SOLUTIONS OF $\phi(n) = \phi(n+k)$ AND $\sigma(n) = \sigma(n+k)$

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ABSTRACT. We show that for some even $k \leqslant 3570$ and *all* k with $442720643463713815200|k$, the equation $\phi(n) = \phi(n+k)$ has infinitely many solutions n, where ϕ is Euler's totient function. We also show that for a positive proportion of all k, the equation $\sigma(n) = \sigma(n+k)$ has infinitely many solutions n. The proofs rely on recent progress on the prime k-tuples conjecture by Zhang, Maynard, Tao and PolyMath.

1. INTRODUCTION

We partially solve a longstanding conjecture about the solubility of

$$\phi(n+k) = \phi(n), \tag{1.1}$$

where ϕ is Euler's function and k is a fixed positive integer.

Hypothesis $\mathcal{S}_k$. The equation (1.1) holds for infinitely many n.

Ratat and Goormaghtigh in 1917–18 (see [**5**], p. 140) listed several solutions when $k = 1$. Erdős conjectured in 1945 that for any m, the simultaneous equations

$$\phi(n) = \phi(n+1) = \cdots = \phi(n+m-1) \tag{1.2}$$

has infinitely many solutions n. If true, this would immediately imply hypothesis $\mathcal{S}_k$ for every k. However, there is only one solution of (1.2) known when $m \geqslant 3$, namely $n = 5186$, $m = 3$. In 1956, Sierpiński [**26**] showed that for any k, (1.1) has at least one solution n (e.g. take $n = (p-1)k$, where p is the smallest prime not dividing k). This was extended by Schinzel [**24**] and by Schinzel and Wakulicz [**25**], who showed that for any $k \leqslant 2 \cdot 10^{58}$ there are at least two solutions of (1.1). In 1958, Schinzel [**24**] explicitly conjectured that $\mathcal{S}_k$ is true for every $k \in \mathbb{N}$. There is good numerical evidence for $\mathcal{S}_k$, at least when $k = 1$ or k is even [**1, 2, 3, 15, 9, 13**]. Information about solutions for $k \in \{1,2,3,4,5,6,7,8,9,10,11,12\}$ can also be found in OEIS [**20**] sequences A001274, A001494, A330251, A179186, A179187, A179188, A179189, A179202, A330429, A276503, A276504 and A217139, respectively. Below 10^{11} there are very few solutions of (1.1) when $k \equiv 3 \pmod 6$ [**9**], e.g. only the two solutions $n \in \{3, 5\}$ for $k = 3$ are known. A further search by G. Resta (see [**20**], sequence A330251) reveals 17 more solutions in $[10^{12}, 10^{15}]$.

There is a close connection between Hypothesis $\mathcal{S}_k$ for even k and generalized prime twins.

Hypothesis $\mathscr{P}(a, b)$: there are infinitely many $n \in \mathbb{N}$ such that both $an + 1$ an $bn + 1$ are prime.

Hypothesis $\mathscr{P}(a, b)$ is believed to be true for any pair of positive integers a, b, indeed this is a special case of Dickson's Prime k-tuples conjecture [**4**]. Klee [**14**] and Moser [**18**] noted that Hypothesis $\mathcal{P}(1, 2)$ immediately gives $\mathcal{S}_2$, and Schinzel [**24**] observed that Hypothesis $\mathcal{P}(1, 2)$ implies $\mathcal{S}_k$ for every even k. The proof is simple: if $n + 1$ and $2n + 1$ are prime and larger than k, then $\phi(k(2n + 1)) = \phi((n + 1)2k)$. Graham, Holt and Pomerance [**9**] generalized this idea, showing the following.

Date: March 10, 2020.

The author is supported in part by National Science Foundation grant DMS-1802139.

The author thanks Paul Pollack for bringing papers [**7**], [**11**] and [**21**] to his attention, Sungjin Kim for sharing the preprint [**12**], and Chandra Chekuri for helpful discussion about graph algorithms.

Lemma 1 ([**9**, Theorem 1]). *For any k and any number j such that j and $j+k$ have the same prime factors, Hypothesis $\mathscr{P}(\frac{j}{(j,j+k)}, \frac{j+k}{(j,j+k)})$ implies $\mathcal{S}_k$.*

This also has an easy proof: if $\frac{j}{(j,j+k)}r+1$ and $\frac{j+k}{(j,j+k)}r+1$ are both prime, then $n = j(\frac{j+k}{(j,j+k)}r+1)$ satisfies (1.1). Note that for odd k there are no such numbers j, and for each even k there are finitely many such j (see [**9**], Section 3). Extending a bound of Erdős, Pomerance and Sárkőzy [**8**] in the case $k = 1$, Graham, Holt and Pomerance showed that the solutions of (1.1) *not* generated from Lemma 1 are very rare, with counting function $O_k(x\exp\{-(\log x)^{1/3}\})$. Pollack, Pomerance and Treviño [**22**] proved a version uniform in k, and Yamada [**28**] sharpened this bound to $O_k(x\exp\{-(1/\sqrt{2}+o(1))\sqrt{\log x \log\log\log x}\})$. Assuming the Hardy-Littlewood conjectures [**10**], when k is even we conclude that there are $\sim C_k x/\log^2 x$ solutions $n \leqslant x$ of (1.1), where $C_k > 0$.

At present, Hypothesis $\mathscr{P}(a,b)$ is not known to hold for any pair a, b. However, the work of Zhang, Maynard, Tao and the PolyMath8b project allow us to conclude $\mathscr{P}(a,b)$ for some pairs a, b from a given collection of pairs. To set things up, we say that a collection of linear forms $(a_1n+b_1,\dots,a_kn+b_k)$ is *admissible* if $a_i > 0$ for each i, the forms a_in+b_i are distinct and there is no prime dividing $(a_1n+b_1)\cdots(a_kn+b_k)$ for every integer n.

Lemma 2. *For any $m \geqslant 2$, there is a constant K_m so that if $k \geqslant K_m$ and $(a_1n+b_1,\dots,a_kn+b_k)$ is set of admissible linear forms, then for some distinct $i_1,\dots,i_m \in \{1,\dots,k\}$, there are infinitely many r such that the m numbers $a_{i_1}r+b_{i_1},\dots,a_{i_m}r+b_{i_m}$ are simultaneously prime. Moreover, $K_2 = 50$ is admissible.*

The case $m = 2$ is a generalization of the celebrated theorem of Yitang Zhang [**29**], while Maynard [**16, 17**] proved the existence of K_m for any m. We note that $K_2 \leqslant 50$ and $K_m \ll \exp\{(4-\frac{28}{15})m\}$ by the collaborative PolyMath8b project [**23**][1]. If $a_1,\dots,a_k$ are distinct, then the set of forms $(a_in+1,\dots,a_kn+1)$ are always admissible. Thus, given any set $\{a_1,\dots,a_{50}\}$ of positive integers, there is an $i \neq j$ so that $\mathscr{P}(a_i,a_j)$ holds. In fact [**17**, Theorem 3.4], the number of such $r \leqslant x$ in Lemma 2 is $\gg_{\mathbf{a}} x/\log^k x$.

Theorem 1. *We have*

(a) *For any k that is a multiple of 442720643463713815200, $\mathcal{S}_k$ is true;*
(b) *There is some even $\ell \leqslant 3570$ such that $\mathcal{S}_k$ is true whenever $\ell | k$; consequently, the number of $k \leqslant x$ for which $\mathcal{S}_k$ is true is at least $x/3570$.*

Using Lemma 2, we also make progress toward Erdős' conjecture that (1.2) has infinitely many solutions.

Theorem 2. *For any $m \geqslant 3$ there is a tuple of distinct positive integers $h_1,\dots,h_m$ so that for any $\ell \in \mathbb{N}$, the simultaneous equations*

$$\phi(n+\ell h_1) = \phi(n+\ell h_2) = \cdots = \phi(n+\ell h_m)$$

have infinitely many solutions n.

Maynard [**16, 17**] showed that $K_2 \leqslant 5$ under the assumption of the Elliott-Halberstam Conjecture. Improvements to K_2 allow us to improve significantly on Theorem 1.

Theorem 3. *If $K_2 \leqslant 5$, then $\mathcal{S}_k$ is true for all k with $30|k$. If $K_2 \leqslant 4$, then $\mathcal{S}_k$ is true for all k with $6|k$.*

The author recently learned that Sungjin Kim [**12**] proved weaker statements in the direction of Theorem 1. He used Lemma 2 to show that $\mathcal{S}_k$ holds for some $k \in \{B, 2B, \dots, 50B\}$, with $B = \prod_{p\leqslant 50} p$. He also proved that the set of k for which $\mathcal{S}_k$ holds has counting function $\gg \log\log x$.

[1] We utilize the "general linear forms" version of Maynard [**17**, Theorem 3.1]. Although the results of [**23**] are not stated with this generality, they hold with trivial modifications to the proof; see [**17**], [**23**, Theorem 3.2 (i), Theorem 3.13] for details.

On can ask analogous questions about the sum of divisors function $\sigma(n)$. As $\sigma(p) = p+1$ vs $\phi(p) = p-1$, oftentimes one can port theorems about ϕ over to σ. This is not the case here, since our results depend heavily on the existence of solutions of

$$a\phi(b) = b\phi(a),$$

which is true if and only if a and b have the same set of prime factors. The analogous equation

$$a\sigma(b) = b\sigma(a) \quad \Leftrightarrow \quad \frac{\sigma(a)}{a} = \frac{\sigma(b)}{b}$$

has more sporadic solutions, e.g. if a, b are both perfect numbers or multiply perfect numbers.

Theorem 4. *For a positive proportion of all $k \in \mathbb{N}$, the equation*

$$\sigma(n) = \sigma(n+k)$$

has infinitely many solutions n.

Unfortunately, our methods cannot specify any particular k for which the conclusion holds. Our method requires finding, for $t = K_2$, numbers $a_1, \ldots, a_t$ so that

$$\frac{\sigma(a_1)}{a_1} = \cdots = \frac{\sigma(a_t)}{a_t} = y. \tag{1.3}$$

Such collections of numbers are sometimes referred to as "friends" in the literature, e.g. [**21**]. Finding larger collections of a_i satisfying (1.3) leads to stronger conclusions.

Theorem 5. *Let $m \geqslant 2$, let $t = K_m$ and assume that there is a y and positive integers $a_1, \ldots, a_t$ satisfying* (1.3)*. Then there are positive integers $h_1 < h_2 < \cdots < h_m$ so that for a positive proportion of integers ℓ, there are infinitely many solutions of*

$$\sigma(n + \ell h_1) = \cdots = \sigma(n + \ell h_m).$$

It is known [**19**] that for $y = 9$, there is a set of 2095 integers satisfying (1.3). Also $K_2 \leqslant 50$ [**23**], and hence Theorem 4 follows from the case $m = 2$ of Theorem 5. Even the weaker bound $K_2 \leqslant 105$ from [**16**] suffices. We cannot make the conclusion unconditional when $m \geqslant 3$, since the best know bounds for K_3 is $K_3 \leqslant 35410$ [**23**, Theorem 3.2 (ii)].

Conjecture A. For any t, there is an y such that $\sigma(a)/a = y$ has at least t solutions. That is, there are arbitrarily large circles of friends.

Clearly, Conjecture A implies the conclusion of Theorem 5 for all m. In [**7**], Erdős mentions Conjecture A and states that he doesn't know of any argument that would lead to its resolution. In the opposite direction, Hornfeck and Wirsing [**11**] showed that for any y, there are $\leqslant z^{o(1)}$ solutions of $\sigma(a)/a = y$ with $a \leqslant z$; this was improved by Wirsing [**27**], who showed that the counting function is $O(z^{c/\log\log z})$ for some c, uniformly in y. Pollack and Pomerance [**21**] studied the solutions of (1.3), gathering data on pairs, triples and quadruples of friends, but did not address Conjecture A.

Using (1.3) and prime pairs $an-1$ and $bn-1$, one can generate many solutions of $\sigma(n) = \sigma(n+k)$, analogous to Lemma 1; see Yamada [**28**, Theorem 1.1]. For example, if $\sigma(m)/m = \sigma(m+1)/(m+1)$ (the ratios need not be integers as claimed in [**28**]), $r > m+1$, and $rm-1$ and $r(m+1)-1$ are both prime, then $\sigma(m(r(m+1)-1)) = \sigma((m+1)(mr-1))$. Yamada [**28**, Theorem 1.2] showed that there are $\ll x\exp\{-(1/\sqrt{2}+o(1))\sqrt{\log x \log\log\log x}\}$ solutions $n \leqslant x$ not generated in this way.

2. PROOFS

Throughout, $1 \leqslant a < b$ are integers. We first show that $\mathscr{P}(a,b)$ implies $\mathcal{S}_k$ for certain k, inverting Lemma 1. Define

$$\kappa(a,b) = (b'-a')\prod_{p|a'b'} p, \qquad a' = \frac{a}{(a,b)},\ b' = \frac{b}{(a,b)}. \tag{2.1}$$

We observe that $\kappa(a,b)$ is always even.

Lemma 3. *Assume $\mathscr{P}(a,b)$. Then $\mathcal{S}_k$ holds for every k which is a multiple of $\kappa(a,b)$.*

Proof. Define $a' = \frac{a}{(a,b)}$, $b' = \frac{b}{(a,b)}$ and observe that $\mathscr{P}(a,b) \Rightarrow \mathscr{P}(a',b')$. Let $s = \prod_{p|a'b'} p$, and suppose that $r > \max(a',b')$ such that $a'r+1$ and $b'r+1$ are both prime. Let $\ell \in \mathbb{N}$ and set

$$m_1 = b'\ell s(a'r+1), \quad m_2 = a'\ell s(b'r+1).$$

As all of the prime factors of $a'b'$ divide ℓs, we have $\phi(b'\ell s) = b'\phi(\ell s)$ and $\phi(a'\ell s) = a'\phi(\ell s)$, and it follows than $\phi(m_1) = \phi(m_2)$. Finally, $m_1 - m_2 = (b'-a')\ell s = \ell\kappa(a,b)$. $\square$

Proof of Theorem 1. Let

$$\{a_1,\ldots,a_{50}\} = \{1,2,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,52,56\},$$

By Lemma 2, for some i,j with $1 \leqslant i < j \leqslant 50$, $\mathscr{P}(a_1,a_j)$ is true. We compute

$$\mathrm{lcm}\{\kappa(a_i,a_j) : 1 \leqslant i < j \leqslant 50\} = 442720643463713815200 = 2^5 3^3 5^2 \prod_{7 \leqslant p \leqslant 47} p,$$

and thus (a) follows from Lemma 3.

For part (b), we take

$$\{a_1,\ldots,a_{50}\} = \{15,20,30,36,40,45,60,72,75,80,90,96,100,108,120,135,144,150,180,192,200,216,225,240,250,270,288,300,320,324,360,375,384,400,405,450,480,500,540,600,720,750,810,900,960,1080,1200,1440,1500,1800\},$$

numbers that only have prime factors $2,3,5$. We also compute that

$$\max_{1 \leqslant i < j \leqslant 50} \kappa(a_i,a_j) = 3570,$$

and again invoke Lemma 3. This proves (b). $\square$

Remarks. For any choice of $a_1,\ldots,a_{50}$, $\frac{442720643463713815200}{6} | L(\mathbf{a})$, where $L(\mathbf{a}) = \mathrm{lcm}\{\kappa(a_i,a_j) : i < j\}$. Without loss of generality, assume $(a_1,\ldots,a_{50}) = 1$. For a prime $7 \leqslant p \leqslant 47$, if $p|a_i$ for some i then $p \nmid a_j$ for some j and thus $p|\kappa(a_i,a_j)$. If $p \nmid a_i$ for all i, by the pigeonhole principle, there are two indices with $a_i \equiv a_j \pmod p$. Again, $p|\kappa(a_i,a_j)$. Thus, $p|L(\mathbf{a})$. Now we show that $5^2|L(\mathbf{a})$. Let $S_b = \{a_i : 5^b \| a_i\}$ for $b \geqslant 0$. Then $|S_0| \geqslant 1$. If $|S_b| \geqslant 1$ for some $b \geqslant 2$, then there are i,j with $5^2|a_i$ and $5 \nmid a_j$, and then $5^2|\kappa(a_i,a_j)$. Otherwise, we have $|S_b| \geqslant 21$ for some $b \in \{0,1\}$. By the pigeonhole principle, there is $i \neq j$ with $5^b \| a_i, 5^b \| a_j$ and $5^{b+2}|(a_i - a_j)$. This also implies that $5^2|\kappa(a_i,a_j)$. Simlarly, let $T_b = \{a_i : 3^b \| a_i\}$. Then we have either $|T_b| \geqslant 1$ for some $b \geqslant 2$, or $|T_i| \geqslant 7$ for some $i \in \{0,1\}$. Either way, $3^2|L(\mathbf{a})$. Let $U_b = \{a_i : 2^b \| a_i\}$. Then either $|U_b| \geqslant 1$ for some $b \geqslant 4$ or $|U_b| \geqslant 9$ for some $b \in \{0,1,2,3\}$. Either way, $2^4|L(\mathbf{a})$. It is easy to construct $\mathbf{a} = (a_1,\ldots,a_{50})$ such that $3^3 \nmid L(\mathbf{a})$ and $2^5 \nmid L(\mathbf{a})$. However, such constructions seem to always produce $q|L(\mathbf{a})$ for some prime $q > 50$.

We likewise believe that 3570 is ths smallest number than can be produced for Theorem 1 (b). Using numbers divisible by 4 or more primes always produces some very large $\kappa(a,b)$, thus we limitied our search with numbers composed only of the primes 2,3,5. For a given finite set of integers $\{b_1,\dots,b_r\}$, the problem of minimizing $\max_{i,j\in I}\kappa(b_i,b_j)$ over all 50-element subsets $I\subset\{1,\dots,r\}$, is equivalent to that of finding the largest clique in a graph. Take vertex set $\{1,\dots,r\}$ and draw an edge from i to j if $\kappa(b_i,b_j)\leqslant t$. Using the Sage routing `clique_number()` with $t = 3569$ and $\{b_1,\dots,b_r\}$ being the smallest 800 numbers composed only of primes 2,3,5 (the largest being 12754584), we find that the largest clique has size 49.

Proof of Theorem 2. Let $m \geqslant 2$, $k = K_m$ and consider any set $\{a_1, a_2, \dots, a_k\}$ of k positive integers. By Lemma 2, there are $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant k$ such that for infinitely many r, the m numbers $a_{i_1}r+1,\dots,a_{i_m}r+1$ are all prime. Let r be such a number. Define

$$h_j = \frac{(a_{i_1}\cdots a_{i_m})^2}{a_{i_j}} \qquad (1\leqslant j\leqslant m).$$

Let $\ell\in\mathbb{N}$ and set $n = \ell(a_{i_1}\cdots a_{i_m})^2 r$. Then, since $a_{i_j}|h_j$ for all j, it follows that for any j,

$$\phi(n+\ell h_j) = \phi(\ell h_j(a_{i_j}r+1)) = \phi(\ell h_j)a_{i_j}r = \phi(\ell h_j a_{i_j})r. \qquad \square$$

Proof of Theorem 3. Same as the proof of Theorem 1 (a), but take $\{a_1,a_2,a_3,a_4,a_5\} = \{1,2,3,4,6\}$ if $K_2\leqslant 5$ and $\{a_1,\dots,a_4\} = \{1,2,3,4\}$ if $K_2 \leqslant 4$. $\square$

Proof of Theorem 5. Let $t = K_m$ and $a_1,\dots,a_t$ satisfy (1.3). Put $A = \mathrm{lcm}[a_1,\dots,a_t]$ and for each i define $b_i = A/a_i$. By Lemma 2 applied to the collection of linear forms b_in-1, $1\leqslant i\leqslant t$, there exist $i_1,\dots,i_m$ such that for infinitely many $r\in\mathbb{N}$, the m numbers $b_{i_j}r-1$ are all prime. Let $r > A$ be such a number, and let $\ell\in\mathbb{N}$ such that $(\ell, A) = 1$ (this holds for a positive proportion of all ℓ). Let

$$t_j = \ell a_{i_j}(b_{i_j}r-1) = A\ell r - \ell a_{i_j} \qquad (1\leqslant j\leqslant m).$$

By (1.3), for every j we have

$$\sigma(t_j) = \sigma(\ell)\sigma(a_{i_j})b_{i_j}r = y\sigma(\ell)Ar. \qquad \square$$

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