# Flexible circuits in the d-dimensional rigidity matroid

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#### Abstract

A bar-joint framework (G, p) in  $\mathbb{R}^d$  is rigid if the only edge-length preserving continuous motions of the vertices arise from isometries of  $\mathbb{R}^d$ . It is known that, when (G, p) is generic, its rigidity depends only on the underlying graph G, and is determined by the rank of the edge set of G in the generic d-dimensional rigidity matroid  $\mathcal{R}_d$ . Complete combinatorial descriptions of the rank function of this matroid are known when d=1,2, and imply that all circuits in  $\mathcal{R}_d$  are generically rigid in  $\mathbb{R}^d$  when d=1,2. Determining the rank function of  $\mathcal{R}_d$  is a long standing open problem when  $d\geq 3$ , and the existence of non-rigid circuits in  $\mathcal{R}_d$  for  $d\geq 3$  is a major contributing factor to why this problem is so difficult. We begin a study of non-rigid circuits by characterising the non-rigid circuits in  $\mathcal{R}_d$  which have at most d+6 vertices.

### 1 Introduction

A bar-joint framework (G, p) in  $\mathbb{R}^d$  is the combination of a finite graph G = (V, E) and a realisation  $p: V \to \mathbb{R}^d$ . The framework is said to be rigid if the only edge-length preserving continuous motions of its vertices arise from isometries of  $\mathbb{R}^d$ , and otherwise it is said to be flexible. The study of the rigidity of frameworks has its origins in work of Cauchy and Euler on Euclidean polyhedra [5] and Maxwell [14] on frames.

Abbot [1] showed that it is NP-hard to determine whether a given d-dimensional framework is rigid whenever  $d \geq 2$ . The problem becomes more tractable for generic frameworks (G, p) since we can linearise the problem and consider 'infinitesimal rigidity' instead. We define the rigidity matrix R(G, p) as the  $|E| \times d|V|$  matrix in which, for  $e = v_i v_i \in E$ ,

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the submatrices in row e and columns  $v_i$  and  $v_j$  are  $p(v_i) - p(v_j)$  and  $p(v_j) - p(v_i)$ , respectively, and all other entries are zero. We say that (G, p) is infinitesimally rigid if rankR $(G, p) = d|V| - {d+1 \choose 2}$ . Asimow and Roth [2] showed that infinitesimal rigidity is equivalent to rigidity for generic frameworks (and hence that generic rigidity depends only on the underlying graph of the framework).

The d-dimensional rigidity matroid of a graph G = (V, E) is the matroid  $\mathcal{R}_d(G)$  on E in which a set of edges  $F \subseteq E$  is independent whenever the corresponding rows of R(G, p) are independent, for some (or equivalently every) generic p. We denote the rank function of  $\mathcal{R}_d(G)$  by  $r_d$  and put  $r_d(G) = r_d(E)$ . We say that G is:  $\mathcal{R}_d$ -independent if  $r_d(G) = |E|$ ;  $\mathcal{R}_d$ -rigid if G is a complete graph on at most d+1 vertices or  $r_d(G) = d|V| - \binom{d+1}{2}$ ; minimally  $\mathcal{R}_d$ -rigid if G is  $\mathcal{R}_d$ -rigid and  $\mathcal{R}_d$ -independent; and an  $\mathcal{R}_d$ -circuit if G is not  $\mathcal{R}_d$ -independent but G - e is  $\mathcal{R}_d$ -independent for all  $e \in E$ .

It is not difficult to see that the 1-dimensional rigidity matroid of a graph G is equal to its cycle matroid. Landmark results of Pollaczek-Geiringer [12, 15], and Lovász and Yemini [13] characterise independence and the rank function in  $\mathcal{R}_2$ . These results imply that every  $\mathcal{R}_d$ -circuit is rigid when d=1,2. This is no longer true when  $d\geq 3$  (see Figures 1 and 2 below), and the existence of flexible circuits is a fundamental obstuction to obtaining a combinatorial characterisation of independence in  $\mathcal{R}_d$ .

Previous work on flexible  $\mathcal{R}_d$ -circuits has concentrated on constructions, see Tay [16], and Cheng, Sitharam and Streinu [6]. We will adopt a different approach: that of characterising the flexible  $\mathcal{R}_d$ -circuits in which the number of vertices is small compared to the dimension. To state our theorem we will have to define two families of graphs.

For  $d \geq 3$  and  $2 \leq t \leq d-1$ , the graph  $B_{d,t}$  is defined by putting  $B_{d,t} = (G_1 \cup G_2) - e$  where  $G_i \cong K_{d+2}$ ,  $G_1 \cap G_2 \cong K_t$  and  $e \in E(G_1 \cap G_2)$ . The family  $\mathcal{B}_{d,d-1}^+$  consists of all graphs  $B_{d,d-1}^+ = (G_1 \cup G_2) - \{e, f, g\}$  where:  $G_1 \cong K_{d+3}$  and  $e, f, g \in E(G_1)$ ;  $G_2 \cong K_{d+2}$  and  $e \in E(G_2)$ ;  $G_1 \cap G_2 \cong K_{d-1}$ ; e, f, g do not all have a common end-vertex; if  $\{f, g\} \subset E(G_1) \setminus E(G_2)$  then f, g do not have a common end-vertex. See Figure 1 for an illustration of the general construction and Figure 2 for specific examples.

**Theorem 1.** Suppose G is a flexible  $\mathcal{R}_d$ -circuit with at most d+6 vertices. Then either

(a) 
$$d = 3$$
 and  $G \in \{B_{3,2}\} \cup \mathcal{B}_{3,2}^+$  or

(b) 
$$d \ge 4$$
 and  $G \in \{B_{d,d-1}, B_{d,d-2}\} \cup \mathcal{B}_{d,d-1}^+$ .

A recent preprint of Jordán [11] characterises  $\mathcal{R}_d$ -rigid graphs with at most d+4 vertices. His characterisation implies that every  $\mathcal{R}_d$ -circuit with at most d+4 vertices is  $\mathcal{R}_d$ -rigid. Theorem 1 immediately gives the following characterisation of  $\mathcal{R}_d$ -rigid graphs with at most d+6 vertices in terms of d-tight subgraphs (which are defined in the next section).

**Corollary 2.** Let G = (V, E) be a graph with  $|V| \le d + 6$ . Then G is  $\mathcal{R}_d$ -rigid if and only if G has a d-tight, d-connected spanning subgraph H such that  $B_{d,d-1}, B_{d,d-2} \nsubseteq H$ .

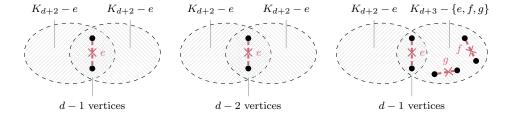


Figure 1:  $B_{d,d-1}$  on the left,  $B_{d,d-2}$  in the middle and  $B_{d,d-1}^+$  on the right.

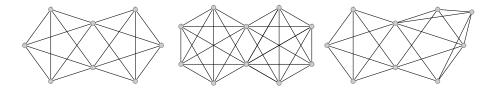


Figure 2:  $B_{3,2}$  on the left,  $B_{4,2}$  in the middle and  $B_{3,2}^+$  on the right.

### 2 Preliminary Lemmas

Given a vertex v in a graph G=(V,E), we will use  $d_G(v)$  and  $N_G(v)$  to denote the degree and neighbour set respectively of v. For a set  $V'\subseteq V$ , we define by  $N_G(V')=(\bigcup_{v\in V'}N_G(v))-V'$ . We will use  $\delta(G)$  and  $\Delta(G)$  to denote the minimum and maximum degree, respectively, in G, and  $\mathrm{dist}_G(x,y)$  to denote the length of a shortest path between two vertices  $x,y\in V$ . We will suppress the subscript in these notations whenever the graph is clear from the context. The graph G is d-sparse if  $|E'|\leq d|V'|-\binom{d+1}{2}$  for all subgraphs G'=(V',E') of G with  $|V'|\geq d+2$ . It is d-tight if it is d-sparse and has  $d|V|-\binom{d+1}{2}$  edges. We will need the following standard results from rigidity theory.

**Lemma 3.** [19, Lemma 11.1.3] Let G = (V, E) be  $\mathcal{R}_d$ -independent with  $|V| \ge d + 2$ . Then  $r_d(G) \le d|V| - \binom{d+1}{2}$ .

Lemma 3 implies that every  $\mathcal{R}_d$ -independent graph is d-sparse. The characterisations of  $\mathcal{R}_d$ -independence when  $d \leq 2$  show that the converse holds for these values of d. The existence of flexible  $\mathcal{R}_d$ -circuits implies that the converse fails for all  $d \geq 3$ .

A graph G' is said to be obtained from another graph G by: a  $\theta$ -extension if G = G' - v for a vertex  $v \in V(G')$  with  $d_{G'}(v) = d$ ; or a 1-extension if G = G' - v + xy for a vertex  $v \in V(G')$  with  $d_{G'}(v) = d + 1$  and  $x, y \in N(v)$ .

**Lemma 4.** [19, Lemma 11.1.1, Theorem 11.1.7] Let G be  $\mathcal{R}_d$ -independent and let G' be obtained from G by a 0-extension or a 1-extension. Then G' is  $\mathcal{R}_d$ -independent.

A vertex split of a graph G = (V, E) is defined as follows: choose  $v \in V, x_1, x_2, \ldots, x_{d-1} \in N(v)$  and a partition  $N_1, N_2$  of  $N(v) \setminus \{x_1, x_2, \ldots, x_{d-1}\}$ ; then delete v from G and add two new vertices  $v_1, v_2$  joined to  $N_1, N_2$ , respectively; finally add new edges  $v_1v_2, v_1x_1, v_2x_1, v_1x_2, v_2x_2, \ldots, v_1x_{d-1}, v_2x_{d-1}$ .

**Lemma 5.** [18, Proposition 10] Let G be  $\mathcal{R}_d$ -independent and let G' be obtained from G by a vertex split. Then G' is  $\mathcal{R}_d$ -independent.

**Lemma 6.** [17] Let  $d \geq 1$  be an integer, G be a graph and let G' be obtained from G by adding a new vertex adjacent to every vertex of G. Then G is  $\mathcal{R}_d$ -independent if and only if G' is  $\mathcal{R}_{d+1}$ -independent.

Lemma 6 immediately implies that G is  $\mathcal{R}_d$ -rigid if and only if G' is  $\mathcal{R}_{d+1}$ -rigid and G is an  $\mathcal{R}_d$ -circuit if and only if G' is an  $\mathcal{R}_{d+1}$ -circuit.

**Lemma 7.** [19, Lemma 11.1.9] Let  $G_1$ ,  $G_2$  be subgraphs of a graph G and suppose that  $G = G_1 \cup G_2$ .

- (a) If  $|V(G_1) \cap V(G_2)| \ge d$  and  $G_1, G_2$  are  $\mathcal{R}_d$ -rigid then G is  $\mathcal{R}_d$ -rigid.
- (b) If  $G_1 \cap G_2$  is  $\mathcal{R}_d$ -rigid and  $G_1, G_2$  are  $\mathcal{R}_d$ -independent then G is  $\mathcal{R}_d$ -independent.
- (c) If  $|V(G_1) \cap V(G_2)| \leq d-1$ ,  $u \in V(G_1) V(G_2)$  and  $v \in V(G_2) V(G_1)$  then  $r_d(G + uv) = r_d(G) + 1$ .

We also require some new lemmas. Lemma 7(b) immediately implies that every  $\mathcal{R}_{d}$ -circuit G = (V, E) is 2-connected and that, if  $G - \{u, v\}$  is disconnected for some  $u, v \in V$ , then  $uv \notin E$ . Our first new lemma gives more structural information when  $G - \{u, v\}$  is disconnected.

Given three graphs G = (V, E),  $G_1 = (V_1, E_1)$ , and  $G_2 = (V_2, E_2)$ , we say that G is a 2-sum of  $G_1$  and  $G_2$  along a pair of vertices u, v if  $V_1 \cap V_2 = \{u, v\}$ ,  $E_1 \cap E_2 = \{uv\}$ ,  $V = V_1 \cup V_2$  and  $E = (E_1 \cup E_2) - uv$ .

**Lemma 8.** Suppose that G = (V, E) is the 2-sum of  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ . Then G is an  $\mathcal{R}_d$ -circuit if and only if  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -circuits.

*Proof.* We first prove necessity. Suppose that G is an  $\mathcal{R}_d$ -circuit. If  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -independent then G + uv is  $\mathcal{R}_d$ -independent by Lemma 7(b), a contradiction since G is an  $\mathcal{R}_d$ -circuit. If exactly one of  $G_1$  and  $G_2$ , say  $G_1$ , is  $\mathcal{R}_d$ -independent then uv belongs to the unique  $\mathcal{R}_d$ -circuit contained in  $G_2$ . We may extend uv to a base of  $E_i$ , for i = 1, 2, and then apply Lemma 7(b) to obtain  $r_d(G + uv) = r_d(G_1) + r_d(G_2) - 1$ . Thus we have

 $r_d(G) = r_d(G + uv) = |E_1| + |E_2| - 2 = |E|$ , a contradiction since G is an  $\mathcal{R}_d$ -circuit. Hence  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -dependent. Then the matroid circuit elimination axiom combined with the fact that G is an  $\mathcal{R}_d$ -circuit imply that  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -circuits.

We next prove sufficiency. Suppose that  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -circuits. The circuit elimination axiom implies that G is  $\mathcal{R}_d$ -dependent and hence that G contains an  $\mathcal{R}_d$ -circuit G' = (V', E'). Since  $G_i - uv$  is  $\mathcal{R}_d$ -independent for i = 1, 2, we have  $E' \cap E_i \neq \emptyset$ . This implies that G' is a 2-sum of  $G'_1 = (G_1 \cap G') + uv$  and  $G'_2 = (G_2 \cap G') + uv$ . The proof of necessity in the previous paragraph now tells us that  $G'_1$  and  $G'_2$  are both  $\mathcal{R}_d$ -circuits. Since  $G_i$  is an  $\mathcal{R}_d$ -circuit and  $G'_i \subseteq G_i$  we must have  $G'_i = G_i$  for i = 1, 2 and hence G = G'.  $\square$ 

The special cases of Lemma 8 when d=2,3 were proved by Berg and Jordán [3] and Tay [16], respectively.

We may apply Lemma 8 to the  $\mathcal{R}_3$ -circuit  $K_5$  to deduce that  $B_{3,2}$  is an  $\mathcal{R}_3$ -circuit. The same argument applied to the  $\mathcal{R}_4$ -circuit  $K_6$  implies that  $B_{4,2}$  is an  $\mathcal{R}_4$ -circuit. We can now use Lemma 6 to deduce that  $B_{d,d-1}$  and  $B_{d,d-2}$  are  $\mathcal{R}_d$ -circuits for all  $d \geq 4$ . Similarly, we may apply Lemma 8 to the  $\mathcal{R}_3$ -circuits  $K_5$  and  $K_6 - \{f,g\}$ , for two non-adjacent edges f,g, to deduce that  $B_{3,2}^+$  is an  $\mathcal{R}_3$ -circuit, and then use Lemma 6 to deduce that  $B_{d,d-1}^+$  is an  $\mathcal{R}_d$ -circuit unless f or g has an end-vertex in  $V_1 \cap V_2$ . Our next result extends this to all graphs in  $\mathcal{B}_{d,d-1}^+$ .

**Lemma 9.** Every graph in  $\mathcal{B}_{d,d-1}^+$  is an  $\mathcal{R}_d$ -circuit.

Proof. Let  $B_{d,d-1}^+ \in \mathcal{B}_{d,d-1}^+$  and suppose that  $G_1 = (V_1, E_1)$ ,  $G_2 = (V_2, E_2)$ , and e, f, g are as in the definition of  $\mathcal{B}_{d,d-1}^+$ . Since  $B_{d,d-1}^+$  is d-tight and not  $\mathcal{R}_d$ -rigid (since it is not d-connected), it is  $\mathcal{R}_d$ -dependent.

We will complete the proof by showing that  $B_{d,d-1}^+ - h$  is  $\mathcal{R}_d$ -independent for all edges h of  $B_{d,d-1}^+$ . If h is incident with a vertex  $x \in V_2 \setminus V_1$ , then we can reduce  $B_{d,d-1}^+ - h$  to  $G_1 - \{e, f, g\}$  by recursively deleting vertices of degree at most d (starting from x). Since  $G_1 - \{e, f, g\}$  is  $\mathcal{R}_d$ -independent, Lemma 4 and the fact that edge deletion preserves independence now imply that  $B_{d,d-1}^+ - h$  is  $\mathcal{R}_d$ -independent. Thus we may assume that  $h \in E_2$ .

Suppose that f, g, h do not have a common end-vertex. Choose a vertex  $x \in V_2 \setminus V_1$  and let  $H = B_{d,d-1}^+ - h - x + e$  be the graph obtained by applying a 1-reduction at x. We can reduce H to  $G_1 - \{f, g, h\}$  by recursively deleting vertices of degree at most d. Since f, g, h do not have a common end-vertex,  $G_1 - \{f, g, h\}$  is  $\mathcal{R}_d$ -independent. We can now use Lemma 4 to deduce that  $B_{d,d-1}^+ - h$  is  $\mathcal{R}_d$ -independent.

Hence we may assume that f, g, h have a common end-vertex u. The definition of  $\mathcal{B}_{d,d-1}$  now implies that at least one of f and g, say f, is an edge of  $G_1 \cap G_2$ . Since e, f, g do not have a common end-vertex, e is not incident with u and hence e, g, h do not have a common end-vertex. We can now apply the argument in the previous paragraph with the roles of e and f reversed to deduce that  $B_{d,d-1}^+ - h$  is  $\mathcal{R}_d$ -independent.

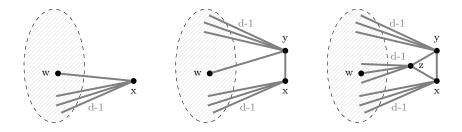


Figure 3: Construction of G in the proof of Lemma 10.

**Lemma 10.** Suppose  $G = G_1 \cup G_2$  where:  $G_1 \cap G_2 = \emptyset$ ;  $G_1$  is minimally  $\mathcal{R}_d$ -rigid;  $G_2 = K_3$ ; each vertex of  $G_2$  has d-1 neighbours in  $G_1$ ; the set of all neighbours of the vertices of  $G_2$  in  $G_1$  has size at least d. Then G is minimally  $\mathcal{R}_d$ -rigid.

Proof. Let  $V(G_2) = \{x, y, z\}$ . Since the set of all neighbours in  $G_1$  of the vertices in  $G_2$  has size at least d, we may suppose that some vertex  $w \in G_1$  is a neighbour of z, but not x or y. Then G can be obtained from  $G_1$  by a 0-extension adding x and edges from x to its d-1 neighbours in  $G_1$  as well as w, followed by two 1-extensions adding y and deleting yw and then adding y and deleting yw. (See Figure 3.) Hence y is y-independent by Lemma 4. Minimal rigidity follows by a simple edge count.

**Lemma 11.** (a) Every 6-regular graph on 10 vertices is  $\mathcal{R}_4$ -independent.

(b) Every 12-regular graph on 15 vertices is  $\mathcal{R}_9$ -independent.

*Proof.* There are 21 6-regular graphs on 10 vertices (see OEIS sequence A165627) and 17 12-regular graphs on 15 vertices (by direct computation). That they are  $\mathcal{R}_d$ -independent for the stated dimensions can now be checked by any computer algebra systems.

Our final lemma is purely graph theoretic.

**Lemma 12.** Suppose that G = (V, E) is a graph with  $|V| \ge 11$ , minimum degree two and maximum degree three. Then there exist vertices  $x, y \in V$  with d(x) = 2, d(y) = 3 and  $dist(x, y) \ge 3$ .

*Proof.* Assume G = (V, E) is a counterexample to the lemma. Choose a vertex  $v \in V$  of degree 2. Then there are at most 6 vertices at distance 1 or 2 from v. Hence there are at most 6 vertices of degree 3. Now choose a vertex  $u \in V$  of degree 3. Each neighbour of u is either a vertex of degree 2 which has at most one other neighbour of degree 2 or a vertex of degree 3 which has at most two other neighbours of degree 2. Therefore we have at most 6 vertices of degree 2. If there does not exist 6 vertices of degree 3 then the number of

vertices of degree 3 is at most 4, and we would have  $|V| \leq 10$ . Hence there are exactly 6 vertices of degree 3 and v is adjacent to two vertices of degree 3. Since v is an arbitrary vertex of degree two, every vertex of degree 2 is adjacent to two vertices of degree 3. Now choose w to be a vertex of degree 3 at distance 2 from v and a vertex  $y \neq v$ , of degree 2, not adjacent to w. Then  $dist(w, y) \geq 3$ .

### 3 Main results

We will prove Theorem 1 and then use it to obtain a lower bound on the number of edges in a flexible  $\mathcal{R}_d$ -circuit.

#### Proof of Theorem 1

We proceed by contradiction. Suppose the theorem is false and choose a counterexample G = (V, E) such that d is as small as possible and, subject to this condition, |V| is as small as possible. Since all  $\mathcal{R}_d$ -circuits are  $\mathcal{R}_d$ -rigid when  $d \leq 2$ , we have  $d \geq 3$ . Since G is an  $\mathcal{R}_d$ -circuit, G - v is  $\mathcal{R}_d$ -independent for all  $v \in V$ , and we can now use the fact that 0-extension preserves  $\mathcal{R}_d$ -independence (by Lemma 4) to deduce that  $\delta(G) \geq d + 1$ . Since G is a flexible  $\mathcal{R}_d$ -circuit, G is d-sparse by Lemma 3.

Case 1. d(v) = d + 1 for some  $v \in V$ .

Since G does not contain the rigid  $\mathcal{R}_d$ -circuit  $K_{d+2}$ , v has two non-adjacent neighbours  $v_1, v_2$ . If  $H = G - v + v_1 v_2$  was  $\mathcal{R}_d$ -independent then G would be  $\mathcal{R}_d$ -independent by Lemma 4. Hence H contains an  $\mathcal{R}_d$ -circuit C. The minimality of G implies that C is  $\mathcal{R}_{d}$ -rigid or  $C = B_{d,d-1}$  which implies that  $G \in \mathcal{B}_{d,d-1}^+$ . The latter alternative contradicts the choice of G, hence G contains a minimally  $\mathcal{R}_d$ -rigid subgraph,  $C - v_1 v_2$ , with at least d+2 vertices. Let G' be a minimally  $\mathcal{R}_d$ -rigid subgraph of G with at least d+2 vertices which is maximal with respect to inclusion, and put  $X = V \setminus V(G')$ . Then  $1 \leq |X| \leq 4$ . If some vertex  $x \in X$  had at least d neighbours in G', then we could create a larger  $\mathcal{R}_d$ -rigid subgraph by adding x to G'. Hence each  $x \in X$  has at most d-1 neighbours in G' and since G has minimum degree at least d+1, x has at least 2 neighbours in X. Thus  $3 \le |X| \le 4$ . Suppose |X| = 3. Then  $G[X] = K_3$  and  $G' = K_{d+2} - e$  for some edge e, or G' = $K_{d+3} - \{e, f, g\}$  for some edges e, f, g which are not incident with the same vertex. If  $|N_G(X)| \geq d$  then we could construct an  $\mathcal{R}_d$ -rigid spanning subgraph of G by Lemma 10. Hence  $|N_G(X)| = d - 1$ , and at least one edge, say e, with its end-vertices in  $N_G(X)$  is missing from G, since otherwise G would contain a copy of  $K_{d+2}$ . This gives  $G = B_{d,d-1}$ when  $G' = K_{d+2} - e$ , so we must have  $G' = K_{d+3} - \{e, f, g\}$ . If f, g are adjacent and neither of them have both their end-vertices in  $N_G(X)$  then G' would contain one of the  $\mathcal{R}_d$ -circuits  $K_{d+2}$  or  $B_{d,d-2}$ . Hence  $G \in \mathcal{B}_{d,d-1}^+$ .

It remains to consider the case |X|=4. Then  $C_4\subseteq G[X]\subseteq K_4$  and  $G'=K_{d+2}-e$ .

Claim 13. N(X) = V(G').

Proof of claim. Suppose not. Let  $Y = X \cup N_G(X)$ . Then  $G[Y] \subsetneq G$  and hence is independent. If  $G[N_G(X)]$  was complete, then G would be independent by Lemma 7(b), since  $G = G' \cup G[Y]$ , G' and G[Y] are independent, and  $G' \cap G[Y]$  is complete. Hence both end-vertices of e belong to  $N_G(X)$ . Choose a vertex  $w \in V(G') \setminus N_G(X)$  and an edge  $f \in G'$  which is incident with w. Consider the graph G'' = G + e - f.

Suppose G''[Y] is  $\mathcal{R}_d$ -independent. Since  $G''[N_G(X)]$  induces a complete graph, we can use Lemma 7(b) as above to deduce that G'' is  $\mathcal{R}_d$ -independent. Then  $G' + e \cong K_{d+2}$  is the unique  $\mathcal{R}_d$ -circuit in G'' + f and hence G = G'' + f - e is  $\mathcal{R}_d$ -independent. This contradiction implies that G''[Y] is  $\mathcal{R}_d$ -dependent.

Let C be an  $\mathcal{R}_d$ -circuit in G''[Y]. Since  $G' \cong K_{d+2} - e$ , w has degree degree d in G'' and hence  $w \notin V(C)$ . If  $C = B_{d,d-1}$  then we can construct G from C by a 1-extension which adds w and deletes e. This would imply that  $G \in \mathcal{B}_{d,d-1}^+$  and contradict the choice of G. Hence  $C \neq B_{d,d-1}$  and the minimality of G now implies that C is rigid.

Since  $G'+e\cong K_{d+2}$  and  $e\in E(C)\cap E(G'+e)$ , we may use the circuit elimination axiom to deduce that  $(C-e)\cup G'$  is  $\mathcal{R}_d$ -dependent. Since  $(C-e)\cup G'\subseteq G$ , we must have  $(C-e)\cup G'=G$ . This implies that X and all edges of G incident to X are contained in C. Thus  $N_G(X)\subset V(C)$ . If  $|N(X)|\geq d$ , then  $G=G'\cup (C-e)$  would be rigid by Lemma 7(a). Hence  $|N_G(X)|\leq d-1$ . If  $|N_G(X)|=d-2$ , then  $C=K_{d+2}$  and  $G=B_{d,d-2}$ . Hence  $|N_G(X)|=d-1$ . Then  $C=K_{d+3}-f-g$  for two non-adjacent edges f,g and  $G\in \mathcal{B}_{d,d-1}^+$ .

Suppose  $G[X] = C_4$ . Since  $\delta(G) = d+1$  and no vertex of X has more than d-1 neighbours in G', each vertex of X has degree d+1 in G. By Claim 13, we can choose  $u \in X$  such that  $|N(X-u) \cap V(G')| \geq d$ . We can perform a 1-reduction of G at u which adds an edge between its two neighbours in X. We can now apply Lemma 10 to the resulting graph H on d+5 vertices to deduce that H is  $\mathcal{R}_d$ -rigid. This would imply that G is  $\mathcal{R}_d$ -rigid, contradicting the choice of G.

Suppose  $G[X] = C_4 + f$ . Then each vertex in X has degree d + 1 or d + 2 in G and the two vertices which are not incident to f have degree d + 1. If both of the vertices incident to f have degree d + 2 then G has more than  $d|V| - {d+1 \choose 2}$  edges, so cannot be a flexible  $\mathcal{R}_d$ -circuit. Hence we may choose an end-vertex w of f with degree d + 1 in G. Construct H from G by performing a 1-reduction at w which adds an edge between its two non-adjacent neighbours in X. If all vertices in X have degree d + 1 in G, then we can reduce H to G' by recursively deleting the remaining 3 vertices of X in such a way that every deleted vertex has degree at most d. Since G' is  $\mathcal{R}_d$ -independent this would imply that G is  $\mathcal{R}_d$ -independent. Hence we may assume that the end-vertex of f distinct from w has has degree d+2 in G. We can now apply Lemma 10 to deduce that either H is  $\mathcal{R}_d$ -rigid

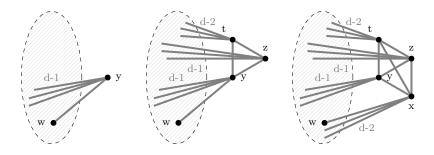


Figure 4: Construction of  $\hat{G}$  in the proof of Case 1.

or  $|N(X-w)\cap V(G')|=d-1$  and H is  $B_{d,d-1}$ . The first alternative would imply that G is  $\mathcal{R}_d$ -rigid, and the second alternative would imply that either G is  $\mathcal{R}_d$ -rigid or  $G \in \mathcal{B}_{d,d-1}^+$ .

It remains to consider the subcase when  $G[X] = K_4$ . Then each vertex in X has degree at least d+1, and at most two of them have degree d+2 otherwise G would have more than  $d|V|-\binom{d+1}{2}$  edges. Let  $\hat{G}$  be obtained from G by adding edges from vertices in X to vertices in G' in such a way that X has exactly two vertices of degree d+1 and exactly two vertices of degree d+2 in  $\hat{G}$ . We will show that G is  $\mathcal{R}_d$ -independent by proving that  $\hat{G}$  is minimally  $\mathcal{R}_d$ -rigid.

Since  $N_{\hat{G}}(X) = V(G')$  by Claim 13, we may choose vertices  $x, y \in X$  such that x has degree d+1, y has degree d+2 and some vertex  $w \in V(G')$  is a neighbour of x in  $\hat{G}$  but not y. Let  $X = \{x, y, z, t\}$  where z has degree d+2 and t has degree d+1 in  $\hat{G}$ . We can construct  $\hat{G}$  from G' by first performing a 0-extension which adds y and all edges from y to its neighbours in G' as well as to w, then add z and then t by successive 0-extensions, and finally add x by a 1-extension which removes the edge yw. (See Figure 4.)

Since G' is minimally  $\mathcal{R}_d$ -rigid this implies that  $\hat{G}$  is also minimally  $\mathcal{R}_d$ -rigid. This contradicts the fact that G is an  $\mathcal{R}_d$ -circuit and completes the proof of Case 1.

Case 2.  $\delta(G) \ge d + 2$ .

Choose  $v \in V$  with  $d(v) = \Delta(G)$ . If G - v was  $\mathcal{R}_{d-1}$ -independent then G would be  $\mathcal{R}_{d}$ -independent by Lemma 6. This is impossible since G is an  $\mathcal{R}_{d}$ -circuit. Hence G - v contains an  $\mathcal{R}_{d-1}$ -circuit C. By the minimality of d, C is  $\mathcal{R}_{d-1}$ -rigid or  $C \in \{B_{d-1,d-2}, B_{d-1,d-3}\} \cup \mathcal{B}_{d-1,d-2}^+$ .

Claim 14. G - v is  $\mathcal{R}_{d-1}$ -rigid.

Proof of Claim. Suppose  $C \in \{B_{d-1,d-2}, B_{d-1,d-3}\} \cup \mathcal{B}_{d-1,d-2}^+$ . Then C has d+4, d+5 or d+5 vertices respectively, whereas G-v has at most d+5 vertices. If C spans G-v then the facts that C contains vertices of degree d+1 and  $\delta(G-v) \geq d+1$  imply that we can

add edges of G-v to C to make it  $\mathcal{R}_{d-1}$ -rigid. Hence we may suppose that  $C=B_{d-1,d-2}$  and  $(G-v)\setminus C$  has exactly one vertex u. Since  $d_{G-v}(u)\geq d+1$ , G-v is  $\mathcal{R}_{d-1}$ -rigid unless all neighbours of u belong to the same copy of  $K_{d+1}-e$  in  $B_{d-1,d-2}$ . Suppose the second alternative occurs and let H be the spanning subgraph of G-v obtained by adding u and all its incident edges to  $B_{d-1,d-2}$ . Since the other copy of  $K_{d+1}-e$  in  $B_{d-1,d-2}$  contains vertices of degree d in H, and degree at least d+1 in G-v, we can now add an edge of G-v to H to make it  $\mathcal{R}_{d-1}$ -rigid.

Suppose C is  $\mathcal{R}_{d-1}$ -rigid. Then  $|V(C)| \geq d+1$ . Let H be a maximal  $\mathcal{R}_{d-1}$ -rigid subgraph of G-v. Suppose  $H \neq G-v$  and note that (G-v)-H has at most 4 vertices. Since each vertex of (G-v)-H has at most d-2 neighbours in H and  $\delta(G-v) \geq d+1$  we have  $(G-v)-H=K_4$  and  $H=C=K_{d+1}$ . We can now apply Lemma 10 to a minimally rigid spanning subgraph of H, and to each  $K_3$  in (G-v)-H, in order to deduce that all vertices of (G-v)-H are adjacent to the same set of d-2 vertices of H. This cannot occur since every vertex of H which is not joined to a vertex of G-v-H would have degree at most d+1 in G, contradicting the assumption of Case 2. Hence H=G-v and G-v is  $\mathcal{R}_{d-1}$ -rigid.

Let  $(G - v)^*$ , respectively  $C^*$ , be obtained from G - v, respectively C, by adding v and all edges from v to G - v, respectively C. Then  $(G - v)^*$  is  $\mathcal{R}_d$ -rigid by Claim 14 and Lemma 6, and, when C is  $\mathcal{R}_{d-1}$ -rigid,  $C^*$  is an  $\mathcal{R}_d$ -circuit again by Lemma 6.

Let S be the set of all edges of  $G^*$  which are not in G. Since  $C^*$  is rigid or  $C^* \in \{B_{d,d-1}, B_{d,d-2}\} \cup \mathcal{B}_{d,d-1}^+$ ,  $C^*$  is not an  $\mathcal{R}_d$ -circuit in G. Hence  $E(C^*) \cap S \neq \emptyset$ . If  $S = \{f\}$  then  $G = (G - v)^* - f$  would be  $\mathcal{R}_d$ -rigid since  $(G - v)^*$  is  $\mathcal{R}_d$ -rigid and  $f \in E(C^*)$ . Hence  $|S| \geq 2$  and  $\Delta(G) = d(v) \leq |V| - 3$ . Let  $\bar{G}$  be the complement of G.

Suppose  $|V| \leq d+5$ . Then |V| = d+5 and G is (d+2)-regular. This implies that  $\bar{G}$  is a 2-regular graph on  $d+5 \geq 8$  vertices and we may choose two non-adjacent vertices  $v_1, v_2$  with no common neighbours in  $\bar{G}$ . Then  $v_1v_2 \in E$  and  $|N_G(v_1) \cap N_G(v_2)| = d-1$ . We can use the facts that G is d-sparse, (d+2)-regular and |V| = d+5 to deduce that  $G/v_1v_2$  is d-sparse. (If not, then some set  $X \subseteq V(G/v_1v_2)$  would induce more that  $d|X| - {d+1 \choose 2}$  edges. Then  $|X| \geq d+2$  and the fact that each vertex of  $V(G/v_1v_2) \setminus X$  has degree at least d+1 will imply that  $G/v_1v_2$  has more that  $d|V(G/v_1v_2)| - {d+1 \choose 2}$  edges. This will contradict the fact that G has at most  $d|V(G)| - {d+1 \choose 2}$  edges.) Since  $G/v_1v_2$  has no flexible  $\mathcal{R}_d$ -circuits (by the minimality of G),  $G/v_1v_2$  is  $\mathcal{R}_d$ -independent. We can now use Lemma 5 to deduce that G is  $\mathcal{R}_d$ -independent. Hence |V| = d+6. Since  $\delta(G) \geq d+2$  and  $\Delta(G) \leq d+3$  we have  $\delta(\bar{G}) \geq 2$  and  $\Delta(\bar{G}) \leq 3$ .

Suppose  $\delta(\bar{G}) = 2$  and  $\Delta(\bar{G}) = 3$ . Then we can find two vertices  $x, y \in V$  with  $d_{\bar{G}}(x) = 2$ ,  $d_{\bar{G}}(y) = 3$  and  $\mathrm{dist}_{\bar{G}}(x,y) \geq 3$  by Lemma 12. We can deduce as in the previous paragraph that G/xy is d-sparse. If G/xy contains an  $\mathcal{R}_d$ -circuit then  $G/xy = B_{d,d-1}$  by the minimality and d-sparsity of G. Since  $B_{d,d-1}$  has d-3 vertices of degree d+4 and six vertices of degree d+1, this would contradict the fact that G has minimum degree d+2

(when  $d \leq 6$ ) and maximum degree d+3 (when  $d \geq 5$ ). Hence  $G/v_1v_2$  is  $\mathcal{R}_d$ -independent. We can now use Lemma 5 to deduce that G is  $\mathcal{R}_d$ -independent.

Next we consider the case when  $\bar{G}$  is 2-regular. Then |S| = 2 and G is (d+3)-regular. The fact that  $(G-v)^*$  is  $\mathcal{R}_d$ -rigid and contains at least two  $\mathcal{R}_d$ -circuits  $(G \text{ and } C^*)$  tells us that  $|E((G-v)^*)| \geq d|V| - {d+1 \choose 2} + 2$ . Since  $|E| = |E((G-v)^*)| - |S|$  and G is d-sparse this gives

$$\frac{(d+3)(d+6)}{2} = |E| = d|V| - \binom{d+1}{2} = \frac{d(d+11)}{2}.$$

This implies that d = 9 and |V| = 15. We can now use Lemma 11(b) to deduce that G is  $\mathcal{R}_9$ -independent, contradicting the fact that G is an  $\mathcal{R}_9$ -circuit.

It remains to consider the final subcase when  $\bar{G}$  is 3-regular. Then |S| = 3 and G is (d+2)-regular. Since  $(G-v)^*$  is  $\mathcal{R}_d$ -rigid and contains at least two  $\mathcal{R}_d$ -circuits we have  $|E(G)| \geq d|V| - {d+1 \choose 2} - 1$ . The fact that G is d-sparse now gives

$$\frac{(d+2)(d+6)}{2} = |E| = d|V| - \binom{d+1}{2} - \alpha = \frac{d(d+11)}{2} - \alpha$$

for some  $\alpha = 0, 1$ . This implies that  $\alpha = 0$  and d = 4. We can now use Lemma 11(a) to deduce that G is  $\mathcal{R}_4$ -independent, contradicting the fact that G is an  $\mathcal{R}_4$ -circuit.

We can use Theorem 1 to obtain a lower bound on the number of edges in a flexible  $\mathcal{R}_d$ -circuit. For G = (V, E) and  $X \subset V$ , we use the notation  $E(X, V \setminus X)$  to denote the set of edges with one endvertex in X and one in  $V \setminus X$ .

**Corollary 15.** Suppose G = (V, E) is a flexible  $\mathcal{R}_d$ -circuit. Then  $|E| \geq d(d+9)/2$ , with equality if and only if  $G = B_{d,d-1}$ .

Proof. The corollary follows immediately from Theorem 1 if  $|V| \leq d+6$ . Since  $\delta(G) \geq d+1$  we have |E| > d(d+9)/2 when either  $|V| \geq d+8$ , or |V| = d+7 and  $\delta(G) \geq d+2$ . Hence we may assume that |V| = d+7 and  $\delta(G) = d+1$ . Choose a vertex v with d(v) = d+1. Then v has two non-adjacent neighbours  $v_1, v_2$  since otherwise G would contain the rigid  $\mathcal{R}_d$ -circuit  $K_{d+2}$ . Let  $H = G - v + v_1v_2$ . If H was  $\mathcal{R}_d$ -independent then G would be  $\mathcal{R}_d$ -independent by Lemma 4. Hence H contains an  $\mathcal{R}_d$ -circuit C. If C is flexible then Theorem 1 implies that  $C \in \{B_{d,d-2}\} \cup \mathcal{B}_{d,d-1}^+$  and hence |E| > |E(C)| > d(d+9)/2. Thus we may assume that C is  $\mathcal{R}_d$ -rigid. Then  $C - v_1v_2$  is an  $\mathcal{R}_d$ -rigid subgraph with at least d+2 vertices. Let

 $X = V(G) \setminus V(C)$ . Then  $1 \le |X| \le 5$ . Since  $\delta(G) = d + 1$  and  $|X| \le 5$  we have

$$|E| = |E(C - v_1 v_2)| + |E(X)| + |E(X, V \setminus X)|$$

$$\geq d|V \setminus X| - \binom{d+1}{2} + \binom{|X|}{2} + |X|(d+1-|X|+1)$$

$$= d|V| - \binom{d+1}{2} - \frac{|X|(|X|-3)}{2}$$

$$\geq \frac{d(d+13)}{2} - 5.$$

We can now use the fact that  $d \ge 3$  to deduce that |E| > d(d+9)/2.

## 4 Closing Remarks

#### 4.1 Generalised 2-sums

Let G = (V, E),  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs. We say that G is a t-sum of  $G_1, G_2$  along an edge e if  $G = (G_1 \cup G_2) - e$ ,  $G_1 \cap G_2 = K_t$  and  $e \in E_1 \cap E_2$ . We conjecture that Lemma 8 can be extended to t-sums.

**Conjecture 16.** Suppose that G is a t-sum of  $G_1, G_2$  along an edge e for some  $2 \le t \le d+1$ . Then G is an  $\mathcal{R}_d$ -circuit if and only if  $G_1, G_2$  are  $\mathcal{R}_d$ -circuits.

Our proof technique for Lemma 8 gives the following partial result.

**Lemma 17.** Let G = (V, E),  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs such that G is a t-sum of  $G_1, G_2$  along an edge e for some  $2 \le t \le d+1$ .

- (a) If G is an  $\mathcal{R}_d$ -circuit, then  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -circuits.
- (b) If  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -circuits, then G contains a unique  $\mathcal{R}_d$ -circuit G' and  $E \setminus (E_1 \cap E_2) \subseteq E(G')$ .

Proof. (a) If  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -independent, then Lemma 7(b) implies that  $G_1 \cup G_2$  is  $\mathcal{R}_d$ -independent. This contradicts the facts that G is a  $\mathcal{R}_d$ -circuit and  $G \subseteq G_1 \cup G_2$ . If exactly one of  $G_1$  and  $G_2$ , say  $G_1$ , is  $\mathcal{R}_d$ -independent then e belongs to the unique  $\mathcal{R}_d$ -circuit in  $G_2$  and Lemma 7(b) gives  $r_d(G) = r_d(G+e) = |E_1| + |E_2| - {t \choose 2} - 1 = |E|$ . This again contradicts the hypothesis that G is an  $\mathcal{R}_d$ -circuit. Hence  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -dependent. Then the matroid circuit elimination axiom combined with the fact that G is an  $\mathcal{R}_d$ -circuit imply that  $G_1$  and  $G_2$  are both  $\mathcal{R}_d$ -circuit.

(b) The circuit elimination axiom implies that G is  $\mathcal{R}_d$ -dependent and hence that G contains an  $\mathcal{R}_d$ -circuit G' = (V', E'). Since  $G_i - e$  is  $\mathcal{R}_d$ -independent for i = 1, 2, we have  $E' \setminus E_i \neq \emptyset$ .

Let  $G_i'$  be obtained from  $G_i \cap G'$  by adding an edge between every pair of non-adjacent vertices in  $V' \cap V_1 \cap V_2$ . If  $G_i'$  is a proper subgraph of  $G_i$  for i = 1, 2 then each  $G_i'$  is  $\mathcal{R}_d$ -independent and we can use Lemma 7(b) to deduce that  $G_1' \cup G_2'$  is  $\mathcal{R}_d$ -independent. This gives a contradiction since  $G' \subseteq G_1' \cup G_2'$ . Relabelling if necessary we have  $G_1' = G_1$ . If  $G_2' \neq G_2$  then we may deduce similarly that  $G_1' \cup G_2' - e$  is independent. This again gives a contradiction since  $G' \subseteq G_1' \cup G_2' - e$ . Hence  $G_2' = G_2$ . It remains to show uniqueness. For i = 1, 2, let  $B_i$  be a base of  $\mathcal{R}_d(G_i)$  which contains  $E(G_1) \cap E(G_2)$ . Then  $|B_i| = |E_i| - 1$  and Lemma 7(b) gives

$$r_d(G) = r_d(G_1 \cup G_2 - e) = r_d(G_1 \cup G_2) = |B_1| + |B_2| - {t \choose 2} = |E| - 1.$$

Hence, G contains a unique  $\mathcal{R}_d$ -circuit.

We can also use a result of Connelly [8] to deduce that Conjecture 16 holds when t = d+1 and  $G_1, G_2$  are both globally rigid in  $\mathbb{R}^d$ .

### 4.2 Highly connected flexible circuits

Bolker and Roth [4] determined  $r_d(K_{s,t})$  for all complete bipartite graphs  $K_{s,t}$ . Their result implies that  $K_{d+2,d+2}$  is a (d+2)-connected  $\mathcal{R}_d$ -circuit for all  $d \geq 3$  and is flexible when  $d \geq 4$ , see [9, Theorem 5.2.1]. We know of no (d+3)-connected flexible  $\mathcal{R}_d$ -circuits and it is tempting to conjecture that they do not exist.

For the case when d=3, Tay [16] gives examples of 4-connected flexible  $\mathcal{R}_3$ -circuits and Jackson and Jordán [10] conjecture that all 5-connected  $\mathcal{R}_3$ -circuits are rigid. An analogous statement has recently been verified for circuits in the closely related  $C_2^1$ -cofactor matroid by Clinch, Jackson and Tanigawa [7].

#### 4.3 Extending Theorem 1

We saw in the previous subsection that  $K_{d+2,d+2}$  is a flexible  $\mathcal{R}_d$ -circuit with 2d+4 vertices for all  $d \geq 4$ . We can use Lemma 6 to obtain a smaller flexible  $\mathcal{R}_d$ -circuit: we can recursively apply the coning operation to the flexible  $\mathcal{R}_4$ -circuit  $K_{6,6}$  to obtain a flexible  $\mathcal{R}_d$ -circuit on d+8 vertices. This suggests that it may be difficult to extend Theorem 1 to graphs on d+8 vertices, but it is conceivable that all flexible  $\mathcal{R}_d$ -circuits on d+7 vertices have the form  $(G_1 \cup G_2) - S$  where  $G_i \in \{K_{d+2}, K_{d+3}, K_{d+4}\}$ ,  $G_1 \cap G_2 \in \{K_{d-3}, K_{d-2}, K_{d-1}\}$  and S is a suitably chosen set of edges.

For the case when d = 3, Tay [16] gives examples of 3-connected flexible  $\mathcal{R}_3$ -circuits with 13 vertices but it is possible that all flexible circuits on at most 12 vertices can be obtained by taking 2-sums of rigid circuits on at most 9 vertices.

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