# Flexible circuits in the $d$-dimensional rigidity matroid 

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#### Abstract

A bar-joint framework ( $G, p$ ) in $\mathbb{R}^{d}$ is rigid if the only edge-length preserving continuous motions of the vertices arise from isometries of $\mathbb{R}^{d}$. It is known that, when $(G, p)$ is generic, its rigidity depends only on the underlying graph $G$, and is determined by the rank of the edge set of $G$ in the generic $d$-dimensional rigidity matroid $\mathcal{R}_{d}$. Complete combinatorial descriptions of the rank function of this matroid are known when $d=1,2$, and imply that all circuits in $\mathcal{R}_{d}$ are generically rigid in $\mathbb{R}^{d}$ when $d=1,2$. Determining the rank function of $\mathcal{R}_{d}$ is a long standing open problem when $d \geq 3$, and the existence of non-rigid circuits in $\mathcal{R}_{d}$ for $d \geq 3$ is a major contributing factor to why this problem is so difficult. We begin a study of non-rigid circuits by characterising the non-rigid circuits in $\mathcal{R}_{d}$ which have at most $d+6$ vertices.


## 1 Introduction

A bar-joint framework $(G, p)$ in $\mathbb{R}^{d}$ is the combination of a finite graph $G=(V, E)$ and a realisation $p: V \rightarrow \mathbb{R}^{d}$. The framework is said to be rigid if the only edge-length preserving continuous motions of its vertices arise from isometries of $\mathbb{R}^{d}$, and otherwise it is said to be flexible. The study of the rigidity of frameworks has its origins in work of Cauchy and Euler on Euclidean polyhedra [5] and Maxwell [14] on frames.

Abbot [1] showed that it is NP-hard to determine whether a given $d$-dimensional framework is rigid whenever $d \geq 2$. The problem becomes more tractable for generic frameworks $(G, p)$ since we can linearise the problem and consider 'infinitesimal rigidity' instead. We define the rigidity matrix $R(G, p)$ as the $|E| \times d|V|$ matrix in which, for $e=v_{i} v_{j} \in E$,

[^0]the submatrices in row $e$ and columns $v_{i}$ and $v_{j}$ are $p\left(v_{i}\right)-p\left(v_{j}\right)$ and $p\left(v_{j}\right)-p\left(v_{i}\right)$, respectively, and all other entries are zero. We say that $(G, p)$ is infinitesimally rigid if $\operatorname{rankR}(\mathrm{G}, \mathrm{p})=\mathrm{d}|\mathrm{V}|-\binom{\mathrm{d}+1}{2}$. Asimow and Roth [2] showed that infinitesimal rigidity is equivalent to rigidity for generic frameworks (and hence that generic rigidity depends only on the underlying graph of the framework).

The d-dimensional rigidity matroid of a graph $G=(V, E)$ is the matroid $\mathcal{R}_{d}(G)$ on $E$ in which a set of edges $F \subseteq E$ is independent whenever the corresponding rows of $R(G, p)$ are independent, for some (or equivalently every) generic $p$. We denote the rank function of $\mathcal{R}_{d}(G)$ by $r_{d}$ and put $r_{d}(G)=r_{d}(E)$. We say that $G$ is: $\mathcal{R}_{d}$-independent if $r_{d}(G)=|E| ; \mathcal{R}_{d^{-}}$ rigid if $G$ is a complete graph on at most $d+1$ vertices or $r_{d}(G)=d|V|-\binom{d+1}{2}$; minimally $\mathcal{R}_{d}$-rigid if $G$ is $\mathcal{R}_{d}$-rigid and $\mathcal{R}_{d}$-independent; and an $\mathcal{R}_{d}$-circuit if $G$ is not $\mathcal{R}_{d}$-independent but $G-e$ is $\mathcal{R}_{d}$-independent for all $e \in E$.

It is not difficult to see that the 1-dimensional rigidity matroid of a graph $G$ is equal to its cycle matroid. Landmark results of Pollaczek-Geiringer [12, 15], and Lovász and Yemini [13] characterise independence and the rank function in $\mathcal{R}_{2}$. These results imply that every $\mathcal{R}_{d}$-circuit is rigid when $d=1,2$. This is no longer true when $d \geq 3$ (see Figures 1 and 2 below), and the existence of flexible circuits is a fundamental obstuction to obtaining a combinatorial characterisation of independence in $\mathcal{R}_{d}$.

Previous work on flexible $\mathcal{R}_{d}$-circuits has concentrated on constructions, see Tay [16], and Cheng, Sitharam and Streinu [6]. We will adopt a different approach: that of characterising the flexible $\mathcal{R}_{d}$-circuits in which the number of vertices is small compared to the dimension. To state our theorem we will have to define two families of graphs.

For $d \geq 3$ and $2 \leq t \leq d-1$, the graph $B_{d, t}$ is defined by putting $B_{d, t}=\left(G_{1} \cup G_{2}\right)-e$ where $G_{i} \cong K_{d+2}, G_{1} \cap G_{2} \cong K_{t}$ and $e \in E\left(G_{1} \cap G_{2}\right)$. The family $\mathcal{B}_{d, d-1}^{+}$consists of all graphs $B_{d, d-1}^{+}=\left(G_{1} \cup G_{2}\right)-\{e, f, g\}$ where: $G_{1} \cong K_{d+3}$ and $e, f, g \in E\left(G_{1}\right) ; G_{2} \cong K_{d+2}$ and $e \in E\left(G_{2}\right) ; G_{1} \cap G_{2} \cong K_{d-1} ; e, f, g$ do not all have a common end-vertex; if $\{f, g\} \subset$ $E\left(G_{1}\right) \backslash E\left(G_{2}\right)$ then $f, g$ do not have a common end-vertex. See Figure 1 for an illustration of the general construction and Figure 2 for specific examples.
Theorem 1. Suppose $G$ is a flexible $\mathcal{R}_{d}$-circuit with at most $d+6$ vertices. Then either
(a) $d=3$ and $G \in\left\{B_{3,2}\right\} \cup \mathcal{B}_{3,2}^{+}$or
(b) $d \geq 4$ and $G \in\left\{B_{d, d-1}, B_{d, d-2}\right\} \cup \mathcal{B}_{d, d-1}^{+}$.

A recent preprint of Jordán [11] characterises $\mathcal{R}_{d}$-rigid graphs with at most $d+4$ vertices. His characterisation implies that every $\mathcal{R}_{d}$-circuit with at most $d+4$ vertices is $\mathcal{R}_{d}$-rigid. Theorem 1 immediately gives the following characterisation of $\mathcal{R}_{d}$-rigid graphs with at most $d+6$ vertices in terms of $d$-tight subgraphs (which are defined in the next section).
Corollary 2. Let $G=(V, E)$ be a graph with $|V| \leq d+6$. Then $G$ is $\mathcal{R}_{d}$-rigid if and only if $G$ has a d-tight, d-connected spanning subgraph $H$ such that $B_{d, d-1}, B_{d, d-2} \nsubseteq H$.


Figure 1: $B_{d, d-1}$ on the left, $B_{d, d-2}$ in the middle and $B_{d, d-1}^{+}$on the right.


Figure 2: $B_{3,2}$ on the left, $B_{4,2}$ in the middle and $B_{3,2}^{+}$on the right.

## 2 Preliminary Lemmas

Given a vertex $v$ in a graph $G=(V, E)$, we will use $d_{G}(v)$ and $N_{G}(v)$ to denote the degree and neighbour set respectively of $v$. For a set $V^{\prime} \subseteq V$, we define by $N_{G}\left(V^{\prime}\right)=$ $\left(\bigcup_{v \in V^{\prime}} N_{G}(v)\right)-V^{\prime}$. We will use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree, respectively, in $G$, and $\operatorname{dist}_{\mathrm{G}}(\mathrm{x}, \mathrm{y})$ to denote the length of a shortest path between two vertices $x, y \in V$. We will suppress the subscript in these notations whenever the graph is clear from the context. The graph $G$ is $d$-sparse if $\left|E^{\prime}\right| \leq d\left|V^{\prime}\right|-\binom{d+1}{2}$ for all subgraphs $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ of $G$ with $\left|V^{\prime}\right| \geq d+2$. It is $d$-tight if it is $d$-sparse and has $d|V|-\binom{d+1}{2}$ edges.

We will need the following standard results from rigidity theory.
Lemma 3. [19, Lemma 11.1.3] Let $G=(V, E)$ be $\mathcal{R}_{d}$-independent with $|V| \geq d+2$. Then $r_{d}(G) \leq d|V|-\binom{d+1}{2}$.

Lemma 3 implies that every $\mathcal{R}_{d}$-independent graph is $d$-sparse. The characterisations of $\mathcal{R}_{d}$-independence when $d \leq 2$ show that the converse holds for these values of $d$. The existence of flexible $\mathcal{R}_{d}$-circuits implies that the converse fails for all $d \geq 3$.

A graph $G^{\prime}$ is said to be obtained from another graph $G$ by: a 0 -extension if $G=G^{\prime}-v$ for a vertex $v \in V\left(G^{\prime}\right)$ with $d_{G^{\prime}}(v)=d$; or a 1 -extension if $G=G^{\prime}-v+x y$ for a vertex $v \in V\left(G^{\prime}\right)$ with $d_{G^{\prime}}(v)=d+1$ and $x, y \in N(v)$.

Lemma 4. [19, Lemma 11.1.1, Theorem 11.1.7] Let $G$ be $\mathcal{R}_{d}$-independent and let $G^{\prime}$ be obtained from $G$ by a 0 -extension or a 1-extension. Then $G^{\prime}$ is $\mathcal{R}_{d}$-independent.

A vertex split of a graph $G=(V, E)$ is defined as follows: choose $v \in V, x_{1}, x_{2}, \ldots, x_{d-1} \in$ $N(v)$ and a partition $N_{1}, N_{2}$ of $N(v) \backslash\left\{x_{1}, x_{2}, \ldots, x_{d-1}\right\}$; then delete $v$ from $G$ and add two new vertices $v_{1}, v_{2}$ joined to $N_{1}, N_{2}$, respectively; finally add new edges $v_{1} v_{2}, v_{1} x_{1}, v_{2} x_{1}$, $v_{1} x_{2}, v_{2} x_{2}, \ldots, v_{1} x_{d-1}, v_{2} x_{d-1}$.

Lemma 5. [18, Proposition 10] Let $G$ be $\mathcal{R}_{d}$-independent and let $G^{\prime}$ be obtained from $G$ by a vertex split. Then $G^{\prime}$ is $\mathcal{R}_{d}$-independent.

Lemma 6. [17] Let $d \geq 1$ be an integer, $G$ be a graph and let $G^{\prime}$ be obtained from $G$ by adding a new vertex adjacent to every vertex of $G$. Then $G$ is $\mathcal{R}_{d}$-independent if and only if $G^{\prime}$ is $\mathcal{R}_{d+1}$-independent.

Lemma 6 immediately implies that $G$ is $\mathcal{R}_{d}$-rigid if and only if $G^{\prime}$ is $\mathcal{R}_{d+1}$-rigid and $G$ is an $\mathcal{R}_{d}$-circuit if and only if $G^{\prime}$ is an $\mathcal{R}_{d+1}$-circuit.

Lemma 7. [19, Lemma 11.1.9] Let $G_{1}, G_{2}$ be subgraphs of a graph $G$ and suppose that $G=G_{1} \cup G_{2}$.
(a) If $\left|V\left(G_{1}\right) \cap V\left(G_{2}\right)\right| \geq d$ and $G_{1}, G_{2}$ are $\mathcal{R}_{d}$-rigid then $G$ is $\mathcal{R}_{d}$-rigid.
(b) If $G_{1} \cap G_{2}$ is $\mathcal{R}_{d}$-rigid and $G_{1}, G_{2}$ are $\mathcal{R}_{d}$-independent then $G$ is $\mathcal{R}_{d}$-independent.
(c) If $\left|V\left(G_{1}\right) \cap V\left(G_{2}\right)\right| \leq d-1$, $u \in V\left(G_{1}\right)-V\left(G_{2}\right)$ and $v \in V\left(G_{2}\right)-V\left(G_{1}\right)$ then $r_{d}(G+u v)=r_{d}(G)+1$.

We also require some new lemmas. Lemma $7(\mathrm{~b})$ immediately implies that every $\mathcal{R}_{d^{-}}$ circuit $G=(V, E)$ is 2-connected and that, if $G-\{u, v\}$ is disconnected for some $u, v \in V$, then $u v \notin E$. Our first new lemma gives more structural information when $G-\{u, v\}$ is disconnected.

Given three graphs $G=(V, E), G_{1}=\left(V_{1}, E_{1}\right)$, and $G_{2}=\left(V_{2}, E_{2}\right)$, we say that $G$ is a 2-sum of $G_{1}$ and $G_{2}$ along a pair of vertices $u, v$ if $V_{1} \cap V_{2}=\{u, v\}, E_{1} \cap E_{2}=\{u v\}$, $V=V_{1} \cup V_{2}$ and $E=\left(E_{1} \cup E_{2}\right)-u v$.

Lemma 8. Suppose that $G=(V, E)$ is the 2-sum of $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$. Then $G$ is an $\mathcal{R}_{d}$-circuit if and only if $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-circuits.

Proof. We first prove necessity. Suppose that $G$ is an $\mathcal{R}_{d}$-circuit. If $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-independent then $G+u v$ is $\mathcal{R}_{d}$-independent by Lemma $7(\mathrm{~b})$, a contradiction since $G$ is an $\mathcal{R}_{d}$-circuit. If exactly one of $G_{1}$ and $G_{2}$, say $G_{1}$, is $\mathcal{R}_{d}$-independent then $u v$ belongs to the unique $\mathcal{R}_{d}$-circuit contained in $G_{2}$. We may extend $u v$ to a base of $E_{i}$, for $i=1,2$, and then apply Lemma $7(\mathrm{~b})$ to obtain $r_{d}(G+u v)=r_{d}\left(G_{1}\right)+r_{d}\left(G_{2}\right)-1$. Thus we have
$r_{d}(G)=r_{d}(G+u v)=\left|E_{1}\right|+\left|E_{2}\right|-2=|E|$, a contradiction since $G$ is an $\mathcal{R}_{d}$-circuit. Hence $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-dependent. Then the matroid circuit elimination axiom combined with the fact that $G$ is an $\mathcal{R}_{d}$-circuit imply that $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-circuits.

We next prove sufficiency. Suppose that $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-circuits. The circuit elimination axiom implies that $G$ is $\mathcal{R}_{d}$-dependent and hence that $G$ contains an $\mathcal{R}_{d}$-circuit $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$. Since $G_{i}-u v$ is $\mathcal{R}_{d}$-independent for $i=1,2$, we have $E^{\prime} \cap E_{i} \neq \emptyset$. This implies that $G^{\prime}$ is a 2-sum of $G_{1}^{\prime}=\left(G_{1} \cap G^{\prime}\right)+u v$ and $G_{2}^{\prime}=\left(G_{2} \cap G^{\prime}\right)+u v$. The proof of necessity in the previous paragraph now tells us that $G_{1}^{\prime}$ and $G_{2}^{\prime}$ are both $\mathcal{R}_{d}$-circuits. Since $G_{i}$ is an $\mathcal{R}_{d}$-circuit and $G_{i}^{\prime} \subseteq G_{i}$ we must have $G_{i}^{\prime}=G_{i}$ for $i=1,2$ and hence $G=G^{\prime}$.

The special cases of Lemma 8 when $d=2,3$ were proved by Berg and Jordán [3] and Tay [16], respectively.

We may apply Lemma 8 to the $\mathcal{R}_{3}$-circuit $K_{5}$ to deduce that $B_{3,2}$ is an $\mathcal{R}_{3}$-circuit. The same argument applied to the $\mathcal{R}_{4}$-circuit $K_{6}$ implies that $B_{4,2}$ is an $\mathcal{R}_{4}$-circuit. We can now use Lemma 6 to deduce that $B_{d, d-1}$ and $B_{d, d-2}$ are $\mathcal{R}_{d}$-circuits for all $d \geq 4$. Similarly, we may apply Lemma 8 to the $\mathcal{R}_{3}$-circuits $K_{5}$ and $K_{6}-\{f, g\}$, for two non-adjacent edges $f, g$, to deduce that $B_{3,2}^{+}$is an $\mathcal{R}_{3}$-circuit, and then use Lemma 6 to deduce that $B_{d, d-1}^{+}$is an $\mathcal{R}_{d}$-circuit unless $f$ or $g$ has an end-vertex in $V_{1} \cap V_{2}$. Our next result extends this to all graphs in $\mathcal{B}_{d, d-1}^{+}$.
Lemma 9. Every graph in $\mathcal{B}_{d, d-1}^{+}$is an $\mathcal{R}_{d^{-}}$-circuit.
Proof. Let $B_{d, d-1}^{+} \in \mathcal{B}_{d, d-1}^{+}$and suppose that $G_{1}=\left(V_{1}, E_{1}\right), G_{2}=\left(V_{2}, E_{2}\right)$, and $e, f, g$ are as in the definition of $\mathcal{B}_{d, d-1}^{+}$. Since $B_{d, d-1}^{+}$is $d$-tight and not $\mathcal{R}_{d}$-rigid (since it is not $d$-connected), it is $\mathcal{R}_{d}$-dependent.

We will complete the proof by showing that $B_{d, d-1}^{+}-h$ is $\mathcal{R}_{d}$-independent for all edges $h$ of $B_{d, d-1}^{+}$. If $h$ is incident with a vertex $x \in V_{2} \backslash V_{1}$, then we can reduce $B_{d, d-1}^{+}-h$ to $G_{1}-\{e, f, g\}$ by recursively deleting vertices of degree at most $d$ (starting from $x$ ). Since $G_{1}-\{e, f, g\}$ is $\mathcal{R}_{d}$-independent, Lemma 4 and the fact that edge deletion preserves independence now imply that $B_{d, d-1}^{+}-h$ is $\mathcal{R}_{d}$-independent. Thus we may assume that $h \in E_{2}$.

Suppose that $f, g, h$ do not have a common end-vertex. Choose a vertex $x \in V_{2} \backslash V_{1}$ and let $H=B_{d, d-1}^{+}-h-x+e$ be the graph obtained by applying a 1 -reduction at $x$. We can reduce $H$ to $G_{1}-\{f, g, h\}$ by recursively deleting vertices of degree at most $d$. Since $f, g, h$ do not have a common end-vertex, $G_{1}-\{f, g, h\}$ is $\mathcal{R}_{d}$-independent. We can now use Lemma 4 to deduce that $B_{d, d-1}^{+}-h$ is $\mathcal{R}_{d}$-independent.

Hence we may assume that $f, g, h$ have a common end-vertex $u$. The definition of $\mathcal{B}_{d, d-1}$ now implies that at least one of $f$ and $g$, say $f$, is an edge of $G_{1} \cap G_{2}$. Since $e, f, g$ do not have a common end-vertex, $e$ is not incident with $u$ and hence $e, g, h$ do not have a common end-vertex. We can now apply the argument in the previous paragraph with the roles of $e$ and $f$ reversed to deduce that $B_{d, d-1}^{+}-h$ is $\mathcal{R}_{d}$-independent.


Figure 3: Construction of $G$ in the proof of Lemma 10.

Lemma 10. Suppose $G=G_{1} \cup G_{2}$ where: $G_{1} \cap G_{2}=\emptyset ; G_{1}$ is minimally $\mathcal{R}_{d}$-rigid; $G_{2}=K_{3}$; each vertex of $G_{2}$ has $d-1$ neighbours in $G_{1}$; the set of all neighbours of the vertices of $G_{2}$ in $G_{1}$ has size at least d. Then $G$ is minimally $\mathcal{R}_{d}$-rigid.

Proof. Let $V\left(G_{2}\right)=\{x, y, z\}$. Since the set of all neighbours in $G_{1}$ of the vertices in $G_{2}$ has size at least $d$, we may suppose that some vertex $w \in G_{1}$ is a neighbour of $z$, but not $x$ or $y$. Then $G$ can be obtained from $G_{1}$ by a 0 -extension adding $x$ and edges from $x$ to its $d-1$ neighbours in $G_{1}$ as well as $w$, followed by two 1 -extensions adding $y$ and deleting $x w$ and then adding $z$ and deleting $y w$. (See Figure 3.) Hence $G$ is $\mathcal{R}_{d}$-independent by Lemma 4. Minimal rigidity follows by a simple edge count.

Lemma 11. (a) Every 6 -regular graph on 10 vertices is $\mathcal{R}_{4}$-independent.
(b) Every 12-regular graph on 15 vertices is $\mathcal{R}_{9}$-independent.

Proof. There are 216 -regular graphs on 10 vertices (see OEIS sequence A165627) and 17 12 -regular graphs on 15 vertices (by direct computation). That they are $\mathcal{R}_{d}$-independent for the stated dimensions can now be checked by any computer algebra systems.

Our final lemma is purely graph theoretic.
Lemma 12. Suppose that $G=(V, E)$ is a graph with $|V| \geq 11$, minimum degree two and maximum degree three. Then there exist vertices $x, y \in V$ with $d(x)=2, d(y)=3$ and $\operatorname{dist}(\mathrm{x}, \mathrm{y}) \geq 3$.

Proof. Assume $G=(V, E)$ is a counterexample to the lemma. Choose a vertex $v \in V$ of degree 2. Then there are at most 6 vertices at distance 1 or 2 from $v$. Hence there are at most 6 vertices of degree 3 . Now choose a vertex $u \in V$ of degree 3 . Each neighbour of $u$ is either a vertex of degree 2 which has at most one other neighbour of degree 2 or a vertex of degree 3 which has at most two other neighbours of degree 2 . Therefore we have at most 6 vertices of degree 2. If there does not exist 6 vertices of degree 3 then the number of
vertices of degree 3 is at most 4 , and we would have $|V| \leq 10$. Hence there are exactly 6 vertices of degree 3 and $v$ is adjacent to two vertices of degree 3 . Since $v$ is an arbitrary vertex of degree two, every vertex of degree 2 is adjacent to two vertices of degree 3 . Now choose $w$ to be a vertex of degree 3 at distance 2 from $v$ and a vertex $y \neq v$, of degree 2 , not adjacent to $w$. Then $\operatorname{dist}(\mathrm{w}, \mathrm{y}) \geq 3$.

## 3 Main results

We will prove Theorem 1 and then use it to obtain a lower bound on the number of edges in a flexible $\mathcal{R}_{d}$-circuit.

## Proof of Theorem 1

We proceed by contradiction. Suppose the theorem is false and choose a counterexample $G=(V, E)$ such that $d$ is as small as possible and, subject to this condition, $|V|$ is as small as possible. Since all $\mathcal{R}_{d}$-circuits are $\mathcal{R}_{d}$-rigid when $d \leq 2$, we have $d \geq 3$. Since $G$ is an $\mathcal{R}_{d}$-circuit, $G-v$ is $\mathcal{R}_{d}$-independent for all $v \in V$, and we can now use the fact that 0 -extension preserves $\mathcal{R}_{d}$-independence (by Lemma 4) to deduce that $\delta(G) \geq d+1$. Since $G$ is a flexible $\mathcal{R}_{d}$-circuit, $G$ is $d$-sparse by Lemma 3 .

Case 1. $d(v)=d+1$ for some $v \in V$.
Since $G$ does not contain the rigid $\mathcal{R}_{d}$-circuit $K_{d+2}$, $v$ has two non-adjacent neighbours $v_{1}, v_{2}$. If $H=G-v+v_{1} v_{2}$ was $\mathcal{R}_{d}$-independent then $G$ would be $\mathcal{R}_{d}$-independent by Lemma 4. Hence $H$ contains an $\mathcal{R}_{d}$-circuit $C$. The minimality of $G$ implies that $C$ is $\mathcal{R}_{d}$-rigid or $C=B_{d, d-1}$ which implies that $G \in \mathcal{B}_{d, d-1}^{+}$. The latter alternative contradicts the choice of $G$, hence $G$ contains a minimally $\mathcal{R}_{d}$-rigid subgraph, $C-v_{1} v_{2}$, with at least $d+2$ vertices. Let $G^{\prime}$ be a minimally $\mathcal{R}_{d}$-rigid subgraph of $G$ with at least $d+2$ vertices which is maximal with respect to inclusion, and put $X=V \backslash V\left(G^{\prime}\right)$. Then $1 \leq|X| \leq 4$. If some vertex $x \in X$ had at least $d$ neighbours in $G^{\prime}$, then we could create a larger $\mathcal{R}_{d}$-rigid subgraph by adding $x$ to $G^{\prime}$. Hence each $x \in X$ has at most $d-1$ neighbours in $G^{\prime}$ and since $G$ has minimum degree at least $d+1, x$ has at least 2 neighbours in $X$. Thus $3 \leq|X| \leq 4$.

Suppose $|X|=3$. Then $G[X]=K_{3}$ and $G^{\prime}=K_{d+2}-e$ for some edge $e$, or $G^{\prime}=$ $K_{d+3}-\{e, f, g\}$ for some edges $e, f, g$ which are not incident with the same vertex. If $\left|N_{G}(X)\right| \geq d$ then we could construct an $\mathcal{R}_{d}$-rigid spanning subgraph of $G$ by Lemma 10. Hence $\left|N_{G}(X)\right|=d-1$, and at least one edge, say $e$, with its end-vertices in $N_{G}(X)$ is missing from $G$, since otherwise $G$ would contain a copy of $K_{d+2}$. This gives $G=B_{d, d-1}$ when $G^{\prime}=K_{d+2}-e$, so we must have $G^{\prime}=K_{d+3}-\{e, f, g\}$. If $f, g$ are adjacent and neither of them have both their end-vertices in $N_{G}(X)$ then $G^{\prime}$ would contain one of the $\mathcal{R}_{d}$-circuits $K_{d+2}$ or $B_{d, d-2}$. Hence $G \in \mathcal{B}_{d, d-1}^{+}$.

It remains to consider the case $|X|=4$. Then $C_{4} \subseteq G[X] \subseteq K_{4}$ and $G^{\prime}=K_{d+2}-e$.
Claim 13. $N(X)=V\left(G^{\prime}\right)$.
Proof of claim. Suppose not. Let $Y=X \cup N_{G}(X)$. Then $G[Y] \subsetneq G$ and hence is independent. If $G\left[N_{G}(X)\right]$ was complete, then $G$ would be independent by Lemma 7(b), since $G=G^{\prime} \cup G[Y], G^{\prime}$ and $G[Y]$ are independent, and $G^{\prime} \cap G[Y]$ is complete. Hence both end-vertices of $e$ belong to $N_{G}(X)$. Choose a vertex $w \in V\left(G^{\prime}\right) \backslash N_{G}(X)$ and an edge $f \in G^{\prime}$ which is incident with $w$. Consider the graph $G^{\prime \prime}=G+e-f$.

Suppose $G^{\prime \prime}[Y]$ is $\mathcal{R}_{d}$-independent. Since $G^{\prime \prime}\left[N_{G}(X)\right]$ induces a complete graph, we can use Lemma 7 (b) as above to deduce that $G^{\prime \prime}$ is $\mathcal{R}_{d}$-independent. Then $G^{\prime}+e \cong K_{d+2}$ is the unique $\mathcal{R}_{d}$-circuit in $G^{\prime \prime}+f$ and hence $G=G^{\prime \prime}+f-e$ is $\mathcal{R}_{d}$-independent. This contradiction implies that $G^{\prime \prime}[Y]$ is $\mathcal{R}_{d}$-dependent.

Let $C$ be an $\mathcal{R}_{d}$-circuit in $G^{\prime \prime}[Y]$. Since $G^{\prime} \cong K_{d+2}-e, w$ has degree degree $d$ in $G^{\prime \prime}$ and hence $w \notin V(C)$. If $C=B_{d, d-1}$ then we can construct $G$ from $C$ by a 1-extension which adds $w$ and deletes $e$. This would imply that $G \in \mathcal{B}_{d, d-1}^{+}$and contradict the choice of $G$. Hence $C \neq B_{d, d-1}$ and the minimality of $G$ now implies that $C$ is rigid.

Since $G^{\prime}+e \cong K_{d+2}$ and $e \in E(C) \cap E\left(G^{\prime}+e\right)$, we may use the circuit elimination axiom to deduce that $(C-e) \cup G^{\prime}$ is $\mathcal{R}_{d^{\prime}}$-dependent. Since $(C-e) \cup G^{\prime} \subseteq G$, we must have $(C-e) \cup G^{\prime}=G$. This implies that $X$ and all edges of $G$ incident to $X$ are contained in $C$. Thus $N_{G}(X) \subset V(C)$. If $|N(X)| \geq d$, then $G=G^{\prime} \cup(C-e)$ would be rigid by Lemma 7 (a). Hence $\left|N_{G}(X)\right| \leq d-1$. If $\left|N_{G}(X)\right|=d-2$, then $C=K_{d+2}$ and $G=B_{d, d-2}$. Hence $\left|N_{G}(X)\right|=d-1$. Then $C=K_{d+3}-f-g$ for two non-adjacent edges $f, g$ and $G \in \mathcal{B}_{d, d-1}^{+}$.

Suppose $G[X]=C_{4}$. Since $\delta(G)=d+1$ and no vertex of $X$ has more than $d-1$ neighbours in $G^{\prime}$, each vertex of $X$ has degree $d+1$ in $G$. By Claim 13, we can choose $u \in X$ such that $\left|N(X-u) \cap V\left(G^{\prime}\right)\right| \geq d$. We can perform a 1-reduction of $G$ at $u$ which adds an edge between its two neighbours in $X$. We can now apply Lemma 10 to the resulting graph $H$ on $d+5$ vertices to deduce that $H$ is $\mathcal{R}_{d}$-rigid. This would imply that $G$ is $\mathcal{R}_{d}$-rigid, contradicting the choice of $G$.

Suppose $G[X]=C_{4}+f$. Then each vertex in $X$ has degree $d+1$ or $d+2$ in $G$ and the two vertices which are not incident to $f$ have degree $d+1$. If both of the vertices incident to $f$ have degree $d+2$ then $G$ has more than $d|V|-\binom{d+1}{2}$ edges, so cannot be a flexible $\mathcal{R}_{d}$-circuit. Hence we may choose an end-vertex $w$ of $f$ with degree $d+1$ in $G$. Construct $H$ from $G$ by performing a 1 -reduction at $w$ which adds an edge between its two non-adjacent neighbours in $X$. If all vertices in $X$ have degree $d+1$ in $G$, then we can reduce $H$ to $G^{\prime}$ by recursively deleting the remaining 3 vertices of $X$ in such a way that every deleted vertex has degree at most $d$. Since $G^{\prime}$ is $\mathcal{R}_{d}$-independent this would imply that $G$ is $\mathcal{R}_{d}$-independent. Hence we may assume that the end-vertex of $f$ distinct from $w$ has has degree $d+2$ in $G$. We can now apply Lemma 10 to deduce that either $H$ is $\mathcal{R}_{d}$-rigid


Figure 4: Construction of $\hat{G}$ in the proof of Case 1.
or $\left|N(X-w) \cap V\left(G^{\prime}\right)\right|=d-1$ and $H$ is $B_{d, d-1}$. The first alternative would imply that $G$ is $\mathcal{R}_{d}$-rigid, and the second alternative would imply that either $G$ is $\mathcal{R}_{d}$-rigid or $G \in \mathcal{B}_{d, d-1}^{+}$.

It remains to consider the subcase when $G[X]=K_{4}$. Then each vertex in $X$ has degree at least $d+1$, and at most two of them have degree $d+2$ otherwise $G$ would have more than $d|V|-\binom{d+1}{2}$ edges. Let $\hat{G}$ be obtained from $G$ by adding edges from vertices in $X$ to vertices in $G^{\prime}$ in such a way that $X$ has exactly two vertices of degree $d+1$ and exactly two vertices of degree $d+2$ in $\hat{G}$. We will show that $G$ is $\mathcal{R}_{d}$-independent by proving that $\hat{G}$ is minimally $\mathcal{R}_{d}$-rigid.

Since $N_{\hat{G}}(X)=V\left(G^{\prime}\right)$ by Claim 13, we may choose vertices $x, y \in X$ such that $x$ has degree $d+1, y$ has degree $d+2$ and some vertex $w \in V\left(G^{\prime}\right)$ is a neighbour of $x$ in $\hat{G}$ but not $y$. Let $X=\{x, y, z, t\}$ where $z$ has degree $d+2$ and $t$ has degree $d+1$ in $\hat{G}$. We can construct $\hat{G}$ from $G^{\prime}$ by first performing a 0 -extension which adds $y$ and all edges from $y$ to its neighbours in $G^{\prime}$ as well as to $w$, then add $z$ and then $t$ by successive 0 -extensions, and finally add $x$ by a 1 -extension which removes the edge $y w$. (See Figure 4.)

Since $G^{\prime}$ is minimally $\mathcal{R}_{d}$-rigid this implies that $\hat{G}$ is also minimally $\mathcal{R}_{d}$-rigid. This contradicts the fact that $G$ is an $\mathcal{R}_{d}$-circuit and completes the proof of Case 1.

Case 2. $\delta(G) \geq d+2$.
Choose $v \in V$ with $d(v)=\Delta(G)$. If $G-v$ was $\mathcal{R}_{d-1}$-independent then $G$ would be $\mathcal{R}_{d^{-}}$ independent by Lemma 6. This is impossible since $G$ is an $\mathcal{R}_{d}$-circuit. Hence $G-v$ contains an $\mathcal{R}_{d-1}$-circuit $C$. By the minimality of $d, C$ is $\mathcal{R}_{d-1}$-rigid or $C \in\left\{B_{d-1, d-2}, B_{d-1, d-3}\right\} \cup$ $\mathcal{B}_{d-1, d-2}^{+}$.

Claim 14. $G-v$ is $\mathcal{R}_{d-1}$-rigid.
Proof of Claim. Suppose $C \in\left\{B_{d-1, d-2}, B_{d-1, d-3}\right\} \cup \mathcal{B}_{d-1, d-2}^{+}$. Then $C$ has $d+4, d+5$ or $d+5$ vertices respectively, whereas $G-v$ hast at most $d+5$ vertices. If $C$ spans $G-v$ then the facts that $C$ contains vertices of degree $d+1$ and $\delta(G-v) \geq d+1$ imply that we can
add edges of $G-v$ to $C$ to make it $\mathcal{R}_{d-1}$-rigid. Hence we may suppose that $C=B_{d-1, d-2}$ and $(G-v) \backslash C$ has exactly one vertex $u$. Since $d_{G-v}(u) \geq d+1, G-v$ is $\mathcal{R}_{d-1}$-rigid unless all neighbours of $u$ belong to the same copy of $K_{d+1}-e$ in $B_{d-1, d-2}$. Suppose the second alternative occurs and let $H$ be the spanning subgraph of $G-v$ obtained by adding $u$ and all its incident edges to $B_{d-1, d-2}$. Since the other copy of $K_{d+1}-e$ in $B_{d-1, d-2}$ contains vertices of degree $d$ in $H$, and degree at least $d+1$ in $G-v$, we can now add an edge of $G-v$ to $H$ to make it $\mathcal{R}_{d-1}$-rigid.

Suppose $C$ is $\mathcal{R}_{d-1}$-rigid. Then $|V(C)| \geq d+1$. Let $H$ be a maximal $\mathcal{R}_{d-1}$-rigid subgraph of $G-v$. Suppose $H \neq G-v$ and note that $(G-v)-H$ has at most 4 vertices. Since each vertex of $(G-v)-H$ has at most $d-2$ neighbours in $H$ and $\delta(G-v) \geq d+1$ we have $(G-v)-H=K_{4}$ and $H=C=K_{d+1}$. We can now apply Lemma 10 to a minimally rigid spanning subgraph of $H$, and to each $K_{3}$ in $(G-v)-H$, in order to deduce that all vertices of $(G-v)-H$ are adjacent to the same set of $d-2$ vertices of $H$. This cannot occur since every vertex of $H$ which is not joined to a vertex of $G-v-H$ would have degree at most $d+1$ in $G$, contradicting the assumption of Case 2. Hence $H=G-v$ and $G-v$ is $\mathcal{R}_{d-1}$-rigid.

Let $(G-v)^{*}$, respectively $C^{*}$, be obtained from $G-v$, respectively $C$, by adding $v$ and all edges from $v$ to $G-v$, respectively $C$. Then $(G-v)^{*}$ is $\mathcal{R}_{d}$-rigid by Claim 14 and Lemma 6 , and, when $C$ is $\mathcal{R}_{d-1}$-rigid, $C^{*}$ is an $\mathcal{R}_{d}$-circuit again by Lemma 6.

Let $S$ be the set of all edges of $G^{*}$ which are not in $G$. Since $C^{*}$ is rigid or $C^{*} \in$ $\left\{B_{d, d-1}, B_{d, d-2}\right\} \cup \mathcal{B}_{d, d-1}^{+}, C^{*}$ is not an $\mathcal{R}_{d}$-circuit in $G$. Hence $E\left(C^{*}\right) \cap S \neq \emptyset$. If $S=\{f\}$ then $G=(G-v)^{*}-f$ would be $\mathcal{R}_{d}$-rigid since $(G-v)^{*}$ is $\mathcal{R}_{d}$-rigid and $f \in E\left(C^{*}\right)$. Hence $|S| \geq 2$ and $\Delta(G)=d(v) \leq|V|-3$. Let $\bar{G}$ be the complement of $G$.

Suppose $|V| \leq d+5$. Then $|V|=d+5$ and $G$ is $(d+2)$-regular. This implies that $\bar{G}$ is a 2-regular graph on $d+5 \geq 8$ vertices and we may choose two non-adjacent vertices $v_{1}, v_{2}$ with no common neighbours in $\bar{G}$. Then $v_{1} v_{2} \in E$ and $\left|N_{G}\left(v_{1}\right) \cap N_{G}\left(v_{2}\right)\right|=d-1$. We can use the facts that $G$ is $d$-sparse, $(d+2)$-regular and $|V|=d+5$ to deduce that $G / v_{1} v_{2}$ is $d$-sparse. (If not, then some set $X \subseteq V\left(G / v_{1} v_{2}\right)$ would induce more that $d|X|-\binom{d+1}{2}$ edges. Then $|X| \geq d+2$ and the fact that each vertex of $V\left(G / v_{1} v_{2}\right) \backslash X$ has degree at least $d+1$ will imply that $G / v_{1} v_{2}$ has more that $d\left|V\left(G / v_{1} v_{2}\right)\right|-\binom{d+1}{2}$ edges. This will contradict the fact that $G$ has at most $d|V(G)|-\binom{d+1}{2}$ edges.) Since $G / v_{1} v_{2}$ has no flexible $\mathcal{R}_{d}$-circuits (by the minimality of $G$ ), $G / v_{1} v_{2}$ is $\mathcal{R}_{d}$-independent. We can now use Lemma 5 to deduce that $G$ is $\mathcal{R}_{d}$-independent. Hence $|V|=d+6$. Since $\delta(G) \geq d+2$ and $\Delta(G) \leq d+3$ we have $\delta(\bar{G}) \geq 2$ and $\Delta(\bar{G}) \leq 3$.

Suppose $\delta(\bar{G})=2$ and $\Delta(\bar{G})=3$. Then we can find two vertices $x, y \in V$ with $d_{\bar{G}}(x)=2, d_{\bar{G}}(y)=3$ and $\operatorname{dist}_{\bar{G}}(x, y) \geq 3$ by Lemma 12. We can deduce as in the previous paragraph that $G / x y$ is $d$-sparse. If $G / x y$ contains an $\mathcal{R}_{d}$-circuit then $G / x y=B_{d, d-1}$ by the minimality and $d$-sparsity of $G$. Since $B_{d, d-1}$ has $d-3$ vertices of degree $d+4$ and six vertices of degree $d+1$, this would contradict the fact that $G$ has minimum degree $d+2$
(when $d \leq 6$ ) and maximum degree $d+3$ (when $d \geq 5$ ). Hence $G / v_{1} v_{2}$ is $\mathcal{R}_{d}$-independent. We can now use Lemma 5 to deduce that $G$ is $\mathcal{R}_{d}$-independent.

Next we consider the case when $\bar{G}$ is 2 -regular. Then $|S|=2$ and $G$ is ( $d+3$ )-regular. The fact that $(G-v)^{*}$ is $\mathcal{R}_{d}$-rigid and contains at least two $\mathcal{R}_{d}$-circuits ( $G$ and $C^{*}$ ) tells us that $\left|E\left((G-v)^{*}\right)\right| \geq d|V|-\binom{d+1}{2}+2$. Since $|E|=\left|E\left((G-v)^{*}\right)\right|-|S|$ and $G$ is $d$-sparse this gives

$$
\frac{(d+3)(d+6)}{2}=|E|=d|V|-\binom{d+1}{2}=\frac{d(d+11)}{2} .
$$

This implies that $d=9$ and $|V|=15$. We can now use Lemma 11(b) to deduce that $G$ is $\mathcal{R}_{9}$-independent, contradicting the fact that $G$ is an $\mathcal{R}_{9}$-circuit.

It remains to consider the final subcase when $\bar{G}$ is 3-regular. Then $|S|=3$ and $G$ is $(d+2)$-regular. Since $(G-v)^{*}$ is $\mathcal{R}_{d}$-rigid and contains at least two $\mathcal{R}_{d}$-circuits we have $|E(G)| \geq d|V|-\binom{d+1}{2}-1$. The fact that $G$ is $d$-sparse now gives

$$
\frac{(d+2)(d+6)}{2}=|E|=d|V|-\binom{d+1}{2}-\alpha=\frac{d(d+11)}{2}-\alpha
$$

for some $\alpha=0,1$. This implies that $\alpha=0$ and $d=4$. We can now use Lemma 11(a) to deduce that $G$ is $\mathcal{R}_{4}$-independent, contradicting the fact that $G$ is an $\mathcal{R}_{4}$-circuit.

We can use Theorem 1 to obtain a lower bound on the number of edges in a flexible $\mathcal{R}_{d}$-circuit. For $G=(V, E)$ and $X \subset V$, we use the notation $E(X, V \backslash X)$ to denote the set of edges with one endvertex in $X$ and one in $V \backslash X$.

Corollary 15. Suppose $G=(V, E)$ is a flexible $\mathcal{R}_{d}$-circuit. Then $|E| \geq d(d+9) / 2$, with equality if and only if $G=B_{d, d-1}$.

Proof. The corollary follows immediately from Theorem 1 if $|V| \leq d+6$. Since $\delta(G) \geq d+1$ we have $|E|>d(d+9) / 2$ when either $|V| \geq d+8$, or $|V|=d+7$ and $\delta(G) \geq d+2$. Hence we may assume that $|V|=d+7$ and $\delta(G)=d+1$. Choose a vertex $v$ with $d(v)=d+1$. Then $v$ has two non-adjacent neighbours $v_{1}, v_{2}$ since otherwise $G$ would contain the rigid $\mathcal{R}_{d}$-circuit $K_{d+2}$. Let $H=G-v+v_{1} v_{2}$. If $H$ was $\mathcal{R}_{d}$-independent then $G$ would be $\mathcal{R}_{d}$-independent by Lemma 4. Hence $H$ contains an $\mathcal{R}_{d}$-circuit $C$. If $C$ is flexible then Theorem 1 implies that $C \in\left\{B_{d, d-2}\right\} \cup \mathcal{B}_{d, d-1}^{+}$and hence $|E|>|E(C)|>d(d+9) / 2$. Thus we may assume that $C$ is $\mathcal{R}_{d}$-rigid. Then $C-v_{1} v_{2}$ is an $\mathcal{R}_{d}$-rigid subgraph with at least $d+2$ vertices. Let
$X=V(G) \backslash V(C)$. Then $1 \leq|X| \leq 5$. Since $\delta(G)=d+1$ and $|X| \leq 5$ we have

$$
\begin{aligned}
|E| & =\left|E\left(C-v_{1} v_{2}\right)\right|+|E(X)|+|E(X, V \backslash X)| \\
& \geq d|V \backslash X|-\binom{d+1}{2}+\binom{|X|}{2}+|X|(d+1-|X|+1) \\
& =d|V|-\binom{d+1}{2}-\frac{|X|(|X|-3)}{2} \\
& \geq \frac{d(d+13)}{2}-5 .
\end{aligned}
$$

We can now use the fact that $d \geq 3$ to deduce that $|E|>d(d+9) / 2$.

## 4 Closing Remarks

### 4.1 Generalised 2-sums

Let $G=(V, E), G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ be graphs. We say that $G$ is a $t$-sum of $G_{1}, G_{2}$ along an edge $e$ if $G=\left(G_{1} \cup G_{2}\right)-e, G_{1} \cap G_{2}=K_{t}$ and $e \in E_{1} \cap E_{2}$. We conjecture that Lemma 8 can be extended to $t$-sums.

Conjecture 16. Suppose that $G$ is a $t$-sum of $G_{1}, G_{2}$ along an edge e for some $2 \leq t \leq d+1$. Then $G$ is an $\mathcal{R}_{d}$-circuit if and only if $G_{1}, G_{2}$ are $\mathcal{R}_{d}$-circuits.

Our proof technique for Lemma 8 gives the following partial result.
Lemma 17. Let $G=(V, E), G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ be graphs such that $G$ is a $t$-sum of $G_{1}, G_{2}$ along an edge $e$ for some $2 \leq t \leq d+1$.
(a) If $G$ is an $\mathcal{R}_{d}$-circuit, then $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-circuits.
(b) If $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-circuits, then $G$ contains a unique $\mathcal{R}_{d}$-circuit $G^{\prime}$ and $E \backslash\left(E_{1} \cap E_{2}\right) \subseteq E\left(G^{\prime}\right)$.

Proof. (a) If $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-independent, then Lemma 7(b) implies that $G_{1} \cup G_{2}$ is $\mathcal{R}_{d}$-independent. This contradicts the facts that $G$ is a $\mathcal{R}_{d}$-circuit and $G \subseteq G_{1} \cup G_{2}$. If exactly one of $G_{1}$ and $G_{2}$, say $G_{1}$, is $\mathcal{R}_{d}$-independent then $e$ belongs to the unique $\mathcal{R}_{d^{-}}$ circuit in $G_{2}$ and Lemma $7(\mathrm{~b})$ gives $r_{d}(G)=r_{d}(G+e)=\left|E_{1}\right|+\left|E_{2}\right|-\binom{t}{2}-1=|E|$. This again contradicts the hypothesis that $G$ is an $\mathcal{R}_{d}$-circuit. Hence $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-dependent. Then the matroid circuit elimination axiom combined with the fact that $G$ is an $\mathcal{R}_{d}$-circuit imply that $G_{1}$ and $G_{2}$ are both $\mathcal{R}_{d}$-circuits.
(b) The circuit elimination axiom implies that $G$ is $\mathcal{R}_{d}$-dependent and hence that $G$ contains an $\mathcal{R}_{d}$-circuit $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$. Since $G_{i}-e$ is $\mathcal{R}_{d}$-independent for $i=1,2$, we have $E^{\prime} \backslash E_{i} \neq \emptyset$.

Let $G_{i}^{\prime}$ be obtained from $G_{i} \cap G^{\prime}$ by adding an edge between every pair of non-adjacent vertices in $V^{\prime} \cap V_{1} \cap V_{2}$. If $G_{i}^{\prime}$ is a proper subgraph of $G_{i}$ for $i=1,2$ then each $G_{i}^{\prime}$ is $\mathcal{R}_{d}$-independent and we can use Lemma $7(\mathrm{~b})$ to deduce that $G_{1}^{\prime} \cup G_{2}^{\prime}$ is $\mathcal{R}_{d}$-independent. This gives a contradiction since $G^{\prime} \subseteq G_{1}^{\prime} \cup G_{2}^{\prime}$. Relabelling if necessary we have $G_{1}^{\prime}=G_{1}$. If $G_{2}^{\prime} \neq G_{2}$ then we may deduce similarly that $G_{1}^{\prime} \cup G_{2}^{\prime}-e$ is independent. This again gives a contradiction since $G^{\prime} \subseteq G_{1}^{\prime} \cup G_{2}^{\prime}-e$. Hence $G_{2}^{\prime}=G_{2}$. It remains to show uniqueness. For $i=1,2$, let $B_{i}$ be a base of $\mathcal{R}_{d}\left(G_{i}\right)$ which contains $E\left(G_{1}\right) \cap E\left(G_{2}\right)$. Then $\left|B_{i}\right|=\left|E_{i}\right|-1$ and Lemma 7(b) gives

$$
r_{d}(G)=r_{d}\left(G_{1} \cup G_{2}-e\right)=r_{d}\left(G_{1} \cup G_{2}\right)=\left|B_{1}\right|+\left|B_{2}\right|-\binom{t}{2}=|E|-1
$$

Hence, $G$ contains a unique $\mathcal{R}_{d}$-circuit.
We can also use a result of Connelly [8] to deduce that Conjecture 16 holds when $t=d+1$ and $G_{1}, G_{2}$ are both globally rigid in $\mathbb{R}^{d}$.

### 4.2 Highly connected flexible circuits

Bolker and Roth [4] determined $r_{d}\left(K_{s, t}\right)$ for all complete bipartite graphs $K_{s, t}$. Their result implies that $K_{d+2, d+2}$ is a $(d+2)$-connected $\mathcal{R}_{d}$-circuit for all $d \geq 3$ and is flexible when $d \geq 4$, see [9, Theorem 5.2.1]. We know of no $(d+3)$-connected flexible $\mathcal{R}_{d}$-circuits and it is tempting to conjecture that they do not exist.

For the case when $d=3$, Tay [16] gives examples of 4 -connected flexible $\mathcal{R}_{3}$-circuits and Jackson and Jordán [10] conjecture that all 5-connected $\mathcal{R}_{3}$-circuits are rigid. An analogous statement has recently been verified for circuits in the closely related $C_{2}^{1}$-cofactor matroid by Clinch, Jackson and Tanigawa [7].

### 4.3 Extending Theorem 1

We saw in the previous subsection that $K_{d+2, d+2}$ is a flexible $\mathcal{R}_{d}$-circuit with $2 d+4$ vertices for all $d \geq 4$. We can use Lemma 6 to obtain a smaller flexible $\mathcal{R}_{d}$-circuit: we can recursively apply the coning operation to the flexible $\mathcal{R}_{4}$-circuit $K_{6,6}$ to obtain a flexible $\mathcal{R}_{d}$-circuit on $d+8$ vertices. This suggests that it may be difficult to extend Theorem 1 to graphs on $d+8$ vertices, but it is conceivable that all flexible $\mathcal{R}_{d}$-circuits on $d+7$ vertices have the form $\left(G_{1} \cup G_{2}\right)-S$ where $G_{i} \in\left\{K_{d+2}, K_{d+3}, K_{d+4}\right\}, G_{1} \cap G_{2} \in\left\{K_{d-3}, K_{d-2}, K_{d-1}\right\}$ and $S$ is a suitably chosen set of edges.

For the case when $d=3$, Tay [16] gives examples of 3 -connected flexible $\mathcal{R}_{3}$-circuits with 13 vertices but it is possible that all flexible circuits on at most 12 vertices can be obtained by taking 2 -sums of rigid circuits on at most 9 vertices.

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