Some Modular Considerations Regarding Odd Perfect Numbers

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February 28, 2020

Abstract

Let $p^k m^2$ be an odd perfect number with special prime p. In this article, we provide an alternative proof for the biconditional that $\sigma(m^2) \equiv 1 \pmod 4$ holds if and only if $p \equiv k \pmod 8$. We then give an application of this result to the case when $\sigma(m^2)/p^k$ is a square.

Keywords: Sum of divisors, Sum of aliquot divisors, Deficiency, Odd perfect number, Special prime.
2010 Mathematics Subject Classification: 11A05, 11A25.

1 Introduction

Let $\sigma(z)$ denote the sum of the divisors of $z \in \mathbb{N}$, the set of positive integers. Denote the deficiency [5] of z by $D(z) = 2z - \sigma(z)$, and the sum of the aliquot divisors [6] of z by $s(z) = \sigma(z) - z$. Note that we have the identity $D(z) + s(z) = z$.

If n is odd and $\sigma(n) = 2n$, then n is said to be an odd perfect number [8]. Euler proved that an odd perfect number, if one exists, must have the form $n = p^k m^2$, where p is the special prime satisfying $p \equiv k \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$.

Chen and Luo [2] gave a characterization of the forms of odd perfect numbers $n = p^k m^2$ such that $p \equiv k \pmod 8$. Starni [7] proved that there is no odd perfect number decomposable into primes all of the type $\equiv 1 \pmod 4$ if $n = p^k m^2$ and $p \not\equiv k \pmod 8$. Starni used a congruence from Ewell [3] to prove this result.

Note that, in general, since m^2 is a square, we get

$$\sigma(m^2) \equiv 1 \pmod 2.$$

This paper provides an alternative proof for Theorem 3.3, equation 3.1 in Chen and Luo's article titled "Odd multiperfect numbers" [2]:

Theorem 1.1. *Let $n = \pi^\alpha M^2$ be an odd 2-perfect number, with π prime, $\gcd(\pi, M) = 1$ and $\pi \equiv \alpha \equiv 1 \pmod 4$. Then*

$$\sigma(M^2) \equiv 1 \pmod 4 \iff \pi \equiv \alpha \pmod 8.$$

The method presented in this paper may potentially be used to extend the arguments to consider $\sigma(m^2)$ modulo 8.

2 Preliminaries

Starting from the fundamental equality

$$\frac{\sigma(m^2)}{p^k} = \frac{2m^2}{\sigma(p^k)}$$

(which follows from the facts that $\sigma(n) = 2n$, σ is multiplicative, and $\gcd(p^k, \sigma(p^k)) = 1$) one can derive

$$\frac{\sigma(m^2)}{p^k} = \frac{2m^2}{\sigma(p^k)} = \gcd(m^2, \sigma(m^2))$$

so that we ultimately have

$$\frac{D(m^2)}{s(p^k)} = \frac{2m^2 - \sigma(m^2)}{\sigma(p^k) - p^k} = \gcd(m^2, \sigma(m^2))$$

and

$$\frac{s(m^2)}{D(p^k)/2} = \frac{\sigma(m^2) - m^2}{p^k - \frac{\sigma(p^k)}{2}} = \gcd(m^2, \sigma(m^2)),$$

whereby we obtain

$$\frac{D(p^k)D(m^2)}{s(p^k)s(m^2)} = 2.$$

Note that we also have the following equation:

$$\frac{2D(m^2)s(m^2)}{D(p^k)s(p^k)} = \left(\gcd(m^2, \sigma(m^2))\right)^2. \qquad (*)$$

Lastly, notice that we can easily get

$$\sigma(p^k) \equiv k + 1 \equiv 2 \pmod 4$$

(since $p \equiv k \equiv 1 \pmod 4$) so that it remains to consider the possible equivalence classes for $\sigma(m^2)$ modulo 4. Since $\sigma(m^2)$ is odd, we only need to consider two.

We ask: Which equivalence class of $\sigma(m^2)$ modulo 4 makes Equation $(*)$ untenable?

3 Discussion and Results

We know that the answer to the question we posed in the previous section must somehow depend on the equivalence class of p and k modulo 8, but as we only know that $p \equiv k \equiv 1 \pmod 4$, we need to consider the following cases separately and thereby prove the corresponding results:

Remark 3.1. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p. We claim the truth of the following propositions, which we will need to treat separately later:*

1. *If $p \equiv k \equiv 1 \pmod 8$, then $\sigma(m^2) \equiv 3 \pmod 4$ is impossible.*
2. *If $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, then $\sigma(m^2) \equiv 1 \pmod 4$ is impossible.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $\sigma(m^2) \equiv 1 \pmod 4$ is impossible.*
4. *If $p \equiv k \equiv 5 \pmod 8$, then $\sigma(m^2) \equiv 3 \pmod 4$ is impossible.*

First, we prove the following lemmas:

Lemma 3.2. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $p \equiv 1 \pmod 8$, then $\sigma(p^k) \equiv k + 1 \pmod 8$.*
2. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $\sigma(p^k) \equiv 6 \pmod 8$.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 5 \pmod 8$, then $\sigma(p^k) \equiv 2 \pmod 8$.*

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime p. It follows that $p \equiv 1 \pmod 4$.

We consider two cases:

Case 1: $p \equiv 1 \pmod 8$ We obtain

$$\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv 1 + \sum_{i=1}^{k} p^i \equiv 1 + \sum_{i=1}^{k} 1^i \equiv k + 1 \pmod 8,$$

as desired.

Case 2: $p \equiv 5 \pmod 8$ We get

$$\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv \sum_{i=0}^{k} 5^i \equiv \begin{cases} 6 \pmod 8, \text{ if } k \equiv 1 \pmod 8 \\ 2 \pmod 8, \text{ if } k \equiv 5 \pmod 8 \end{cases}$$

□

Lemma 3.3. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $p \equiv 1 \pmod 8$ and $k \equiv 1 \pmod 8$, then $D(p^k) \equiv 0 \pmod 8$.*
2. *If $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, then $D(p^k) \equiv 4 \pmod 8$.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $D(p^k) \equiv 4 \pmod 8$.*
4. *If $p \equiv 5 \pmod 8$ and $k \equiv 5 \pmod 8$, then $D(p^k) \equiv 0 \pmod 8$.*

Proof. The proof is trivial and follows directly from Lemma 3.2, using the formula $D(p^k) = 2p^k - \sigma(p^k)$. □

Lemma 3.4. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $p \equiv 1 \pmod 8$ and $k \equiv 1 \pmod 8$, then $s(p^k) \equiv 1 \pmod 8$.*
2. *If $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, then $s(p^k) \equiv 5 \pmod 8$.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $s(p^k) \equiv 1 \pmod 8$.*
4. *If $p \equiv 5 \pmod 8$ and $k \equiv 5 \pmod 8$, then $s(p^k) \equiv 5 \pmod 8$.*

Proof. The proof is trivial and follows directly from Lemma 3.3, using the formula $s(p^k) = p^k - D(p^k)$. □

Lemma 3.5. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $\sigma(m^2) \equiv 1 \pmod 4$, then $D(m^2) \equiv 1 \pmod 4$.*
2. *If $\sigma(m^2) \equiv 3 \pmod 4$, then $D(m^2) \equiv 3 \pmod 4$.*

Proof. The proof is trivial and follows directly from the fact that $m^2 \equiv 1 \pmod 4$ (since m is odd), using the underlying assumptions and the formula $D(m^2) = 2m^2 - \sigma(m^2)$. □

Lemma 3.6. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $\sigma(m^2) \equiv 1 \pmod 4$, then $s(m^2) \equiv 0 \pmod 4$.*
2. *If $\sigma(m^2) \equiv 3 \pmod 4$, then $s(m^2) \equiv 2 \pmod 4$.*

Proof. The proof is trivial and follows directly from Lemma 3.5, using the formula $s(m^2) = m^2 - D(m^2)$. □

We are now ready to prove our main result.

Theorem 3.7. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $p \equiv k \equiv 1 \pmod 8$, then $\sigma(m^2) \equiv 3 \pmod 4$ is impossible.*
2. *If $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, then $\sigma(m^2) \equiv 1 \pmod 4$ is impossible.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $\sigma(m^2) \equiv 1 \pmod 4$ is impossible.*
4. *If $p \equiv k \equiv 5 \pmod 8$, then $\sigma(m^2) \equiv 3 \pmod 4$ is impossible.*

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime p.

Notice that the right-hand side of Equation $(*)$

$$\frac{2D(m^2)s(m^2)}{D(p^k)s(p^k)} = \left(\gcd(m^2, \sigma(m^2))\right)^2. \qquad (*)$$

is odd. (Furthermore, it is congruent to 1 modulo 8.)

First, suppose that $p \equiv k \equiv 1 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 3 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 0 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 3 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 1 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_1 + 3)(4b_1 + 2) = (8x_1 + 1)(8c_1)(8d_1 + 1)$$

which does not have any integer solutions.

Next, suppose that $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 1 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 4 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 1 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 5 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 0 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_2 + 1)(4b_2) = (8x_2 + 1)(8c_2 + 4)(8d_2 + 5)$$

which does not have any integer solutions.

Now, suppose that $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 1 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 4 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 1 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 1 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 0 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_3 + 1)(4b_3) = (8x_3 + 1)(8c_3 + 4)(8d_3 + 1)$$

which does not have any integer solutions.

Finally, suppose that $p \equiv k \equiv 5 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 3 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 0 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 3 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 5 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_4 + 3)(4b_4 + 2) = (8x_4 + 1)(8c_4)(8d_4 + 5)$$

which does not have any integer solutions.

This concludes the proof. □

Remark 3.8. *To summarize, Theorem 3.7 just states that if $n = p^k m^2$ is an odd perfect number with special prime p, then $\sigma(m^2) \equiv 1 \pmod 4$ holds if and only if $p \equiv k \pmod 8$. Our argument provides an alternative proof for Theorem 3.3, equation 3.1 in [2] (as reproduced above in Theorem 1.1).*

4 An Application

Let $n = p^k m^2$ be an odd perfect number with special prime p, and let $\sigma(m^2)/p^k$ be a square. Since $\sigma(m^2)/p^k$ is odd, it follows that $\sigma(m^2)/p^k \equiv 1 \pmod 4$. But it is known that $p \equiv k \equiv 1 \pmod 4$. In particular, we know that $p^k \equiv 1 \pmod 4$. This implies that $\sigma(m^2) \equiv 1 \pmod 4$, if $\sigma(m^2)/p^k$ is a square. By Theorem 3.7, we know that $p \equiv k \pmod 8$.

Moreover, Broughan, Delbourgo, and Zhou proved in [1] (Lemma 8, page 7) that if $\sigma(m^2)/p^k$ is a square, then $k = 1$ holds.

Thus, under the assumption that $\sigma(m^2)/p^k$ is a square, we have

$$p \equiv k = 1 \pmod 8.$$

This implies that the lowest possible value for the special prime p is 17.

We state this result as our next theorem.

Theorem 4.1. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p. If $\sigma(m^2)/p^k$ is a square, then $p \geq 17$.*

Remark 4.2. *Let $n = p^k m^2$ be an odd perfect number with special prime p.*

Note that if

$$\frac{\sigma(m^2)}{p^k} = \frac{m^2}{\sigma(p^k)/2}$$

is a square, then $k = 1$ and $\sigma(p^k)/2 = (p+1)/2$ is also a square.

The possible values for the special prime satisfying $p < 100$ and $p \equiv 1 \pmod 8$ are 17, 41, 73, 89, and 97.

For each of these values:

$$\frac{p_1+1}{2} = \frac{17+1}{2} = 9 = 3^2.$$

$$\frac{p_2+1}{2} = \frac{41+1}{2} = 21 \textit{ which is not a square.}$$

$$\frac{p_3+1}{2} = \frac{73+1}{2} = 37 \textit{ which is not a square.}$$

$$\frac{p_4+1}{2} = \frac{89+1}{2} = 45 \textit{ which is not a square.}$$

$$\frac{p_5+1}{2} = \frac{97+1}{2} = 49 = 7^2.$$

A quick way to rule out 41, 73 and 89, as remarked by Ochem [4] over at Mathematics StackExchange, is as follows: "If $(p+1)/2$ is an odd square, then $(p+1)/2 \equiv 1 \pmod 8$, so that $p \equiv 1 \pmod{16}$. This rules out 41, 73, and 89."

5 Conclusion

Additional tools are required if we are to push the analysis from $\sigma(m^2)$ modulo 4 to consider $\sigma(m^2)$ modulo 8. The authors have tried to check Equation $(*)$ by considering $m^2 \equiv 1 \pmod 8$, and the various corresponding cases for $\sigma(m^2)$ modulo 8 (which are determined by Theorem 3.7), but so far all their attempts have not resulted in any contradictions.

Acknowledgements

The authors are indebted to the anonymous referees whose valuable feedback improved the overall presentation and style of this manuscript.

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