The degree of symmetry of lattice paths

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Abstract

The degree of symmetry of a combinatorial object, such as a lattice path, is a measure of how symmetric the object is. It typically ranges from zero, if the object is completely asymmetric, to its size, if it is completely symmetric. We study the behavior of this statistic on Dyck paths and grand Dyck paths, with symmetry described by reflection along a vertical line through their midpoint; partitions, with symmetry given by conjugation; and certain compositions interpreted as bargraphs. We find expressions for the generating functions for these objects with respect to their degree of symmetry, and their semilength or semiperimeter. The resulting generating functions are algebraic in most cases, with the notable exception of Dyck paths, for which we apply bijections to walks in the plane in order to find a functional equation for the corresponding generating function, which we conjecture to be D-finite but not algebraic.

1 Introduction

For combinatorial objects with a standard reflection operation, it is natural to study the subset of those that are *symmetric*, that is, invariant under such reflection. Examples of symmetric combinatorial objects include symmetric Dyck paths [6], symmetric grand Dyck paths, self-conjugate partitions [22, Prop. 1.8.4], palindromic compositions [16], and symmetric binary trees, all of which appear in the literature.

In this paper we refine the concept of symmetric objects by introducing a type of combinatorial statistic that we call the *degree of symmetry*, which measures how close the object is to being symmetric. To the best of our knowledge, the notion of degree of symmetry seems to be new. In some instances, it is related to other statistics studied in the literature, such as the number of *centered tunnels* in Dyck paths (introduced in [10, 12, 13]) or the number of transpositions in permutations.

Let us start by defining the degree of symmetry of certain lattice paths. Let $\mathcal{GD}_n$ be the set of all lattice paths in the plane with up-steps $U = (1, 1)$ and down-steps $D = (1, -1)$ from $(0, 0)$ to $(2n, 0)$. These are called *grand Dyck paths*, and n is called the *semilength*. Let $\mathcal{D}_n$ be the subset of those that do not go below the x-axis. These are called *Dyck paths*. We use the notation $[n] = \{1, 2, \dots, n\}$.

Given a path $P \in \mathcal{GD}_n$, we view its steps as segments in the plane, which we denote by $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_{2n}$ from left to right. For example, $\bar{p}_1$ has endpoints $(0, 0)$ and $(1, \pm 1)$, and $\bar{p}_{2n}$ has endpoints $(2n - 1, \pm 1)$ and $(2n, 0)$. For $i \in [n]$, we say that P is symmetric in position i (or that $\bar{p}_i$

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is a *symmetric step*) if $\bar{p}_i$ and $\bar{p}_{2n+1-i}$ are mirror images of each other with respect to the reflection along the vertical line $x = n$. We define the *degree of symmetry* of P, denoted by $\mathrm{ds}(P)$, as the number of $i \in [n]$ such that P is symmetric in position i. See Figure 1 for an example.

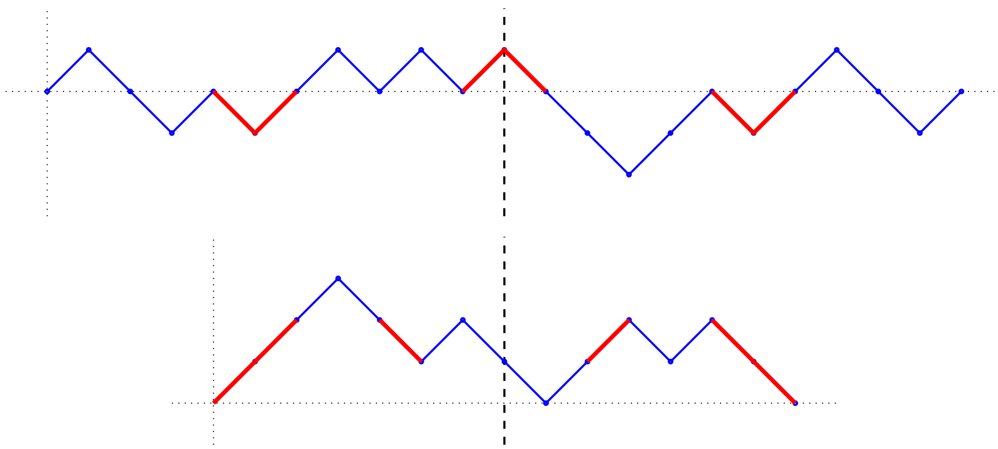

Figure 1: A grand Dyck path and a Dyck path, both with degree of symmetry 3. The symmetric steps and their mirror images are highlighted in red.

In Section 2 we give some background on generalized Motzkin paths, along with some enumerative results that will be needed in later sections. In Section 3 we derive an expression for the generating function of grand Dyck paths with respect to semilength and degree of symmetry. Section 4 focuses on the degree of symmetry of partitions, giving generating functions with respect to different definitions of size and degree of symmetry. In Section 5 we consider unimodal compositions whose maximum is in the middle, and we enumerate them with respect to semiperimeter (by viewing compositions as bargraphs) and degree of symmetry. Section 6 deals with the enumeration of Dyck paths by the degree of symmetry. Using bijections for walks in the quarter plane, we derive a functional equation for the corresponding generating function. The behavior of the degree of symmetry on Dyck paths is significantly more complex than on grand Dyck paths. In Section 7 try to understand this phenomenon by looking at other statistics on these paths.

2 Bicolored (grand) Motzkin paths

In this section we present some results about bicolored Motzkin paths that will be used in the proof of our main theorems in Sections 3, 4 and 5.

Like a grand Dyck path, a *bicolored grand Motzkin path* starts at the origin, ends on the x-axis, and has steps $U = (1, 1)$, $D = (1, -1)$, but it also may contain horizontal steps $(1, 0)$ of two kinds (or colors), denoted by H_1 and H_2. If such a path does not go below the x-axis, it is called a *bicolored Motzkin path*. Denote by $\mathcal{GM}^2$ the set of bicolored grand Motzkin paths, and by $\mathcal{M}^2$ the set of bicolored Motzkin paths. We often identify a path with its sequence of steps. For a path $M \in \mathcal{GM}^2$, let $u(M)$ denote its number of U steps (which also equals its number $d(M)$ of D steps), and define $h_1(M)$ and $h_2(M)$ analogously. Additionally, let $h_1^0(M)$ denote the number of H_1 steps of M on the x-axis, and define $h_2^0(M)$ similarly.

Define the *length* of a path M to be its total number of steps, which we denote by $|M|$. Note that $|M| = u(M) + d(M) + h_1(M) + h_2(M)$. Let $\mathcal{M}^2_n \subset \mathcal{M}^2$ and $\mathcal{GM}^2_n \subset \mathcal{GM}^2$ denote the subsets consisting of paths of length n in each case.

Lemma 2.1. *Let $F(x,y) = \sum_{M\in\mathcal{M}^2} x^{d(M)+h_1(M)} y^{u(M)+h_2(M)}$. Then*

$$F(x,y) = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2xy}.$$

Proof. A non-empty path $M \in \mathcal{M}^2$ can be decomposed uniquely as H_1M', H_2M' or $UM'DM''$, where $M', M'' \in \mathcal{M}^2$ (translated appropriately). This decomposition yields the following equation for the generating function:

$$F(x,y) = 1 + (x+y)F(x,y) + xyF(x,y)^2.$$

Solving for $F(x,y)$ and taking the sign of the square root that results in a formal power series, we get the stated expression. $\square$

Lemma 2.2. *Let $G(x,y,s_1,s_2) = \sum_{M\in\mathcal{GM}^2} x^{d(M)+h_1(M)} y^{u(M)+h_2(M)} s_1^{h_1^0(M)} s_2^{h_2^0(M)}$. Then*

$$G(x,y,s_1,s_2) = \frac{1}{(1-s_1)x + (1-s_2)y + \sqrt{(1-x-y)^2-4xy}}.$$

Proof. For $M' \in \mathcal{M}^2$, denote by $\overline{M'}$ the path in $\mathcal{GM}^2$ obtained by reflecting M' with respect to the x-axis; equivalently, by changing the U steps into D steps and viceversa.

A path $M \in \mathcal{GM}^2$ can be written uniquely as a sequence of steps H_1 on the x-axis, steps H_2 on the x-axis, paths of the form $UM'D$, and paths of the form $D\overline{M'}U$, where $M' \in \mathcal{M}^2$. It follows that

$$G(x,y,s_1,s_2) = \frac{1}{1-s_1x-s_2y-2xyF(x,y)}.$$

Using the expression for $F(x,y)$ given by Lemma 2.1, we obtain the desired formula. $\square$

3 Symmetry of grand Dyck paths

The main result of this section is the following surprisingly simple formula.

Theorem 3.1. *The generating function for grand Dyck paths with respect to their degree of symmetry is*

$$\sum_{n\geq 0}\sum_{P\in\mathcal{GD}_n} s^{\mathrm{ds}(P)} z^n = \frac{1}{2(1-s)z+\sqrt{1-4z}}.$$

Before we prove this theorem, let us introduce some notation. Given $P \in \mathcal{GD}_n$, construct two paths as follows. Let P_L denote the left half of P, and let P_R be the path obtained by reflecting the right half of P along the vertical line $x = n$. Note that P_L and P_R are paths with steps U and D from $(0,0)$ to some common endpoint on the line $x = n$. Denote the ith step of P_L by $\bar{\ell}_i$ when viewed as a segment in the plane, and let $\ell_i \in \{U, D\}$ be the direction of this step. Define $\bar{r}_i$ and r_i similarly for the path P_R. Next we describe a bijection ϕ from $\mathcal{GD}_n$ to $\mathcal{GM}^2_n$.

Definition 3.2. *For $P \in \mathcal{GD}_n$ with the above notation, let $\phi(P) \in \mathcal{GM}_n^2$ be the path whose ith step is equal to*

$$\begin{cases} U & \text{if } \ell_i = U \text{ and } r_i = D, \\ D & \text{if } \ell_i = D \text{ and } r_i = U, \\ H_1 & \text{if } \ell_i = r_i = D, \\ H_2 & \text{if } \ell_i = r_i = U. \end{cases}$$

Figure 2 shows an example of this construction.

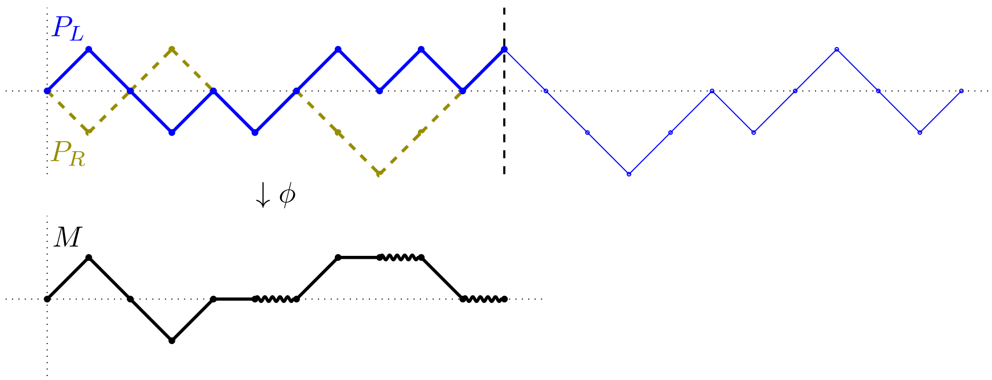

Figure 2: The bijection $\phi : \mathcal{GD}_n \to \mathcal{GM}_n^2$. The path $P \in \mathcal{GD}_n$, which is the same from the top of Figure 1, is drawn in blue, and its reflected right half P_R is drawn in olive color with dahes. The steps H_2 in $\phi(P)$ are drawn with wavy lines.

Lemma 3.3. *The map $\phi : \mathcal{GD}_n \to \mathcal{GM}_n^2$ is a bijection with the property that, if $M = \phi(P)$, then* $\mathrm{ds}(P) = h_1^0(M) + h_2^0(M)$.

Proof. A path $P \in \mathcal{GD}_n$ is symmetric in position i if and only if the steps $\bar{\ell}_i$ and $\bar{r}_i$ coincide as segments. Thus, the degree of symmetry of P equals the number of common (i.e. overlapping) steps of P_L and P_R.

By construction of ϕ, the height (i.e., y-coordinate) of each vertex of M is obtained by subtracting the height of the corresponding vertex of P_R from the corresponding vertex of P_L and dividing by two. Thus, steps where P_L and P_R coincide become horizontal steps of M at height 0. □

Proof of Theorem 3.1. Combining Lemmas 3.3 and 2.2,

$$\sum_{n\geq 0} \sum_{P\in\mathcal{GD}_n} s^{\mathrm{ds}(P)} z^n = \sum_{M\in\mathcal{GM}^2} s^{h_1^0(M)+h_2^0(M)} z^{|M|} = G(z,z,s,s) = \frac{1}{2(1-s)z + \sqrt{1-4z}}.$$

□

When P_L lies strictly above P_R (except at their common endpoints), the pair (P_L, P_R) is called a *parallelogram polyomino* [19, 21, 5], and its semiperimeter is defined to be the length of either of the two paths. The bijection ϕ in Definition 3.2, after removing the first and the last step of the image path, restricts to a bijection between parallelogram polyominos of semiperimeter n and bicolored Motzkin paths of length $n-2$, which are known to be counted by the Catalan number C_{n-1}.

Indeed, using this bijection and Lemma 2.1, the generating function for parallelogram polyominoes where z marks the semiperimeter is

$$z^2F(z,z) = \frac{1-2z-\sqrt{1-4z}}{2} = z(C(z)-1),$$

where

$$C(z) = \sum_{n\geq 0} C_n z^n = \frac{1-\sqrt{1-4z}}{2z} \tag{1}$$

is the generating function for the Catalan numbers. More generally, $xyF(x,y)$ is the generating function for parallelogram polyominoes where x and y mark the number of D and U steps, respectively, of the upper (equivalently, the lower) path.

An alternative measure of the symmetry of a grand Dyck path is its *number of symmetric vertices*, that is, vertices in the first half of the path that are mirror images of vertices in the second half, again with respect to reflection along the vertical line that passes through the midpoint of the path. We do not consider the midpoint itself as a symmetric vertex. For example, the grand Dyck path in Figure 3 has 6 symmetric vertices. Denote by $\mathrm{sv}(P)$ the number of symmetric vertices of the path $P \in \mathcal{GD}_n$.

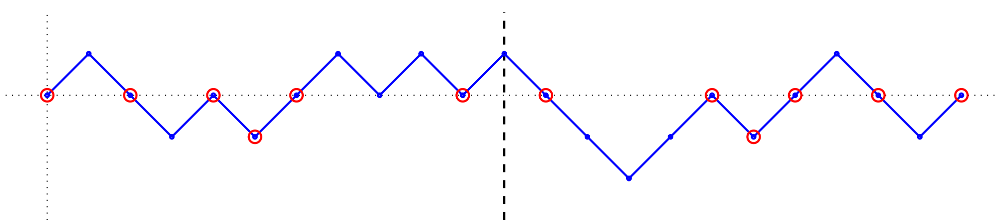

Figure 3: A grand Dyck path with 6 symmetric vertices, highlighted in red along with their mirror images.

Theorem 3.4. *The generating function for grand Dyck paths with respect to their number of symmetric vertices is*

$$\sum_{n\geq 0}\sum_{P\in\mathcal{GD}_n} v^{\mathrm{sv}(P)}z^n = \frac{1}{1-2vzC(z)} = \frac{1}{1-v+v\sqrt{1-4z}}.$$

Proof. Let $P \in \mathcal{GD}_n$ and $M = \phi(P)$, where ϕ is the bijection from Definition 3.2. Then $\mathrm{sv}(P)$ equals the number of vertices of M on the x-axis minus one. As in the proof of Lemma 2.2, we can decompose $M \in \mathcal{GM}^2$ as a sequence of steps H_1 and H_2 on the x-axis, paths of the form $UM'D$, and paths of the form $D\overline{M'}U$, where $M' \in \mathcal{M}^2$. Since each one of these four types of blocks contributes one new symmetric vertex, it follows that

$$\sum_{n\geq 0}\sum_{P\in\mathcal{GD}_n} v^{\mathrm{sv}(P)}z^n = \frac{1}{1-2vz-2vz^2F(z,z)} = \frac{1}{1-2vzC(z)},$$

with F given by Lemma 2.1. $\square$

For $P \in \mathcal{GD}_n$, denote by $\mathrm{ret}(P)$ the number of returns of P to the x-axis, that is, the number of steps that end on the x-axis.

Corollary 3.5. *The statistics* sv *and* ret *are equidistributed on $\mathcal{GD}_n$; that is, for all $n, k \geq 0$,*

$$|\{P \in \mathcal{GD}_n : \mathrm{sv}(P) = k\}| = |\{P \in \mathcal{GD}_n : \mathrm{ret}(P) = k\}|.$$

Proof. It is easy to see directly that the generating function in Theorem 3.4 enumerates grand Dyck paths with respect to the number of returns to the x-axis [24, A108747], from where the result follows.

Alternatively, one can give a simple bijective proof of this equality. Given $P \in \mathcal{GD}_n$, first apply ϕ and let $M = \phi(P) \in \mathcal{GM}^2_n$. Then replace each step U of M with UU, each step D with DD, each step H_1 with UD, and each step H_2 with DU. Let Q be the resulting grand Dyck path. This map $P \mapsto Q$ is bijection from $\mathcal{GD}_n$ to itself with the property that $\mathrm{sv}(P) = \mathrm{ret}(Q)$. As an example, the image of the path in Figure 3 is given in Figure 4. □

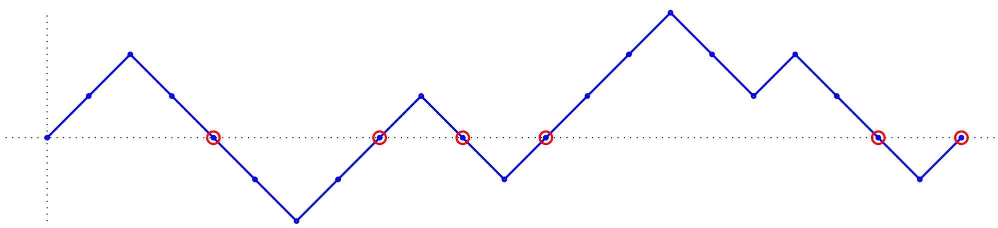

Figure 4: A grand Dyck path with 6 returns to the x-axis, highlighted in red.

The first few coefficients of the generating functions from Theorems 3.1 and 3.4 are given in Table 1.

$|\{P \in \mathcal{GD}_n : \mathrm{ds}(P) = k\}|$

$n \setminus k$	0	1	2	3	4	5	6
1	0	2					
2	2	0	4				
3	4	8	0	8			
4	14	16	24	0	16		
5	44	64	48	64	0	32	
6	148	208	216	128	160	0	64

$|\{P \in \mathcal{GD}_n : \mathrm{sv}(P) = k\}|$

$n \setminus k$	0	1	2	3	4	5	6
1	0	2					
2	2	0	4				
3	4	8	0	8			
4	10	20	24	0	16		
5	28	56	72	64	0	32	
6	84	168	224	224	160	0	64

Table 1: The number of grand Dyck paths of length $n \leq 6$ with a given degree of symmetry (left, see Theorem 3.1) and with a given number of symmetric vertices (right, see Theorem 3.4).

4 Symmetry of partitions

Let $\mathcal{P}$ denote the set of integer partitions, that is, sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $k \geq 0$ (called the number of parts) and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. We draw the Young diagram of λ in English notation, by arranging boxes (unit squares) into k left-justified rows, where the ith row from the top has λ_i boxes for each i; see Figure 5 for an example. The *conjugate* of λ, denoted by λ', is the partition defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$ for $1 \leq i \leq \lambda_1$. The Young diagram of λ' is obtained by transposing the Young diagram of λ. Note that λ'_1 equals the number of parts of λ.

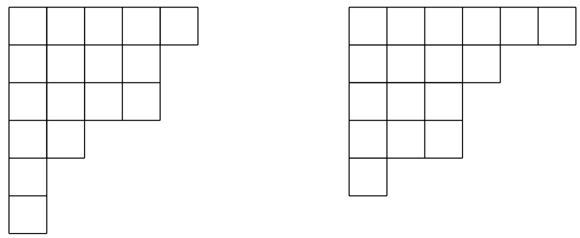

Figure 5: The Young diagrams of the partition $\lambda = (5, 4, 4, 2, 1, 1)$ and its conjugate $\lambda' = (6, 4, 3, 3, 1)$.

Each one of the following subsections considers a different measure of the symmetry of a partition. The first one views partitions inside a square and relates them to grand Dyck paths. The second one is perhaps the most natural measure of symmetry: the number of parts that equal the corresponding part in the conjugate partition. The third measure involves a decomposition of partitions into diagonal hooks.

4.1 Partitions inside a square

Let $\mathcal{P}^{\square}_n \subset \mathcal{P}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $k \leq n$ and $\lambda_1 \leq n$. These can be thought of as partitions whose Young diagram fits inside an $n \times n$ square. For such $\lambda \in \mathcal{P}^{\square}_n$, let $\tilde{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0)$ denote the sequence of length n obtained by appending $n - k$ zeros to λ. Let $\tilde{\lambda}'$ be the sequence of length n obtained by conjugating $\tilde{\lambda}$, that is, $\tilde{\lambda}'_i = |\{j : \tilde{\lambda}_j \geq i\}|$ for $1 \leq i \leq n$.

Viewing λ as a partition inside a square, one can define the following measure of symmetry, where the zeros in $\tilde{\lambda}$ are allowed to contribute. Let

$$\mathrm{ds}^{\square}_n(\lambda) = |\{i \in [n] : \tilde{\lambda}_i = \tilde{\lambda}'_i\}|.$$

For example, if $\lambda = (5, 4, 4, 2, 1, 1)$, then $\mathrm{ds}^{\square}_6(\lambda) = 2$ but $\mathrm{ds}^{\square}_7(\lambda) = 3$, since in the second case, $\tilde{\lambda} = (5, 4, 4, 2, 1, 1, 0)$ and $\tilde{\lambda}' = (6, 4, 3, 3, 1, 0, 0)$ coincide in positions 2, 5, and 7.

To relate partitions inside a square and grand Dyck paths, we define a straightforward bijection $\partial_n : \mathcal{P}^{\square}_n \to \mathcal{GD}_n$ as follows.

Definition 4.1. *For $\lambda \in \mathcal{P}^{\square}_n$, let*

$$\partial_n(\lambda) = D^{\tilde{\lambda}_n} U D^{\tilde{\lambda}_{n-1} - \tilde{\lambda}_n} U D^{\tilde{\lambda}_{n-2} - \tilde{\lambda}_{n-1}} U \dots D^{\tilde{\lambda}_1 - \tilde{\lambda}_2} U D^{n - \tilde{\lambda}_1} \in \mathcal{GD}_n.$$

This bijection can be visualized by placing the Young diagram of λ inside an $n \times n$ square (aligned with the top and left edges), reading the south-east boundary of the diagram from the south-west corner of the square to the north-east corner, and then translating north steps to U steps and east steps to D steps. An example is given in Figure 6. The following is a consequence of Theorem 3.1.

Corollary 4.2. *The generating function for partitions whose Young diagram fits inside a square with respect to the side length of the square and the statistic* $\mathrm{ds}_n^{\square}$ *is*

$$\sum_{n\geq 0}\sum_{\lambda\in\mathcal{P}_n^{\square}} s^{\mathrm{ds}_n^{\square}(\lambda)} z^n = \frac{1}{2(1-s)z+\sqrt{1-4z}}.$$

Proof. Letting $\lambda \in \mathcal{P}_n^{\square}$ and $P = \partial_n(\lambda) \in \mathcal{GD}_n$ as in Definition 4.1, we have that $\tilde{\lambda}_i = \tilde{\lambda}'_i$ if and only if the i-th D step of P from the left is a mirror image (with respect to the reflection along $x = n$) of its i-th U step from the right. Thus,

$$\mathrm{ds}_n^{\square}(\lambda) = \mathrm{ds}(P). \tag{2}$$

The result now follows from Theorem 3.1. $\square$

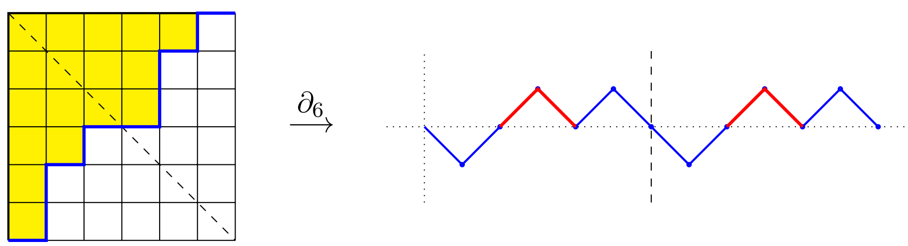

Figure 6: The bijection ∂_n for $n = 6$ applied to $\lambda = (5, 4, 4, 2, 1, 1) \in \mathcal{P}_6^{\square}$.

4.2 Symmetry by self-conjugate parts

Another notion of symmetry, which we call simply the degree of symmetry of $\lambda \in \mathcal{P}$, is defined as

$$\mathrm{ds}(\lambda) = |\{i : \lambda_i = \lambda'_i\}|,$$

that is, the number of parts of λ that equal the corresponding parts of its conjugate. For example, if $\lambda = (5, 4, 4, 2, 1, 1)$, then $\mathrm{ds}(\lambda) = 2$ because $\lambda_2 = \lambda'_2 = 4$ and $\lambda_5 = \lambda'_5 = 1$, but $\lambda_i \neq \lambda'_i$ for every other i for which these quantities are defined.

The following straightforward observation relates the two measures of symmetry for partitions defined so far.

Lemma 4.3. *Let* $\lambda \in \mathcal{P}$, *and let* $m = \max\{\lambda_1, \lambda'_1\}$ *be the side length of the smallest square where its Young diagram fits. Then* $\mathrm{ds}_m^{\square}(\lambda) = \mathrm{ds}(\lambda)$.

Proof. For this choice of m, at least one of $\tilde{\lambda}$ and $\tilde{\lambda}'$ has no zero entries, and so no zero entries contribute to $\mathrm{ds}_m^{\square}$. $\square$

In the generating function from Corollary 4.2, each partition $\lambda \in \mathcal{P}$ contributes infinitely many terms, since it belongs to all $\mathcal{P}_n^{\square}$ for n large enough. Next we modify this generating function so that each partition is weighted by $s^{\mathrm{ds}(\lambda)}$ and contributes exactly once.

Corollary 4.4. *The generating function for partitions with respect to the side length of the smallest square containing their Young diagram and their degree of symmetry is*

$$\sum_{\lambda\in\mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\max\{\lambda_1,\lambda'_1\}} = \frac{1-sz}{2(1-s)z+\sqrt{1-4z}}.$$

Proof. Let $\lambda \in \mathcal{P}$, and let $m = \max\{\lambda_1, \lambda'_1\}$. Then $\mathrm{ds}^{\square}_m(\lambda) = \mathrm{ds}(\lambda)$ by Lemma 4.3. It follows that

$$\sum_{\lambda\in\mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\max\{\lambda_1,\lambda'_1\}} = \sum_{\lambda\in\mathcal{P}} \sum_{n\geq\max\{\lambda_1,\lambda'_1\}} s^{\mathrm{ds}^{\square}_n(\lambda)} z^n - \sum_{\lambda\in\mathcal{P}} \sum_{n>\max\{\lambda_1,\lambda'_1\}} s^{\mathrm{ds}^{\square}_n(\lambda)} z^n \tag{3}$$

By letting $j = n-1$ and noting that $\mathrm{ds}^{\square}_{j+1}(\lambda) = \mathrm{ds}^{\square}_j(\lambda)+1$ for $j \geq \max\{\lambda_1, \lambda'_1\}$, the subtracting term in Equation (3) can be written as

$$\sum_{\lambda\in\mathcal{P}} \sum_{j\geq\max\{\lambda_1,\lambda'_1\}} s^{\mathrm{ds}^{\square}_j(\lambda)+1} z^{j+1},$$

and so the right-hand side of Equation (3) equals

$$(1-sz)\sum_{\lambda\in\mathcal{P}} \sum_{n\geq\max\{\lambda_1,\lambda'_1\}} s^{\mathrm{ds}^{\square}_n(\lambda)} z^n = (1-sz)\sum_{n\geq 0} \sum_{\lambda\in\mathcal{P}^{\square}_n} s^{\mathrm{ds}^{\square}_n(\lambda)} z^n,$$

using that each partition $\lambda \in \mathcal{P}$ belongs to $\mathcal{P}^{\square}_n$ for all $n \geq \max\{\lambda_1, \lambda'_1\}$. The result now follows from Corollary 4.2. $\square$

For $\lambda \in \mathcal{P}$, let $\mathrm{sp}(\lambda) = \lambda_1 + \lambda'_1$ denote the semiperimeter of its Young diagram. Next we count partitions by semiperimeter.

Theorem 4.5. *The generating function for partitions with respect to their semiperimeter and their degree of symmetry is*

$$\sum_{\lambda\in\mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\mathrm{sp}(\lambda)} = 1 + \frac{z^2\left((1-s)(1-2z)-\sqrt{1-4z^2}\right)}{(2z-1)\left(2(1-s)z^2+\sqrt{1-4z^2}\right)}. \tag{4}$$

Proof. Let $\lambda \in \mathcal{P}$, and let $m = \max\{\lambda_1, \lambda'_1\}$. Let $P = \partial_m(\lambda) \in \mathcal{GD}_m$, given by Definition 4.1, and let $M = \phi(P)$, where $\phi : \mathcal{GD}_m \to \mathcal{GM}^2_m$ is the bijection from Definition 3.2. Then $\mathrm{ds}(\lambda) = \mathrm{ds}^{\square}_m(\lambda) = \mathrm{ds}(P) = h^0_1(M) + h^0_2(M)$, using Lemma 4.3, Equation (2), and Lemma 3.3.

Because of the choice of m, it is not possible for both P_L and P_R to begin with a U step. The semiperimeter $\mathrm{sp}(\lambda)$ equals the combined number of steps of P_L and P_R, not counting any initial U steps before the first D. To see how this statistic translates to the path M, let us consider three cases:

- If $\lambda_1 = \lambda'_1 > 0$, then both P_L and P_R begin with a D step. In this case, M begins with an H_2 step, and $\mathrm{sp}(\lambda)$ simply equals twice the length of M. Using Lemma 2.2, the generating function for such paths is $sz^2G(z^2, z^2, s, s)$.

- If $\lambda_1 > \lambda'_1$, then P_R begins with a D step and P_L begins with a U step. In this case, M begins with a U step as well, and the U steps in P_L before the first D correspond to U and H_2 steps in M before the first D or H_1. We want to find the generating function $K(s,z)$ for paths $M \in \mathcal{GM}^2$ that begin with a U, where s marks $h_1^0(M)+h_2^0(M)$, and z marks twice the number of steps of M, minus the number of U and H_2 steps before the first D or H_1. This is equivalent to defining the weight of M to be the product of its step weights, where each step is assigned weight z^2, except for steps U and H_2 before the first D or H_1, which are assigned weight z.

  Let $J(z) = F(z^2, z^2)$, with F given by Lemma 2.1, be the generating function for $\mathcal{M}^2$ where all steps have weight z^2. Let $\tilde{J}(z)$ be the generating function for $\mathcal{M}^2$ where all steps have weight z^2 except for steps U and H_2 before the first D or H_1, which have weight z. The usual decomposition of bicolored Motzkin paths into one of H_1M', H_2M' or $UM'DM''$, where $M', M'' \in \mathcal{M}^2$, gives
$$\tilde{J}(z) = 1 + z^2 J(z) + z\tilde{J}(z) + z^3 \tilde{J}(z) J(z),$$
  from where
$$\tilde{J}(z) = \frac{1 + z^2 J(z)}{1 - z - z^3 J(z)} = \frac{1 - 2z - \sqrt{1-4z^2}}{2z(2z-1)}. \tag{5}$$

  Finally, since every $M \in \mathcal{GM}^2$ that begins with a U can be decomposed uniquely as $UM'DM''$, where $M' \in \mathcal{M}^2$ and $M'' \in \mathcal{GM}^2$, we have that $K(s,z) = z^3 \tilde{J}(z) G(z^2, z^2, s, s)$.

- If $\lambda_1 < \lambda'_1$, then P_L begins with a D step and P_R begins with a U step. By symmetry, the corresponding generating function is again $K(s,z) = z^3 \tilde{J}(z) G(z^2, z^2, s, s)$.

Combining the above three cases and adding the empty partition yields
$$\sum_{\lambda \in \mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\mathrm{sp}(\lambda)} = 1 + \left(sz^2 + 2z^3 \tilde{J}(z) \right) G(z^2, z^2, s, s).$$

Using Equation (5) and Lemma 2.2, we obtain Equation (4). □

It is interesting to note that, while the generating function for partitions by semiperimeter is rational, namely $\frac{z^2}{1-2z}$ (setting $s=1$ in Equation (4)), the generating function by semiperimeter and degree of symmetry is not. The first few coefficients of this algebraic generating function, given by Theorem 4.5, are shown in Table 2.

4.3 Symmetry by self-conjugate hooks

Next we consider a third notion of symmetry for partitions. As in [1], the boxes in the Young diagram of $\lambda \in \mathcal{P}$ can be decomposed into *diagonal hooks* as follows: the first hook is the largest hook, consisting of the first row and the first column; the second hook is the largest hook after the first hook has been removed, and so on (see Figure 7 for an example). The number of hooks in this decomposition equals the largest δ such that $\lambda_\delta \geq \delta$ (also known as the side length of Durfee square of λ). Let $\mathrm{ds}^{\ulcorner}(\lambda)$ be the number of diagonal hooks in the Young diagram of λ that are self-conjugate, that is, they have the same number of boxes in the row than in the column.

$n \setminus k$	0	1	2	3	4
2	0	1	0	0	0
3	2	0	0	0	0
4	2	0	2	0	0
5	4	4	0	0	0
6	6	6	0	4	0
7	16	8	8	0	0
8	24	16	16	0	8

Table 2: The number of partitions with semiperimeter n and degree of symmetry k, for $2 \leq n \leq 8$ (see Theorem 4.5).

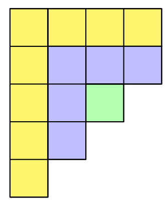

Figure 7: The Young diagram of the partition $\lambda = (4, 4, 3, 2, 1)$ has two symmetric diagonal hooks, and so $\mathrm{ds}^{\ulcorner}(\lambda) = 2$. Note that $\mathrm{ds}(\lambda) = 3$ in this case, since $\lambda' = (5, 4, 3, 2)$.

Proposition 4.6. *The generating functions for partitions with respect to the statistic* $\mathrm{ds}^{\ulcorner}$ *and the side length of any (in the first formula) or the smallest (in the second formula) square containing their Young diagram are*

$$\sum_{n\geq 0} \sum_{\lambda \in \mathcal{P}_n^{\square}} s^{\mathrm{ds}^{\ulcorner}(\lambda)} z^n = \frac{1}{(1-s)z + \sqrt{1-4z}}, \tag{6}$$

$$\sum_{\lambda \in \mathcal{P}} s^{\mathrm{ds}^{\ulcorner}(\lambda)} z^{\max\{\lambda_1, \lambda'_1\}} = \frac{1-z}{(1-s)z + \sqrt{1-4z}}.$$

Proof. Let $\lambda \in \mathcal{P}_n^{\square}$ and $P = \partial_n(\lambda)$, where $\partial_n : \mathcal{P}_n^{\square} \to \mathcal{GD}_n$ is the bijection from Definition 4.1. Then $\mathrm{ds}^{\ulcorner}(\lambda)$ equals the number of D steps in the first half of P that are a mirror images (with respect reflection along $x = n$) of U steps in the second half; equivalently, the number of D steps of P_L and P_R that coincide as segments. Via the bijection $\phi : \mathcal{GD}_n \to \mathcal{GM}_n^2$ from Definition 3.2, such steps become H_1 steps at height 0 of the bicolored grand Motzkin path $\phi(P)$.

Thus, composing the two bijections, we have that $\mathrm{ds}^{\ulcorner}(\lambda) = h_1^0(\phi(\partial_n(\lambda)))$, and using Lemma 2.2,

$$\sum_{n\geq 0} \sum_{\lambda \in \mathcal{P}_n^{\square}} s^{\mathrm{ds}^{\ulcorner}(\lambda)} z^n = \sum_{M \in \mathcal{GM}^2} s^{h_1^0(M)} z^{|M|} = G(z, z, s, 1) = \frac{1}{(1-s)z + \sqrt{1-4z}}.$$

Finally, using the same argument as in the proof of Corollary 4.4 and noting that $\mathrm{ds}^{\ulcorner}(\lambda)$ does not depend on the square where the Young diagram of λ is placed, we obtain the second formula. $\square$

For $P \in \mathcal{GD}_n$, denote by $\mathrm{ph}_1(P)$ the number of peaks of P at height 1, that is, occurrences of UD whose middle vertex has y-coordinate equal to 1.

Corollary 4.7. *For $n, k \geq 0$,*

$$|\{\lambda \in \mathcal{P}_n^{\square} : \mathrm{ds}^{\ulcorner}(P) = k\}| = |\{P \in \mathcal{GD}_n : \mathrm{ph}_1(P) = k\}|.$$

Proof. We will give two proofs of this equality. The first one consists of showing that

$$\sum_{n\geq 0} \sum_{P\in\mathcal{GD}_n} s^{\mathrm{ph}_1(P)} z^n = \frac{1}{(1-s)z + \sqrt{1-4z}}, \tag{7}$$

from where the result follows by Equation (6). Making the substitution $s = t+1$ in the left-hand side of Equation (7), we obtain

$$\sum_{n\geq 0} \sum_{P\in\mathcal{GD}_n} (t+1)^{\mathrm{ph}_1(P)} z^n = \sum_{n\geq 0} \sum_{P\in\mathcal{GD}_n} \sum_{T} t^{|T|} z^n, \tag{8}$$

where T ranges over all subsets of the set of peaks of height one in P. One can think of t as keeping track of *marked* peaks, which are an arbitrary subset of all peaks at height 1. Recall that the generating function for grand Dyck paths by semilength is simply $\frac{1}{\sqrt{1-4z}}$. Since every grand Dyck path with marked peaks can be decomposed as a sequence of blocks consisting of a grand Dyck path followed by a marked peak (at height 1), plus a grand Dyck path at the end, the generating function (8) equals

$$\frac{1}{1-\frac{tz}{\sqrt{1-4z}}} \frac{1}{\sqrt{1-4z}} = \frac{1}{\sqrt{1-4z}-tz}.$$

Setting $t = s-1$ we obtain the right-hand side of Equation (7).

The second proof is bijective. Consider the bijection $\psi : \mathcal{P}_n^{\square} \to \mathcal{GD}_n$ from [1, Lemma 3.5], which can be defined as follows. Given $\lambda \in \mathcal{P}_n^{\square}$, let δ be the number of hooks in its diagonal hook decomposition described above. For $1 \leq i \leq \delta$, let a_i and ℓ_i denote the arm length and the leg length of the i-th diagonal hook, respectively. Let

$$\psi(\lambda) = D^{a_\delta} U^{\ell_\delta+1} D^{a_{\delta-1}-a_\delta} U^{\ell_{\delta-1}-\ell_\delta} \dots D^{a_1-a_2} U^{\ell_1-\ell_2} D^{n-a_1} U^{n-1-\ell_1}.$$

Then $\mathrm{ds}^{\ulcorner}(\lambda) = \mathrm{ph}_1(\psi(\lambda))$. Indeed, the peaks of $\psi(\lambda)$ occur at heights $1+\ell_\delta - a_\delta, 1+\ell_{\delta-1} - a_{\delta-1}, \dots, 1+\ell_1 - a_1$, and so $\mathrm{ph}_1(\psi(\lambda)) = |\{i : a_i = \ell_i\}| = \mathrm{ds}^{\ulcorner}(\lambda)$. Figure 8 shows an example of this construction. $\square$

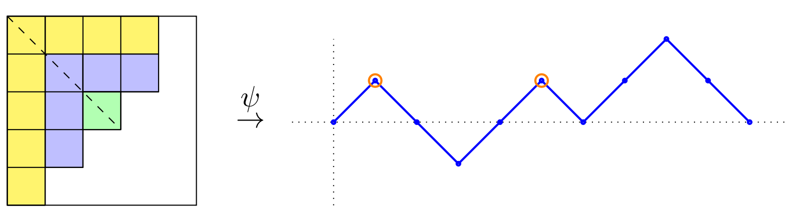

Figure 8: The bijection ψ applied to $\lambda = (4,4,3,2,1) \in \mathcal{P}_5^{\square}$. Here $\delta = 3$, $a_1 = 3$, $\ell_1 = 4$, $a_2 = \ell_2 = 2$, $a_3 = \ell_3 = 0$. The peaks at height 1 in $\phi(\lambda)$ are highlighted in orange.

5 Symmetry of unimodal compositions

The degree of symmetry of compositions, namely, sequences of positive integers $(a_1, a_2, \dots, a_k)$ for some $k \geq 1$, is studied in [9]. The degree of symmetry of such a composition is the number of indices $i \leq k/2$ such that $a_i = a_{k+1-i}$. Similarly to how partitions are represented as Young diagrams, compositions can be represented as *bargraphs*, by arranging boxes (unit squares) into k bottom-justified columns, where column i from the left has a_i boxes for each i; see Figure 9 for an example. Bargraphs have been studied in the literature as a special case of column-convex polyominoes (see e.g. [14, 4, 20, 7, 8]), and they are used in statistical physics to model polymers.

A bargraph can be identified with the lattice path determined by its upper boundary, namely, a self-avoiding path with steps $N = (0, 1)$, $E = (1, 0)$ and $S = (0, -1)$ starting at the origin and returning to the x-axis only at the end. For a bargraph B, let $e(B)$ denote its number of E steps (also called the width of B), and define $n(B)$ similarly. The semiperimeter of B is defined as $\mathrm{sp}(B) = e(B) + n(B)$, and its degree of symmetry is defined as the degree of symmetry of the composition determined by its column heights, and denoted by $\mathrm{ds}(B)$.

Interpreting partitions as weakly decreasing compositions, it is natural to consider the related notion of unimodal compositions; equivalently, unimodal bargraphs. Let $\mathcal{U}$ denote the set of *unimodal bargraphs* with a centered maximum, defined as those whose column heights satisfy $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{\lfloor (k+1)/2 \rfloor}$ and $a_{\lceil (k+1)/2 \rceil} \geq \cdots \geq a_{k-1} \geq a_k \geq 1$. The bargraph in Figure 9 is unimodal.

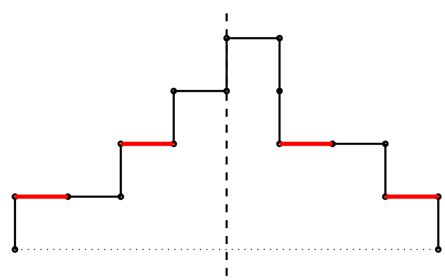

Figure 9: A unimodal bargraph B with $\mathrm{ds}(B) = 2$, $e(B) = 8$ and $n(B) = 4$, corresponding to the composition $(1, 1, 2, 3, 4, 2, 2, 1)$. In this case, $B_L = EENENEN$ and $B_R = ENEENNE$.

Theorem 5.1. *The generating function for unimodal bargraphs with a centered maximum with respect to the number of E and N steps and the degree of symmetry is*

$$\sum_{B \in \mathcal{U}} s^{\mathrm{ds}(B)} x^{e(B)} y^{n(B)} = \frac{y(1 + x - y)}{(1 - s)x^2 + \sqrt{((x+1)^2 - y)((x-1)^2 - y)}} - y.$$

Proof. Let $B \in \mathcal{U}$. If B has odd width, say $2j + 1$, splitting B at the middle E step we can write $B = NB_LEB'_RS$. Let B_R be the path obtained by reflecting B'_R along the vertical line $x = j + 1/2$, which passes through the center of this middle E step. Then B_L and B_R are lattice paths with N and E steps from the origin to some common endpoint.

Suppose now that B has even width, say $2j$. We can uniquely split B in the middle as $B = NB_LB'_RS$ in such a way that, if B_R is the path obtained by reflecting B'_R along the vertical line $x = j$, then B_L and B_R are lattice paths with N and E steps from the origin to some common endpoint, with the caveat that now they cannot both end with an N step.

Along the lines of Definition 3.2, we describe a bijection between pairs of paths with N and E steps from the origin to a common endpoint and bicolored grand Motzkin paths. Denoting by l_i and r_i be the ith step of B_L and B_R, respectively, we construct $M \in \mathcal{GM}^2$ by letting its ith step be equal to

$$\begin{cases} U & \text{if } l_i = N \text{ and } r_i = E, \\ D & \text{if } l_i = E \text{ and } r_i = N, \\ H_1 & \text{if } l_i = r_i = E, \\ H_2 & \text{if } l_i = r_i = N. \end{cases}$$

Then $\mathrm{ds}(B)$ is equal to the number of common (i.e. coinciding as segments) E steps of B_L and B_R, which in turn equals $h_1^0(M)$. If B has even width, then

$$e(B) = e(B_L) + e(B_R) = u(M) + d(M) + 2\#H_1(M) = 2(d(M) + h_1(M)).$$

As similar equality, but with the left-hand side replaced with $e(B) - 1$, holds when B has odd width. Finally,

$$n(B) = n(B_L) + 1 = u(M) + h_2(M) + 1.$$

With $G(x, y, s_1, s_2)$ defined as in Lemma 2.2, it follows that the generating function for bargraphs in $\mathcal{U}$ of odd width is

$$xyG(x^2, y, s, 1),$$

and the one for those of even width is

$$y[(1 - y)G(x^2, y, s, 1) - 1].$$

In the last formula, the $-y$ term subtracts pairs of paths B_L and B_R ending both with an N, and the -1 at the end removes the possibility that both B_L and B_R are empty. Summing these two expressions, we obtain

$$\sum_{B\in\mathcal{U}} s^{\mathrm{ds}(B)} x^{e(B)} y^{n(B)} = y(1 + x - y)G(x^2, y, s, 1) - y,$$

which, after using Lemma 2.2, equals the stated formula. $\square$

Setting $x = z$ and $y = z$ in Theorem 5.1, we obtain the generating function for unimodal bargraphs with their maximum in the middle where z marks the semiperimeter:

$$\sum_{B\in\mathcal{U}} s^{\mathrm{ds}(B)} z^{\mathrm{sp}(B)} = \frac{z}{(1 - s)z^2 + \sqrt{1 - 2z - z^2 - 2z^3 + z^4}} - z.$$

6 Symmetry of Dyck paths

Let

$$D(s, z) = \sum_{n\geq 0} \sum_{P\in\mathcal{D}_n} s^{\mathrm{ds}(P)} z^n$$

denote the generating function for Dyck paths with respect to their degree of symmetry. In contrast to the simplicity of the generating function in Theorem 3.1 for grand Dyck paths, the generating function $D(s,z)$ is unwieldy. To study the statistic ds on Dyck paths, we will first rephrase the problem in terms of walks in the plane. Then we will apply some transformations on the walks that will allow us to obtain a functional equation for a refinement of $D(s,z)$.

Let $\mathcal{W}_n^1$ denote the set of walks in the first quadrant $\{(x,y) : x,y \geq 0\}$ starting at the origin, ending on the diagonal $y=x$, and having n steps in $\{\mathrm{NE},\mathrm{NW},\mathrm{SE},\mathrm{SW}\}$, where we use the notation $\mathrm{NE}=(1,1)$, $\mathrm{NW}=(-1,1)$, $\mathrm{SE}=(1,-1)$, $\mathrm{SW}=(-1,-1)$.

We start by describing a standard bijection $\omega : \mathcal{D}_n \to \mathcal{W}_n^1$, which, in a similar form, has been used in [11, 15, 3]. Given $P \in \mathcal{D}_n$, first define two paths as in Section 3: P_L is the left half of P, and P_R is the path obtained by reflecting the right half of P along the vertical line $x=n$. Both P_L and P_R are paths with steps U and D from $(0,0)$ to the line $x=n$, not going below the x-axis. Denote the ith step of P_L, P_R by $\ell_i, r_i \in \{U, D\}$, respectively. Now let $\omega(P) \in \mathcal{W}_n^1$ be the walk whose ith step is equal to

$$\begin{cases} \mathrm{NE} & \text{if } \ell_i = r_i = U, \\ \mathrm{NW} & \text{if } \ell_i = U \text{ and } r_i = D, \\ \mathrm{SE} & \text{if } \ell_i = D \text{ and } r_i = U, \\ \mathrm{SW} & \text{if } \ell_i = r_i = D. \end{cases}$$

Under this bijection, symmetric steps of P, which correspond to common steps of P_L and P_R, become steps of $\omega(P)$ lying entirely on the diagonal $y=x$. An equation for the generating function of walks with a variable keeping track of the number of such steps would be troublesome, since it would contain a term corresponding to walks ending on the diagonal $y=x$, which would involve taking diagonals of generating functions.

To circumvent this problem, we modify the walks so that the steps that we need to keep track of lie on the boundary of the region. Folding walks in $\mathcal{W}_n^1$ along the diagonal $y=x$ (that is, reflecting the steps above the diagonal onto steps below it), we obtain walks in the first octant $\{(x,y) : x \geq y \geq 0\}$. In order not to lose information while folding, we have to allow the resulting walks in the octant to use two colors for steps SE leaving the diagonal. These colors keep track of whether the portion of the walk between the colored step and the next return to the diagonal was above or below the diagonal on the original quadrant walk. We obtain a bijection between $\mathcal{W}_n^1$ and the set $\mathcal{W}_n^2$ of walks in the first octant starting at the origin, ending on the diagonal $y=x$, having n steps in $\{\mathrm{NE},\mathrm{NW},\mathrm{SE},\mathrm{SW}\}$, and where steps SE leaving the diagonal $y=x$ can have two colors.

Next we apply the linear transformation $(x,y) \mapsto (y, \frac{x-y}{2})$ from the first octant to the first quadrant. This transformation gives a bijection between $\mathcal{W}_n^2$ and the set $\mathcal{W}_n^3$ of walks in the first quadrant starting at the origin, ending on the x-axis, having n steps in $\{E, W, \mathrm{NW}, \mathrm{SE}\}$ (denoting $E=(1,0)$ and $W=(-1,0)$), and where steps NW leaving the x-axis can have 2 colors. Under this bijection, steps of walks in $\mathcal{W}_n^2$ lying on the diagonal become steps of walks in $\mathcal{W}_n^3$ lying on the x-axis. Table 3 summarizes the above sequence of bijections from $\mathcal{D}_n$ to $\mathcal{W}_n^3$. It follows that $D(s,z)$ is the generating function for walks in $\mathcal{W}_n^3$ where z marks the length and s marks the number of steps on the x-axis.

To enumerate walks in $\mathcal{W}_n^3$, we consider more general walks where any endpoint is allowed. Let $R(x,y,s,z)$ be the generating function where the coefficient of $x^i y^j s^k z^n$ is the number of walks in the first quadrant with n steps in $\{E, W, \mathrm{NW}, \mathrm{SE}\}$, starting at the origin, ending at (i,j), having k steps entirely on the x-axis, and where steps NW leaving the x-axis can have 2 colors. The generating function for Dyck paths with respect to the statistic ds is the specialization $D(s,z) = R(1,0,s,z)$. By considering the different possibilities for the last step of the walk, we obtain the following

$$\mathcal{D}_n \xrightarrow{\omega} \mathcal{W}_n^1 \xrightarrow{\text{fold along y=x}} \mathcal{W}_n^2 \xrightarrow{(x,y)\mapsto(y,\frac{x-y}{2})} \mathcal{W}_n^3$$

	$\mathcal{D}_n$	$\mathcal{W}_n^1$	$\mathcal{W}_n^2$	$\mathcal{W}_n^3$
walks in	first octant	first quadrant	first octant	first quadrant
allowed steps	$U=\mathrm{NE}, D=\mathrm{SE}$	$\mathrm{NE},\mathrm{NW},\mathrm{SE},\mathrm{SW}$	$\mathrm{NE},\mathrm{NW},\mathrm{SE},\mathrm{SW}$	$E,W,\mathrm{NW},\mathrm{SE}$
length	$2n$	n	n	n
ending on	x-axis	diagonal	diagonal	x-axis
2 colors for			SE leaving diagonal	NW leaving x-axis
ds counts	symmetric steps	steps on diagonal	steps on diagonal	steps on x-axis

Table 3: A summary of the bijections between Dyck paths and walks.

functional equation for $R(x,y) := R(x,y,s,z)$.

$$\begin{aligned}
R(x,y) = 1 &+ z\left(x+\frac{1}{x}+\frac{x}{y}+\frac{y}{x}\right)R(x,y) \\
&- z\left(\frac{1}{x}+\frac{y}{x}\right)R(0,y) && \text{(no steps } W \text{ or NW when on } y\text{-axis)} \\
&- z\,\frac{x}{y}R(x,0) && \text{(no steps SE when on } x\text{-axis)} \\
&+ z\,\frac{y}{x}\left(R(x,0)-R(0,0)\right) && \text{(second color for steps NW when on positive } x\text{-axis)} \\
&+ z\,(s-1)\left(x+\frac{1}{x}\right)R(x,0) && \text{(steps } E \text{ and } W \text{ on } x\text{-axis have weight } s) \\
&- z\,(s-1)\frac{1}{x}R(0,0). && \text{(but no step } W \text{ when at the origin)}
\end{aligned}$$

Collecting terms in this equation, we can summarize our result as follows.

Theorem 6.1. *The generating function for Dyck paths with respect to the statistic* ds *is* $D(s,z) = R(1,0,s,z)$, *where* $R(x,y) := R(x,y,s,z)$ *satisfies the functional equation*

$$\begin{aligned}
&\left(xy - z(y+x^2)(1+y)\right)R(x,y) \\
&\qquad = xy - zy(1+y)R(0,y) + z\left(y^2 - x^2 + (s-1)y(x^2+1)\right)R(x,0) - zy(y+s-1)R(0,0).
\end{aligned}$$

Even though we have been unable to solve this functional equation using the kernel method, the equation suggests that the generating function $D(s,z) = R(1,0,s,z)$ is D-finite. Computations by Alin Bostan [2] using Theorem 6.1 have led to the following conjecture.

Conjecture 6.2. *The generating function* $D(s,z)$ *is D-finite in* z *but not algebraic. Specifically, it satisfies a fifth order linear differential equation with polynomial coefficients with maximum degree 27 in* z.

Note that by removing the choice of colors for steps NW leaving the x-axis, and setting the weight $s = 1$ for steps on the x-axis, the resulting walks in the first quadrant with steps in $\{E, W, \mathrm{NW}, \mathrm{SE}\}$ are in bijection with walks in the first octant with steps in $\{N, S, E, W\}$, which are counted by sequence A005558 in [24].

In analogy with Theorem 3.4 for grand Dyck paths, we consider an alternative measure of the symmetry of a Dyck path, given by its number of *symmetric vertices*. For example, the Dyck path at the bottom of Figure 1 has 5 symmetric vertices.

$|\{P \in \mathcal{D}_n : \mathrm{ds}(P) = k\}|$

$n \setminus k$	1	2	3	4	5	6	7
1	1						
2	0	2					
3	2	0	3				
4	2	6	0	6			
5	8	8	16	0	10		
6	16	32	24	40	0	20	
7	52	84	108	60	90	0	35

$|\{P \in \mathcal{D}_n : \mathrm{sv}(P) = k\}|$

$n \setminus k$	1	2	3	4	5	6	7
1	1						
2	0	2					
3	0	2	3				
4	0	2	6	6			
5	0	4	12	16	10		
6	0	8	24	40	40	20	
7	0	20	60	104	120	90	35

Table 4: The number of Dyck paths of length $n \leq 7$ with a given degree of symmetry (left, see Theorem 6.1) and with a given number of symmetric vertices (right, see Theorem 6.3).

Let

$$\widehat{D}(v,z) = \sum_{n\geq 0} \sum_{P\in\mathcal{D}_n} v^{\mathrm{sv}(P)} z^n$$

denote be the generating function for Dyck paths with respect to the number of symmetric vertices. Through the sequence of bijections in Table 3, symmetric vertices of Dyck paths become vertices on the x-axis of walks in $\mathcal{W}^3_n$, not counting the endpoint of the walk. The argument that we used to obtain an equation for $R(x,y,s,z)$ can be modified to obtain the following functional equation for the generating function $\widehat{R}(x,y) := \widehat{R}(x,y,v,z)$, where the coefficient of $x^i y^j v^k z^n$ is the number of walks in the first quadrant with n steps in $\{E, W, \mathrm{NW}, \mathrm{SE}\}$, starting at the origin, ending at (i,j), having k steps starting on the x-axis, and where steps NW leaving the x-axis can have 2 colors.

$$\begin{aligned}
\widehat{R}(x,y) = 1 &+ z\left(x + \frac{1}{x} + \frac{x}{y} + \frac{y}{x}\right)\widehat{R}(x,y) \\
&- z\left(\frac{1}{x} + \frac{y}{x}\right)\widehat{R}(0,y) && \text{(no steps } W \text{ or NW when on } y\text{-axis)} \\
&- z\,\frac{x}{y}\widehat{R}(x,0) && \text{(no steps SE when on } x\text{-axis)} \\
&+ z\,\frac{y}{x}\left(\widehat{R}(x,0) - \widehat{R}(0,0)\right) && \text{(second color for steps NW when on positive } x\text{-axis)} \\
&+ z\,(v-1)\left(x + \frac{1}{x} + \frac{2y}{x}\right)\widehat{R}(x,0) && \text{(steps } E, W, \text{NW leaving } x\text{-axis have weight } v) \\
&- z\,(v-1)\left(\frac{1}{x} + \frac{2y}{x}\right)\widehat{R}(0,0). && \text{(but no steps } W, \text{NW when at the origin)}
\end{aligned}$$

Theorem 6.3. *The generating function for Dyck paths with respect to the statistic* sv *is* $\widehat{D}(v,z) = \widehat{R}(1,0,v,z)$*, where* $\widehat{R}(x,y) := \widehat{R}(x,y,s,z)$ *satisfies the functional equation*

$$\begin{aligned}
&\left(xy - z(y+x^2)(1+y)\right)\widehat{R}(x,y) \\
&= xy - zy(1+y)\widehat{R}(0,y) + z\left(y^2 - x^2 + (v-1)y(x^2+1+2y)\right)\widehat{R}(x,0) - zy\left(y + (v-1)(1+2y)\right)\widehat{R}(0,0).
\end{aligned}$$

The first few coefficients of the generating functions $D(s,z)$ and $\widehat{D}(v,z)$ appear in Table 4.

7 Other path statistics

Given the simple algebraic generating function in Theorem 3.1 for the number of grand Dyck paths with respect to their degree of symmetry, it may seem surprising that the enumeration of Dyck paths with respect to the same statistic is significantly more complicated.

In this section we try to understand this phenomenon by considering another path statistic related to the degree of symmetry, namely the height of the midpoint of the path. We first consider generalizations of Dyck paths that start and end at arbitrary heights.

7.1 Paths with arbitrary start and ending heights

For $a, b \geq 0$, let $\mathcal{B}_n^{(a,b)}$ denote the set of lattice paths with steps U and D from $(0, a)$ to (n, b) that do not go below the x-axis. Let $\mathcal{B}^{(a,b)} = \bigcup_{n\geq 0} \mathcal{B}_n^{(a,b)}$. By definition, $\mathcal{B}_{2n}^{(0,0)} = \mathcal{D}_n$. In the formulas in this subsection, we will assume that $n \equiv a + b \bmod 2$, since otherwise $\mathcal{B}_n^{(a,b)} = \emptyset$. Define the generating functions

$$B^{(a,b)}(z) = \sum_{n\geq 0} |\mathcal{B}_n^{(a,b)}| z^n \quad \text{and} \quad B(x, y, z) = \sum_{a,b\geq 0} B^{(a,b)}(z)\, x^a y^b.$$

Let us first consider the case $a = b$. Since $|\mathcal{B}_n^{(a,a)}| = 0$ for odd n, we use $n = 2m$ in the following lemma, and we define $C^{(a)}(z) = \sum_{m\geq 0} |\mathcal{B}_{2m}^{(a,a)}| z^m$, noting that $C^{(a)}(z^2) = B^{(a,a)}(z)$. In particular, $C^{(0)}(z) = C(z)$, as defined in Equation (1).

Lemma 7.1. *(i) For $m, a \geq 0$,*

$$|\mathcal{B}_{2m}^{(a,a)}| = \binom{2m}{m} - \binom{2m}{m-a-1}.$$

(ii) For $a \geq 0$,

$$C^{(a)}(z) = C(z)\,\frac{1 - z^{a+1}C(z)^{2a+2}}{1 - zC(z)^2}.$$

(iii)

$$\sum_{m,a\geq 0} |\mathcal{B}_{2m}^{(a,a)}| x^a z^m = \sum_{a\geq 0} C^{(a)}(z) x^a = \frac{2}{(1-x)\left(1 - x + (1 + x)\sqrt{1-4z}\right)}.$$

Proof. Part (i) is obtained using the reflection principle, since paths from $(0, a)$ to $(2m, a)$ that go below the x-axis are in bijection with paths from $(0, a)$ to $(2m, -a - 2)$.

For part (ii), note that every path $P \in \mathcal{B}^{(a,a)}$ can be decomposed uniquely as

$$P = A_1 D A_2 D \dots D A_{i+1} U A_{i+2} U \dots U A_{2i+1} \tag{9}$$

for some $0 \leq i \leq a$, where the A_j are Dyck paths for $1 \leq j \leq 2i + 1$. It follows that

$$C^{(a)}(z) = \sum_{i=0}^{a} z^i C(z)^{2i+1} = C(z)\,\frac{1 - z^{a+1}C(z)^{2a+2}}{1 - zC(z)^2}.$$

To obtain part (iii), one can multiply part (ii) by x^a and sum over $a \geq 0$. Alternatively, we can first consider paths starting and ending at the same height (marked by x) that do not go below the x-axis but touch the x-axis. Since each such path can be decomposed as in Equation (9) (where now i is the beginning and ending height), their generating function is

$$\frac{C(z)}{1 - xzC(z)^2} = \frac{2}{1 - x + (1 + x)\sqrt{1 - 4z}}.$$

Multiplying by $\frac{1}{1-x}$ to account for vertical translations of these paths (so they do not necessarily touch the x-axis) gives the desired expression. $\square$

Next we state the formulas for arbitrary a and b.

Lemma 7.2. *(i) For $n, a, b \geq 0$ with $n \equiv a + b \bmod 2$,*

$$|\mathcal{B}_n^{(a,b)}| = \binom{n}{\frac{n-b+a}{2}} - \binom{n}{\frac{n-b-a-2}{2}}.$$

(ii) For $0 \leq a \leq b$,

$$B^{(a,b)}(z) = z^{b-a}C(z^2)^{b-a+1}\, \frac{1 - \left(zC(z^2)\right)^{2a+2}}{1 - \left(zC(z^2)\right)^2}.$$

(iii)

$$B(x, y, z) = \frac{2}{(1 - xy)\left(1 + xy - 2(x + y)z + (1 - xy)\sqrt{1 - 4z^2}\right)}.$$

Proof. Again, part (i) is obtained using the reflection principle.

For part (ii), note that for $a \leq b$, every path $P \in \mathcal{B}^{(a,b)}$ can be decomposed uniquely as

$$P = QUA_1UA_2 \dots UA_{b-a},$$

where $Q \in \mathcal{B}^{(a,a)}$, and the A_j are Dyck paths for $1 \leq j \leq b - a$. It follows that

$$B^{(a,b)}(z) = C^{(a)}(z^2)\left(zC(z^2)\right)^{b-a} = z^{b-a}C(z^2)^{b-a+1}\, \frac{1 - \left(zC(z^2)\right)^{2a+2}}{1 - \left(zC(z^2)\right)^2},$$

using Lemma 7.1(ii).

For part (iii), first consider paths that start at an arbitrary height (marked by x), end at an arbitrary height (marked by y), do not go below the x-axis, but touch the x-axis. Since each such path starting at height i and ending at height j can be decomposed as

$$A_iD \dots DA_1DA_0UA'_1U \dots UA'_j,$$

their generating function is

$$\frac{C(z^2)}{(1 - uzC(z^2))(1 - vzC(z^2))} = \frac{2}{1 + xy - 2(x + y)z + (1 - xy)\sqrt{1 - 4z^2}}.$$

Multiplying by $\frac{1}{1-xy}$ to account for vertical translations of these paths gives the desired expression.
$\square$

7.2 The height of the midpoint

For $P \in \mathcal{GD}_n$, let $\mathrm{hm}(P)$ denote the height of the midpoint of P. Note that the midpoint of P has coordinates $(n, \mathrm{hm}(P))$.

To enumerate Dyck paths with respect to hm, recall from Lemma 7.2(i) that

$$|\mathcal{B}_n^{(0,b)}| = \binom{n}{\frac{n-b}{2}} - \binom{n}{\frac{n-b}{2}-1} = \frac{2b+2}{n+b+2}\binom{n}{\frac{n-b}{2}}$$

if $n-b$ is even and $0 \le b \le n$, and $|\mathcal{B}_n^{(0,b)}| = 0$ otherwise. The generating function of these ballot numbers is

$$\sum_{n,b\ge 0} |\mathcal{B}_n^{(0,b)}| y^b z^n = B(0,y,z) = \frac{2}{1-2yz+\sqrt{1-4z^2}} = \frac{C(z^2)}{1-yzC(z^2)} \tag{10}$$

by Lemma 7.2(iii).

Let now

$$H(y,z) = \sum_{n\ge 0}\sum_{P\in\mathcal{D}_n} y^{\mathrm{hm}(P)} z^n$$

be the generating function of Dyck paths with respect to the height of their midpoint, and note that $H(1,z) = C(z)$. Then

$$H(y,z) = \sum_{n,b\ge 0} |\mathcal{B}_n^{(0,b)}|^2 y^b z^n.$$

However, this generating function is not algebraic, since the coefficients of $H(0,z) = \sum_{m\ge 0} C_m^2 z^{2m}$ grow asymptotically as $C_m^2 \sim \frac{16^m}{m^3\pi}$, which is not a possible asymptotic behavior for coefficients of an algebraic generating function (see [17]). Using Equation (10), we can express $H(y,z)$ as a diagonal of an algebraic generating function, as

$$H(y,z) = \mathrm{diag}_{z_1,z_2}^{z} \frac{C(z_1^2)C(z_2^2)}{1-yz_1z_2C(z_1^2)C(z_2^2)},$$

which implies by [18] that $H(y,z)$ is D-finite.

Extracting the coefficient of y^b, the generating function for Dyck paths whose midpoint has height b is $\sum_{n\ge 0} |\mathcal{B}_n^{(0,b)}|^2 z^n$, which is the Hadamard product of $B^{(0,b)}(z)$ with itself. By Lemma 7.2(ii), $B^{(0,b)}(z) = z^b C(z^2)^{b+1}$ is algebraic, and so its Hadamard product with itself is D-finite, even though it is not algebraic.

For comparison, let us show that the enumeration of grand Dyck paths with respect to the height of their midpoint is straightforward.

Proposition 7.3. *The generating function for grand Dyck paths with respect to their semilength and the height of their midpoint is*

$$\sum_{n\ge 0}\sum_{P\in\mathcal{GD}_n} y^{\mathrm{hm}(P)} z^n = \frac{1}{\sqrt{(1-yz-\frac{z}{y})^2 - 4z^2}} = \sum_{n\ge 0}\sum_{k=0}^{n} \binom{n}{k}^2 y^{n-2k} z^n. \tag{11}$$

Proof. Let $P \in \mathcal{GD}_n$ and let $M = \phi(P) \in \mathcal{GM}^2$, where ϕ is the bijection from Definition 3.2. Then $\mathrm{hm}(P)$ equals the number of U steps minus the number of D steps of P_L, which in turn equals $u(M) + h_2(M) - d(M) - h_2(M)$. It follows that the left-hand side of Equation (11) is equal to

$$\sum_{M \in \mathcal{GM}^2} y^{u(M)+h_2(M)-d(M)-h_2(M)} z^{|M|} = G(z/y, yz, 1, 1) = \frac{1}{\sqrt{(1 - yz - \frac{z}{y})^2 - 4z^2}},$$

by Lemma 2.2.

The coefficient of $y^{n-2k}z^n$ in this generating function can be also computed directly, since in order to construct a path $P \in \mathcal{GD}_n$ with $\mathrm{hm}(P) = n - 2k$, there are $\binom{n}{k}$ choices for the left half and $\binom{n}{k}$ choices for the right half. $\square$

Taking square roots of the coefficients in the generating function (11), which is equivalent to counting left halves of grand Dyck paths according to their ending height, we obtain a rational generating function

$$\sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k} y^{n-2k} z^n = \frac{1}{1 - (y + \frac{1}{y})z}.$$

The fact that the generating function for grand Dyck paths with respect to hm is the diagonal of a rational generating function explains why it is algebraic; see [23, Theorem 6.3.3].

Finally, we remark that the method used to prove Theorem 3.1 and Proposition 7.3 also gives a common generalization of the two generating functions, namely

$$\sum_{n \geq 0} \sum_{P \in \mathcal{GD}_n} s^{\mathrm{ds}(P)} y^{\mathrm{hm}(P)} z^n = \frac{1}{(1-s)(y + \frac{1}{y})z + \sqrt{(1 - yz - \frac{z}{y})^2 - 4z^2}}.$$

8 Future work

In this paper we have given generating functions for different combinatorial objects with respect to their degree of symmetry. In all cases, the size of the object was measured by a "one-dimensional" parameter. Examples of such parameters are the length (or semilength) of a path in Sections 3 and 6, the semiperimeter of a partition or the side length of a square containing its Young diagram in Section 4, and the semiperimeter of a bargraph in Section 5.

For some combinatorial objects, such as partitions and compositions (represented as Young diagrams and bargraphs, respectively), it is also natural to consider another measure of size, namely the sum of the entries (equivalently, the area). Using this "two-dimensional" parameter for size, the degree of symmetry in compositions is studied in [9]. However, in most other cases, including partitions and unimodal compositions, the generating functions by area and degree of symmetry are yet unknown.

Finally, it remains an open problem to solve the functional equations in Theorems 6.1 and 6.3, giving generating functions for Dyck paths with respect to their degree of symmetry and to their number of symmetric vertices.

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