$	No	C_2^{*2}	A_4
	[14]		Yes	$\mathbb{Z}$	
S_4	id	$\{2, 3\}$	No	C_2^{*2}	S_4
	[14]		Yes	$\mathbb{Z}$	
C_{3+2}	id	$\{2, 3\}$	No	C_2^{*2}	C_{3+2}
	[23]		No	$\mathbb{Z}$	
S_3^*	id	$\{2, 3\}$	No	C_2^{*2}	S_3^*
	[23]		No	$\mathbb{Z}$	
S_{3+2}	id	$\{2, 3\}$	No	C_2^{*2}	S_{3+2}
	[23]		No	$\mathbb{Z}$	
C_5	id	$\varnothing$	No	$C_5 * \mathbb{Z}$	1
D_{10}	id	$\{2\}$	No	C_2	D_{10}
$GA(1, 5)$	id	$\{2\}$	No	C_2	$GA(1, 5)$
A_5	id	$\{2, 3\}$	No	C_2	A_5
S_5	id	$\{2, 3\}$	No	C_2	S_5

Table 2: The 40 vertex-transitive (P)-closed actions on trees of degree 5

We should remember that the list of groups we have obtained so far is up to equivalence of action on the tree, not up to isomorphism as groups. For example, the group $\mathbb{Z} * C_2$ appears as a vertex-transitive 1-tree-closed subgroup of both $\mathrm{Aut}(T_3)$ and $\mathrm{Aut}(T_4)$, but clearly the actions are not equivalent. It is not clear what group isomorphisms could exist between the nondiscrete groups in the list (that is, all the groups listed such that the local action is not free).

Recall the class $\mathscr{S}$ of t.d.l.c. groups that are compactly generated, nondiscrete and topologically simple. Of the 70 entries in the tables, only seven have $G^+ \in \mathscr{S}$, namely the Burger–Mozes groups $U(F)$ with transitive local action for $F \in \{S_3, A_4, S_4, D_{10}, GA(1,5), A_5, S_5\}$. However, in a further 37(?) cases (those with nontrivial local prime content where G does not fix an end), G^+ is nondiscrete and simple, but fails to be compactly generated. The latter simple groups have a complicated structure in general, which may merit further investigation. For instance, by [3], in every nondiscrete Burger–Mozes group there is a compactly generated closed subgroup K (where without loss of generality $K \leqslant G^+$), and a discrete normal subgroup D of K, such that $K/D \in \mathscr{S}$. We do not know if any of these 37 non-compactly generated simple groups G^+ are isomorphic to one another as abstract or topological groups.

One motivation for studying vertex-transitive groups acting on trees of small degree is to understand compactly generated t.d.l.c. groups in terms of their degree. The **degree** $\deg(G)$ of a compactly generated t.d.l.c. group G is the smallest degree of a Cayley–Abels graph for G. The degree 0 groups are the compact groups and the degree 2 groups are the compact-by-cyclic groups, but even for degree 3 the structure is not well-understood, except that the degree must be larger than the maximum of the local prime content. What can be said in general, given a group G acting with kernel K on a Cayley–Abels graph Γ of minimal degree d, is that the action lifts to an action of a group $\widetilde{G}$ acting vertex-transitively on T_d, with an associated homomorphism $\theta : \widetilde{G} \to G/K$ with discrete kernel. We can then consider the (P)-closure $\widetilde{G}^{(\mathrm{P})}$ as a first step towards understanding $\widetilde{G}$ and the original group G. Both the local action and the orbit pairing for $\widetilde{G}^{(\mathrm{P})}$ come from the action of G on Γ.

7.2 Building more compactly generated simple groups

Recall that Corollary 1.9 provides conditions that allow us to construct a local action diagram that will yield a group in $\mathscr{S}$. There are easy ways to meet most of these conditions. For instance, to ensure Δ is irreducible it is sufficient to have $|X_a| > 1$ for all $a \in A\Gamma$, in other words, none of the groups $G(v)$ has a fixed point. Given a compactly generated t.d.l.c. group G, then every compact open subgroup U of G will give rise to a closed transitive subdegree-finite action of G on G/U. The only difficulty is in making sure that the action of G on G/U is generated by point stabilizers; but this will certainly be the case if, for example, $U \neq \{1\}$ and G does not admit any proper discrete quotient.

As an illustration, we prove Theorem 1.10, which shows that we can 'combine' finitely many groups in $\mathscr{S}$ (chosen from $\mathscr{S}$ arbitrarily) to make another group in $\mathscr{S}$ of the form $U(\Delta)$ for a suitably chosen Δ.

Proof of Theorem 1.10. Define $\Delta = (\Gamma, (X_a), (G(v)))$ as follows:

Γ is a star with central vertex v_0, leaves $v_1, \ldots, v_n$, and exactly one arc a_i from v_0 to v_i for $1 \leqslant i \leqslant n$.

For $1 \leqslant i \leqslant n$, set $X_{a_i} = \{3i, 3i+1, 3i+2\}$ and $X_{\overline{a_i}}$ is the left coset space G_i/U_i.

$G(v_0)$ is the direct product of n copies of $\mathrm{Sym}(3)$, where the i-th copy of $\mathrm{Sym}(3)$ acts naturally on X_{a_i}.

For $1 \leqslant i \leqslant n$, $G(v_i)$ is G_i acting by left translation on G_i/U_i.

It is now easy to see that we have defined a valid local action diagram, and taking the action of $U(\Delta)$ on its defining tree, all the conditions of Corollary 1.9(ii) are immediately apparent. Thus $U(\Delta) \in \mathscr{S}$.

Let $(T, \pi, \mathcal{L})$ be the Δ-tree defining $U(\Delta)$. For $1 \leqslant i \leqslant n$ let $v_i^* \in \pi^{-1}(v_i)$ and let $O_i = U(\Delta)_{v_i^*}$. Then the action of O_i on $o^{-1}(v_i^*)$ is exactly $G(v_i)$, which is isomorphic to G_i; indeed, by Proposition 4.5, the action homomorphism of O_i on $o^{-1}(v_i^*)$ splits, so $O_i \cong K_i \rtimes G_i$ where K_i is the kernel of the action. In turn, K_i fixes an arc, hence is compact by Proposition 6.2. ☐

This theorem suggests an interesting preorder on $\mathscr{S}$: say that $G_1 \prec_{OK} G_2$ if there is an open subgroup O of G_2 and a compact normal subgroup K of O such that $O/K \cong G_1$. The corollary shows that $\prec_{OK}$ is a directed preorder on $\mathscr{S}$, that is, any finite subset has an upper bound, and moreover, within $\mathscr{S}$, the groups admitting faithful (P)-closed actions on trees are cofinal. On the other hand, every element of $\mathscr{S}$ is 'close to the bottom' in the following sense: given $G \in \mathscr{S}$, there are only $\aleph_0$ compactly generated open subgroups, each of which has at most finitely many quotients in $\mathscr{S}$, so for each $G \in \mathscr{S}$ there are at most $\aleph_0$ different isomorphism types of $H \in \mathscr{S}$ such that $H \prec_{OK} G$. On the other hand, by [11], $\mathscr{S}$ as a whole has $2^{\aleph_0}$ isomorphism classes. In particular, writing $\mathscr{S}/OK$ for the poset generated by $\prec_{OK}$, it follows that $\mathscr{S}/OK$ has infinite ascending chains. We are naturally led to a 'well-foundedness' question:

Question 1. Does $\mathscr{S}/OK$ have infinite descending chains? That is, does there exist a sequence $G_0, G_1, \ldots$ in $\mathscr{S}$ such that $G_{i+1} \prec_{OK} G_i$ but $G_i \nprec_{OK} G_{i+1}$ for all i?

It would also be interesting to find a non-trivial $\prec_{OK}$-equivalence class, i.e. a set $\mathcal{X}$ of two or more pairwise non-isomorphic groups in $\mathscr{S}$ such that for any $G, H \in \mathcal{X}$, then G can be realized as a quotient with compact kernel of an open subgroup of H and *vice versa*.

A A GAP implementation

The following GAP code [5] implements the classification of vertex-transitive (P)-closed actions on regular trees explained in Section 7.1. Given $d \in \mathbb{N}_{\geqslant 2}$ it outputs a list of representatives of all associated local action diagrams. Here, a local action diagram takes the form

[local action, arcs, edge-reversal]

where the arcs are given as the list of orbits of the local action and the edge-reversal as an element of order 2 of the symmetric group on said list.

```
LocalActionDiagrams:=function(d)
local list, G, cSubG, cGv, Gv, arcs, i, NGv, S, actNGv, R, cr, r;
# list to contain all the relevant local action diagrams
list:=[];
# initialize Sym(d) and its subgroup conjugacy classes
G:=SymmetricGroup(d);
cSubG:=ConjugacyClassesSubgroups(G);
# for each conjugacy class of subgroups of G...
for cGv in cSubG do
        # ...choose a representative and find orbits as a list of sets
        Gv:=cGv[1];
        arcs:=ShallowCopy(Orbits(Gv,[1..d]));
        for i in [1..Length(arcs)] do arcs[i]:=Set(arcs[i]); od;

        # initialize the normalizer of Gv in G
        NGv:=Normalizer(G,Gv);

        # choose edge-reversal, i.e. an element of order at most 2 of Sym(arcs), up to the action of NGv on arcs (Gv-orbits)
        S:=SymmetricGroup(Size(arcs));
        actNGv:=ActionHomomorphism(NGv,arcs,OnSets);
        R:=Image(actNGv);
        # in the line below, 'OrbitsDomain(R,S)' returns the list of orbits for the conjugation action of R on S
        for cr in OrbitsDomain(R,S) do
                r:=cr[1];
                if not r*r=() then continue; fi;
                Add(list,[Gv,arcs,r]);
```

```
        od;
od;
return list;
end;
```

For example, we obtain the following output for the case $d = 3$, which corresponds to the entries with $d = 3$ in Table 1.

```
LocalActionDiagrams(3);
[ [ Group(()), [ [ 1 ], [ 2 ], [ 3 ] ], () ],
  [ Group(()), [ [ 1 ], [ 2 ], [ 3 ] ], (2,3) ],
  [ Group([ (2,3) ]), [ [ 1 ], [ 2, 3 ] ], () ],
  [ Group([ (2,3) ]), [ [ 1 ], [ 2, 3 ] ], (1,2) ],
  [ Group([ (1,2,3) ]), [ [ 1, 2, 3 ] ], () ],
  [ Group([ (1,2,3), (2,3) ]), [ [ 1, 2, 3 ] ], () ] ]
```

The following table records the number of vertex-transitive (P)-closed actions on the regular tree T_d for $d \in \{3, \ldots, 11\}$ in comparison to the number of conjugacy classes $|S_d \backslash \mathrm{Sub}(S_d)|$ of subgroups of S_d, which is also the number of Burger–Mozes groups for T_d.

d	$\lvert S_d \backslash \mathrm{Sub}(S_d)\rvert$	vertex-transitive, (P)-closed actions on T_d
3	4	6
4	11	19
5	19	40
6	56	125
7	96	285
8	296	904
9	554	2240
10	1593	7213
11	3094	19326
12	10723	?

Table 3: Number of vertex-transitive (P)-closed actions up to degree 11

We remark that according to the Online Encyclopedia of Integer Sequences [10, A000638], the number of conjugacy classes of subgroups of S_d has been calculated up to $d \leqslant 18$, with the most recent reference being to work of D. Holt [6].

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