ON ZERO-SUM FREE SEQUENCES CONTAINED IN RANDOM SUBSETS OF FINITE CYCLIC GROUPS

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Abstract. Let C_n be a cyclic group of order n. A *sequence* S of length ℓ over C_n is a sequence $S = a_1 \cdot a_2 \cdot \ldots \cdot a_\ell$ of ℓ elements in C_n, where a repetition of elements is allowed and their order is disregarded. We say that S is a zero-sum sequence if $\Sigma_{i=1}^{\ell} a_i = 0$ and that S is a zero-sum free sequence if S contains no zero-sum subsequence.

Let R be a random subset of C_n obtained by choosing each element in C_n independently with probability p. Let N_{n-1-k}^R be the number of zero-sum free sequences of length $n-1-k$ in R. Also, let $N_{n-1-k,d}^R$ be the number of zero-sum free sequences of length $n-1-k$ having d distinct elements in R. We obtain the expectation of N_{n-1-k}^R and $N_{n-1-k,d}^R$ for $0 \le k \le \lfloor n/3 \rfloor$. We also show a concentration result on N_{n-1-k}^R and $N_{n-1-k,d}^R$ when k is fixed.

1. Introduction

Let C_n be a cyclic group of order n. A *sequence* S of length ℓ over C_n is a sequence

$$S = a_1 \cdot a_2 \cdot \ldots \cdot a_\ell$$

of ℓ elements in C_n, where a repetition of elements is allowed and their order is disregarded. We say that a sequence S over C_n is contained in $A \subset C_n$ if each element in S is contained in A. For $a \in C_n$, let

$$\mathsf{v}_a(S) = |\{i \in [1, \ell] \mid a_i = a\}|$$

be the *multiplicity* of a in S. A *subsequence* T of S is a sequence over C_n satisfying $\mathsf{v}_a(T) \le \mathsf{v}_a(S)$ for all $a \in C_n$. We say that S is a *zero-sum sequence* if $a_1 + a_2 + \ldots + a_\ell = 0$. A sequence is called *zero-sum free* if it contains no zero-sum subsequence.

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An initial study of zero-sum sequences dates back to 1961 when Erdős, Ginzburg, and Ziv [6] proved that $2n-1$ is the smallest positive integer ℓ such that every sequence of length ℓ over C_n has a zero-sum subsequence of length n. Since that time, zero-sum sequences over a finite group have actively studied in additive combinatorics. For more details, see a survey paper by Gao and Geroldinger [8]. Although earlier works often focused on finite abelian groups, an application to factorization theory and invariant theory pushed the object forward to non-abelian groups. The reader can refer to Geroldinger, Grynkiewicz, Zhong, and the second author [9] for recent progress with respect to factorization theory and to Cziszter, Domokos, and Szöllősi [3, 5] for connection with invariant theory.

In this paper, we focus on zero-sum free sequences over a cyclic group. Well-known problems about zero-sum free sequences over a finite group are to determine the maximum length of zero-sum free sequences, which is a combinatorial group invariant known as the *Davenport constant*, and to characterize the structure of zero-sum free sequences. Observe that the maximum length of all zero-sum free sequences over C_n is $n-1$. Also, we have that S is a zero-sum free sequence of length $n-1$ over C_n if and only if

$$S = \underbrace{g \boldsymbol{\cdot} g \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g}_{n-1}$$

for a generator $g \in C_n$. Gao [7] proved the following result on the structure of long zero-sum free sequences over C_n.

Theorem 1 (Theorem 4.3 in [8], Lemma 2.5 in [7]). *Let $n \geq 2$ and $0 \leq k \leq \lfloor \frac{n}{3} \rfloor$. Then S is a zero-sum free sequence of length $n-1-k$ over C_n if and only if*

$$S = \underbrace{g \boldsymbol{\cdot}\ g \boldsymbol{\cdot}\ \ldots \boldsymbol{\cdot}\ g}_{n-1-2k} \boldsymbol{\cdot}\ (x_1 g) \boldsymbol{\cdot}\ (x_2 g) \boldsymbol{\cdot}\ \ldots \boldsymbol{\cdot}\ (x_k g),$$

where g is a generator of C_n and $x_1, x_2, \ldots, x_k$ are positive integers such that

$$1 \leq x_1 \leq x_2 \leq \ldots \leq x_k \quad \textit{and} \quad x_1 + x_2 + \ldots + x_k \leq 2k. \tag{1}$$

Theorem 1 was generalized by Savchev and Chen [17] on the zero-sum free sequences of length at least $(n+1)/2$ over C_n. Theorem 1 and the result by Savchev and Chen were applied to the number of minimal zero-sum sequences of long length by Ponomarenko [15] and Cziszter, Domokos, and Geroldinger [4], respectively.

Remark that Theorem 4.3 in [8] only gives the statement in Theorem 1 from the left-hand side to the right-hand side. The proof from the right-hand side to the left-hand side is obvious since g is a generator of C_n and all subsequences T of S satisfy $\sigma(T) = \ell g \neq 0$ for some integer $0 < \ell < n$, where $\sigma(T)$ is the sum of all elements in T.

In this paper, we are interested in zero-sum free sequences of a given length contained in a random subset of C_n. Investigating how classical extremal results in *dense* environments transfer to *sparse* settings has become a deep line of research. For example, Roth's theorem on 3-term arithmetic progressions [16] was generalized for random subsets of integers [13], and there are recent generalizations about various classical extremal results by Schacht [18] and Conlon and Gowers [2].

Let R be a random subset of C_n obtained by choosing each element in C_n independently with probability p. Let N_{n-1-k} be the number of zero-sum free sequences of length $n-1-k$ over C_n. Also, let N^R_{n-1-k} be the number of zero-sum free sequences of length $n-1-k$ in R. The result on the expectation of N^R_{n-1-k} is as follows.

Theorem 2. *Let $n \geq 2$ and $0 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$. The expected number of zero-sum free sequences of length $n-1-k$ in a random subset R of C_n is*

$$\mathbb{E}\left(N^R_{n-1-k}\right) = \varphi(n)\left[p + \sum_{d=2}^{D} p^d \left(\sum_{j=\frac{(d-1)d}{2}}^{k} q(j, d-1)\right)\right],$$

where

- $D = \left\lfloor \frac{1+\sqrt{1+8k}}{2} \right\rfloor$,
- *$\varphi(n)$ denotes the number of generators in C_n, and*
- *$q(j, d-1)$ is the number of partitions of j having $d-1$ distinct parts.*

The number $q(j, d-1)$ can be computed in two ways: The first way is based on its generating function (see Section 2.1 for details). Second, we provide a recursive formula for computing $X_{k,d-1} = \sum_{j=(d-1)d/2}^{k} q(j, d-1)$ (see Section 2.2).

If k is fixed, then we can obtain a simpler statement as follows.

Corollary 3. *If k is fixed and $p = o(1)$ as $n \to \infty$, then*

$$\mathbb{E}\left(N^R_{n-1-k}\right) = p\varphi(n)\left(1 + O_k(p)\right),$$

where the constant in O_k depends only on k.

Next, we have a concentration result on N^R_{n-1-k} when k is fixed.

Theorem 4. *Let k be fixed, and let p be such that*

$$\frac{(\log n)^{2d} \log\log n}{n} \ll p \ll 1.$$

Then, asymptotically almost surely (a.a.s.)

$$N^R_{n-1-k} = p\varphi(n) + O_k\left(p^2\varphi(n) + \sqrt{p\varphi(n)}(\log n)^d\right),$$

where the constant in O_k depends only on k.

Moreover, we have a refined result. Let $N_{n-1-k,d}$ be the number of zero-sum free sequences of length $n-1-k$ having d distinct elements over C_n. Also, let $N^R_{n-1-k,d}$ be the number of zero-sum free sequences of length $n-1-k$ having d distinct elements contained in a random subset R of C_n. We show a concentration result on $N^R_{n-1-k,d}$.

Theorem 5. *If* $0 \le k \le \left\lfloor \frac{n}{3} \right\rfloor$ *and*

$$p \gg \frac{\log\log n}{n},$$

then we have that a.a.s.

$$p\varphi(n) - \omega\sqrt{p\varphi(n)} \le N^R_{n-1-k,1} \le p\varphi(n) + \omega\sqrt{p\varphi(n)},$$

where ω *tends to* ∞ *arbitrarily slowly as* $n \to \infty$.

Let $d \ge 2$. *If* k *is fixed and*

$$p \gg \frac{(\log n)^2(\log\log n)^{1/d}}{n^{1/d}},$$

then we have that a.a.s.

$$N^R_{n-1-k,d} = p^d\varphi(n)\left(\sum_{j=\frac{(d-1)d}{2}}^{k} q(j,d-1)\right) + O_k\left(\sqrt{p^d\varphi(n)}(\log n)^d\right).$$

The organization of this paper is as follows. In Section 2, we consider expectations and prove Theorem 2 and Corollary 3. Then, we deal with our concentration results and prove Theorems 4 and 5 in Section 3.

2. Expectation

In this section, we prove Theorem 2 and Corollary 3. Also, we provide a recursive formula to compute the important value $X_{k,d-1} = \sum_{j=(d-1)d/2}^{k} q(j,d-1)$ given in Theorem 2.

2.1. Proofs of Theorem 2 and Corollary 3.

It turns out that the number of distinct elements in a zero-sum free sequence plays an important role since each element in C_n is contained in a random set R with probability p. Recall that $N_{n-1-k,d}$ is the number of zero-sum free sequences of length $n-1-k$ having d distinct elements over C_n, and $N^R_{n-1-k,d}$ is the number of zero-sum free sequences over C_n of length $n-1-k$ having d distinct elements contained in a random set R.

Clearly, the expectation of $N^R_{n-1-k,d}$ is

$$\mathbb{E}\left(N^R_{n-1-k,d}\right) = p^d N_{n-1-k,d}.$$

Based on Theorem 1, the numbers $N_{n-1-k,d}$ and $N^R_{n-1-k,d}$ are related to the number of

$$(x_1, x_2, \dots, x_k)$$

satisfying that $x_1, x_2, \dots, x_k$ are positive integers such that (1) holds and the number of distinct $x_i \neq 1$ is $d-1$. With $x'_i := x_i - 1$, the number can be simplified as follows.

Definition 6. Let $X_{k,d}$ be the number of $(x'_1, x'_2, \dots x'_k)$ such that

$$0 \leq x'_1 \leq x'_2 \leq \dots \leq x'_k, \quad x'_1 + x'_2 + \dots + x'_k \leq k,$$

and the number of distinct positive x'_i is d.

Theorem 1 and Definition 6 give that

$$N_{n-1-k,d} = \varphi(n) X_{k,d-1}, \tag{2}$$

where $\varphi(n)$ is the number of generators in C_n. Therefore, the expectation of $N^R_{n-1-k,d}$ is

$$\mathbb{E}(N^R_{n-1-k,d}) = p^d \varphi(n) X_{k,d-1}. \tag{3}$$

From now on, we focus on estimating $X_{k,d-1}$. To this end, we use the definition of a partition of an integer. A *partition* of a positive integer k is a non-decreasing sequence whose sum equals k. A partition λ of k can be shortly expressed by

$$1^{r_1}\ 2^{r_2}\ \cdots\ t^{r_t}$$

meaning that

$$k = (\underbrace{1 + 1 + \ldots + 1}_{r_1}) + (\underbrace{2 + 2 + \ldots + 2}_{r_2}) + \ldots + (\underbrace{t + t + \ldots + t}_{r_t}).$$

If λ is a partition of k, then we denote $\lambda \vdash k$. Let $|\lambda| = k$ if $\lambda \vdash k$.

Let $q(k,d)$ be the number of partitions of k having d distinct parts. For example, all partitions of 7 are as follows:

- $1^7,\ 7^1,$
- $1^5 2^1,\ 1^3 2^2,\ 1^1 2^3,\ 1^4 3^1,\ 1^1 3^2,\ 2^2 3^1,\ 1^3 4^1,\ 3^1 4^1,\ 1^2 5^1,\ 2^1 5^1,\ 1^1 6^1,$
- $1^2 2^1 3^1,\ 1^1 2^1 4^1.$

We have that $q(7,1) = 2$, $q(7,2) = 11$, $q(7,3) = 2$, and $q(7,d) = 0$ for $d \geq 4$.

Recalling Definition 6, we have that $X_{k,d}$ is the same as the number of partitions λ of at most k having d distinct parts. Observe that if λ is counted for $X_{k,d}$, then

$$\frac{d(d+1)}{2} \leq |\lambda| \leq k \tag{4}$$

because λ contains parts with at least $1, 2, \ldots, d$. Thus, we have

$$X_{k,d} = \sum_{j=\frac{d(d+1)}{2}}^{k} q(j, d). \tag{5}$$

Remark that the number $q(j, d)$ can be found in A116608 of the on-line encyclopedia of integer sequences (OEIS), and it can be computed from its generating function

$$Q(x, t) = -1 + \prod_{i=1}^{\infty} \left(1 + \frac{tx^i}{1 - x^i}\right),$$

where

$$Q(x, t) = \sum_{j,d \geq 1} q(j, d) x^j t^d.$$

There are related results on $q(j, d)$. Kim [11] constructed a generating function with one variable for $q(j, d)$ when d is fixed. Also, Goh and Schmutz [10] obtained the asymptotic distribution of the number of distinct part sizes in a random integer partition. On the other hand, $X_{k,d}$ is not found in OEIS.

We are ready to prove Theorem 2.

Proof of Theorem 2. Trivially, the expected number of zero-sum free sequences of length $n - 1 - k$ with same elements in R is

$$\mathbb{E}\left(N^R_{n-1-k,1}\right) = \varphi(n)p.$$

Next, for $d \geq 2$, we infer that

$$\mathbb{E}\left(N^R_{n-1-k,d}\right) = p^d N_{n-1-k,d} \overset{(2)}{=} p^d \varphi(n) X_{k,d-1} \overset{(5)}{=} p^d \varphi(n) \left(\sum_{j=\frac{(d-1)d}{2}}^{k} q(j, d-1) \right).$$

Next we consider the range of d. If $X_{k,d-1}$ is positive, then (4) gives that

$$k \geq \frac{(d-1)d}{2}.$$

Hence, let D be the lagest integer d satisfying $(d-1)d/2 \leq k$, and then,

$$d \leq D = \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor.$$

This completes our proof of Theorem 2. □

Now we are ready to prove Corollary 3 using Theorem 2.

Proof of Corollary 3. For a fixed k, Theorem 2 gives that

$$\begin{aligned}\mathbb{E}\left(N^R_{n-1-k}\right) &= \varphi(n)\left[p+\sum_{d=2}^{D}p^d\left(\sum_{j=\frac{(d-1)d}{2}}^{k}q(j,d-1)\right)\right]\\ &= p\varphi(n)\left(1+\sum_{d=2}^{D}O_k(p^{d-1})\right)\\ &= p\varphi(n)\left(1+O_k(p)\right),\end{aligned}$$

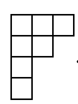
where the constant in O_k depends only on k, which completes the proof of Corollary 3. □

2.2. **Recursive formula for $X_{k,d}$.** Here, we give another way to compute the important value

$$X_{k,d-1}=\sum_{j=\frac{(d-1)d}{2}}^{k}q(j,d-1)$$

given in Theorem 2 using a recursive formula.

A partition of an integer can be illustrated by a *Young diagram* (also called a *Ferrers diagram*), which is a useful way to understand a partition in combinatorics. A Young diagram corresponding to a partition $\lambda \vdash k$ is a collection of left-justified rows of k boxes piled up in non-decreasing order of row lengths from parts. For example, the partition $1^2 2^1 3^1 \vdash 7$ corresponds to the Young diagram


.

Let $Y_{b,c,d}$ be the number of partitions of at most b with at most c parts having d distinct parts. Equivalently, $Y_{b,c,d}$ is the number of Young diagrams with at most b boxes, at most c rows, and d distinct rows. See Figure 1 (a).

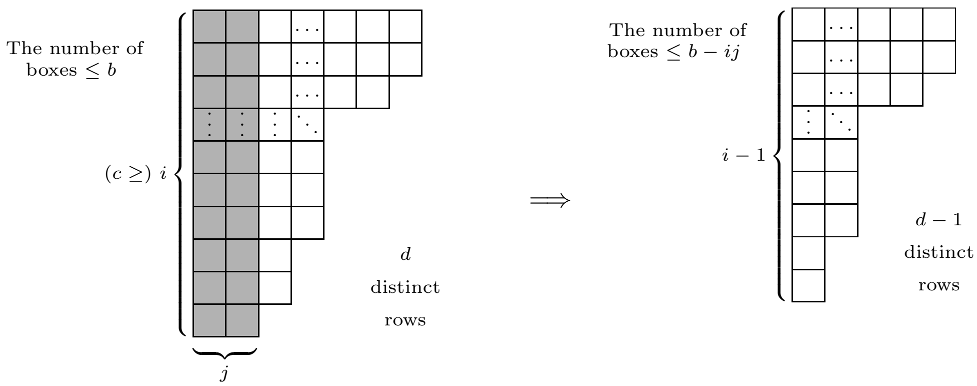
Note that $Y_{b,c,d}>0$ if and only if $b \geq \frac{d(d+1)}{2}$ and $c \geq d$, where the first inequaity follows from (4). Observe that

$$X_{k,d}=Y_{k,k,d}.$$

A recursive formula for $Y_{b,c,d}$ is as follows. (Hence we have a recursive formula for $X_{k,d}$.)

Lemma 7. *We have that, for $b \geq \frac{d(d+1)}{2}$ and $c \geq d \geq 2$,*

$$Y_{b,c,d}=\sum_{i=1}^{c}\sum_{j=1}^{\left\lfloor\frac{b-(d-1)d/2}{i}\right\rfloor}Y_{b-ij,i-1,d-1} \tag{6}$$



(a) $Y_{b,c,d}$ (b) $Y_{b-ij,i-1,d-1}$

FIGURE 1. Definition of $Y_{b,c,d}$ and the deletion process

and, for $b \geq 1$ *and* $c \geq 1$*,*

$$Y_{b,c,1} = \sum_{i=1}^{c} \left\lfloor \frac{b}{i} \right\rfloor. \tag{7}$$

Proof. We first show (6). We delete the gray retangle in Figure 1 from a Young diagram counted for $Y_{b,c,d}$, and then we have a Young diagram with at most $b - ij$ boxes, at most $i-1$ rows, and $d-1$ distinct rows.

We consider the ranges of i and j. Clearly, the range of i is $1 \leq i \leq c$. Then the remaining Young diagram after the deletion has $d-1$ distinct rows, and hence, it has at least $(d-1)d/2$ boxs. Thus,

$$ij + \frac{(d-1)d}{2} \leq b.$$

So the range of j is

$$1 \leq j \leq \left\lfloor \frac{b-(d-1)d/2}{i} \right\rfloor.$$

Next, we show (7). The number $Y_{b,c,1}$ is the same as the number of rectangles with at most b boxes and at most c rows. Let i and j be the numbers of rows and columns, respectively, of such a rectangle. Clearly, $1 \leq i \leq c$. Since $ij \leq b$, we have $j \leq \left\lfloor \frac{b}{i} \right\rfloor$. □

3. CONCENTRATION

Recall that $N^R_{n-1-k,d}$ be the number of zero-sum free sequences of length $n-1-k$ having d distinct elements in a random subset R. From (3), recall that

$$\mathbb{E}(N^R_{n-1-k,d}) = p^d \varphi(n) X_{k,d-1}.$$

From now on, we consider a concentration of $N^R_{n-1-k,d}$ and N^R_{n-1-k} using a graph theoretical approach called the Kim–Vu polynomial concentration result.

3.1. **Kim–Vu polynomial concentration result.** Let $\mathcal{H} = (V, E)$ be a weighted hypergraph with $V = [n] := \{0, 1, \ldots, n-1\}$. Recall that R is a random subset of $[n]$ obtained by selecting each $v \in [n]$ independently with probability p. Let $\mathcal{H}[R]$ be the sub-hypergraph of $\mathcal{H}$ induced on R, and we let Z be the sum of weights of hyperedges in $\mathcal{H}[R]$. Kim and Vu [12] obtained a result that provides a concentration of Z around its mean $\mathbb{E}(Z)$ with high probability. For more details, see Alon and Spencer [1]. To state the result, we need some definitions.

Definition 8. Let ℓ be the maximum size of hyperedges in $\mathcal{H}$, and let $A \subset [n]$ be such that $|A| \leq \ell$. We let

- $Z_A :=$ the sum of weights of hyperedges in $\mathcal{H}[R]$ containing A,
- $\mathbb{E}_A := \mathbb{E}\left(Z_A \mid A \subset R\right)$,
- $\mathbb{E}_i :=$ the maximum of $\mathbb{E}_A$ for $A \subset [n]$ with $|A| = i$,
- $$\mathbb{E}' := \max_{1 \leq i \leq \ell} \mathbb{E}_i \quad \text{and} \quad \mathbb{E}^* := \max\{\mathbb{E}', \mathbb{E}(Z)\}.$$

The concentration result by Kim and Vu [12] is as follows.

Theorem 9 (**Kim–Vu polynomial concentration inequality**). *With the notation as above, we have that, for each $\lambda > 1$,*

$$\Pr\left[|Z - \mathbb{E}(Z)| > a_\ell \sqrt{\mathbb{E}' \cdot \mathbb{E}^*} \lambda^\ell\right] < 2e^{-\lambda+2} n^{\ell-1},$$

where $a_\ell = 8^\ell (\ell!)^{1/2}$.

3.2. **Hypergraph and example.** For a given positive integer k, we define the hypergraph $\mathcal{H}_{n-1-k} = \mathcal{H}_{n-1-k}(C_n) = ([n], E)$ such that $a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1-k}$ is a zero-sum free sequence over C_n if and only if the corresponding set $\{b_1, b_2, \ldots, b_\ell\} = \{a_1, a_2, \ldots, a_{n-1-k}\}$, with $1 \leq \ell \leq n-1-k$, is contained in E. The weight of an hyperedge $\{b_1, b_2, \ldots, b_\ell\}$ of $\mathcal{H}_{n-1-k}$ is the number of zero-sum free sequences over C_n consisting of $b_1, b_2, \ldots, b_\ell$.

Then $\mathbb{E}_{d,A}$ defined above is the expected number of zero-sum free sequences of length $n-1-k$ having d distinct elements that contains $A \subset C_n$ and is contained in R under the condition that $A \subset R$. Also, for $1 \leq i \leq d$, let

$$\mathbb{E}_{d,i} = \max\{\mathbb{E}_{d,A} \mid A \subset C_n \text{ with } |A| = i\}.$$

We will estimate $\mathbb{E}_{d,A}$ and $\mathbb{E}_{d,i}$.

For an easier understanding, we give an example in C_8 before estimating $\mathbb{E}_{d,A}$ and $\mathbb{E}_{d,i}$ in a general C_n. Let $C_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and we consider the case where $k = 2$. In this case, the length of zero-sum free sequences is $n - 1 - k = 8-1-2 = 5$. All generators in C_8 are $1, 3, 5, 7$, and all possible (x_1, x_2) in Theorem 1 are $(1, 1), (1, 2), (1, 3)$, and $(2, 2)$. Thus, Theorem 1 gives that all zero-sum free sequences of length 5 over C_8 are

$$\begin{array}{llll} 1\cdot1\cdot1\cdot1\cdot1 & 3\cdot3\cdot3\cdot3\cdot3 & 5\cdot5\cdot5\cdot5\cdot5 & 7\cdot7\cdot7\cdot7\cdot7 \\ 1\cdot1\cdot1\cdot1\cdot2 & 3\cdot3\cdot3\cdot3\cdot6 & 5\cdot5\cdot5\cdot5\cdot2 & 7\cdot7\cdot7\cdot7\cdot6 \\ 1\cdot1\cdot1\cdot1\cdot3 & 3\cdot3\cdot3\cdot3\cdot1 & 5\cdot5\cdot5\cdot5\cdot7 & 7\cdot7\cdot7\cdot7\cdot5 \\ 1\cdot1\cdot1\cdot2\cdot2 & 3\cdot3\cdot3\cdot6\cdot6 & 5\cdot5\cdot5\cdot2\cdot2 & 7\cdot7\cdot7\cdot6\cdot6. \end{array}$$

Hence, the hypergraph $\mathcal{H}_5(C_8)$ has hyperedges as follows:

Hyperedge	$\{1\}$	$\{3\}$	$\{5\}$	$\{7\}$	$\{1,2\}$	$\{1,3\}$	$\{3,6\}$	$\{5,2\}$	$\{5,7\}$	$\{7,6\}$
Weight	1	1	1	1	2	2	2	2	2	2

As an example, we estimate $\mathbb{E}_{2,1}$ by considering $\mathbb{E}_{2,\{a\}}$ for $a \in C_8$. First, let $a = 1$. Note that our goal here is not to get the exact value of $\mathbb{E}_{2,\{1\}}$ but to obtain a uniform upper bound of $\mathbb{E}_{2,\{a\}}$ for all $a \in C_8$. For a generator g, there are several cases we need to deal with:

• **Case 1 ($a = 1 = g$):** Trivially, $g = 1$. Since $(x'_1, x'_2) = (0,1), (0,2)$, or $(1,1)$, we have $(x_1, x_2) = (1,2), (1,3)$, or $(2,2)$, and hence, all zero-sum free sequences of this case in C_8 are

$$1\cdot1\cdot1\cdot1\cdot(2\cdot1) \qquad 1\cdot1\cdot1\cdot1\cdot(3\cdot1) \qquad 1\cdot1\cdot1\cdot(2\cdot1)\cdot(2\cdot1).$$

Thus, the expected number of all zero-sum free sequences of this case in R is

$$X_{2,1}\cdot p.$$

• **Case 2 ($a = 1 = 2g$):** There is no such g, but we go forward to get a uniform upper bound. Since $a = 2g$, we have $x'_\ell = 1$ for some ℓ. Hence, the number of all zero-sum free sequences of this case in C_8 is at most $X_{2-1,1} + X_{2-1,0}$, where the first term is from the situation when all other x' are different from x'_ℓ and the second term is from the other situation. Thus, the expected number of all zero-sum free sequences of this case in R is at most

$$(X_{2-1,1} + X_{2-1,0})\, p.$$

• **Case 3 ($a = 1 = 3g$):** We infer that $g = 3$ and $(x'_1, x'_2) = (0,2)$. Hence, every zero-sum free sequences of this case in C_8 is

$$3\cdot3\cdot3\cdot3\cdot(3\cdot3) = 3\cdot3\cdot3\cdot3\cdot1.$$

The number of all zero-sum free sequences of this case over C_8 is

$$X_{2-2,1} + X_{2-2,0} \le X_{2-1,1} + X_{2-1,0}.$$

Thus, the expected number of all zero-sum free sequences of this case in R is at most

$$(X_{2-1,1} + X_{2-1,0})\, p.$$

Therefore,

$$\mathbb{E}_{2,\{1\}} \leq (X_{2,1} + 2\,(X_{2-1,1} + X_{2-1,0}))\, p.$$

By the same argument, for every $a \in C_8$, we have that $\mathbb{E}_{2,\{a\}}$ has the same upper bound, and hence,

$$\mathbb{E}_{2,1} \leq (X_{2,1} + 2\,(X_{2-1,1} + X_{2-1,0}))\, p.$$

In a similar way, one can estimate $\mathbb{E}_2$ in C_8, which gives $\mathbb{E}'$ and $\mathbb{E}^*$.

3.3. **Estimating $\mathbb{E}_{d,i}$.** We are ready to estimate $\mathbb{E}_{d,i}$ in a general C_n. First, we consider the case where $i = 1$.

Lemma 10. *For $2 \leq d \leq \left\lfloor \frac{1+\sqrt{1+8k}}{2} \right\rfloor$, we have that*

$$\begin{aligned} \mathbb{E}_{d,1} &\leq p^{d-1}\,(X_{k,d-1} + k(X_{k-1,d-1} + X_{k-1,d-2})) \quad \textit{and} \\ \mathbb{E}_{1,1} &= 1. \end{aligned}$$

Proof. Fix $a \in C_n$. We estimate the expected number of zero-sum free sequences

$$g \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g \boldsymbol{\cdot} (x_1 g) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (x_k g)$$

in R containing $\{a\}$ with two cases separately: for a generator g, the first case is when $a = g$, and the second case is when $a = jg$ for $2 \leq j \leq k+1$.

- **Case 1 ($a = g$):** The number of zero-sum free sequences over C_n containing $a = g$ is $X_{k,d-1}$. Hence, the expected number of zero-sum free sequences in R containing $a = g$ is

$$X_{k,d-1} \cdot p^{d-1}. \tag{8}$$

- **Case 2 ($a = jg$ for $2 \leq j \leq k+1$):** We first estimate the number of zero-sum free sequences over C_n containing $a = jg = x_\ell g$ for some ℓ. Since $x'_\ell = x_\ell - 1 \geq 1$, the remaining $x'_1, \ldots, x'_{\ell-1}, x'_{\ell+1}, \ldots, x'_k$ satisfy $\sum_{\substack{1 \leq i \leq k \\ i \neq \ell}} x'_i \leq k-1$. If $x'_1, \ldots, x'_{\ell-1}, x'_{\ell+1}, \ldots, x'_k$ are different from x'_ℓ, then the number of zero-sum free sequences over C_n is at most $X_{k-1,d-2}$. Otherwise, the number of zero-sum free sequences over C_n is at most $X_{k-1,d-1}$. Since $2 \leq j \leq k+1$, the expected number of zero-sum free sequences of this case in R is

$$k(X_{k-1,d-1} + X_{k-1,d-2})p^{d-1}. \tag{9}$$

From (8) and (9), we have that

$$\mathbb{E}_{d,1} \leq \max_{\{a\}} \mathbb{E}_{d,\{a\}} \leq p^{d-1}\,(X_{k,d-1} + k(X_{k-1,d-1} + X_{k-1,d-2}))\,,$$

which completes our proof of the lemma. □

Next, we consider a general i with $|A| = i$.

Lemma 11. *For* $1 \le i < d \le \left\lfloor \frac{1+\sqrt{1+8k}}{2} \right\rfloor$, *we have that*

$$\begin{aligned} \mathbb{E}_{d,i} &\le p^{d-i}\left[i\binom{k}{i-1}+\binom{k}{i}\right]\left(\sum_{j=0}^{i} X_{k-i+1,d-1-j}\right) \quad \textit{and} \\ \mathbb{E}_{d,d} &= 1. \end{aligned}$$

Proof. Fix $a_1, a_2, \dots, a_i \in C_n$. We estimate the number of zero-sum free sequences

$$g \cdot \ldots \cdot g \cdot (x_1 g) \cdot \ldots \cdot (x_k g)$$

containing $\{a_1, a_2, \dots, a_i\}$ with two cases separately: for a generator g, the first case is when $g = a_\ell$ for some ℓ, and the second case is when $g \neq a_\ell$ for all ℓ.

• **Case 1 (**$a_1 = g$ **and** $\{a_2, \dots, a_i\} = \{j_2 g, \dots, j_i g\}$ **for** $2 \le j_2 < \cdots < j_i \le k+1$**):** For fixed g and $j_2, \dots, j_i$, the number of zero-sum free sequences over C_n containing $\{a_1, a_2, \dots, a_i\} = \{g, j_2 g, \dots, j_i g\}$ is at most $X_{k-i+1,d-1} + \cdots + X_{k-i+1,d-i}$. The number of choices $(g, j_2, \dots, j_i)$ such that $g = a_\ell$ for some ℓ and $2 \le j_2 < \cdots < j_i \le k+1$ is at most $i\binom{k}{i-1}$. Hence, the expected number of zero-sum free sequences in R containing $\{a_1, a_2, \dots, a_i\} = \{g, j_2 g, \dots, j_i g\}$ is

$$i\binom{k}{i-1}(X_{k-i+1,d-1} + \cdots + X_{k-i+1,d-i})p^{d-i}. \tag{10}$$

• **Case 2 (**$\{a_1, a_2, \dots, a_i\} = \{j_1 g, j_2 g, \dots, j_i g\}$ **for** $2 \le j_1 < \cdots < j_i \le k+1$**):** For fixed $j_1 < \cdots < j_i$, we first consider the number of zero-sum free sequences over C_n containing $\{a_1, a_2, \dots, a_i\} = \{j_1 g, j_2 g, \dots, j_i g\}$. Without loss of generality, we let $x_1 = j_1, \dots, x_i = j_i$. Since $x'_\ell = x_\ell - 1 \ge 1$, the remaining $x'_{i+1}, \dots, x'_k$ satisfy $\sum_{i+1 \le \ell \le k} x'_\ell \le k - i$. The number of distinct $x'_{i+1}, \dots x'_k$ from $x'_1, \dots, x'_i$ are possibly $d-1$, $d-2, \dots,$ or $d-1-i$, and hence, the number of zero-sum free sequences over C_n containing $\{a_1, a_2, \dots, a_i\} = \{j_1 g, j_2 g, \dots, j_i g\}$ is at most $X_{k-i,d-1} + X_{k-i,d-2} + \cdots + X_{k-i,d-1-i}$. From the choices of $2 \le j_1 < \cdots < j_i \le k+1$, the expected number of zero-sum free sequences in R containing $\{a_1, \dots, a_i\}$ is at most

$$\binom{k}{i}(X_{k-i,d-1} + X_{k-i,d-2} + \cdots + X_{k-i,d-1-i})p^{d-i}. \tag{11}$$

From (10) and (11), we have that

$$\begin{aligned} \mathbb{E}_{d,i} &\le \max_{\{a_1,\dots,a_i\}} \mathbb{E}_{d,\{a_1,\dots,a_i\}} \\ &\le \left[i\binom{k}{i-1}+\binom{k}{i}\right]\left(\sum_{j=0}^{i} X_{k-i+1,d-1-j}\right) p^{d-i}, \end{aligned}$$

which completes our proof of the lemma. ☐

3.4. **Proofs of Theorems 4 and 5.**

Proof of Theorem 4. Let $X = N^R_{n-1-k}$. Under the assumption that k is fixed and $p \ll 1$, Corollary 3 gives that

$$\mathbb{E}(X) = p\varphi(n)\left(1 + O_k(p)\right).$$

Since k is fixed, Lemmas 10 and 11 yield that

$$\begin{aligned}
\mathbb{E}_1 &= \mathbb{E}_{1,1} + \mathbb{E}_{2,1} + \cdots + \mathbb{E}_{D,1} = O_k(1),\\
\mathbb{E}_2 &= \mathbb{E}_{2,2} + \mathbb{E}_{3,2} + \cdots + \mathbb{E}_{D,2} = O_k(1),\\
&\vdots\\
\mathbb{E}_D &= \mathbb{E}_{D,D} = 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}' &= \max_{1\le i\le D}\{\mathbb{E}_i\} = O_k(1) \text{ and}\\
\mathbb{E}^* &= \max\{\mathbb{E}', \mathbb{E}\} = p\varphi(n),
\end{aligned}$$

provided that $p\varphi(n) \gg 1$, i.e., $p \gg \frac{\log\log n}{n}$.

Set $\lambda = d\log n$, then $e^{-\lambda}n^{d-1} = 1/n = o(1)$, and hence, the Kim–Vu polynomial concentration result (Theorem 9) gives that a.a.s.

$$|X - \mathbb{E}(X)| = O_k\left(\sqrt{p\varphi(n)}(\log n)^d\right),$$

that is,

$$X = p\varphi(n) + O_k\left(p^2\varphi(n) + \sqrt{p\varphi(n)}(\log n)^d\right).$$

Note that $p\varphi(n) \gg \sqrt{p\varphi(n)}(\log n)^d$ is equivalent to $p \gg \frac{(\log n)^{2d}\log\log n}{n}$, and hence, our assumption on p is

$$\frac{(\log n)^{2d}\log\log n}{n} \ll p \ll 1.$$

Thus, we complete the proof of Theorem 4. ☐

For the proof of Theorem 5, we use the following version of Chernoff's bound.

Lemma 12 (**Chernoff's bound**, Corollary 4.6 in [14])**.** *Let X_i be independent random variables such that*

$$\Pr[X_i = 1] = p_i \quad \textit{and} \quad \Pr[X_i = 0] = 1 - p_i,$$

and let $X = \sum_{i=1}^n X_i$. For $0 < \lambda < 1$,

$$\Pr\Big[|X - \mathbb{E}(X)| \ge \lambda\mathbb{E}(X)\Big] \le 2\exp\Big(-\frac{\lambda^2}{3}\mathbb{E}(X)\Big).$$

We are ready to prove Theorem 5.

Proof of Theorem 5. Let $X = N^R_{n-1-k,d}$. First, we consider the case where $d = 1$. Observe that $X \sim Bin(\varphi(n), p)$, and hence, Chernoff's bound with $\lambda = \omega/\sqrt{p\varphi(n)}$ implies that a.a.s.

$$|X - p\varphi(n)| < \omega(p\varphi(n))^{1/2},$$

provided that $p\varphi(n) \gg 1$, i.e., $p \gg \frac{\log\log n}{n}$, where ω tends to ∞ arbitrarily slowly as $n \to \infty$.

Next we consider the case when $d \geq 2$. It follows from (3) that

$$\mathbb{E}(X) = p^d\varphi(n)X_{k,d-1}.$$

Lemma 11 gives that for a fixed k,

$$\begin{aligned} \mathbb{E}_{d,i} &= O_k(p^{d-i}) \text{ for } 1 \leq i \leq d-1 \text{ and} \\ \mathbb{E}_{d,d} &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}' &= \max_{1\leq i\leq d} \mathbb{E}_{d,i} = O_k(1) \text{ and} \\ \mathbb{E}^* &= O_k(\max\{1, p^d\varphi(n)\}) = O_k(p^d\varphi(n)), \end{aligned}$$

provided that $p^d\varphi(n) \gg 1$, i.e., $p \gg \left(\frac{\log\log n}{n}\right)^{1/d}$.

Set $\lambda = d\log n$, then $e^{-\lambda}n^{d-1} = 1/n = o(1)$, and hence, the Kim–Vu polynomial concentration result (Theorem 9) implies that a.a.s.

$$|X - \mathbb{E}(X)| \leq a_d(\mathbb{E}'\mathbb{E}^*)^{1/2}\lambda^d = O_k\left(p^{d/2}\varphi(n)^{1/2}(\log n)^d\right).$$

Note that $p^d\varphi(n) \gg \sqrt{p^d\varphi(n)}(\log n)^d$ is equivalent to $p \gg \frac{(\log n)^2(\log\log n)^{1/d}}{n^{1/d}}$, which is our assumption on p. This completes our proof of Theorem 5. □

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