CONTINUED FRACTIONS OF TAILS OF HYPERGEOMETRIC SERIES

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1. Motivation

The tails of the Taylor series for many standard functions such as arctan and log can be expressed as continued fractions in a variety of ways. A surprising side effect is that some of these continued fractions provide a dramatic acceleration for the underlying power series. These investigations were motivated by a surprising observation about Gregory's series. Gregory's series for π, truncated at $500,000$ terms gives to forty places

$$(1)\quad 4\sum_{k=1}^{500,000}\frac{(-1)^{k-1}}{2k-1} = 3.14159\underline{0}6535897932\underline{40}4626433832\underline{6}9502884197\cdots.$$

To one's initial surprise only the underlined digits are wrong — differ from those of π. This is explained, ex post facto, by setting N equal to one million in the result below:

Theorem 1. *For integer N divisible by 4 the following asymptotic expansion holds:*

$$(2)\qquad \frac{\pi}{2} - 2\sum_{k=1}^{N/2}\frac{(-1)^{k-1}}{2k-1} \sim \sum_{m=0}^{\infty}\frac{E_{2m}}{N^{2m+1}} = \frac{1}{N} - \frac{1}{N^3} + \frac{5}{N^5} - \frac{61}{N^7} + \cdots,$$

where the numerators $1, -1, 5, -61, 1385, -50521, \cdots$ *are the Euler numbers* E_0, E_2, E_4, E_6, E_8, $E_{10}, \cdots$.

The observation (1) arrived in the mail from Roy North in 1987. After verifying its truth numerically (which is much quicker today), it was an easy matter to generate a large number of the "errors" to high precision. The authors of [1] then recognized the sequence of errors in (1) as the Euler numbers — with the help of Sloane's 'Handbook of Integer Sequences'. The presumption that (1) is a form of Euler-Maclaurin summation is now formally verifiable for any fixed N in Maple. This allowed them to determine that (1) is equivalent to a set of identities between Bernoulli and Euler numbers that could with considerable effort have been established. Secure in the knowledge that (1) holds it is easier, however, to use the *Boole Summation formula* which applies directly to alternating series and *Euler*

Date: March 24, 2003.
1991 *Mathematics Subject Classification.* Primary .
Research supported by NSERC and by the Canada Research Chair Programme.

numbers (see [1]). Because N was a power of ten, the asymptotic expansion was obvious on the computer screen.

This is a good example of a phenomenon which really does not become apparent without working to reasonably high precision (who recognizes 2, -2, 10 ?), and which highlights the role of pattern recognition and hypothesis validation in experimental mathematics.

It was an amusing additional exercise to compute Pi to $5,000$ digits from (1). Indeed, with $N = 200,000$ and correcting using the first thousand even Euler numbers, Borwein and Limber [2] obtained $5,263$ digits of Pi (plus 12 guard digits). Thus, while the alternating Gregory series is very slowly convergent, the errors are highly predictable.

2. Three Continued Fraction Classes

We will discuss three classes of continued fractions: Euler, Gauss and Perron in this section.

2.1. **Euler's Continued Fraction.** Using the following notation for continued fraction:

$$\frac{a_1}{b_1} \pm \frac{a_2}{b_2} \pm \frac{a_3}{b_3} \pm \cdots = \cfrac{a_1}{b_1 \pm \cfrac{a_2}{b_2 \pm \cfrac{a_3}{b_3 \pm \ddots}}},$$

identities such as

$$a_0 + a_1 + a_1a_2 + a_1a_2a_3 + a_1a_2a_3a_4 = a_0 + \frac{a_1}{1} - \frac{a_2}{1+a_2} - \frac{a_3}{1+a_3} - \frac{a_4}{1+a_4}$$

are easily verified symbolically. The general form

$$(3) \quad a_0 + a_1 + a_1a_2 + a_1a_2a_3 + \cdots + a_1a_2a_3\cdots a_N = a_0 + \frac{a_1}{1} - \frac{a_2}{1+a_2} - \frac{a_3}{1+a_3} - \cdots - \frac{a_N}{1+a_N}$$

can then be obtained by substituting $a_N + a_N\, a_{N+1}$ for a_N and checking that the shape of the right hand side is preserved. This allows many series to be re-expressed as continued fractions. For example, with $a_0 = 0, a_1 = z, a_2 = -z^2/3, a_3 = -3z^2/5, \cdots$,

$$\arctan(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} - \cdots$$

we obtain, in the limit, the continued fraction for arctan due to Euler:

$$\arctan(z) = \frac{z}{1} + \frac{z^2}{3-z^2} + \frac{9z^2}{5-3z^2} + \frac{25z^2}{7-5z^2} + \cdots.$$

When $z = 1$, this becomes the first infinite continued fraction, given by Lord Brouncker (1620-1684):

$$\frac{4}{\pi} = 1 + \frac{1}{2\,+}\;\frac{9}{2\,+}\;\frac{25}{2\,+}\;\frac{49}{2\,+}\;\cdots. \tag{4}$$

If we let $a_0 = \sum_1^N b_k$ be the initial segment of a similar series we may use (3) to replace the remaining terms by a continued fraction. For example, if we put

$$a_0 = \sum_{n=1}^{N} \frac{(-1)^{n-1} z^{2n-1}}{2n-1}, a_1 = \frac{(-1)^N z^{2N+1}}{2N+1}, a_2 = -\frac{2N+1}{2N+3} z^2, a_3 = -\frac{2N+3}{2N+5} z^2, \cdots$$

then we get

$$\arctan(z) = \sum_{n=1}^{N} (-1)^{n-1} \frac{z^{2n-1}}{2n-1} + \frac{(-1)^N z^{2N+1}}{2N+1\,+}\;\frac{(2N+1)^2 z^2}{(2N+3)-(2N+1)z^2\,+}\;\frac{(2N+3)^2 z^2}{(2N+5)-(2N+3)z^2\,+}\;\frac{(2N+5)^2 z^2}{(2N+7)-(2N+5)z^2\,+}\;\cdots. \tag{5}$$

2.2. **Gauss's Continued Fraction.** A rich vein lies in Gauss's continued fraction for the ratio of two hypergeometric functions $\dfrac{\mathrm{F}(a,b+1;c+1;z)}{\mathrm{F}(a,b;c;z)}$, see [5]. Recall that within its radius of convergence, the Gaussian hypergeometric function is defined by

$$\begin{aligned} \mathrm{F}(a,b;c;z) &= 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 \\ &+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} z^3 + \cdots. \end{aligned} \tag{6}$$

The general continued fraction is developed by a reworking of the *contiguity relation*

$$\mathrm{F}(a,b;c;z) = \mathrm{F}(a,b+1;c+1;z) - \frac{a(c-b)}{c(c+1)} z\, \mathrm{F}(a+1,b+1;c+2;z), \tag{7}$$

and formally at least is quite easy to derive. Convergence and convergence estimates are more delicate. We therefore have

$$\frac{\mathrm{F}(a,b+1;c+1;z)}{\mathrm{F}(a,b;c;z)} = \left(1 - \frac{a(c-b)}{c(c+1)} z \frac{\mathrm{F}(a+1,b+1;c+2;z)}{\mathrm{F}(a,b+1;c+1;z)}\right)^{-1}$$

and this yields the recursive process for the continued fraction. In the limit, for $b = 0$ and replacing c by $c-1$, this process yields

$$\mathrm{F}(a,1;c;z) = \frac{1}{1\,-}\;\frac{a_1 z}{1\,-}\;\frac{a_2 z}{1\,-}\;\frac{a_3 z}{1\,-}\;\cdots \tag{8}$$

which is the case of present interest. Here

$$a_{2l+1} = \frac{(a+l)(c-1+l)}{(c+2l-1)(c+2l)} \qquad a_{2l+2} = \frac{(l+1)(c-a+l)}{(c+2l)(c+2l+1)}$$

for $l = 0, 1, \cdots$. We also let

$$F_M(a,1;c;z) = \frac{1}{1\,-}\;\frac{a_1 z}{1\,-}\;\frac{a_2 z}{1\,-}\;\cdots\;-\;\frac{a_{M-1} z}{1}$$

denote the Mth convergent of the continued fraction to $F(a,1;c;z)$.

It is well known and easy to verify that $\log(1+z) = z\,\mathrm{F}(1,1;2;-z)$. It is then a pleasant surprise to discover that $\log(1+z) - z = -\frac{1}{2}z^2\,\mathrm{F}(2,1;3;-z)$, $\log(1+z) - z + \frac{1}{2}z^2 = \frac{1}{3}z^3\,\mathrm{F}(3,1;4;-z)$ and to conjecture that

$$\log(1+z) + \sum_{n=1}^{N-1} \frac{(-1)^n z^n}{n} = -\frac{(-1)^N z^N}{N}\,\mathrm{F}(N,1;N+1;-z). \tag{9}$$

This is easy to first verify for a few cases and then confirm rigorously. As always, a formula for log leads correspondingly to one for arctan:

$$\arctan(z) - \sum_{n=0}^{N-1} \frac{(-1)^n z^{2n+1}}{2n+1} = \frac{(-1)^N z^{2N+1}}{2N+1}\,\mathrm{F}\left(N+\frac{1}{2},1;N+\frac{3}{2};-z^2\right). \tag{10}$$

Happily, in both cases (8) is applicable — as it is for a variety of other functions such as $\log\left(\frac{1+z}{1-z}\right)$, $(1+z)^k$, and $\int_0^z (1+t^n)^{-1}\,dt = z\,\mathrm{F}\left(\frac{1}{n},1;1+\frac{1}{n};-z^n\right)$. Note that this last function recaptures $\log(1+z)$ and $\arctan(z)$ for $n=1$ and 2 respectively.

We next give the explicit continued fractions for (9) and (10).

Theorem 2. *Gauss's continued fractions for (9) and (10) are:*

$$\log(1+z) + \sum_{n=1}^{N-1} \frac{(-1)^n z^n}{n} \tag{11}$$

$$= \frac{(-1)^{N+1} z^N}{N} \; \frac{N^2 z}{+\; N+1} \; \frac{1^2 z}{+\; N+2} \; \frac{(N+1)^2 z}{+\; N+3} \; \frac{2^2 z}{+\; N+4} \; + \cdots$$

and

$$\arctan(z) - \sum_{n=0}^{N-1} \frac{(-1)^n z^{2n+1}}{2n+1} \tag{12}$$

$$= \frac{(-1)^N z^{2N+1}}{2N+1} \; \frac{(2N+1)^2 z^2}{+\; 2N+3} \; \frac{2^2 z^2}{+\; 2N+5} \; \frac{(2N+3)^2 z^2}{+\; 2N+7} \; \frac{4^2 z^2}{+\; 2N+9} \; + \cdots.$$

Suppose we return to Gregory's series, but add a few terms of the continued fraction for (10). One observes numerically that if the results are with $N = 500,000$, adding only six terms of the continued fraction has the effect of increasing the precision by 40 digits.

Example 3.

Let

$$E_1(N,M,z) := \log(1+z) - \left(-\sum_{n=1}^{N} \frac{(-z)^n}{n} - \frac{(-z)^{N+1}}{N+1}\mathrm{F}_M(N+1,1;N+2;-z)\right)$$

and

$$E_2(N,M,z) := \arctan(z) - \left(\sum_{n=0}^{N-1} \frac{(-1)^n z^{2n+1}}{2n+1} + \frac{(-1)^N z^{2N+1}}{2N+1}\mathrm{F}_M(N+\frac{1}{2},1;N+\frac{3}{2};-z^2)\right).$$

Then $E_1(N,M,z)$ and $E_2(N,M,z)$ measure the precision of the approximations to $\log(1+z)$ and $\arctan(x)$ obtained by computing the first N terms of Taylor series and then adding M terms of their continued fractions respectively. Tables 1, 2,

		5×10	5×10^2	5×10^3	5×10^4
M	0	0.48×10^{-4}	0.13×10^{-25}	0.15×10^{-232}	0.13×10^{-2292}
	1	0.43×10^{-4}	0.11×10^{-25}	0.14×10^{-232}	0.11×10^{-2292}
	2	0.40×10^{-8}	0.11×10^{-31}	0.14×10^{-240}	0.11×10^{-2302}
	3	0.34×10^{-8}	1.00×10^{-32}	0.12×10^{-240}	0.10×10^{-2302}
	4	0.12×10^{-11}	0.40×10^{-37}	0.50×10^{-248}	0.41×10^{-2312}
	5	0.10×10^{-11}	0.35×10^{-37}	0.45×10^{-248}	0.37×10^{-2312}
	6	0.78×10^{-15}	0.31×10^{-42}	0.40×10^{-255}	0.33×10^{-2321}

TABLE 1. Error $|E_1(N,M,0.9)|$ for $N = 5\times 10^k (1\le k\le 4)$ and $0\le M\le 6$.

		5×10	5×10^2	5×10^3	5×10^4	5×10^5	5×10^6
M	0	0.99×10^{-2}	1.00×10^{-3}	1.00×10^{-4}	1.00×10^{-5}	1.00×10^{-6}	1.00×10^{-7}
	1	0.97×10^{-2}	1.00×10^{-3}	1.00×10^{-4}	1.00×10^{-5}	1.00×10^{-6}	1.00×10^{-7}
	2	0.91×10^{-6}	1.00×10^{-9}	1.00×10^{-12}	1.00×10^{-15}	1.00×10^{-18}	1.00×10^{-21}
	3	0.86×10^{-6}	1.00×10^{-9}	1.00×10^{-12}	1.00×10^{-15}	1.00×10^{-18}	1.00×10^{-21}
	4	0.31×10^{-9}	0.39×10^{-14}	0.40×10^{-19}	0.40×10^{-24}	0.40×10^{-29}	0.40×10^{-34}
	5	0.28×10^{-9}	0.39×10^{-14}	0.40×10^{-19}	0.40×10^{-24}	0.40×10^{-29}	0.40×10^{-34}
	6	0.22×10^{-12}	0.34×10^{-19}	0.36×10^{-26}	0.36×10^{-33}	0.36×10^{-40}	0.36×10^{-47}

TABLE 2. Error $|E_1(N,M,1)|$ for $N = 5\times 10^k (1\le k\le 6)$ and $0\le M\le 6$.

		5×10	5×10^2	5×10^3	5×10^4	5×10^5	5×10^6
M	0	0.50×10^{-2}	0.50×10^{-3}	0.50×10^{-4}	0.50×10^{-5}	0.50×10^{-6}	0.50×10^{-7}
	1	0.49×10^{-2}	0.50×10^{-3}	0.50×10^{-4}	0.50×10^{-5}	0.50×10^{-6}	0.50×10^{-7}
	2	0.47×10^{-6}	0.50×10^{-9}	0.50×10^{-12}	0.50×10^{-15}	0.50×10^{-18}	0.50×10^{-21}
	3	0.44×10^{-6}	0.49×10^{-9}	0.50×10^{-12}	0.50×10^{-15}	0.50×10^{-18}	0.50×10^{-21}
	4	0.16×10^{-9}	0.20×10^{-14}	0.20×10^{-19}	0.20×10^{-24}	0.20×10^{-29}	0.20×10^{-34}
	5	0.15×10^{-9}	0.19×10^{-14}	0.20×10^{-19}	0.20×10^{-24}	0.20×10^{-29}	0.20×10^{-34}
	6	0.12×10^{-12}	0.17×10^{-19}	0.18×10^{-26}	0.18×10^{-33}	0.18×10^{-40}	0.18×10^{-47}

TABLE 3. Error $|E_2(N,M,1)|$ for $N = 5\times 10^k (1\le k\le 6)$ and $0\le M\le 6$.

3 and 4 record those data for the approximations to $\log(1.9), \log(2), \arctan(1)$ and $\arctan(1/2)+\arctan(1/5)+\arctan(1/8)$ respectively. Note that

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{5}) + \arctan(\frac{1}{8})$$

is a formula of Machin type used by Johann Dase to compute 205 digits of π in his head in 1844.

After some further numerical experimentation it is clear that for large a, c the continued fraction $\mathrm{F}(a,1,c;z)$ is rapidly convergent. And indeed the rough rate is apparent.

This is part of the content of the next theorem:

		5×10	5×10^2
M	0	0.31×10^{-32}	0.37×10^{-304}
	1	0.19×10^{-33}	0.23×10^{-305}
	2	0.11×10^{-37}	0.15×10^{-311}
	3	0.26×10^{-38}	0.37×10^{-312}
	4	0.56×10^{-42}	0.92×10^{-318}
	5	0.13×10^{-42}	0.23×10^{-318}
	6	0.59×10^{-46}	0.13×10^{-323}

TABLE 4. Error $|E_2(N+1,M,1/2)+E_2(N+1,M,1/5)+E_2(N+1,M,1/8)|$ for $N=5\times 10^k (1\le k\le 2)$ and $0\le M\le 6$.

Theorem 4. *Suppose $2\le a, a+1\le c\le 2a$ and $M\ge 2$. Then for $-1\le z<0$ one has*

$$|\mathrm{F}(a,1;c;z)-\mathrm{F}_M(a,1;c;z)| \le \frac{\Gamma(n+1)(n+a)\Gamma(n+c-a)\Gamma(a)\Gamma(c)}{\Gamma(n+a)\Gamma(n+c)a\Gamma(c-a)}\left(\frac{2a}{(c-2)\left(1-\frac{2}{z}\right)+(2a-c)}\right)^M$$

where $n=[M/2]$ and $\mathrm{F}_M(a,1;c;z)$ is the M-th convergent of the continued fraction to $F(a,1,c;z)$.

The proof of Theorem 4 will be given in the Appendix below.

In [5] one can find listed many explicit continued fractions which can be derived from Gauss's continued fraction or various of its limiting cases. These include exp, tanh, tan and various less elementary functions. One especially attractive fraction is that for $J_{n-1}(z)/J_n(z)$ and $I_{n-1}(z)/I_n(z)$ where J and I are *Bessel functions of the first kind.* In particular,

$$\frac{J_{n-1}(2z)}{J_n(2z)}=\frac{n}{z}-\frac{\frac{z}{(n+1)}}{1}\,\underset{-}{}\,\frac{\frac{z^2}{(n+1)(n+2)}}{1}\,\underset{-}{}\,\frac{\frac{z^2}{(n+2)(n+3)}}{1}\,\underset{-}{}\,\cdots. \tag{13}$$

Setting $z=i$ and $n=1$ leads to the very beautiful continued fraction

$$\frac{I_1(2)}{I_0(2)}=[1,2,3,4,\cdots].$$

In general, arithmetic simple continued fractions correspond to such ratios.

An example of a more complicated situation is:

$$\frac{(2z)^{2N+1}\,\mathrm{F}\left(N+\frac{1}{2},\frac{1}{2};N+\frac{3}{2};z^2\right)}{(N+1)\binom{2N+2}{N+1}\mathrm{F}\left(\frac{1}{2},-\frac{1}{2};\frac{1}{2};z^2\right)}=\frac{\arcsin(z)}{\sqrt{1-z^2}}-\sigma_{2N}(z) \tag{14}$$

where σ_{2N} is the $2N$-th Taylor polynomial for $\frac{\arcsin(z)}{\sqrt{1-z^2}}$. Only for $N=0$ is this precisely of the form of Gauss's continued fraction.

2.3. **Perron's Continued Fraction.** Another continued fraction expansion is based on Stieltjes work on the moment problem (see Perron [4]) and leads to similar acceleration. In volume 2, page 18 of [4] one finds a beautiful continued fraction for

$$\frac{1}{z^\mu}\int_0^z \frac{t^\mu}{1+t}\,dt = \frac{z}{\mu+1} + \frac{(\mu+1)^2 z}{(\mu+2)-(\mu+1)z} + \frac{(\mu+2)^2 z}{(\mu+3)-(\mu+2)z} + \cdots. \tag{15}$$

valid for $\mu > -1, -1 < z \le 1$. One may deduce this as a consequence of Euler's continued fraction if we write

$$\frac{1}{z^\mu}\int_0^z \frac{t^\mu}{1+t}\,dt = \frac{z}{\mu+1} - \frac{z^2}{\mu+2} + \frac{z^3}{\mu+3} - \frac{z^4}{\mu+4} + \cdots$$

and observe that (15) follows from (3) in the limit.

Since

$$\frac{z^{\mu+1}}{\mu+1}\,\mathrm{F}\left(\mu+1, 1; \mu+2; -z\right) = \int_0^z \frac{t^\mu}{1+t}\,dt, \tag{16}$$

$$\frac{z^{2\,\mu+1}}{2\,\mu+1}\,\mathrm{F}\left(\mu+\frac{1}{2}, 1; \mu+\frac{3}{2}; -z^2\right) = \int_0^z \frac{t^{2\,\mu}}{1+t^2}\,dt, \tag{17}$$

for $\mu > 0$, on examining (9) and (10) this is immediately applicable to provide Euler continued fractions for the tail of the log and arctan series. Explicitly, we obtain:

Theorem 5. *Perron's continued fractions for (9) and (10) are:*

$$\log(1+z) + \sum_{n=1}^{N-1} \frac{(-1)^n z^n}{n} = \frac{(-1)^{N+1} z^N}{N} + \frac{N^2 z}{(N+1) - Nz} + \frac{(N+1)^2 z}{(N+2)-(N+1)z} + \cdots \tag{18}$$

and

$$\arctan(z) - \sum_{n=0}^{N-1} \frac{(-1)^n z^{2\,n+1}}{2\,n+1} = \frac{(-1)^N z^{2N+1}}{2N+1} + \frac{(2N+1)^2 z^2}{(2N+3)-(2N+1)z^2} + \frac{(2N+3)^2 z^2}{(2N+5)-(2N+3)z} + \cdots. \tag{19}$$

Moreover, while the Gauss and Euler/Perron continued fractions obtained are quite distinct the convergence behaviour is very similar to that of the previous section. Note also the coincidence of (19) and (5). Indeed as we have seen Theorem 5 coincides with a special case of (3).

3. Appendix

Recall that Gauss's continued fraction for $\mathrm{F}(a, 1; c; z)$ is

$$\mathrm{F}(a,1;c;z) = \frac{1}{1} - \frac{a_1 z}{1} - \frac{a_2 z}{1} - \frac{a_3 z}{1} - \cdots$$

where

$$a_{2l+1} = \frac{(a+l)(c-1+l)}{(c+2l-1)(c+2l)} \qquad a_{2l+2} = \frac{(l+1)(c-a+l)}{(c+2l)(c+2l+1)}$$

for $l = 0, 1, \cdots$. Let

$$\frac{A_n(z)}{B_n(z)} = \frac{1}{1} \, {}_{-} \, \frac{a_1 z}{1} \, {}_{-} \, \frac{a_2 z}{1} \, {}_{-} \, \cdots \, {}_{-} \, \frac{a_{n-1} z}{1} = \mathrm{F}_n(a, 1; c; z)$$

be the n-th convergent of the continued fraction. It can be proved by induction that $A_1(z) = A_2(z) = B_1(z) = 1, B_2(z) = 1 - a_1 z$ and

$$A_k(z) = A_{k-1}(z) - a_{k-1} z A_{k-2}(z),$$

and

$$B_k(z) = B_{k-1}(z) - a_{k-1} z B_{k-2}(z),$$

for $k \geq 3$. Hence for $k \geq 2$, we have

$$A_k(z) B_{k-1}(z) - A_{k-1}(z) B_k(z) = a_1 \cdots a_{k-1} z^{k-1}.$$

Using the estimation in Theorem 8.9 of [3], we find that if $a_i > 0$ for all i, then

$$\left| \mathrm{F}(a, 1; c; z) - \frac{A_n(z)}{B_n(z)} \right| \leq \left| \frac{A_n(z)}{B_n(z)} - \frac{A_{n-1}(z)}{B_{n-1}(z)} \right| = \left| \frac{a_1 \cdots a_{n-1} z^{n-1}}{B_n(z) B_{n-1}(z)} \right|$$

One may verify that $B_n(z)$ are hypergeometric polynomials (see [5]) and explicitly

$$B_{2k}(z) = F(-k, 1-a-k, 2-c-2k; z)$$

and

$$B_{2k+1}(z) = F(-k, -a-k, 1-c-2k; z).$$

These may also be written in terms of Jacobi Polynomials so that

$$B_{2k}(z) = \binom{2k+c-2}{k}^{-1} (-z)^k P_k^{(a-1, c-a-1)} \left(1 - \frac{2}{z}\right)$$

and

$$B_{2k+1}(z) = \binom{2k+c-1}{k}^{-1} (-z)^k P_k^{(a, c-a-1)} \left(1 - \frac{2}{z}\right).$$

We let

$$E_n := E_n(a, c, z) = \frac{a_1 a_2 \cdots a_n z^n}{B_n(z) B_{n+1}(z)} \quad \text{and} \quad F_n := \mathrm{F}_n(a, c, z) = \frac{E_{n+1}}{E_n}.$$

Then we get

$$F_{2n} = \frac{a_{2n+1} z B_{2n}(z)}{B_{2n+2}(z)} = \frac{(n+a)}{(n+1)} \frac{P_n^{(a-1, c-a-1)}}{P_{n+1}^{(a-1, c-a-1)}} \left(1 - \frac{2}{z}\right)$$

and

$$F_{2n-1} = \frac{a_{2n} z B_{2n-1}(z)}{B_{2n+1}(z)} = \frac{(n+c-a-1)}{(n+c-1)} \frac{P_n^{(a, c-a-1)}}{P_{n+1}^{(a, c-a-1)}} \left(1 - \frac{2}{z}\right).$$

We need the following estimation. Assume $0 \leq \beta \leq \alpha, 1 \leq \alpha, 1 \leq n$ and $0 < x \leq 1$. We shall show

$$\frac{P_n^{(\alpha,\beta)}}{P_{n-1}^{(\alpha,\beta)}} \left(1 + \frac{2}{x}\right) \geq \frac{(n+\alpha-1)\left((\alpha+\beta)\left(1+\frac{2}{x}\right) + (\alpha-\beta)\right)}{2n\alpha}. \tag{20}$$

The Jacobi polynomials satisfy the recurrence relation

$$(21)\quad 2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x) = (2n+\alpha+\beta-1)\left((2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\right)P_{n-1}^{(\alpha,\beta)}(x) - 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x)$$

for $n = 2, 3, \cdots$ where

$$P_0^{(\alpha,\beta)}(x) \equiv 1 \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).$$

We let

$$R_n := \frac{P_n^{(\alpha,\beta)}}{P_{n-1}^{(\alpha,\beta)}}\left(1+\frac{2}{x}\right)$$

and

$$T_n := \frac{(n+\alpha-1)\left((\alpha+\beta)\left(1+\frac{2}{x}\right)+(\alpha-\beta)\right)}{2n\alpha}.$$

For $n = 1$,

$$\begin{aligned} R_1 &= \frac{1}{2}(\alpha+\beta+2)\left(1+\frac{2}{x}\right)+\frac{1}{2}(\alpha-\beta) \\ &\geq \frac{(\alpha+\beta)\left(1+\frac{2}{x}\right)+(\alpha-\beta)}{2} = T_1. \end{aligned}$$

So (20) is true for $n = 1$. By the recurrence relation (21), we get

$$\begin{aligned} R_n &= \frac{(2n+\alpha+\beta-1)\left\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)\left(1+\frac{2}{x}\right)+\alpha^2-\beta^2\right\}}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)} \\ &\quad -\frac{(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{n(n+\alpha+\beta)(2n+\alpha+\beta-2)}\frac{1}{R_{n-1}} \\ &:= \alpha_n - \beta_n\frac{1}{R_{n-1}} \end{aligned}$$

for $n \geq 2$. Suppose (20) is true for $n-1$. Then

$$R_n \geq \alpha_n - \frac{\beta_n}{T_{n-1}}.$$

For convenience, we write $f(n,\alpha,\beta,x)$ for the numerator of the expression $\alpha_n - \frac{\beta_n}{T_{n-1}} - T_n$ after simplification to a fractional form, that is

$$\begin{aligned} &\frac{f(n,\alpha,\beta,x)}{x(n-2+\alpha)(\alpha x+\alpha+\beta)n(n+\alpha+\beta)(2n+\alpha+\beta-2)\alpha} \\ &:= \alpha_n - \frac{\beta_n}{T_{n-1}} - T_n. \end{aligned}$$

The function $f(n,\alpha,\beta,x)$ is a polynomial in n of degree 4 and can be shown that subject to our conditions on α, β and x, that it is increasing on n and $f(1,\alpha,\beta,x) > 0$. It follows that $\alpha_n - \frac{\beta_n}{T_{n-1}} > T_n$ and $R_n \geq T_n$. This proves (20).

In view of (20), we have

$$F_{2n} \leq \frac{(n+a)}{(n+a-1)}\frac{2(a-1)}{\left((c-2)\left(1-\frac{2}{x}\right)+(2a-c)\right)}$$

and

$$F_{2n-1} \leq \frac{n(n+c-a-1)}{(n+c-1)(n+a-1)} \frac{2a}{\left((c-1)\left(1-\frac{2}{z}\right)+(2a-c+1)\right)}.$$

Thus for $n \geq 1$,

$$F_{2n}F_{2n-1} \leq \frac{(n+a)n(n+c-a-1)}{(n+c-1)(n+a-1)^2} \left\{ \frac{2a}{(c-2)\left(1-\frac{2}{z}\right)+(2a-c)} \right\}^2 .$$

We are now ready to estimate E_n. Note that

$$\begin{aligned} E_{2n+1} &= E_1 F_{2n} \cdots F_1 \\ &= E_1 \left\{ \prod_{i=1}^{n} \frac{(i+a)i(i+c-a-1)}{(i+c-1)(i+a-1)^2} \right\} \left(\frac{2a}{(c-2)\left(1-\frac{2}{z}\right)+(2a-c)} \right)^{2n} \\ &\leq \frac{\Gamma(n+1)(n+a)\Gamma(n+c-a)\Gamma(a)\Gamma(c)}{\Gamma(n+a)\Gamma(n+c)a\Gamma(c-a)} \left(\frac{2a}{(c-2)\left(1-\frac{2}{z}\right)+(2a-c)} \right)^{2n+1} \end{aligned}$$

as claimed, because

$$E_1 = \frac{a_1 z}{B_1(z)B_2(z)} \leq \frac{2a}{(c-2)\left(1-\frac{2}{z}\right)+(2a-c)}.$$

The bound for E_{2n} can be obtained similarly. This proves Theorem 4. **QED**

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