Enumeration of Dumont permutations avoiding certain four-letter patterns

Alexander Burstein
Department of Mathematics
Howard University
Washington, DC 20059
aburstein@howard.edu

Opel Jones
Department of Mathematics
Towson University
Towson, MD 21252
ojones@towson.edu

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Abstract

In this paper, we enumerate Dumont permutations of the fourth kind avoiding or containing certain permutations of length 4. We also conjecture a Wilf-equivalence of two 4-letter patterns on Dumont permutations of the first kind.

1 Preliminaries

This paper is concerned with enumeration of pattern-restricted Dumont permutations, and therefore, we will begin by defining patterns and Dumont permutations of various kinds (the original two kinds defined by Dumont [12] and the more recent third and fourth kinds defined in [10]). We will then discuss earlier results on this topic and present several new avoidance results for Dumont permutations of the fourth kind, as well as a conjecture for the a Wilf-equivalence on restricted Dumont permutations of the first kind.

1.1 Patterns

We begin with an example of pattern containment. Suppose we are given a permutation, say $p = 26483751$, and another permutation usually of shorter or equal length, e.g. $q = 132$. We say that the string 265 is *order-isomorphic* to 132. In such a case, we say that p contains q as a *pattern*, and that 265 is an *instance*, or *occurrence*, of q in p. Notice also that 265 is not the only instance of q in p, so are the subsequences 287 and 475.

On the other hand, for the same p and a pattern $q' = 1234$, we see that there is no length 4 subsequence of p that is order-isomorphic to q'. In this case we say that p *avoids* q'. Let us now define pattern containment and pattern avoidance more formally.

1.2 Pattern avoidance

Simion and Schmidt [25] was the first paper devoted solely to pattern avoidance. To quote the first line of that seminal work, "This paper is concerned with counting permutations which do not contain certain subsequences." Since that time pattern avoidance, or *restricted permutations*, as the work was titled, has been the subject of many papers, including e.g. [2, 4, 6, 7, 10, 11, 19, 20, 21, 24, 27].

Definition 1.1. [?] Let S_n be the set of permutations of $[n] = \{1, 2, 3, ..., n\}$. Let $p = (p_1, p_2, ..., p_n) \in S_n$ and $q = (q_1, q_2, ..., q_k) \in S_k$ be two permutations. Then we say that p *contains* q if there is a k-tuple $(p_{i_1}, p_{i_2}, \dots, p_{i_k})$ in p such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and $p_{i_\alpha} < p_{i_\beta}$ if and only if $q_\alpha < q_\beta$. Otherwise, we say that p *avoids* q.

Notation 1.2. For a set of permutations avoiding a pattern or set of patterns, we let $S_n(\tau)$ be the set of permutations in S_n avoiding a pattern τ, and we let $S_n(T)$ be the set of permutations avoiding a set of patterns T, in other words, simultaneously avoiding all patterns in T.

Notation 1.3. Let τ' and τ'' be subsequences of a permutation τ. We say that $\tau' > \tau''$ if every entry of τ' is greater than every entry of τ''. We define $\tau' < \tau''$ similarly.

Notation 1.4. For a permutation τ and an integer m, the permutation $\tau + m$ (resp. $\tau - m$) is obtained by adding m to (resp. subtracting m from) every entry of τ.

Since $|S_n(12)| = |S_n(21)| = 1$ for all $n \geq 0$, we are interested in patterns of length $n \geq 3$. For permutations avoiding patterns of length $n = 3$, Knuth [18] and Simion and Schmidt [25] showed that permutations avoiding any $\tau \in S_3$ are Wilf-equivalent, i.e. have the same enumeration sequence. More precisely, for all $\tau \in S_3$ and $n \geq 0$, $|S_n(\tau)| = C_n$, where C_n is the n-th Catalan number.

1.3 Permutation symmetries and Wilf-equivalence

Permutation symmetries are quite convenient, in that they help us find equinumerously avoided patterns (or sets of patterns, respectively).

Definition 1.5. Let patterns τ_1 and τ_2 be such that $|S_n(\tau_1)| = |S_n(\tau_2)|$ for any $n \geq 0$. Also, let sets of patterns T_1 and T_2 be such that $|S_n(T_1)| = |S_n(T_2)|$ for any $n \geq 0$. Then such patterns (or sets of patterns, respectively) are called *Wilf-equivalent* and said to belong to the same *Wilf class*. [7]

Definition 1.6. Let $\pi \in S_n$ be a permutation, and let $\pi(i)$ be the i-th entry of π from left to right. Define a *permutation diagram of* π as the set of dots $\{(i, \pi(i)) \,|\, 1 \leq i \leq n\}$.

Visually, we can represent π by placing dots on an $n \times n$ board in the following fashion: place a dot inside the square in the i-th column from the left and $(\pi(i))$-th row from the bottom. Thus, we read π from left-to-right and bottom-to-top, as mentioned earlier. Note, the origin of our $n \times n$ board is at the bottom-left corner. See Figure 1 on page 11.

Now, let us define a few symmetry operations on S_n that map every pattern onto a Wilf-equivalent pattern on the ambient set S_n.

Definition 1.7. Let $\pi \in S_n$ be a permutation, read left-to-right and bottom-to-top.

- The *reverse* of π, denoted π^{r}, is π read right-to-left and bottom-to-top.
- The *complement* of π, denoted π^{c}, is π read left-to-right but top-to-bottom.
- The *composition of the reverse and the complement* of π, denoted π^{rc}, is π read right-to-left and top-to-bottom. Note that $\pi^{\mathrm{rc}} = \pi^{\mathrm{cr}}$.
- The *inverse* of π, denoted π^{-1}, is the usual inverse of a permutation. That is, the position of each entry and the entry itself of π are switched in π^{-1}.

Definition 1.8. The symmetries above generate the group of symmetries of the square an yield the set of patterns $\{\pi, \pi^{\mathrm{r}}, \pi^{\mathrm{c}}, \pi^{\mathrm{rc}}, \pi^{-1}, (\pi^{-1})^{\mathrm{r}}, (\pi^{-1})^{\mathrm{c}}, (\pi^{-1})^{\mathrm{rc}}\}$, called the *symmetry class* of π.

Example 1.9. Let $\pi = 263541$. Then the symmetry class of π is the set of patterns

$$\{263541, \quad 145362, \quad 514236, \quad 632415, \quad 613542, \quad 245316, \quad 164235, \quad 532461\},$$

Note that each element of the symmetry class of π is Wilf-equivalent to π. Also note that the symmetry class of π, or the symmetry class of any permutation for that matter, yields an interesting fact about pattern avoidances. That is if π avoids a pattern ρ, then each symmetry of π avoids that same symmetry of ρ. In other words, if π avoids ρ, then π^{r} avoids ρ^{r}, π^{c} avoids ρ^{c}, and so on.

Definition 1.10. A *fixed point* (resp. *excedance*, *deficiency*) of a permutation π is a position i such that $\pi(i) = i$ (resp. $\pi(i) > i$, $\pi(i) < i$).

2 Dumont permutations

Dumont permutations are a certain class of permutations shown by Dumont [12] to be counted by the Genocchi numbers, a multiple of the Bernoulli numbers. Let the Genocchi and Bernoulli numbers be denoted G_{2n} and B_{2n}, respectively, then we have $G_{2n} = 2(1-2^{2n})B_{2n}$.

The exponential generating functions for the unsigned and signed Genocchi numbers are given by,

$$\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} (-1)^n G_{2n} \frac{x^{2n}}{(2n)!} = \frac{2x}{e^x+1} - x = -x \tanh \frac{x}{2}.$$

The definition of Dumont permutations of the first and third kinds involves descents and ascents, and thus, the linear structure of permutations. The definition of Dumont permutations of the second and fourth kinds involves fixed points, excedances, and deficiencies, and thus the cyclic structure of permutations.

Definition 2.1. We say that $i \in [n-1]$ is a *descent* of π if $\pi(i) > \pi(i+1)$. In this case, we call $\pi(i)$ the *descent top* and $\pi(i+1)$ the *descent bottom*. We also say that $i \in [n-1]$ is an *ascent* of π if $\pi(i) < \pi(i+1)$. In this case, we call $\pi(i)$ the *ascent bottom* and $\pi(i+1)$ the *ascent top*.

Definition 2.2. A *fixed point* (resp. *excedance*, *deficiency*) of a permutation π is a position i such that $\pi(i) = i$ (resp. $\pi(i) > i$, $\pi(i) < i$).

2.1 Dumont permutations of the first and second kinds

Definition 2.3. A *Dumont permutation of the first kind* (or *Dumont-1 permutation* for short) is a permutation wherein each even entry must be immediately followed by a smaller entry, and each odd entry must be immediately followed by a larger entry, or ends the permutation (i.e. the last entry must be odd). In other words, for all i, $1 \le i \le 2n$, and some k, $1 \le k < n$,

$$\begin{aligned} \pi(i) = 2k &\implies i < 2n \quad \text{and} \quad \pi(i) > \pi(i+1), \\ \pi(i) = 2k-1 &\implies \pi(i) < \pi(i+1) \quad \text{or} \quad i = 2n. \end{aligned}$$

Example 2.4. Let $\pi = 435621 \in S_6$. Then 2, 4, and 6, the descent tops of π, are all even, while the descent bottoms of π are 3 and 5, and its final entry is 1, all odd. Thus, π is a Dumont permutation of the first kind.

Definition 2.5. A *Dumont permutation of the second kind* (or *Dumont-2 permutation* for short) is a permutation wherein each entry at an even position is a deficiency, and each entry at an odd position is a fixed point or an excedance. In other words, for all i, $1 \le i \le n$,

$$\pi(2i) < 2i, \qquad \pi(2i-1) \ge 2i-1.$$

Example 2.6. Let $\rho = 614352 \in S_6$. We see that the deficiencies of ρ occur at positions 2, 4 and 6, all even while excedances occur at positions 1 and 3, and a fixed point is at position 5, all odd. Thus, ρ is a Dumont permutation of the second kind.

Notation 2.7. The Dumont permutations of the first (resp. second) kind shall be denoted by $\mathfrak{D}^1_{2n}$ (resp. $\mathfrak{D}^2_{2n}$).

Thus, from Examples 2.4, and 2.6, we have $\pi \in \mathfrak{D}^1_6$, and $\rho \in \mathfrak{D}^2_8$. As stated earlier, Dumont [12] showed that $|\mathfrak{D}^1_{2n}| = |\mathfrak{D}^2_{2n}| = G_{2n+2}$. There is a simple natural bijection [13] between Dumont permutations of the first and second kinds, referred to as *Foata's fundamental transformation* [14] that maps left-to-right maxima to cycle maxima.

Definition 2.8. An entry $\pi(i)$ of a permutation π is a *left-to-right maximum* if $\pi(i) > \pi(j)$ for all $j < i$.

In particular, the first entry of π is always a left-to-right maximum π.

The map f, called the *Foata's fundamental transformation* [14], is defined as follows:

- Start with a Dumont permutation of the first kind, say π.
- Insert parentheses to display the permutation in cycle notation, so that each cycle starts with a left-to-right maximum.

Then f is a bijection, and $\pi \in \mathfrak{D}^1_{2n}$ if and only if $f(\pi) \in \mathfrak{D}^2_{2n}$.

Example 2.9. For $\pi = 435621 \in \mathfrak{D}^1_6$, we have the left-to-right maxima $\pi(1) = 4$, $\pi(3) = 5$, and $\pi(4) = 6$, so

$$f(\pi) = (43)(5)(621) \;=\; 614352.$$

Notice that $f(\pi) = 614352 \in \mathfrak{D}^2_6$.

2.2 Dumont permutations of the third and fourth kinds

With respect to the next two types of Dumont permutations, Kitaev and Remmel [16, 17] first conjectured that sets of permutations where each descent is from an even value to an even value are also counted by the Genocchi numbers. Burstein and Stromquist [11] proved this conjecture and called those sets of permutations Dumont permutations of the third kind. As with Dumont-1 and Dumont-2 permutations, the Foata's fundamental transformation maps Dumont permutations of the third kind onto a related equinumerous set that Burstein and Stromquist [11] called the Dumont permutations of the fourth kind. We now introduce these two sets.

Definition 2.10. A *Dumont permutation of the third kind* (or *Dumont-3 permutation* for short) is a permutation where each descent is from an even value to an even value, that is both descent tops and descent bottoms are even. In other words, for all i, $1 \le i \le 2n-1$,

$$\pi(i) > \pi(i+1) \implies \pi(i) = 2l \text{ and } \pi(i+1) = 2k$$

for some k and l, $1 \le k < l \le n$.

Example 2.11. Let $\pi = 16238457 \in S_8$. Then the descent tops of π are 6 and 8 are and the descent bottoms are 2 and 4, all even entries. Thus, π is a Dumont permutation of the third kind.

Definition 2.12. A *Dumont permutation of the fourth kind* (or *Dumont-4 permutation* for short) is a permutation where deficiencies must be even values at even positions. In other words, for all i, $1 \le i \le 2n$,

$$\pi(i) < i \implies i = 2u \text{ and } \pi(i) = 2v$$

for some $1 \le v < u \le n$.

Example 2.13. Let $\rho = 13657284 \in S_8$. Then the deficiencies of ρ are $\pi(6) = 2$, $\pi(8) = 4$, where all the positions and entries involved are even. Thus, ρ is a Dumont permutation of the fourth kind.

Notation 2.14. Following the same notation as earlier, the Dumont permutations of the third (resp. fourth) kind shall be denoted by $\mathfrak{D}^3_{2n}$ (resp. $\mathfrak{D}^4_{2n}$).

Thus, from Examples 2.11, and 2.13, we have $\pi \in \mathfrak{D}^3_8$, and $\rho \in \mathfrak{D}^4_8$. The same bijection, Foata's fundamental transformation, yields $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^4_{2n}| = |G_{2n+2}|$. (In fact, notice that $f(16238457) = 13657284$ in Examples 2.11 and 2.13 above.)

We now reference the proof [11] that $|\mathfrak{D}^1_{2n}| = |\mathfrak{D}^3_{2n}| = |G_{2(n+1)}|$, thus showing that all four kinds of Dumont permutations are counted by the Genocchi numbers.

2.3 Pattern avoidance in Dumont-1 and Dumont-2 permutations

There have been several enumerations of pattern-restricted Dumont-1 and Dumont-2 permutations, mostly by Mansour, Burstein, Elizalde, and Ofodile [19, 6, 7, 23]. We will first reproduce their theorems for enumerations of Dumont-1 permutations avoiding patterns of length three.

2.3.1 Patterns of length three

For the six patterns of length three, all the corresponding pattern-restricted sets have been enumerated, both in Dumont-1 and Dumont-2 permutations.

Theorem 2.15. *[19] For all $n \geq 0$,*

$$|\mathfrak{D}^1_{2n}(132)| = |\mathfrak{D}^1_{2n}(231)| = |\mathfrak{D}^1_{2n}(312)| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 2.16. *[6] For all $n \geq 1$,*

$$|\mathfrak{D}^1_{2n}(213)| = C_{n-1}.$$

It is important to note here that although reverses, complements, and inverses of patterns create symmetry classes and Wilf-equivalences in the set of all permutations, Dumont permutations are not closed under any symmetry operations, so that applying a symmetry operation to a pattern does not necessarily yield a pattern-avoiding set of the same cardinality. For example, patterns 132 and 312 are complements of each other, and $|\mathfrak{D}^1_{2n}(132)| = |\mathfrak{D}^1_{2n}(312)| = C_n$; on the other hand, patterns 213 and 231 are also complements of each other, but $|\mathfrak{D}^1_{2n}(213)| = C_{n-1} \neq C_n = |\mathfrak{D}^1_{2n}(231)|$.

Theorem 2.17. *[6] For all $n \geq 0$,*

$$|\mathfrak{D}^1_{2n}(321)| = 1, \quad \textit{namely,} \quad \mathfrak{D}^1_{2n}(321) = \{2, 1, 4, 3, \ldots, 2n, 2n-1\}.$$

Theorem 2.18. *[6] For all $n \geq 3$,*

$$|\mathfrak{D}_{2n}^1(123)| = 4, \quad \textit{namely,} \quad \mathfrak{D}_{2n}^1(123) = \{(2n-1, 2n, 2n-3, 2n-2, \ldots, 7, 8, \pi) \mid \pi \in \mathfrak{D}_6^1(123)\},$$

where

$$\mathfrak{D}_6^1(123) = \{436215, 562143, 563421, 564213\}.$$

Now, with respect to enumerations of Dumont-2 permutations avoiding patterns of length three, we can start with the makeup of a Dumont-2 permutation. It is straightforward that for all $n \geq 3$,

$$|\mathfrak{D}_{2n}^2(123)| = |\mathfrak{D}_{2n}^2(132)| = |\mathfrak{D}_{2n}^2(213)| = 0.$$

This follows from the fact that every even position is a deficiency, every odd position is an excedance or a fixed point, and for any Dumont-2 permutation $\pi(2) = 1$ and $\pi(2n-1) = 2n$ or $2n-1$, which eventually leads to the conclusion that it is impossible to avoid 123, 132, and 213.

Theorem 2.19. *[6] For all $n \geq 1$,*

$$|\mathfrak{D}_{2n}^2(231)| = 2^{n-1}.$$

Theorem 2.20. *[6] For all $n \geq 0$,*

$$|\mathfrak{D}_{2n}^2(312)| = 1, \quad \textit{namely} \quad \mathfrak{D}_{2n}^2(312) = \{2, 1, 4, 3, \ldots, 2n, 2n-1\}.$$

Theorem 2.21. *[19] For all $n \geq 0$,*

$$|\mathfrak{D}_{2n}^2(321)| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

2.3.2 Patterns of length four

Now, for Dumont-1 and Dumont-2 permutations avoiding patterns of length four, there are several cases which are open, as well as several enumerations of permutations that avoid two patterns of length four simultaneously. We begin with Dumont-2 permutations avoiding a single pattern of length four.

Theorem 2.22. *[6] For all $n \geq 0$,*

$$|\mathfrak{D}_{2n}^2(3142)| = C_n.$$

Theorem 2.23. *[19] For all $n \geq 0$,*

$$|\mathfrak{D}_{2n}^2(4132)| = |\mathfrak{D}_{2n}^2(321)| = C_n.$$

Here, we note that 321 is a subsequence of 4132, therefore $\mathfrak{D}_{2n}^2(321) \subseteq \mathfrak{D}_{2n}^2(4132)$. Mansour proved that $\mathfrak{D}_{2n}^2(4132)$ consists exactly of the permutations in $\mathfrak{D}_{2n}^2(321)$.

Theorem 2.24. *[7] For all $n \geq 0$,*

$$|\mathfrak{D}^2_{2n}(2143)| = a_n a_{n+1},$$

where

$$a_{2m} = \frac{1}{2m+1}\binom{3m}{m},$$

and

$$a_{2m+1} = \frac{1}{2m+1}\binom{3m+1}{m+1} = \frac{1}{m+1}\binom{3m+1}{m}.$$

To date, there are no other enumerations of Dumont-1 and Dumont-2 permutations avoiding a single pattern of length four, however Burstein and Jones conjecture Wilf-equivalence on Dumont-1 permutations avoiding 2143 and 3412. This will be further explained in Section 5.1.

Now, we will consider simultaneous avoidance of a pair of patterns. We begin with Dumont-1 permutations.

Theorem 2.25. *[7] For all $n \geq 0$,*

$$|\mathfrak{D}^1_{2n}(1342, 1423)| = |\mathfrak{D}^1_{2n}(2341, 2413)| = |\mathfrak{D}^1_{2n}(1342, 2413)| = s_{n+1},$$

where s_n is the n-th little Schröder number ([26], A001003).

Note that (s_n) starts s_n $(1, 1, 3, 11, 45, 197, 903, \ldots)$ and is recursively given by $s_1 = 1$ and

$$s_{n+1} = -s_n + 2\sum_{k=1}^{n} s_k s_{n-k}, \quad n \geq 2,$$

with the generating function

$$s(x) = \sum_{n=1}^{\infty} s_n x^n = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4}.$$

Theorem 2.26. *[7] For all $n \geq 1$,*

$$|\mathfrak{D}^1_{2n}(231, 4213)| = 1, \quad \textit{namely} \quad \mathfrak{D}^1_{2n}(231, 4213) = \{(2, 1, 4, 3, \ldots 2n, 2n-1)\}.$$

Theorem 2.27. *[7] For all $n \geq 1$, $|\mathfrak{D}^1_{2n}(1342, 4213)| = 2^{n-1}$.*

The last enumeration of Dumont permutations avoiding a pair of patterns we introduce has an interesting sequence.

Theorem 2.28. *[7] For all $n \geq 3$, $|\mathfrak{D}^1_{2n}(2341, 1423)| = b_n$, where b_n satisfies the recurrence relation*

$$b_n = 3b_{n-1} + 2b_{n-2} \quad \textit{for} \quad n \geq 3, \quad \textit{with} \quad b_0 = 1, b_1 = 1, b_2 = 3.$$

Note that this sequence $(1, 1, 3, 11, 39, 139, 495...)$ is ([26], A007482) shifted one position to the right. That is, $|\mathfrak{D}^1_{2n}(2341, 1423)|$ is the number of subsets of $[2n-2]$ where each odd element m has an even neighbor $(m-1$ or $m+1)$.

3 Avoidance on Dumont permutations of the fourth kind

In this section, we will consider pattern avoidance on Dumont-4 permutations. In previous pattern-avoidance literature, the first nontrivial cases to be analyzed were patterns of length 3. Since all Dumont-4 permutations start with 1, we will also consider avoiding patterns $\pi = (1, \pi'+1)$, where π' is a permutation in S_3, i.e. $\pi \in \{1234, 1243, 1324, 1342, 1423, 1432\}$. Note that in all but the first two cases, i.e. if π' does not start with 1, we have

$$\mathfrak{D}^4_{2n}(\pi) = \mathfrak{D}^4_{2n}(\pi').$$

3.1 Enumerating Dumont-4 permutations avoiding Dumont-4 permutations of length four

We will begin by considering Dumont-4 permutations avoiding patterns of length 4, which are themselves Dumont-4 permutations, that is $\pi \in \{1234, 1342, 1432\}$.

Then in the next section we will consider the remaining cases, where $\pi \in \{1243, 1324, 1423\}$.

3.1.1 $\mathfrak{D}^4_{2n}(1234)$

Theorem 3.1. $|\mathfrak{D}^4_{2n}(1234)| = 0$*, for* $n \geq 4$.

Note that $|\mathfrak{D}^4_{2n}(1234)| = 1, 1, 2, 4$, for $n = 0, 1, 2, 3$, respectively. The Dumont-4 permutations of length at most 6 avoiding pattern 1234 are ϵ, 12, 1342, 1432, 132654, 136254, 143265, 143652, where ϵ denotes the empty permutation.

Proof. Recall that in Dumont-4 permutations, $\pi(1) = 1$, and $\pi(2n-1) = 2n-1$ or $2n$. In addition to that, for $n \geq 2$, we have $\pi(2) = 3$ or $\pi(3) = 3$.

Since $\pi(1) = 1$, the three conditions above mean that, of the $2n-5$ entries in $[4, 2n-2]$, at most one is to the left of 3 and at most one is to the right of $\pi(2n-1)$. This leaves at least $2n-7 \geq 1$ entries in $[4, 2n-2]$ that are to the right of 3 and to the left of $\pi(2n-1)$. Any such entry, together with 1, 3, and $\pi(2n-1)$ would form an occurrence of pattern 1234. □

3.1.2 $\mathfrak{D}^4_{2n}(1342)$

As noted earlier, since the entry following 1 is not 2, it follows that $\mathfrak{D}^4_{2n}(1342) = \mathfrak{D}^4_{2n}(231)$.

Theorem 3.2. $|\mathfrak{D}^4_{2n}(1342)| = 2^{n-1}$*, for* $n \geq 1$.

We will first prove that all odd entries are fixed points.

Lemma 3.3. *Let* $\pi \in \mathfrak{D}^4_{2n}(1342)$*. Then* $\pi(2k-1) = 2k-1$ *for all* k*,* $1 \leq k \leq n$.

Proof. By definition of a Dumont-4 permutation $\pi(1) = 1$, in other words 1 is a fixed point.

Now assume that all odd entries from 1 through $2j-1$ are fixed points. Consider the entry in the next odd position, $\pi(2j+1)$, and as before, suppose $\pi(2j+1) \neq 2j+1$. It follows that

- the entry $2j+1$ must be to the left of $\pi(2j+1)$,
- $\pi(2j+1) > 2j+1$ as an odd position cannot be a deficiency, and
- at least one even entry $2l \leq 2j$ must be to the right of $\pi(2j+1)$. This is because if all odd entries at most $2j+1$ are to the left of $\pi(2j+1)$, so if all even entries less than $2j+1$ are also to the left of $2j+1$, then there will be $2j+1$ entries to the left of position $2j+1$, which is impossible.

This yields a 1342-occurrence $(1, 2j+1, \pi(2j+1), 2l)$. Therefore $\pi(2j+1) = 2j+1$, that is $2j+1$ is fixed.

By induction, the lemma is proved. □

Next, we will prove that if we have a deficiency at position $2k$, then it must be the entry $2k-2$.

Lemma 3.4. *Let $\pi \in \mathfrak{D}^4_{2n}(1342)$. If $\pi(2k) < 2k$ for all k, $1 \leq k \leq n$, then $\pi(2k) = 2k-2$.*

Proof. We will prove by contradiction that no deficiency at position $2k$ can have an entry less than or equal to $2k-4$. So, let $\pi(2k) \leq 2k-4$. Since all odd entries are fixed points by Lemma 3.3, the two odd entries $2k-3$ and $2k-1$ are above and to the left of $\pi(2k)$, the triple $(2k-3, 2k-1, \pi(2k))$ at positions $(2k-3, 2k-1, 2k)$ would yield a 231-occurrence.

By contradiction, the lemma is proved. □

Now we will prove Theorem 3.2.

Proof. Given Lemma 3.3, all odd entries are fixed, thus the even values are the only entries of interest. Moreover, since all odd entries are fixed, all even values occur in even positions. Additionally, due to Lemma 3.4, a deficiency at position $2k$ must be the entry $2k-2$.

Now, let $\pi \in \mathfrak{D}^4_{2n}(1342)$ and consider $\pi(2)$. If $\pi(2) = 2$, then $\pi = (1, 2, \pi' + 2)$ where $\pi' \in \mathfrak{D}^4_{2n-2}(1342)$. So let $\pi(2) = 2k \geq 4$, for some $2 \leq k \leq n$. Therefore all entries in $[2, 2k-1]$ must precede all entries larger than $2k$, otherwise, we encounter a 231-occurrence. Then the block immediately to the right of $\pi(2) = 2k$ consists of entries belonging to the interval $[2, 2k-1]$ which are in positions $[3, 2k]$. Since the entries of interest are even values at even positions, we focus on even entries belonging to the set $[2, 2k-2]$ which are in even positions in $[4, 2k]$.

Now, since $\pi(2) = 2k$ is an excedance, and since the entries in our block belong to the interval $[2, 2k-2]$, the entry in position $2k$ must be a deficiency, namely the entry $2k-2$ by Lemma 3.4, i.e. $\pi(2k) = 2k-2$. Therefore $\pi(2k-2)$ cannot be the entry $2k-2$, nor can it be an excedance since our entries belong to the interval $[2, 2k-2]$. Hence $\pi(2k-2)$ must be a deficiency, namely the entry $2k-4$, i.e. $\pi(2k-2) = 2k-4$. Continuing in this fashion, we have $\pi(2l) = 2l-2$ for $2 \leq l \leq k$ for the entire block with entries belonging to the set $[2, 2k-2]$, which are in positions $[4, 2k]$. More specifically, $\pi = (1, 2k, \pi', \pi'' + 2k)$ where $\pi' = (3, 2, 5, 4, 7, 6, \ldots, 2k-1, 2k-2)$ is uniquely determined by the placement of $2k$, and $\pi'' \in \mathfrak{D}^4_{2n-2k}(1342)$. Notice also that if $k = 1$, then π' is empty. Moreover, the next even nondeficiency (a fixed point or an excedance) occurs at position $2k+2$.

Continuing recursively in this fashion, we see that the entire structure of π is determined by the values of its even nondeficiencies. Note that one of the nondeficiencies must be the entry $2n$, since $2n \geq \pi^{-1}(2n)$, the position of $2n$. Thus, π is determined by the choice of a subset $S \subseteq [1, n-1]$ such that if $k \in S$, then $2k$ is a nondeficiency. All of these choices are unrestricted, therefore the number of such choices is 2^{n-1}. Moreover, it is clear that the semilengths of the resulting blocks of the same form as $(1, 2k, \pi')$ yield a composition of n into nonzero parts. See Figure 1 corresponding to $\pi = 16325478 \in \mathfrak{D}_8^4(1342)$, which corresponds to the composition $4 = 3 + 1$. ☐

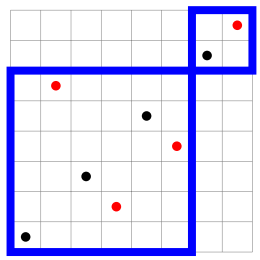

Figure 1: Permutation diagram of $\pi = 16325478 \in \mathfrak{D}_8^4(1342)$

Example 3.5. Below are all eight Dumont-4 permutations of length 8 avoiding 1342, and the corresponding compositions of the integer 4.

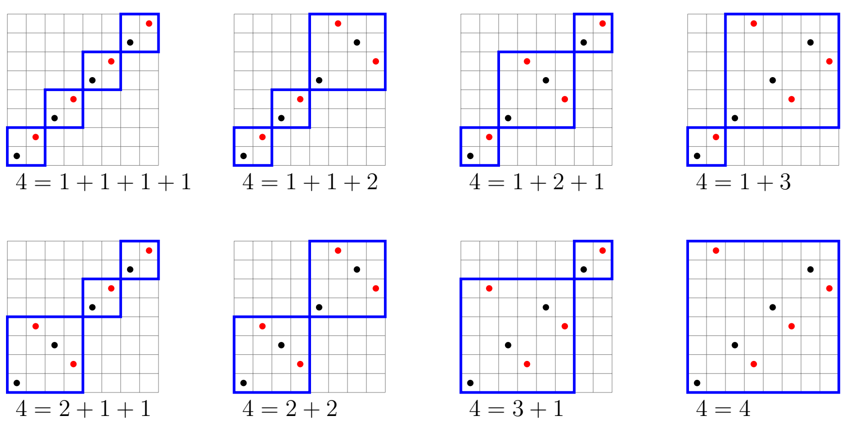

Figure 2: The eight Dumont-4 permutations of length 8 avoiding 1342

3.1.3 $\mathfrak{D}_{2n}^4(1432)$

As noted earlier, since the entry following 1 is not 2, it follows that $\mathfrak{D}_{2n}^4(1432) = \mathfrak{D}_{2n}^4(321)$.

Theorem 3.6. *$|\mathfrak{D}_{2n}^4(1432)| = C_n$, for $n \geq 0$, where C_n is the n-th Catalan number.*

Proof. Let $\pi \in \mathfrak{D}^4_{2n}(1432) = \mathfrak{D}^4_{2n}(321)$, and consider right-to-left minima (an entry $\pi(i)$ for which $\pi(i) < \pi(j)$ whenever $i < j$). If an odd fixed point is a right-to-left minimum, then every entry to the left must have a smaller value, and every entry to the right must have a larger value, otherwise we encounter a 321-occurrence with the entry $2k+1$ serving as the "2". In other words, if $\pi(2k+1) = 2k+1$, for some $0 \le k \le n-1$, then $\pi = (\pi', 2k+1, \pi''+2k+1)$ where $\pi' \in \mathfrak{D}^4_{2k}(321)$, and $(1, \pi''+1) \in \mathfrak{D}^4_{2n-2k}(321)$. And, since the odd entry $\pi(2k+1) = 2k+1$ is a right-to-left minimum, the entry $2k+2$, which must be to the right of $2k+1$, is also a right-to-left minimum.

Consider a $2n \times 2n$ board with the dots in the i-th column from the left being in $\pi(i)$-th row from the bottom, for $1 \le i \le 2n$. Now consider the dots which represent even right-to-left minima (solid red dots in Figure 3.1.3), and lower them one cell down (hollow red dots in Figure 3.1.3). Notice that, in particular, this will place a hollow red dot in each row with an odd right-to-left minimum.

Now travel along the cell boundaries from $(0,0)$ to $(2n,2n)$ in an *East-North* fashion, using steps $(1,0)$ (*east*) and $(0,1)$ (*north*) and keeping all dots to the left of the path, staying as close to the diagonal as possible. Equivalently, this is the path where the peaks (instances where an east step is followed directly by a north step) are exactly the bottom and right boundaries of the cells with hollow red dots.

For example, consider Figure 3.1.3 corresponding to $\pi = 1\ 3\ 5\ 2\ 6\ 4\ 7\ 8\ 9\ 11\ 12\ 10 \in \mathfrak{D}^4_{12}(1432)$. Note that red dots are even right-to-left minima, the blue dots are odd right-to-left minima (which are fixed points), and the black dots are excedances.

Notice that in this path (see Figure 4), all *runs* (maximal consecutive segments) of east/north steps are of even length. Dividing the lengths of these runs in half, we obtain a Dyck path of semilength n from $(0,0)$ to (n,n) (see Figure 5).

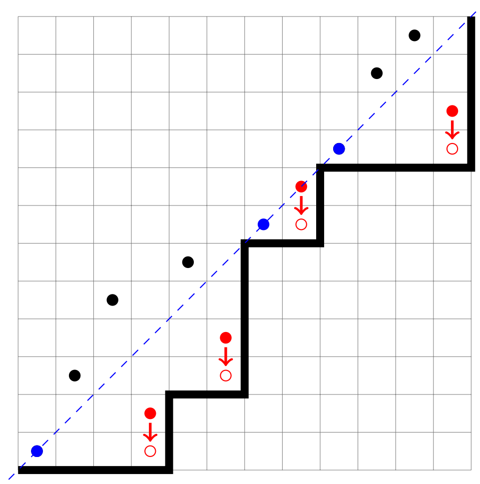

Figure 3: Permutation diagram of $\pi = 1\ 3\ 5\ 2\ 6\ 4\ 7\ 8\ 9\ 11\ 12\ 10 \in \mathfrak{D}^4_{12}(1432)$

We one can read each east step as a **u** and each north step as a **d**. Note, the diagonal of the permutation diagram in Figure corresponds to the x-axis in the Dyck path in Figure 4, while **u** corresponds to step $(1,1)$ and **d** corresponds to step $(1,-1)$.

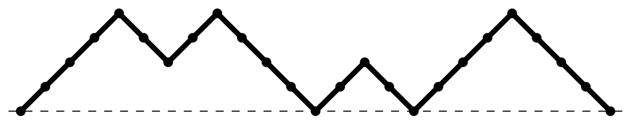

Figure 4: Dyck path corresponding to $\pi = 1\ 3\ 5\ 2\ 6\ 4\ 7\ 8\ 9\ 11\ 12\ 10 \in \mathfrak{D}^4_{12}(1432)$

Note that the corresponding Dyck word is **uuuu dd uu dddd uu dd uuuu dddd**. Dividing each run length in half, we obtain the string **uu d u dd u d uu dd** as in Figure 5.

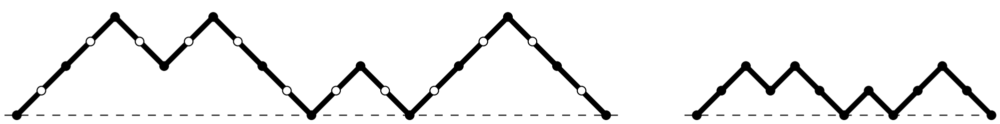

Figure 5: Dyck path of semilength $2n$ mapping to Dyck path of semilength n

Therefore, the number of Dumont-4 permutations of length $2n$ avoiding 321 is the same as the number of Dyck paths of semilength n, which is the n-th Catalan number, C_n. This ends the proof. □

3.2 Enumerating Dumont-4 permutations avoiding certain permutations of length four

Now that all three Dumont-4 permutations of length four have been avoided by Dumont-4 permutations of length $2n$, we will look at three other permutations of length four starting with 1, namely 1324, 1243, and 1423. Also note, excluding the entry "1", the three permutations in the previous section coupled with the three permutations in this section constitute all of S_3.

3.2.1 $\mathfrak{D}^4_{2n}(1324)$

As noted earlier, since the entry following 1 is not 2, it follows that $\mathfrak{D}^4_{2n}(1324) = \mathfrak{D}^4_{2n}(213)$.

Theorem 3.7. $|\mathfrak{D}^4_{2n}(1324)| = 2\binom{n}{2} + 1 = n^2 - n + 1$, *for* $n \geq 0$.

Proof. For this enumeration, we will start "from the right". If the last entry is the maximal element, $2n$, and by definition of a Dumont-4 permutation, the first entry is 1, then the only Dumont-4 permutation that would avoid 213 is the increasing permutation. That is, if the entry $2n$ occurs in the last position, then $\pi = 1234\ldots 2n$. This one permutation takes care of the last summand, in other words, the "1", in $2\binom{n}{2} + 1$.

Now, if the last entry is not $2n$, then it must be a deficiency, so by definition of a Dumont-4 permutation, it must be even, say $2k$ where $1 \le k \le n-1$. Using the block decomposition method, which was first proposed by Mansour and Vainshtein in [20] and referenced and defined in [21] by them as well, we have $\pi = (1, \pi', \pi'', 2k)$ where,

- π' is a permutation of the set $[2, 2k-1]$,
- π'' is a permutation of the set $[2k+1, 2n]$, and
- both π' and π'' avoid 213.

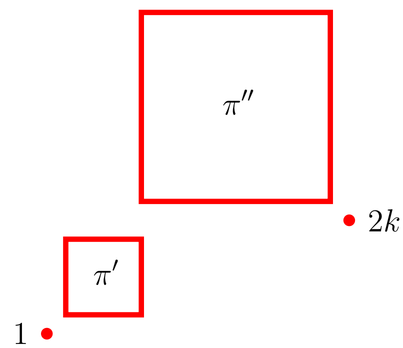

Figure 6: Block decomposition for $\pi \in \mathfrak{D}^4_{2n}(1324)$ when $2k$ is in position $2n$

Note, that the block representing π'' must contain $2n$. Now, reconsidering the entry $2n$, again, if it is the last entry of the block representing π'' the resulting permutation would be increasing on the diagonal of that block. The block representing π' would be increasing as well. Thus we have,

$$\pi = (\pi', \pi'', 2k) = \{1, 2, 3, \ldots, 2k-1, 2k+1, 2k+2, \ldots, 2n-1, 2n, 2k\}.$$

Now, since $2n$ is an excedance, it could be in any position in the block representing π'', say position $l \in [2k+1, 2n-1]$.

Note that $\pi(2n-1) = 2n-1$ or $2n$ and that all entries below and to the left of $\pi(2n-1)$ must be in increasing order to avoid 213. Therefore, all entries in $[1, 2n-1] \setminus \{2k\}$, i.e. all entries of π except $2n$ and $2k$, are in increasing order, which determines their positions uniquely.

This forces another decomposition inside the block representing π''. In other words, we have $\pi'' = (\rho, 2n, \sigma)$ where,

- ρ is an increasing permutation of the set $\{2k+1, 2k+2, \ldots, l\}$, and
- σ is an increasing permutation of the set $\{l+1, l+2, \ldots, 2n-1\}$.

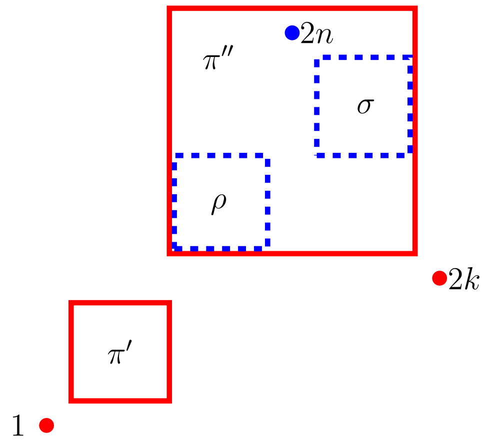

Figure 7: Block decomposition for $\pi \in \mathfrak{D}^4_{2n}(1324)$ when $2n$ is in position l

Given that $\pi(2n) = 2k < 2n$, the number of possible choices for the position l of $2n$ is the number of elements in $[2k+1, 2n]$, i.e. $2n - 2k$. Since $1 \le k \le n-1$, the total number of Dumont-4 permutations avoiding 1324 is

$$|\mathfrak{D}^4_{2n}(1324)| = 1 + \sum_{k=1}^{n-1}(2n-2k) = 1 + 2\sum_{k=1}^{n-1}(n-k) = 1 + 2\sum_{k=1}^{n-1} k$$
$$= 1 + 2\binom{n}{2} = n^2 - n + 1. \qquad \square$$

Example 3.8. We can create a Dumont-4 permutation of length 16 avoiding 1324 by placing the entry 6 in the last position, and the entry 16 in position 10. All other entries must be in their respective blocks, and in increasing order on the diagonal of each block. See Figure 8.

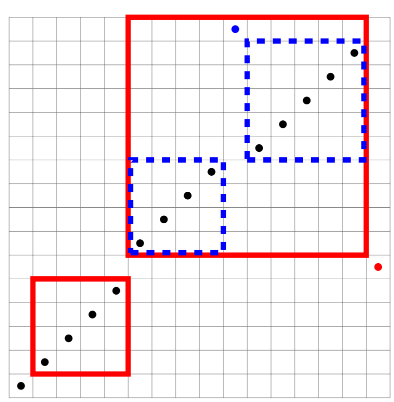

Figure 8: Permutation diagram of $\pi = 1\ 2\ 3\ 4\ 5\ 7\ 8\ 9\ 10\ 16\ 11\ 12\ 13\ 14\ 15\ 6 \in \mathfrak{D}^4_{16}(1324)$

3.2.2 $\mathfrak{D}^4_{2n}(1243)$

Theorem 3.9. $|\mathfrak{D}^4_{2n}(1243)| = |\mathfrak{D}^4_{2n}(1324)| = 2\binom{n}{2} + 1 = n^2 - n + 1$, *for* $n \geq 0$.

In other words, patterns 1324 and 1243 are Wilf-equivalent on Dumont permutations of the fourth kind.

Proof. This enumeration is the same as that of $\mathfrak{D}^4_{2n}(1324)$ since the removal of the entry 1 from patterns 1324 and 1243 yields the patterns 213 and 132 that are each other's reflections with respect to the antidiagonal. That is, 213 and 132 are inverses of reversals of complements of each other, and thus their respective avoidance classes are enumerated by the same sequence. □

Example 3.10. As in Example 3.8 we can create a Dumont-4 permutation of length 16 avoiding 1243. Here we remove the entry "1" and reflect the remaining entries from Example 3.8 about the antidiagonal. This maps blocks onto blocks, with their diagonals mapping onto the diagonals of the images of those blocks. Lastly, re-insert the "1" that was removed, which will then add 1 to every entry. See Figure 9.

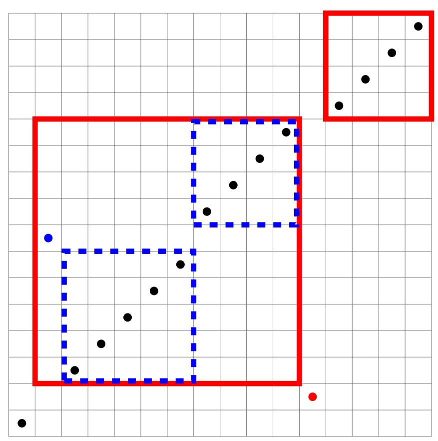

Figure 9: Permutation diagram of $\rho = 1\ 8\ 3\ 4\ 5\ 6\ 7\ 9\ 10\ 11\ 12\ 2\ 13\ 14\ 15\ 16 \in \mathfrak{D}^4_{16}(1243)$

3.2.3 $\mathfrak{D}^4_{2n}(1423)$

Since our theorem involves a continued fraction, we need a remark on notation.

Notation 3.11. For visual convenience, the following notation will be used for continued fractions: instead of

$$\cfrac{a}{\alpha \pm \cfrac{b}{\beta \pm \cfrac{c}{\gamma \pm \ddots}}}$$

we write

$$\frac{a}{\alpha} \pm \frac{b}{\beta} \pm \frac{c}{\gamma} \pm \cdots$$

Theorem 3.12. *For all $k \geq 1$, the ordinary generating function for $|\mathfrak{D}^4_{2n}(1423)|$ is $R_1/z = zR_1/z^2$, where the sequence of functions zR_{2k+1}, $k \geq 0$, satisfies the recurrence relation*

$$zR_{2k+1} = \frac{z^2 C_{e,2k}}{(1 - zC_{o,2k-1})^2} - \frac{z^2 C_{e,2k}}{1 - zC_{o,2k+1}} - \frac{z^2 C_{e,2k} C_{e,2k+2}}{1} - zR_{2k+3}.$$

We consider a Dumont-4 permutation avoiding 1423, and we are interested in the parity of the smallest entry in each block using block decomposition after the "1". Note, since every Dumont-4 permutation begins with the entry "1", avoiding 1423 is the same as avoiding 312 by the Dumont-4 permutation with the "1" removed from the beginning of the permutation. Therefore we analyze the block decomposition of the resulting permutation diagram, ignoring the initial block of "1".

To analyze the blocks resulting from the iterations of the block decompositon, we will need an auxiliary parameter, namely, the length m of the maximal contiguous segment of allowed cells in the bottom row. Our cases are further subdivided according to the parity of m and $n - m$, where n is the dimension of the (square) block. We will refer to the cells on the line $y = x - m$ as the *m-th subdiagonal* of a board and call the cells below the m-th subdiagonal *m-deficiencies* (so, deficiencies as defined earlier are 0-deficiencies in this terminology).

Now, if the bottom row of a block is an odd row in the starting diagram of a Dumont-4 permutation then there is no m-deficiency in that bottom row. If the bottom row of a block is an even row in the starting diagram, then there may be an m-deficiency in that bottom row. Given the definition of the Dumont-4 permutations, in the boards resulting from the repeated block decomposition, all cells above or to the left of the m-th subdiagonal are allowed, whereas the positions and values of the possible m-deficiencies are parity-restricted as described below:

EE blocks: $m = 2k$, $n - m$ is even (so n is even), and all m-deficiencies are even values in odd positions;

NE blocks: $m = 2k + 1$, $n - m$ is even (so n is odd), and all m-deficiencies are even values in even positions;

EN blocks: $m = 2k$, $n - m$ is odd (so n is odd), and all m-deficiencies are odd values in even positions;

NN blocks: $m = 2k + 1$, $n - m$ is odd (so n is even), and all m-deficiencies are odd values in odd positions.

We notate the blocks mnemonically using the "E" for even and "N" for "not even," i.e. odd. However, for convenience in solving for the generating functions, we will let the *EE* (resp. NE, EN, and NN) block with parity-restricted positions and values of m-deficiencies be represented by the generating function $P_m = P_m(z)$ (resp. $R_m = R_m(z), S_m = S_m(z)$, and $T_m = T_m(z)$), and refer to that block as a P-board (resp. R-board, S-board, and T-board). See Figure 10, where blue dots are odd positions and red dots are even positions.

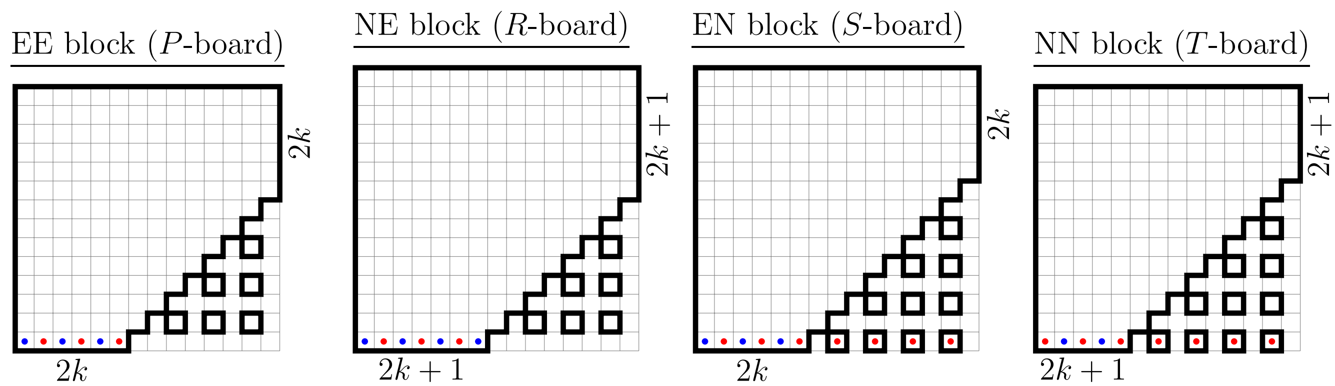

Figure 10: Blocks whose bottom row was odd (even) in the starting $\mathfrak{D}^4$-permutation diagram

Proof. We now produce the generating function which will enumerate $\mathfrak{D}^4_{2n}(1423) = \mathfrak{D}^4_{2n}(312)$. To begin, note that a P-board and T-board may be empty, while an R-board and S-board cannot be empty. This is due to the fact that P-boards and T-boards have even dimensions whereas R-boards and S-boards have odd dimensions. Additionally, we will need to use the truncations of the even and odd parts of the Catalan generating function $C(z)$. Define

$$C_{e,m} = C_{e,m}(z) = \sum_{i \le m} C_{2i} z^{2i} \quad \text{and} \quad C_{o,m} = C_{o,m}(z) = \sum_{i \le m} C_{2i+1} z^{2i+1}.$$

Let us look closer at the origin of the generating functions pertaining to each of the four blocks. For the P-board with $m = 2k$, we have the generating function $P_{2k}(z)$, the recurrence relation for which has three summands:

- The P-board may empty, thus the summand "1".
- If the P-board is nonempty, consider the entry in the bottom row. If it is in an odd position $\le 2k-1$. The factor of z corresponds to the bottom row entry itself, or in this case the blue dot. The block to the left of the blue dot must be a square of even dimension $\le 2k-2$ with all cells allowed, and the block to the right of the blue dot is an S-board with $m = 2k$, thus correponding to the generating function S_{2k}. This takes care of three factors, which we color blue, namely $zC_{e,2k-2}S_{2k}$.
- Lastly, consider the entry in the bottom row that is in an even position $\le 2k$. Again, the factor z corresponds to the bottom row entry, or in this case the red dot.The block to the left of the blue dot must be a square of odd dimension $\le 2k-1$ with all cells allowed, and the block to the right of the blue dot is a P-board with $m = 2k$, thus corresponding to the generating function P_{2k}. This takes care of three factors which we color red, namely $zC_{o,2k-1}P_{2k}$.

Thus, the recurrence formula for our generating function P_{2k} is given by

$$P_{2k} = 1 + zC_{e,2k-2}S_{2k} + zC_{o,2k-1}P_{2k}.$$

We find the remaining recurrence formulas for $R_{2k+1}, S_{2k}, T_{2k+1}$ in the same fashion, which results in the following system of equations:

$$\begin{cases} P_{2k} = 1 + zC_{e,2k-2}S_{2k} + zC_{o,2k-1}P_{2k} \\ R_{2k+1} = zC_{e,2k}T_{2k+1} + zC_{o,2k-1}R_{2k+1} \\ S_{2k} = zC_{e,2k}P_{2k} + zR_{2k+1}S_{2k} \\ T_{2k+1} = 1 + zP_{2k+2}R_{2k+1} + zC_{o,2k-1}T_{2k+1} \end{cases} \tag{1}$$

Now, consider the sequence $\{|\mathfrak{D}^4_{2n}(1423)|\}_{n\ge 0} = \{|\mathfrak{D}^4_{2n}(312)|\}_{n\ge 0}$. We claim that its ordinary generating function is R_1/z. Indeed, removing the top row and the rightmost column of the R-board with $m = 1$ yields exactly the board of allowed cells in a Dumont-4 permutation.

Now consider the function R_1/z. Solving the system of equations (1) for R_{2k+1}, $k \geq 0$, yields the following recursive formula:

$$R_{2k+1} = \frac{zC_{e,2k}}{(1 - zC_{o,2k-1})^2} \; - \; \frac{z^2C_{e,2k}}{1 - zC_{o,2k+1}} \; - \; \frac{z^2C_{e,2k}C_{e,2k+2}}{1} \; - \; zR_{2k+3}.$$

Multiplying both sides by z, we obtain for all $k \geq 0$:

$$zR_{2k+1} = \frac{z^2C_{e,2k}}{(1 - zC_{o,2k-1})^2} \; - \; \frac{z^2C_{e,2k}}{1 - zC_{o,2k+1}} \; - \; \frac{z^2C_{e,2k}C_{e,2k+2}}{1} \; - \; zR_{2k+3}.$$

Note, the term on the left and the last term on the right are in the same form with k increasing by 1. This yields the continued fraction representation for $R_1/z = zR_1/z^2$. □

Also note that the numerators of the other terms in the recurrence formula contain only the truncations of the even part of $C(z)$, and the denominators contain only the truncations of the odd part of $C(z)$.

4 A single occurrence of patterns in Dumont permutations

The results of this section focus on enumeration of Dumont permutations with a single occurrence of certain patterns.

4.1 One occurrence in Dumont-1 and Dumont-2 permutations

We first introduce notation that is useful for going back and forth between sequences and their generating functions. Then we will review results on single occurrences of patterns in Dumont-1 and Dumont-2 permutations. Finally, we will prove a related result on a single occurrence of a pattern in Dumont-4 permutations.

Notation 4.1. For any ordinary generating function, let:

$$A(z) \longleftrightarrow \{a_n\} \quad \text{if} \quad A(z) = \text{ogf}\{a_n\} = \sum_{n=0}^{\infty} a_n z^n.$$

For example, for the Catalan numbers we have $\{C_n\} \longleftrightarrow C = C(z)$ and $C = 1 + zC^2$. Recall that

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

which implies that, for any $k \geq 1$,

$$C^k \longleftrightarrow \frac{k}{n+k}\binom{2n+k-1}{n}.$$

Likewise, for the central binomial coefficients $B_n = \binom{2n}{n}$ we have $\{B_n\} \longleftrightarrow B = B(z)$ and $B = 1 + 2zBC$. This implies that, for any $k \geq 1$,

$$BC^k \longleftrightarrow \binom{2n+k}{n}.$$

Theorem 4.2. *[19] For all $n \geq 0$, there does not exist a Dumont-1 permutation containing 132 exactly once. That is,*

$$|\mathfrak{D}^1_{2n}(132;1)| = 0.$$

The next few theorems are based on the results by Burstein [9] and Ofodile [23] on Dumont-1 and Dumont-2 permutations with a single occurrence of certain patterns.

Theorem 4.3. *[23] For all $n \geq 0$,*

$$|\mathfrak{D}^1_{2n}(312;1)| = 0 \quad \textit{and} \quad |\mathfrak{D}^1_{2n}(231;1)| = \binom{2n-2}{n-3}.$$

Note that $2n-2 = 2(n-3)+4$, so

$$\binom{2n-2}{n-3} = \binom{2(n-3)+4}{n-3} \longleftrightarrow z^3BC^4.$$

Theorem 4.4. *[23] For all $n \geq 4$,*

$$|\mathfrak{D}^1_{2n}(213;1)| = C_{n-2} + \binom{2n-4}{n-4}.$$

Similarly,

$$C_{n-2} + \binom{2n-4}{n-4} = C_{n-2} + \binom{2(n-4)+4}{n-4} \longleftrightarrow z^2C + z^4BC^4.$$

Theorem 4.5. *[9] For all $n \geq 2$,*

$$|\mathfrak{D}^1_{2n}(321;1)| = (n-1)^2.$$

Theorem 4.6. *[9] For all $n \geq 2$,*

$$|\mathfrak{D}^2_{2n}(321;1)| = \frac{5}{n+3}\binom{2n}{n-2} \longleftrightarrow z^2C^5.$$

Theorem 4.7. *[23] For all $n \geq 2$,*

$$|\mathfrak{D}_{2n}^2(3142;1)| = \binom{2n-1}{n-2} \longleftrightarrow z^2BC^3.$$

Theorem 4.8. *[9] For all $n \geq 2$,*

$$|\mathfrak{D}_{2n}^2(2143;1)| = a_n b_{n+1} + b_n a_{n+1} + a_{n-1}a_n,$$

where

$$a_{2k} = \frac{1}{2k+1}\binom{3k}{k}, \quad a_{2k+1} = \frac{1}{k+1}\binom{3k+1}{k},$$

and

$$b_{2k} = \binom{3k-3}{k-2}, \quad b_{2k+1} = 2\binom{3k-2}{k-2}.$$

4.2 A single occurrence of patterns in Dumont-4 permutations

The results of this subsection focus on a single occurrence of patterns in Dumont-4 permutations. To begin, let us consider a theorem.

Theorem 4.9. *[22, 24]*

$$|S_n(123;1)| = \frac{3}{n}\binom{2n}{n-3}$$

Note, this is not only for Dumont permutations, but for all $S_n(123;1)$. The original proof by both Noonan [22] and Noonan and Zeilberger [24] uses a complicated induction and generating functions with multiple auxiliary parameters. Later, Burstein [5] gave a "lovely" combinatorial proof, as stated by Zeilberger [29], who subsequently shortened it. Additionally, Burstein proved it using the reversal 321 of 123 for an additional nice property. Given the methods used in this "lovely" proof, as well as the corresponding permutation diagram of the injection $f : S_n(321;1) \longrightarrow S_{n+2}(321)$[5], we will now prove a theorem first presented in [8], enumerating a single occurrence of 321 in Dumont-4 permutations.

Theorem 4.10. *[8] For all $n \geq 0$,*

$$\begin{aligned}
|\mathfrak{D}_{2n}^4(321;1)| &= |S_n(321;1)| + |S_{n+1}(321;1)| \\
&= \frac{3}{n}\binom{2n}{n-3} + \frac{3}{n+1}\binom{2n+2}{n-2} \\
&= C_{n+3} - 3C_{n+2} - C_{n+1} + 3C_n.
\end{aligned}$$

Note that it follows from Theorems 4.9 and 4.10 that the generating function for the sequence $|\mathfrak{D}_{2n}^4(321;1)|$, $n \geq 0$, is

$$z^3C^6 + \frac{1}{z}\cdot z^3C^6 = (z^2+z^3)C^6.$$

Proof. Considering that $\mathfrak{D}^4_{2n}(321;1) \subseteq S_n(321;1)$, the "2" is fixed and the "3" and "1" have similar properties if we look at moving each to their two respective locations to obtain 321-avoiding permutations which may or may not necessarily be Dumont-4 permutations. We start by temporarily removing the first entry since by definition it is a 1, and we will also remove the "2" of the occurrence of 321 temporarily. We then analyze two cases of the form of that particular permutation with the "2" removed. Next, we add entries to our two permutations so as to result in two new Dumont-4 permutations that avoid 321, which as we know are enumerated by the Catalan numbers. We view these two new Dumont-4 permutations as an ordered pair. We will see that the combined length of the two resulting permutations in $\mathfrak{D}^4(321)$ is either $2n+2$ or $2n+4$, depending on whether the "2" is even or odd, respectively. We will use the map of Theorem 3.6 to produce two permutations that avoid 321 as in the proof in [5] with the combined length of $n+1$ or $n+2$. We then combine them to produce a single Dumont-4 permutation of length n or $n+1$ (depending on the parity of "2") with a single occurrence of pattern 321.

Let $\pi \in \mathfrak{D}^4_{2n}(321;1)$. Now, let π' and π'' be as in the proof in [5] of Theorem 4.9. Then both π' and π'' avoid 321, π' does not end on its top entry, and π'' does not start with its bottom entry.

Now, let us consider two cases based on the parity of the middle entry "2" of the single 321-occurrence in π: (1) the "2" is even, and (2) the "2" is odd.

Case 1. Suppose the entry "2" is even, say, $2k$ for some $k \in [2, n-1]$ (note that the entry $2k$ cannot be either 2 or $2n$ since neither of the two is involved in an occurrence of 321, the former because 1 is a fixed point, the latter because $2n$ is the largest entry).

Then $|\pi'| = 2k$ and $|\pi''| = 2n - 2k + 1$. Moreover, the "1" of the occurrence of 321, which is the last entry of π', is an even value since π is Dumont-4, and it will also occupy the last position in π', i.e. position $2k$, which is even. Therefore, π' is also Dumont-4, and thus $\pi' \in \mathfrak{D}^4_{2k}(321)$. For what follows after this case analysis, let $\widetilde{\pi'} = \pi'$.

Now consider the permutation $\widetilde{\pi''} = (1, \pi''+1)$. Note that the entry 2 in $\widetilde{\pi''}$ is not a fixed point, since π'' does not begin with its lowest entry, thus 2 is a deficiency of $\widetilde{\pi''}$. Therefore, $\widetilde{\pi''}$ is also Dumont-4, and thus $\widetilde{\pi''} \in \mathfrak{D}^4_{2n-2k+2}(321)$.

Note that the sizes of $\widetilde{\pi'}$ and $\widetilde{\pi''}$ add up to $2k + (2n-2k+2) = 2n+2$ in this case.

Case 2. Suppose the entry "2" is odd, say, $2k+1$ for some $k \in [1, n-1]$. Then $|\pi'| = 2k+1$ and $|\pi''| = 2n - 2k$.

Insert the new entry $2k+2$ in the second rightmost position of π' to form a permutation $\widetilde{\pi'}$. Then $|\widetilde{\pi'}| = 2k+2$, which is even. Moreover, as in Case 1, the entry "1" from the the 321 occurrence in π, which is at most $2k$ and even (as a deficiency), now occupies position $2k+2$, also even. Therefore, $\widetilde{\pi'} \in \mathfrak{D}^4_{2k+2}(321)$.

Now form the permutation $\widetilde{\pi''}$ from π'' in the following fashion: if $\pi'' = (\pi_1, 1, \pi_2)$, then $\widetilde{\pi''} = (1, 3, \pi_1 + 2, 2, \pi_2 + 2)$. Note that $|\widetilde{\pi''}| = |\pi''| + 2 = 2n - 2k + 2$ all deficiencies of $\widetilde{\pi''}$ come from deficiencies of π.

The "1" of the 321 occurrence in π (which is a deficiency and thus an even value) becomes the entry 2 of $\widetilde{\pi''}$, and hence also an even deficiency. The entries of $\widetilde{\pi''}$ that are at least 4 are obtained by subtracting $2k-2$ from the corresponding entries of π. However, their position is also shifted to the left by $2k-2$, so the parity of their positions and values as well as the placement relative to the diagonal (i.e. being a fixed point, excedance, or deficiency) remains the same. Therefore, $\widetilde{\pi''}$ is a Dumont-4 permutation, and thus $\widetilde{\pi''} \in \mathfrak{D}_{2n-2k+2}(321)$.

Note that the sizes of $\widetilde{\pi'}$ and $\widetilde{\pi''}$ add up to $(2k+2)+(2n-2k+2) = 2n+4$ in this case.

Now, given permutations $\widetilde{\pi'}$ and $\widetilde{\pi''}$, use the map of Theorem 3.6 to produce two permutations σ' and σ'', respectively, that avoid 321. Note that $\widetilde{\pi'}$ does not end on its top entry, and therefore neither does σ'. Likewise, $\widetilde{\pi''}$ does not start with its bottom entry, and therefore neither does σ''. Thus, the pair σ' and σ'' are as in the proof in [5] with the combined length of $(2n+2)/2 = n+1$ or $(2n+4)/2 = n+2$. Thus, we can combine them as in [5] to produce a single permutation σ of length n or $n+1$ (depending on the parity of "2") with a single occurrence of pattern 321.

Therefore, the number of Dumont-4 permutations of length $2n$ that have a single 321-occurrence is equal to the number of permutations of length n with a single 321-occurrence plus the number of permutations of length $n+1$ with a single 321-occurrence. That is,

$$|\mathfrak{D}^4_{2n}(321;1)| = |S_n(321;1)| + |S_{n+1}(321;1)|.$$

Finally, Zeilberger [29] showed that $|S_n(321;1)| = C_{n+2} - 4C_{n+1} + 3C_n$. It follows that

$$\begin{aligned} |\mathfrak{D}^4_{2n}(321;1)| &= (C_{n+2} - 4C_{n+1} + 3C_n) + (C_{n+3} - 4C_{n+2} + 3C_{n+1}) \\ &= C_{n+3} - 3C_{n+2} - C_{n+1} + 3C_n. \qquad \square \end{aligned}$$

Example 4.11. Let $\pi = 135462 \in \mathfrak{D}^4_6(321;1)$. Here, we see that the occurrence of 321 is the subsequence 542, where the "2" is the entry 4, an even fixed point. The map of the proof in [5] yields $\pi' = 1342$ and $\pi'' = 231$, so that $\widetilde{\pi'} = 1342$ and $\widetilde{\pi''} = 1342$ as well. Now applying the map of Theorem 3.6 yields $\sigma' = 21$ and $\sigma'' = 21$. Combining those again as in [5] yields $\sigma = 321$. Note that $|\sigma| = \dfrac{|\pi|}{2}$.

Similarly, let $\pi = 136254 \in \mathfrak{D}^4_6(321;1)$. Here, we see that the occurrence of 321 is the subsequence 654, where the "2" is the entry 5, an odd fixed point. The map of the proof in [5] yields $\pi' = 13524$ and $\pi'' = 21$, so that $\widetilde{\pi'} = 135624$ and $\widetilde{\pi''} = 1342$. Now applying the map of Theorem 3.6 yields $\sigma' = 312$ and $\sigma'' = 21$. Combining those again as in [5] yields $\sigma = 4132$. Note that $|\sigma| = \dfrac{|\pi|}{2} + 1$.

5 Conjecture and conclusion

5.1 A conjecture on restricted Dumont-1 permutations

In the midst of researching restricted Dumont permutations, we were under the impression that no two Dumont-1 permutations of length four (recall: $\mathfrak{D}^1_4 = \{2143, 3421, 4213\}$) are Wilf-equivalent. Originally, enumerating the sequences of Dumont-1 permutations that avoid 2143 and 3421, it was found that the two sequences were the same up to $n = 4$, then diverged at $n = 5$. After counting very carefully by hand for low n, then using the aid of code as n increased, it was discovered by Burstein and Jones [8] that the two sequences did not diverge at $n = 5$; in fact they were the same for n up to 6. We believed that the sequences were the same for $n \geq 6$; an additional computation by Albert [1] showed that the two sequences are the same for $n \leq 10$, i.e. for Dumont-1 permutations up to length 20. Therefore, we conjecture that the two sequences are equinumerous for all n.

Conjecture 5.1. *[8] For all $n \geq 0$, the following Wilf-equivalence exists on Dumont-1 permutations:*

$$2143 \sim 3421 \text{ on } \mathfrak{D}^1_{2n}, \quad \text{that is} \quad |\mathfrak{D}^1_{2n}(2143)| = |\mathfrak{D}^1_{2n}(3421)| \text{ for all } n \geq 0.$$

Now, it is possible, althought highly unlikely, that the two sequences diverge for some $n > 10$, but it is well known that if a sequence is the same for as long as twice the length of the pattern it avoids, then it will almost certaintly remain the same. In our case, the patterns are of length 4, so since the sequences are the same up to $n = 8$ (and, in fact, up to $n = 10$), we are very confident that they coincide for all n. See Table 1 for the sequence of the enumeration of $|\mathfrak{D}^1_{2n}(2143)| = |\mathfrak{D}^1_{2n}(3421)|$ up to $n = 10$.

n	0	1	2	3	4	5	6	7	8	9	10
$\lvert\mathfrak{D}^1_{2n}(2143)\rvert$	1	1	2	7	36	239	1,892	17,015	168,503	1,799,272	20,409,644
$\lvert\mathfrak{D}^1_{2n}(3421)\rvert$	1	1	2	7	36	239	1,892	17,015	168,503	1,799,272	20,409,644

Table 1: Sequence of the enumeration of $|\mathfrak{D}^1_{2n}(2143)| = |\mathfrak{D}^1_{2n}(3421)|$

This conjecture has turned out to be very hard to prove, and only some partial inroads are made as of this writing. One approach to proving such a conjecture is to refine it by considering a distribution of a certain combinatorial statistic on both $\mathfrak{D}^1_{2n}(2143)$ and $\mathfrak{D}^1_{2n}(3421)$. For this, we will first need a definition originally given in full generality by Babson and Steingrímsson in [3]. They defined it as a generalized permutation pattern; since then the term more commonly used is vincular or dashed pattern.

Definition 5.2. [3] A *vincular* (or *generalized*, or *dashed*) pattern of a permutation is a pattern that allows the additional requirement that two (or more) adjacent letters in a pattern be also adjacent in the containing permutation.

Note, in the pattern 2–31, the "2" and "3" need not be adjacent, however the "3" and the "1" must be adjacent. Now that we have the definition for vincular patterns, we can slightly refine our conjecture. After studying the behavior of distributions of occurrences of various vincular patterns on $\mathfrak{D}^1_{2n}(2143)$ and $\mathfrak{D}^1_{2n}(3421)$, we were able to form the following conjecture.

Conjecture 5.3. *[8] For all $n \geq 0$, we conjecture the following. Let*

$$a_k = |\{\pi \in \mathfrak{D}^1_{2n}(2143) | (2\text{–}31)\pi = k\}|,$$

$$b_k = |\{\pi \in \mathfrak{D}^1_{2n}(3421) | (13\text{–}2)\pi = k\}|,$$

where $(2\text{–}31)\pi$ *(resp.* $(13\text{–}2)\pi$*) is the number of occurrences of* $2\text{–}31$ *(resp.* $13\text{–}2$*) in* π*. Then*

$$a_k = b_k = 1 \ \text{for} \ k = \binom{n}{2},$$
$$a_k = b_k = 0 \ \text{for} \ k > \binom{n}{2}, \ \text{and}$$
$$\sum_{k=0}^{m} a_k \geq \sum_{k=0}^{m} b_k, \ \text{for} \ 0 \leq m \leq \binom{n}{2},$$

with equality for $m = \binom{n}{2}$.

This is less optimal than statistic distributions that are equal to each other; however, of the statistics we studied, this relation is only interesting one we found.

Additionally, we conjecture that both sequences $\{a_k\}$ and $\{b_k\}$ are unimodal for each $n \geq 0$, equal at the tails of the distributions, while in the middle of the distributions we first have a block of $a_k > b_k$ followed by a block of $a_k < b_k$ with the switch in direction of the inequality occurring at approximately $k = 2n - 5$. See Tables 2, 3, and 4 for $n = 5, 6, 7$, respectively.

k	0	1	2	3	4	5	6	7	8	9	10	**Total**
a_k	1	10	30	45	49	42	31	18	9	3	1	**239**
b_k	1	10	29	44	48	43	32	19	9	3	1	**239**
$a_k \square b_k$	$=$	$=$	$>$	$>$	$>$	$<$	$<$	$<$	$=$	$=$	$=$	

Table 2: $|\{\pi \in \mathfrak{D}^1_{10}(2143) | (2\text{–}31)\pi = k\}|$ versus $|\{\pi \in \mathfrak{D}^1_{10}(3421) | (13\text{–}2)\pi = k\}|$

k	0	1	2	3	4	5	6
a_k	1	15	70	160	246	298	303
b_k	1	15	65	147	228	284	302
$a_k \square b_k$	$=$	$=$	$>$	$>$	$>$	$>$	$>$

k	7	8	9	10	11	12	13	14	15	**Total**
a_k	268	208	145	89	49	24	11	4	1	**1892**
b_k	277	223	157	98	53	26	11	4	1	**1892**
$a_k \square b_k$	$<$	$<$	$<$	$<$	$<$	$<$	$=$	$=$	$=$	

Table 3: $|\{\pi \in \mathfrak{D}^1_{12}(2143) | (2\text{–}31)\pi = k\}|$ versus $|\{\pi \in \mathfrak{D}^1_{12}(3421) | (13\text{–}2)\pi = k\}|$

k	0	1	2	3	4	5	6	7	8
a_k	1	21	140	455	945	1497	1956	2215	2226
b_k	1	21	125	388	804	1294	1760	2089	2211
$a_k \square b_k$	$=$	$=$	$>$	$>$	$>$	$>$	$>$	$>$	$>$

k	9	10	11	12	13	14	15	16	17	18	19	20	21	**Total**
a_k	2032	1700	1317	948	641	410	249	140	72	32	13	4	1	**17015**
b_k	2111	1840	1472	1091	750	482	288	158	78	34	13	4	1	**17015**
$a_k \square b_k$	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$=$	$=$	$=$	

Table 4: $|\{\pi \in \mathfrak{D}^1_{14}(2143) | (2\text{–}31)\pi = k\}|$ versus $|\{\pi \in \mathfrak{D}^1_{14}(3421) | (13\text{–}2)\pi = k\}|$

5.2 Conclusion

In conclusion, the research conducted for this paper resulted in enumerations of Dumont-4 permutations avoiding certain patterns, as well as a single occurrence of certain patterns in Dumont-4 permutations. The methods used included proof by induction, block decomposition, Dyck paths, and generating functions. The Catalan numbers and powers of 2 that occur in the enumeration of permutations avoiding 3-letter patterns were also encountered in our results.

We also give an intriguing conjecture regarding the Wilf equivalence of a pair of Dumont-1 patterns. This conjecture has proven very unyielding, and even despite being presented twice at the open problem sessions at Permutation Patterns 2016 and 2018, progress on it is yet to be made.

It would also be interesting to see if other Wilf equivalences exist on Dumont permutations of the first kind avoiding a single 4-letter pattern. The same question may be posed for the other kinds of Dumont permutations.

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