Irrationality Exponents For Even Zeta Constants

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***Abstract*:** Let $k \geq 1$ be a small fixed integer. The rational approximations $\left|p/q - \pi^k\right| > 1/q^{\mu(\pi^k)}$ of the irrational number π^k are bounded away from zero. A general result for the irrationality exponent $\mu(\pi^k)$ will be proved here. The specific results and numerical data for a few cases $k = 2$ and $k = 3$ are also presented and explained. The even parameters $2k$ correspond to the even zeta constants $\zeta(2k)$.

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1 Introduction

Let $k \geq 1$ be a small fixed integer. The rational approximations $\left|p/q - \pi^k\right| > 1/q^{\mu(\pi^k)}$ of the irrational number π^k are bounded away from zero. The earliest result $|p/q - \pi| > 1/q^{42}$ for the irrationality exponent $\mu(\pi)$ was proved by Mahler in 1953, and more recently it was reduced to $|p/q - \pi| > 1/q^{7.6063}$ by Salikhov in 2008. The earliest result for next number $\left|p/q - \pi^2\right| > 1/q^{11.86}$ was proved by Apery in 1979, and more recently it was reduced to $\left|p/q - \pi^2\right| > 1/q^{7.398537}$ by Rhin and Viola in 1996. Therer is no literature for $k \geq 3$. This note introduces elementary techniques to determine the irrationality exponent $\mu(\pi^k)$ of the irrational number π^k. It is shown that the Diophantine inequality $\left|p/q - \pi^k\right| > 1/q^{2+\varepsilon}$, where $\varepsilon > 0$ is an arbitrarily small number, is true for any $k \geq 1$.

1.1 Exponent For The Number π^2

Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the irrational number π^2. The sequence of rational approximations $\{\left|p_n/q_n - \pi^2\right| : n \geq 1\}$ are bounded away from zero. For instance, the 5th and 6th convergents are

$$\text{(i)} \quad \left|\frac{227}{23} - \pi^2\right| \geq \frac{1}{23^{3.236253}}, \qquad \text{(ii)} \quad \left|\frac{10748}{1089} - \pi^2\right| \geq \frac{1}{1089^{2.018434}},$$

respectively, additional data are compiled in Table 2. But, it is difficult to prove a lower bound. The earliest result $\left|p/q - \pi^2\right| \geq 1/q^{11.85}$ was proved by Apery in [1], and more recently it was improved to $\left|p/q - \pi^2\right| \geq 1/q^{5.44}$ by Rhin and Viola in [22]. Basic and elementary ideas are used here to improve it to the followings estimate.

Theorem 1.1. *For all large rational approximations $p/q \to \pi^2$, the Diophantine inequality*

$$\left|\pi^2 - \frac{p}{q}\right| \geq \frac{1}{q^{2+\varepsilon}}, \tag{1}$$

where $\varepsilon > 0$ is a small number, is true.

The proof appears in Section 6.

Table 1: Historical Data For $\mu(\pi^2)$

Irrationality Measure Upper Bound	Reference	Year
$\mu(\pi^2) \leq 11.85078$	Apery, [17]	1976
$\mu(\pi^2) \leq 10.02979$	Dvornicich, Viola, [10]	1987
$\mu(\pi^2) \leq 5.441243$	Rhin, Viola, [22]	2001

1.2 Exponent For The Number π^3

Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the irrational number π^3. The sequence of rational approximations $\{\left|p_n/q_n - \pi^3\right| : n \geq 1\}$ are bounded away from zero. For instance, the 5th and 6th convergents are

$$\text{(i)} \quad \left|\frac{123498}{3983} - \pi^3\right| \geq \frac{1}{3983^{2.320380}}, \qquad \text{(ii)} \quad \left|\frac{1714151}{55284} - \pi^3\right| \geq \frac{1}{55284^{2.096515}},$$

respectively, additional data are compiled Table 3. But, it is difficult to prove a lower bound. The literature does not have any estimate nor numerical data on the irrationality exponent of this number. Basic and elementary ideas are used here to prove the followings estimate.

Theorem 1.2. *For all large rational approximations $p/q \to \pi^3$, the Diophantine inequality*

$$\left|\pi^3 - \frac{p}{q}\right| \geq \frac{1}{q^{2+\varepsilon}}, \tag{2}$$

where $\varepsilon > 0$ is a small number, is true.

The proof appears in Section 7.

1.3 Exponent For The General Case π^k

Theorem 1.3. *Let $k \geq 1$ be a small fixed integer. For all large rational approximations $p/q \to \pi^k$, the Diophantine inequality*

$$\left|\pi^k - \frac{p}{q}\right| \gg \frac{1}{q^{2+\varepsilon}}, \tag{3}$$

where $\varepsilon > 0$ is a small number, is true.

The proof appears in Section 9.

2 Harmonic Summation Kernels

The harmonic summation kernels naturally arise in the partial sums of Fourier series and in the studies of convergences of continuous functions.

Definition 2.1. *The Dirichlet kernel is defined by*

$$\mathcal{D}_x(z) = \sum_{-x \leq n \leq x} e^{i2nz} = \frac{\sin((2x+1)z)}{\sin(z)}, \tag{4}$$

where $x \in \mathbb{N}$ is an integer and $z \in \mathbb{R} - \pi\mathbb{Z}$ is a real number.

Definition 2.2. *The Fejer kernel is defined by*

$$\mathcal{F}_x(z) = \sum_{0 \leq n \leq x,} \sum_{-n \leq k \leq n} e^{i2kz} = \frac{1}{2} \frac{\sin((x+1)z)^2}{\sin(z)^2}, \tag{5}$$

where $x \in \mathbb{N}$ is an integer and $z \in \mathbb{R} - \pi\mathbb{Z}$ is a real number.

These formulas are well known, see [15] and similar references. For $z \neq k\pi$, the harmonic summation kernels have the upper bounds $|\mathcal{K}_x(z)| = |\mathcal{D}_x(z)| \ll |x|$, and $|\mathcal{K}_x(z)| = |\mathcal{F}_x(z)| \ll |x^2|$.

The Dirichlet kernel in Definition 2.1 is a well defined continued function of two variables $x, z \in \mathbb{R}$. Hence, for fixed z, it has an analytic continuation to all the real numbers $x \in \mathbb{R}$.

An important property is the that a proper choice of the parameter $x \geq 1$ can shifts the sporadic large value of the reciprocal sine function $1/\sin z$ to $\mathcal{K}_x(z)$, and the term $1/\sin(2x+1)z$ remains bounded. This principle will be applied to certain lacunary sequences $\{q_n : n \geq 1\}$, which maximize the reciprocal sine function $1/\sin z$, to obtain an effective upper bound of the function $1/\sin z$.

Lemma 2.1. *Let $k \geq 1$ be a small fixed integer, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^k, and $0 \neq z \in \mathbb{Z}$. Then*

$$\frac{1}{|\sin(\pi^{k+1}z)|} \ll \frac{1}{|\sin(\pi^{k+1}q_n)|}. \tag{6}$$

Proof. By the best approximation principle, see Lemma 10.6,

$$\left|m-\pi^k z\right| \geq \left|p_n-\pi^k q_n\right| \tag{7}$$

for any integer $z \leq q_n$. Hence,

$$\begin{aligned} \frac{1}{|\sin(\pi^{k+1}z)|} &= \frac{1}{|\sin(\pi m-\pi^{k+1}z)|} \\ &\leq \frac{1}{|\sin(\pi p_n-\pi^{k+1}q_n)|} \\ &= \frac{1}{|\sin(\pi^{k+1}q_n)|}, \end{aligned} \tag{8}$$

as $n \to \infty$. ∎

3 Upper Bound For $\left|1/\sin \pi^{k+1} z\right|$

As shown in Lemma 2.1, to estimate the upper bound of the function $1/|\sin \pi^{k+1} z|$ over the real numbers $z \in \mathbb{R}$, it is sufficient to fix $z = q_n$, and select a real number $x \in \mathbb{R}$ such that $q_n \asymp x$. This idea is demonstrated below for small integer parameter $k \geq 1$.

Lemma 3.1. *Let $k \geq 1$ be a small fixed integer, let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^k, and define the associated sequence*

$$x_n = \left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}, \tag{9}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Then

(i) $\sin\left(2(x_n-1/2)+1)\pi^{k+1}q_n\right) = \pm 1.$

(ii) $\sin\left(2(x_n+1/2)+1)\pi^{k+1}q_n\right) = \pm\cos 2\pi^{k+1}q_n.$

(iii) $\left|\sin\left(2x_n+1/2)\pi^{k+1}q_n\right)\right| \geq 1-\dfrac{2\pi^2}{q_n^2},$ *as $n \to \infty$.*

Proof. Observe that the value x_n in (9) yields

$$\sin(2\pi^{k+1}q_n x_n) = \sin\left(2\pi^{k+1}q_n\left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}\right) = \sin\left(\frac{\pi}{2}\cdot w_n\right) = \pm 1, \tag{10}$$

and

$$\cos\left(2\pi^{k+1}q_n x_n\right) = \cos\left(2\pi^{k+1}q_n\left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}\right) = \cos\left(\frac{\pi}{2}\cdot w_n\right) = 0, \tag{11}$$

where

$$w_n = \left(\frac{2^{2+2v_2}+1}{2^{2v_2}}\right)q_n^2 \tag{12}$$

is an odd integer. (i) Routine calculations yield this:

$$\begin{aligned} \sin((2(x_n-1/2)+1)\pi^{k+1}q_n) &= \sin\left(2\pi^{k+1}q_n x_n\right) \\ &= \sin\left(2\pi^{k+1}q_n\left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}\right) \\ &= \sin\left(\frac{\pi}{2}\cdot w_n\right) \\ &= \pm 1, \end{aligned} \tag{13}$$

(ii) Routine calculations yield this:

$$\begin{aligned}\sin\left((2(x_n+1/2)+1)\pi^{k+1}q_n\right) &= \sin(2\pi^{k+1}q_nx_n+2\pi^{k+1}q_n) \\ &= \sin(2\pi^{k+1}q_nx_n)\cos(2\pi^{k+1}q_n) \\ &\quad +\cos(2\pi^{k+1}q_nx_n)\sin(2\pi^{k+1}q_n).\end{aligned} \tag{14}$$

Substituting (10) and (11) into (14) return

$$\sin\left(2(x_n+1/2)+1)\pi^{k+1}q_n\right)=\pm\cos\left(2\pi^{k+1}q_n\right). \tag{15}$$

(iii) This follows from the previous result:

$$\begin{aligned}\left|\sin\left(2(x_n+1/2)+1)\pi^{k+1}q_n\right)\right| &= \left|\pm\cos\left(2\pi^{k+1}q_n\right)\right| \\ &= \left|\pm\cos\left(2\pi p_n-2\pi^{k+1}q_n\right)\right| \\ &= \left|\pm\cos\left(2\pi\left(p_n-\pi^kq_n\right)\right)\right| \\ &\asymp 1,\end{aligned} \tag{16}$$

since the sequence of convergents satisfies $\left|p_n-\pi^kq_n\right|\leq 1/q_n$ as $n\to\infty$. ∎

Lemma 3.2. *Let $k\geq 1$ be a small fixed integer, let $\{p_n/q_n : n\geq 1\}$ be the sequence of convergents of the real number π^k, and define the associated sequence*

$$x_n=\left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}, \tag{17}$$

where $v_2=v_2(q_n)=\max\{v:2^v\mid q_n\}$ is the 2-adic valuation, and $n\geq 1$. Then

$$\left|\sin\left(2x^*+1)\pi^{k+1}q_n\right)\right|\asymp 1, \tag{18}$$

where $x^*\in[x_n-1/2,x_n+1/2]$ is an integer.

Proof. Consider the continuous function $f(x)=\left|\sin\left(2x+1)\pi^{k+1}q_n\right)\right|$ over the interval $[x_n-1/2,x_n+1/2]$. By Lemma 3.1, it has a local maximal at $x=x_n-1/2\in\mathbb{R}$:

$$\begin{aligned}\left|\sin\left((2x+1)\pi^{k+1}z\right)\right| &= \left|\sin\left((2(x_n-1/2)+1)\pi^{k+1}q_n\right)\right| \\ &= 1,\end{aligned} \tag{19}$$

and it has a local minimal at $x=x_n+1/2\in\mathbb{R}$:

$$\begin{aligned}\left|\sin\left((2x+1)\pi^{k+1}z\right)\right| &= \left|\sin\left((2(x_n+1/2)+1)\pi^{k+1}q_n\right)\right| \\ &\geq 1-\frac{2\pi^2}{q_n^2}.\end{aligned} \tag{20}$$

Since $f(x)$ is continuous over the interval $[x_n-1/2,x_n+1/2]$, it follows that

$$1-\frac{2\pi^2}{q_n^2}\leq\left|\sin\left((2x^*+1)\pi^{k+1}z\right)\right|\leq 1$$

for any integer $x^*\in[x_n-1/2,x_n+1/2]$ ∎

Theorem 3.1. *If $k\geq 1$ is a small fixed integer, and $z\in\mathbb{N}$ is a large integer, then,*

$$\left|\frac{1}{\sin\pi^{k+1}z}\right|\ll|z|. \tag{21}$$

Proof. Let $\{p_n/q_n : n\geq 1\}$ be the sequence of convergents of the real number π^k. Since the denominators sequence $\{q_n : n\geq 1\}$ maximize the reciprocal sine function $1/\sin\pi^{k+1}z$, see Lemma 2.1, it is sufficient to prove it for $z=q_n$. Define the associated sequence

$$x_n=\left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}, \tag{22}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Let $f(x) = \left|\sin\left((2x+1)\pi^{k+1}z\right)\right|$, and let $z = q_n$. The function $f(x)$ is bounded over the interval $[x_n - 1/2, x_n + 1/2]$, see Lemma 3.1. Replacing the integer parameters $x^* \in [x_n - 1/2, x_n + 1/2]$, $z = q_n$, and applying Lemma 3.1 return

$$\begin{aligned} \left|\sin\left((2x+1)\pi^{k+1}z\right)\right| &= \left|\sin\left((2x^*+1)\pi^{k+1}q_n\right)\right| \\ &\asymp 1. \end{aligned} \tag{23}$$

Rewrite the reciprocal sine function in terms of the harmonic kernel in Definition 2.1, and splice all these information together, to obtain

$$\begin{aligned} \left|\frac{1}{\sin \pi^{k+1}z}\right| &= \left|\frac{\mathcal{D}_x(\pi^{k+1}z)}{\sin((2x+1)\pi^{k+1}z)}\right| \\ &\ll |\mathcal{D}_{x^*}| \left|\frac{1}{\sin((2x^*+1)\pi^{k+1}q_n)}\right| \\ &\ll |x^*| \cdot 1 \\ &\ll |z| \end{aligned} \tag{24}$$

since $|z| \asymp x^* \asymp p_n \asymp q_n$, and the trivial estimate $|\mathcal{D}_x(z)| \ll |x|$. ∎

4 Upper Bound For $\left|1/\sin \pi^3 z\right|$

As shown in Lemma 2.1, to estimate the upper bound of the function $1/|\sin \pi^3 z|$ over the real numbers $z \in \mathbb{R}$, it is sufficient to fix $z = q_n$, and select a real number $x \in \mathbb{R}$ such that $q_n \asymp x$. This idea is demonstrated below.

Lemma 4.1. *Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^2, and define the associated sequence*

$$x_n = \left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^2}, \tag{25}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Then

(i) $\sin\left(2(x_n - 1/2) + 1)\pi^3 q_n\right) = \pm 1.$

(ii) $\sin\left(2(x_n + 1/2) + 1)\pi^3 q_n\right) = \pm \cos 2\pi^3 q_n.$

(iii) $\left|\sin\left(2x_n + 1/2)\pi^3 q_n\right)\right| \geq 1 - \dfrac{2\pi^2}{q_n^2}$, *as $x \to \infty$.*

Proof. Same as Lemma 3.1. ∎

Lemma 4.2. *Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^2, and define the associated sequence*

$$x_n = \left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^k}, \tag{26}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Then

$$\left|\sin\left(2x^* + 1)\pi^3 q_n\right)\right| \asymp 1, \tag{27}$$

where $x^* \in [x_n - 1/2, x_n + 1/2]$ is an integer.

Proof. Same as Lemma 3.2. ∎

Theorem 4.1. *Let $z \in \mathbb{N}$ be a large integer. Then,*

$$\left|\frac{1}{\sin \pi^3 z}\right| \ll |z|. \tag{28}$$

Proof. Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^2. Since the denominators sequence $\{q_n : n \geq 1\}$ maximize the reciprocal sine function $1/\sin \pi^3 z$, it is sufficient to prove it for $z = q_n$. Define the associated sequence

$$x_n = \left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^2}, \tag{29}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Replacing the integer parameters $x^* \in [x_n - 1/2, x_n + 1/2]$, $z = q_n$, and applying Lemma 4.2 return

$$\begin{aligned} \left|\sin\left((2x+1)\pi^3 z\right)\right| &= \left|\sin\left((2x^*+1)\pi^3 q_n\right)\right| \\ &\asymp 1, \end{aligned} \tag{30}$$

since the sequence of convergents satisfies $\left|p_n - \pi^2 q_n\right| \to 0$ as $n \to \infty$. Rewrite the reciprocal sine function in terms of the harmonic kernel in Definition 2.1, and splice all these information together, to obtain

$$\begin{aligned} \left|\frac{1}{\sin \pi^3 z}\right| &= \left|\frac{\mathcal{D}_x(\pi^3 z)}{\sin((2x+1)\pi^3 z)}\right| \\ &\ll |\mathcal{D}_{x^*}|\left|\frac{1}{\sin((2x^*+1)\pi^3 q_n)}\right| \\ &\ll |x^*| \cdot 1 \\ &\ll |z| \end{aligned} \tag{31}$$

since $|z| \asymp x^* \asymp p_n \asymp q_n$, and the trivial estimate $|\mathcal{D}_x(z)| \ll |x|$. ∎

5 Upper Bound For $1/\left|\sin \pi^4 z\right|$

As shown in Lemma 2.1, to estimate the upper bound of the function $1/|\sin \pi^4 z|$ over the real numbers $z \in \mathbb{R}$, it is sufficient to fix $z = q_n$, and select a real number $x \in \mathbb{R}$ such that $q_n \asymp x$. This idea is demonstrated below.

Lemma 5.1. *Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^3, and define the associated sequence*

$$x_n = \left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^3}, \tag{32}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Then

(i) $\sin\left(2(x_n - 1/2) + 1)\pi^4 q_n\right) = \pm 1$.

(ii) $\sin\left(2(x_n + 1/2) + 1)\pi^4 q_n\right) = \pm \cos 2\pi^3 q_n$.

(iii) $\left|\sin\left(2x_n + 1/2)\pi^4 q_n\right)\right| \geq 1 - \dfrac{2\pi^2}{q_n^2}$, *as $x \to \infty$.*

Proof. Same as Lemma 3.1. ∎

Lemma 5.2. *Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^3, and define the associated sequence*

$$x_n = \left(\frac{2^{2+2v_2}+1}{2^{2+2v_2}}\right)\frac{q_n}{\pi^3}, \tag{33}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Then

$$\left|\sin\left(2x^*+1)\pi^4 q_n\right)\right| \asymp 1, \tag{34}$$

where $x^* \in [x_n - 1/2, x_n + 1/2]$ is an integer.

Proof. Same as Lemma 3.2. ∎

Theorem 5.1. *Let $z \in \mathbb{N}$ be a large integer. Then,*

$$\left| \frac{1}{\sin \pi^4 z} \right| \ll |z|. \tag{35}$$

Proof. Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number π^3. Since the denominators sequence $\{q_n : n \geq 1\}$ maximize the reciprocal sine function $1/\sin \pi^4 z$, it is sufficient to prove it for $z = q_n$. Define the associated sequence

$$x_n = \left(\frac{2^{2+2v_2} + 1}{2^{2+2v_2}} \right) \frac{q_n}{\pi^3}, \tag{36}$$

where $v_2 = v_2(q_n) = \max\{v : 2^v \mid q_n\}$ is the 2-adic valuation, and $n \geq 1$. Replacing the integer parameters $x^* \in [x_n - 1/2, x_n + 1/2]$, $z = q_n$, and applying Lemma 5.2 return

$$\begin{aligned} |\sin((2x+1)z)| &= \left|\sin\left((2x^*+1)\pi^4 q_n\right)\right| \\ &\asymp 1, \end{aligned} \tag{37}$$

since the sequence of convergents satisfies $\left|p_n - \pi^3 q_n\right| \to 0$ as $n \to \infty$. Rewrite the reciprocal sine function in terms of the harmonic kernel in Definition 2.1, and splice all these information together, to obtain

$$\begin{aligned} \left| \frac{1}{\sin z} \right| &= \left| \frac{\mathcal{D}_x(z)}{\sin((2x+1)z)} \right| \\ &\ll |\mathcal{D}_{x^*}| \left| \frac{1}{\sin((2x^*+1)\pi^4 q_n)} \right| \\ &\ll |x^*| \cdot 1 \\ &\ll |z| \end{aligned} \tag{38}$$

since $|z| \asymp x^* \asymp p_n \asymp q_n$, and the trivial estimate $|\mathcal{D}_x(z)| \ll |x|$. ■

6 The Exponent Result For π^2

The last estimate for irrationality exponent of the first even zeta constant $\zeta(2) = \pi^2/6$ in Table 1 was derived from the algebraic properties of the cellular integral

$$A_2 + B_2\zeta(2) = \int_0^1 \int_0^1 \frac{x^h(1-x)^i y^j (1-y)^k}{(1-xy)^{i+j-l}} \frac{dx\, dy}{1-xy}, \tag{39}$$

where $A_2, B_2 \in \mathbb{Z}$ are integers. The analysis appears in [23], and an expanded version of the theory of cellular integrals is presented in [2, Section 5.3]. These techniques also rely on rational functions approximations of π^2 and the prime number theorem. Some relevant references are [23], [12], [11], [25], and [6] for an introduction to the rational approximations of π and the various proofs.

Since $\zeta(2)$ and π^2 have the same irrationality exponent, the analysis is done for the simpler number. The proof within is based on an effective upper bound of the reciprocal sine function over the sequence $\{q_n : n \geq 1\}$ as derived in Section 4.

Proof. (Theorem 1.1) Let $\varepsilon > 0$ be an arbitrary small number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the irrational number π^2. By Theorem 4.1, the reciprocal sine function has the upper bound

$$\left| \frac{1}{\sin(\pi^3 q_n)} \right| \ll q_n^{1+\varepsilon}. \tag{40}$$

Moreover, $\sin\left(\pi^3 q_n\right) = \sin\left(\alpha p - \pi^3 q_n\right)$ if and only if $\alpha p = \pi p_n$, where p_n is an integer. These information lead to the following relation.

$$\begin{aligned} \frac{1}{q_n^{1+\varepsilon}} &\ll \left|\sin\left(\pi^3 q_n\right)\right| \\ &\ll \left|\sin\left(\pi^3 q_n - \pi p_n\right)\right| \\ &\ll \left|\sin\left(\pi\left(\pi^2 q_n - p_n\right)\right)\right| \\ &\ll \left|\pi^2 q_n - p_n\right| \end{aligned} \tag{41}$$

for all sufficiently large p_n/q_n. Therefore,

$$\begin{aligned} \left|\pi^2 - \frac{p_n}{q_n}\right| &\gg \frac{1}{q^{2+\varepsilon}} \\ &= \frac{1}{q^{\mu(\pi^2)+\varepsilon}}. \end{aligned} \tag{42}$$

Clearly, this implies that the irrationality measure of the real number π^2 is $\mu(\pi^2) = 2$, see Definition 10.1. Quod erat demontrandum. ■

6.1 Numerical Data For The Exponent $\mu(\pi^2)$

The continued fraction is

$$\pi^2 = [9; 1, 6, 1, 2, 47, 1, 8, 1, 1, 2, 2, 1, 1, 8, 3, 1, 10, 5, 1, 3, 1, 2, 1, 1, 3, 15, \ldots]. \tag{43}$$

The sequence of convergents $\{p_n/q_n : n \geq 1\}$ is computed via the recursive formula provided in Lemma 10.1. The approximation $\mu_n(\pi^2)$ of the exponent in the inequality

$$\left|\pi^2 - \frac{p_n}{q_n}\right| \geq \frac{1}{q^{\mu_n(\pi^2)}} \tag{44}$$

are tabulated in Table 2 for the early stage of the sequence of convergents $p_n/q_n \longrightarrow \pi^2$.

7 The Exponent Result For π^3

The literature seems to offer no information on the irrationality exponent $\mu(\pi^3) \geq 2$ of the irrational number π^3.

Proof. (Theorem 1.2) Let $\varepsilon > 0$ be an arbitrary small number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the irrational number π^3. By Theorem 5.1, the reciprocal sine function has the upper bound

$$\left|\frac{1}{\sin\left(\pi^4 q_n\right)}\right| \ll q_n^{1+\varepsilon}. \tag{45}$$

Moreover, $\sin\left(\pi^4 q_n\right) = \sin\left(\alpha p - \pi^4 q_n\right)$ if and only if $\alpha p = \pi p_n$, where p_n is an integer. These information lead to the following relation.

$$\begin{aligned} \frac{1}{q_n^{1+\varepsilon}} &\ll \left|\sin\left(\pi^4 q_n\right)\right| \\ &\ll \left|\sin\left(\pi^4 q_n - \pi p_n\right)\right| \\ &\ll \left|\sin\left(\pi\left|\pi^3 q_n - p_n\right|\right)\right| \\ &\ll \left|\pi^3 q_n - p_n\right| \end{aligned} \tag{46}$$

for all sufficiently large p_n/q_n. Therefore,

$$\begin{aligned} \left|\pi^3 - \frac{p_n}{q_n}\right| &\gg \frac{1}{q^{2+\varepsilon}} \\ &= \frac{1}{q^{\mu(\pi^3)+\varepsilon}}. \end{aligned} \tag{47}$$

Table 2: Numerical Data For The Exponent $\mu(\pi^2)$

n	p_n	q_n	$\mu_n(\pi^2)$
1	9	1	
2	10	1	
3	69	7	2.253500
4	79	8	2.511334
5	227	23	3.236253
6	10748	1089	2.018434
7	10975	1112	2.321958
8	98548	9985	2.064841
9	109523	11097	2.090224
10	208071	21082	2.107694
11	525665	53261	2.098602
12	1259401	127604	2.071191
13	1785066	180865	2.049770
14	3044467	308469	2.172439
15	26140802	2648617	2.094189
16	81466873	8254320	2.021982
17	107607675	10902937	2.147582
18	1157543623	117283690	2.095357
19	5895325790	597321387	2.018903
20	7052869413	714605077	2.074380
21	27053934029	2741136618	2.023038
22	34106803442	3455741695	2.055226
23	95267540913	9652620008	2.032519
24	129374344355	13108361703	2.031079
25	224641885268	22760981711	2.054176
26	803300000159	81391306836	2.110031
27	12274141887653	1243630584251	2.020459
28	13077441887812	1325021891087	2.030798
29	25351583775465	2568652475338	2.036971
30	63780609438742	6462326841763	2.039154

Clearly, this implies that the irrationality measure of the real number π^3 is $\mu(\pi^3) = 2$, see Definition 10.1. Quod erat demontrandum. ■

7.1 Numerical Data For The Exponent $\mu(\pi^3)$

The continued fraction of the second odd power of π is

$$\pi^3 = [31; 159, 3, 7, 1, 13, 2, 1, 3, 1, 12, 2, 2, 4, 34, 2, 43, 3, 1, 3, 2, \ldots \ldots]. \tag{48}$$

The sequence of convergents $\{p_n/q_n : n \geq 1\}$ is computed via the recursive formula provided in Lemma 10.1. The approximation $\mu_n(\pi^3)$ of the exponent in the inequality

$$\left|\pi^3 - \frac{p_n}{q_n}\right| \geq \frac{1}{q^{\mu_n(\pi^3)}} \tag{49}$$

are tabulated in Table 3 for the early stage of the sequence of convergents $p_n/q_n \longrightarrow \pi^3$.

8 The Exponent Result For The Odd $\zeta(3)$

The last estimate for irrationality exponent of the odd zeta constant $\zeta(3)$ was derived from the algebraic properties of the cellular integral

$$A_3 + B_3\zeta(3) = \int_0^1 \int_0^1 \int_0^1 \frac{x^h(1-x)^l y^s z^j (1-z)^q}{(1-(1-xy)z)^{q+h-r}} \frac{dx\, dy\, dz}{(1-(1-xy)z}, \tag{50}$$

Table 3: Numerical Data For The Exponent $\mu(\pi^3)$

n	p_n	q_n	$\mu_n(\pi^3)$
1	31	1	
2	4930	159	2.225255
3	14821	478	2.342289
4	108677	3505	2.023480
5	123498	3983	2.320380
6	1714151	55284	2.096515
7	3551800	114551	2.047419
8	5265951	169835	2.126720
9	19349653	624056	2.022641
10	24615604	793891	2.189908
11	314736901	10150748	2.057364
12	654089406	21095387	2.059538
13	1622915713	52341522	2.083769
14	7145752258	230461475	2.184225
15	244578492485	7888031672	2.031550
16	496302737228	16006524819	2.160820
17	21585596193289	696168598889	2.048912
18	65253091317095	2104512321486	2.017121
19	86838687510384	2800680920375	2.049611
20	325769153848247	10506555082611	2.034434
21	738376995206878	23813791085597	2.026878
22	1064146149055125	34320346168208	2.020155
23	1802523144262003	58134137253805	2.057247
24	10076761870365140	324991032437233	2.020858
25	11879285014627143	383125169691038	2.021449
26	21956046884992283	708116202128271	2.049213
27	99703472554596275	3215589978204122	2.009654
28	121659519439588558	3923706180332393	2.050107
29	708001069752539065	22834120879866087	2.040614
30	2953663798449744818	95260189699796741	2.023276

where $A_3, B_3 \in \mathbb{Z}$ are integers. The analysis appears in [22], and an expanded version of the theory of cellular integrals is presented in [2, Section 5.3].

Table 4: Historical Data For $\mu(\zeta(3)$

Irrationality Measure Upper Bound	Reference	Year
$\mu(\zeta(3) \leq 13.41782$	Apery, [1]	1979
$\mu(\zeta(3) \leq 7.377956$	Hata, [12]	2000
$\mu(\zeta(3) \leq 5.513891$	Rhin, Viola, [22]	2001

There some relationship between the numbers $\zeta(3)$ and π^3, but is not clear if $\mu(\zeta(3)) = 2$. In [9], it was proved that $\zeta(3) = \alpha\pi^3$, where $\alpha \in \mathbb{R}$ is irrational. The numerical data in Table 5 suggests the followings.

Conjecture 8.1. *The irrationanlity exponent of the first odd zeta constant is* $\mu(\zeta(3)) = \mu(\alpha\pi^3) = 2$*, where* $\alpha \neq 0$ *is a unique irrational number.*

8.1 Numerical Data For The Exponent $\mu(\zeta(3))$

The continued fraction of the first odd zeta constant is

$$\zeta(3) = [1, 2, 0, 2, 0, 5, 6, 9, 0, 3, 1, 5, 9, 5, 9, 4, 2, 8, 5, 3, 9, 9, 7, 3, 8, \ldots], \tag{51}$$

listed as A002117 in OEIS. The sequence of convergents $\{p_n/q_n : n \geq 1\}$ is computed via the recursive formula provided in Lemma 10.1. The approximation $\mu_n(\zeta(3))$ of the exponent in the inequality

$$\left|\pi^3 - \frac{p_n}{q_n}\right| \geq \frac{1}{q^{\mu_n(\pi^3)}} \tag{52}$$

are tabulated in Table 5 for the early stage of the sequence of convergents $p_n/q_n \longrightarrow \pi^3$.

Table 5: Numerical Data For The Exponent $\mu(\zeta(3))$

n	p_n	q_n	$\mu_n(\zeta(3))$
1	1	1	
2	5	4	2.191267
3	6	5	3.843922
4	113	94	2.103378
5	119	99	2.222511
6	232	193	2.102718
7	351	292	2.302278
8	1636	1361	2.038931
9	1987	1653	2.309777
10	19519	16238	2.232018
11	177658	147795	2.084580
12	374835	311828	2.057472
13	552493	459623	2.065833
14	927328	771451	2.053480
15	1479821	1231074	2.072380
16	3886970	3233599	2.138006
17	28688611	23866267	2.041149
18	32575581	27099866	2.041133
19	61264192	50966133	2.114414
20	461424925	383862797	2.124760
21	5136938367	4273456900	2.022499
22	5598363292	4657319697	2.044823
23	10735301659	8930776597	2.025155
24	16333664951	13588096294	2.064764
25	59736296512	49695065479	2.014150
26	76069961463	63283161773	2.082353
27	516156065290	429394036117	2.006174
28	592226026753	492677197890	2.128367
29	18282936867880	15209709972817	2.007412
30	18875162894633	15702387170707	2.056200

9 The Exponent Result For π^k

The method used to prove the irrationality measure $\mu(\pi^k)$ of the number π^k is not based on rational functions approximations of π^k and the prime number theorem. Some relevant references are [23], [12], [17], [18], [8], [11], [25], and [6] for an introduction to the rational approximations of π and the various proofs.

The proof is based on an effective upper bound of the reciprocal sine function over the sequence of $\{q_n : n \geq 1\}$ derived in Section 3.

Proof. (Theorem 1.3) Let $\varepsilon > 0$ be an arbitrary small number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the irrational number π^k, with $k \geq 1$. By Theorem 3.1, the reciprocal sine function has the upper bound

$$\left|\frac{1}{\sin\left(\pi^{k+1}q_n\right)}\right| \ll q_n^{1+\varepsilon}. \tag{53}$$

Moreover, the relation $\sin\left(\pi^{k+1}q_n\right) = \sin\left(\alpha p - \pi^{k+1}q_n\right)$ is true if and only if $\alpha p = \pi p_n$, where p_n is an integer. These information lead to the following inequalities

$$\begin{aligned} \frac{1}{q_n^{1+\varepsilon}} &\ll \left|\sin\left(\pi^{k+1}q_n\right)\right| \\ &\ll \left|\sin\left(\pi^{k+1}q_n - \pi p_n\right)\right| \\ &\ll \left|\sin\left(\pi\left(\pi^k q_n - p_n\right)\right)\right| \\ &\ll \left|\pi^k q_n - p_n\right| \end{aligned} \tag{54}$$

for all sufficiently large p_n/q_n. Therefore,

$$\begin{aligned} \left|\pi^k - \frac{p_n}{q_n}\right| &\gg \frac{1}{q^{2+\varepsilon}} \\ &= \frac{1}{q^{\mu(\pi^k)+\varepsilon}}. \end{aligned} \tag{55}$$

Clearly, this implies that the irrationality measure of the real number π^k is $\mu(\pi^k) = 2$, see Definition 10.1. Quod erat faciendum. ■

10 Basic Diophantine Approximations Results

All the materials covered in this section are standard results in the literature, see [13], [16], [19], [21], [24], [26], et alii.

Lemma 10.1. *Let $\alpha = [a_0, a_1, \ldots, a_n, \ldots,]$ be the continue fraction of the real number $\alpha \in \mathbb{R}$. Then the following properties hold.*

(i) $p_n = a_n p_{n-1} + p_{n-2}$, $\quad p_{-2} = 0, \quad p_{-1} = 1,$ *for all* $n \geq 0$.

(ii) $q_n = a_n q_{n-1} + q_{n-2}$, $\quad q_{-2} = 1, \quad q_{-1} = 0,$ *for all* $n \geq 0$.

(iii) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, *for all* $n \geq 1$.

(iv) $\displaystyle\frac{p_n}{q_n} = a_0 + \sum_{0 \leq k < n} \frac{(-1)^k}{q_k q_{k+1}}$, *for all* $n \geq 1$.

10.1 Rationals And Irrationals Numbers Criteria

A real number $\alpha \in \mathbb{R}$ is called *rational* if $\alpha = a/b$, where $a, b \in \mathbb{Z}$ are integers. Otherwise, the number is *irrational*. The irrational numbers are further classified as *algebraic* if α is the root of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $\deg(f) > 1$, otherwise it is *transcendental*.

Lemma 10.2. *If a real number $\alpha \in \mathbb{R}$ is a rational number, then there exists a constant $c = c(\alpha)$ such that*

$$\frac{c}{q} \leq \left|\alpha - \frac{p}{q}\right| \tag{56}$$

holds for any rational fraction $p/q \neq \alpha$. Specifically, $c \geq 1/b$ if $\alpha = a/b$.

This is a statement about the lack of effective or good approximations for any arbitrary rational number $\alpha \in \mathbb{Q}$ by other rational numbers. On the other hand, irrational numbers $\alpha \in \mathbb{R} - \mathbb{Q}$ have effective approximations by rational numbers. If the complementary inequality $|\alpha - p/q| < c/q$ holds for infinitely many rational approximations p/q, then it already shows that the real number $\alpha \in \mathbb{R}$ is irrational, so it is sufficient to prove the irrationality of real numbers.

Lemma 10.3 (Dirichlet). *Suppose $\alpha \in \mathbb{R}$ is an irrational number. Then there exists an infinite sequence of rational numbers p_n/q_n satisfying*

$$0 < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2} \tag{57}$$

for all integers $n \in \mathbb{N}$.

Lemma 10.4. *Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction of a real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then*

$$0 < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2} \tag{58}$$

for all integers $n \in \mathbb{N}$.

This is standard in the literature, the proof appears in [13, Theorem 171], [24, Corollary 3.7], [14, Theorem 9], and similar references.

Lemma 10.5. *Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction of a real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then*

(i) $\displaystyle \frac{1}{2q_{n+1}q_n} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2},$ (ii) $\displaystyle \frac{1}{2a_{n+1}q_n^2} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2},$

for all integers $n \in \mathbb{N}$.

The recursive relation $q_{n+1} = a_{n+1}q_n + q_{n-1}$ links the two inequalities. Confer [20, Theorem 3.8], [14, Theorems 9 and 13], et alia. The proof of the best rational approximation stated below, appears in [21, Theorem 2.1], and [24, Theorem 3.8].

Lemma 10.6. *Let $\alpha \in \mathbb{R}$ be an irrational real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then, for any rational number $p/q \in \mathbb{Q}^\times$,*

(i) $|\alpha q_n - p_n| \leq |\alpha q - p|,$ (ii) $\displaystyle \left| \alpha - \frac{p_n}{q_n} \right| \leq \left| \alpha - \frac{p}{q} \right|,$

for all sufficiently large $n \in \mathbb{N}$ such that $q \leq q_n$.

10.2 Irrationalities Measures

The concept of measures of irrationality of real numbers is discussed in [26, p. 556], [5, Chapter 11], et alii. This concept can be approached from several points of views.

Definition 10.1. The irrationality measure $\mu(\alpha)$ of a real number $\alpha \in \mathbb{R}$ is the infimum of the subset of real numbers $\mu(\alpha) \geq 1$ for which the Diophantine inequality

$$\left| \alpha - \frac{p}{q} \right| \ll \frac{1}{q^{\mu(\alpha)}} \tag{59}$$

has finitely many rational solutions p and q. Equivalently, for any arbitrary small number $\varepsilon > 0$

$$\left| \alpha - \frac{p}{q} \right| \gg \frac{1}{q^{\mu(\alpha)+\varepsilon}} \tag{60}$$

for all large $q \geq 1$.

Theorem 10.1. ([7, Theorem 2]) *The map $\mu : \mathbb{R} \longrightarrow [2, \infty) \cup \{1\}$ is surjective function. Any number in the set $[2, \infty) \cup \{1\}$ is the irrationality measure of some irrational number.*

Example 10.1. Some irrational numbers of various irrationality measures.

(1) A rational number has an irrationality measure of $\mu(\alpha) = 1$, see [13, Theorem 186].

(2) An algebraic irrational number has an irrationality measure of $\mu(\alpha) = 2$, an introduction to the earlier proofs of Roth Theorem appears in [21, p. 147].

(3) Any irrational number has an irrationality measure of $\mu(\alpha) \geq 2$.

(4) A Champernowne number $\kappa_b = 0.123 \cdots b-1 \cdot b \cdot b+1 \cdot b+2 \cdots$ in base $b \geq 2$, concatenation of the b-base integers, has an irrationality measure of $\mu(\kappa_b) = b$. For example, the decimal number

$$\kappa_{10} = 0.12345678910111213141516 17 \cdots \tag{61}$$

has the irrationality measure of $\mu(\kappa_{10}) = 10$.

(5) A Mahler number $\psi_b = \sum_{n \geq 1} b^{-[\tau]^n}$ in base $b \geq 3$ has an irrationality measure of $\mu(\psi_b) = \tau$, for any real number $\tau \geq 2$, see [7, Theorem 2]. For example, the decimal number

$$\psi_{10} = \frac{1}{10^3} + \frac{1}{10^9} + \frac{1}{10^{27}} + \frac{1}{10^{81}} + \cdots \tag{62}$$

has the irrationality measure of $\mu(\psi_{10}) = 3$.

(6) A Liouville number $\ell_b = \sum_{n \geq 1} b^{-n!}$ parameterized by $b \geq 2$ has an irrationality measure of $\mu(\ell_b) = \infty$, see [13, p. 208]. For example, the decimal number

$$\ell_{10} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \cdots \tag{63}$$

has the irrationality measure of $\mu(\ell_{10}) = \infty$.

Definition 10.2. A measure of irrationality $\mu(\alpha) \geq 2$ of an irrational real number $\alpha \in \mathbb{R}^\times$ is a map $\psi : \mathbb{N} \to \mathbb{R}$ such that for any $p, q \in \mathbb{N}$ with $q \geq q_0$,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{\psi(q)}. \tag{64}$$

Furthermore, any measure of irrationality of an irrational real number satisfies $\psi(q) \geq \sqrt{5} q^{\mu(\alpha)} \geq \sqrt{5} q^2$.

Theorem 10.2. *For all integers $p, q \in \mathbb{N}$, and $q \geq q_0$, the number π satisfies the rational approximation inequality*

$$\left| \pi - \frac{p}{q} \right| \geq \frac{1}{q^{7.6063}}. \tag{65}$$

Proof. Consult the original source [25, Theorem 1]. ■

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