A CRITERION FOR ASYMPTOTIC SHARPNESS IN THE ENUMERATION OF SIMPLY GENERATED TREES

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Abstract. We study the identity $y(x) = xA(y(x))$, from the theory of rooted trees, for appropriate generating functions $y(x)$ and $A(x)$ with non-negative integer coefficients. A problem that has been studied extensively is to determine the asymptotics of the coefficients of $y(x)$ from analytic properties of the complex function $z \mapsto A(z)$, assumed to have a positive radius of convergence R. It is well-known that the vanishing of $A(x) - xA'(x)$ on $(0, R)$ is sufficient to ensure that $y(r) < R$, where r is the radius of convergence of $y(x)$. This result has been generalized in the literature to account for more general functional equations than the one above, and used to determine asymptotics for the Taylor coefficients of $y(x)$. What has not been shown is whether that sufficient condition is also necessary. We show here that it is, thus establishing a criterion for sharpness of the inequality $y(r) \leq R$. As an application, we prove a 1996 conjecture of Kuperberg regarding the asymptotic growth rate of an integer sequence arising in the study of Lie algebra representations.

1. Introduction

1.1. **Motivation.** We explore the sharpness of a certain estimate that occurs naturally in the asymptotic enumeration of rooted trees. As motivation we begin with a particular example from the literature, a conjecture formulated by Kuperberg in his study of the representation theory of simple rank-2 Lie algebras [10, Conjecture 8.2]. One of our main results is a proof of this conjecture. This will follow from Theorem 5 below, which applies to a general class of generating functions that contains the one studied by Kuperberg.

Specifically, set $a_0 = 1$, and for each positive integer n, let a_n denote the number of triangulations of a regular n-gon, such that the minimum degree of each internal vertex is 6. The sequence begins

$$(a_n)_{n=0}^{\infty} = 1, 0, 1, 1, 2, 5, 15, 50, 181, 697, \ldots$$

and is indexed in the On-Line Encyclopedia of Integer Sequences (OEIS, [17]) by A059710. Next, let $b_0 = 1$, and for each positive integer n, let b_n denote the dimension of the vector subspace of invariant tensors in the n-th tensor power of the fundamental 7-dimensional representation of the exceptional simple Lie algebra G_2. The sequence begins

$$(b_n)_{n=0}^{\infty} = 1, 0, 1, 1, 4, 10, 35, 120, 455, 1792, \ldots$$

and is indexed in OEIS as A059710.

The sequence (b_n) is also known to have a combinatorial interpretation as the number of lattice walks in a fundamental Weyl chamber of the root system for G_2 (a 30° sector in the triangular lattice in $\mathbb{R}^2$) that start and end at the origin, subject to certain constraints on the steps [21]. This type of model is not unique to G_2; finite dimensional representations of a large class of Lie algebras (e.g. complex, connected, and semi-simple), in general admit such interpretations. For more information one may consult the work of Littelmann [11, 12, 13] and others, e.g. [8, Thm. 5].

Now let $A(x) = 1+\sum_{n=1}^{\infty} a_n x^n$ and $B(x) = 1+\sum_{n=1}^{\infty} b_n x^n$ be the ordinary generating functions for $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$, respectively. In [10, Section 8], Kuperberg proved the following identity of formal power series:

$$B(x) = A(xB(x)). \tag{1.1}$$

(The connection of (1.1) to the theory of rooted trees will be reviewed in Section 2.) He also observed that $B(x)$ has radius of convergence $\frac{1}{7}$ and satisfies $B\left(\frac{1}{7}\right) < \infty$ (which also follows directly from Proposition 10 below), and that this implies by (1.1) that $A(x)$ has radius of convergence at least $\frac{1}{7}B\left(\frac{1}{7}\right)$ (see Lemma 2 below). He conjectured that this bound is in fact an equality, so that $\limsup_{n\to\infty} \sqrt[n]{a_n} = 7/B(1/7)$, which he estimated to be 6.811. We prove here that this conjecture is true and explicitly identify the value of Kuperberg's constant. The precise result is the following.

Theorem 1. *Let $(a_n)_{n=0}^{\infty}$ and $B(x)$ be as described above. Then,*

$$a_n = \rho^{n+o(n)} \quad \text{as } n \to \infty,$$

where

$$\rho := \frac{7}{B\left(\frac{1}{7}\right)} = \sup_{n\in\mathbb{N}} \sqrt[n]{a_n}\,.$$

Furthermore,

$$\rho = \frac{5\pi}{8575\pi - 15552\sqrt{3}} \approx 6.8211.$$

From (1.1) one can already see that $a_n \le b_n$, for all $n \ge 0$, while the lattice walk model above suggests the probabilistic notion that $b_n \approx 7^n$. As part of our analysis we will deduce substantially more, namely that

$$b_n = K(7^n/n^7)(1+o(1)), \quad \text{as } n \to \infty,$$

for a constant $K \approx 2627.6$ that we determine exactly (see Proposition 10).

More generally, if $A(x) = 1 + \sum_{i\ge1} a_i x^i$ and $B(x) = 1 + \sum_{i\ge1} b_i x^i$ are ordinary generating functions, with radii of convergence R and r respectively, and they satisfy (1.1), then the inequality $rB(r) \le R < 1$ holds under mild conditions that are stated precisely in Theorem 2. A natural question then is to ask when equality holds, and this is the central theme of this paper.

1.2. **Structure of the paper.** In the next section we will describe how the identity (1.1) relates to the theory of rooted trees, and we will use analytic methods to formulate a criterion, Theorem 5, for generating functions $A(x)$ and $B(x)$ satisfying (1.1) that can be used to determine when the radius of convergence of $A(x)$ is as small as possible, as in the above example. After establishing this criterion, we will discuss its connection to previous work. In Section 3, we will use this general result to prove Theorem 1.

2. Analysis of simply generated trees

Let $A(x) = 1 + \sum_{n\geq 1} a_n x^n$ and $B(x) = 1 + \sum_{n\geq 1} b_n x^n$ be power series satisfying (1.1). When $B(x)$ has a positive radius of convergence, we would like to know when the identity (1.1) of formal power series is also an identity of the complex analytic functions defined by these power series in a neighborhood of the origin, since then we may apply analytic methods. A sufficient condition is given by Lemma 2 below. We will adopt a useful convention of setting $y(x) := xB(x)$, whereby the identity (1.1) can be rewritten as

$$y(x) = xA(y(x)). \tag{2.1}$$

The coefficient sequence $(y_n)_{n=1}^{\infty}$ of $y(x)$ is then given by $y_n = b_{n-1}$ for $n \geq 1$. On their disks of convergence, the power series $y(x)$, $B(x)$, and $A(x)$ define analytic functions of a complex variable z, by evaluation at $x = z$. We will refer to these functions as y, B, and A, respectively. There is a minor distinction to be made between the formal power series $y(x)$ and the analytic function it represents on its disk of convergence. Thus, we will always use x as the variable when thinking of $y(x)$ as a power series, whereas when evaluating the function y at points in the complex plane, we will use variables z and w to represents such points (similarly for A and B).

Lemma 2. *Let $A(x) = 1 + \sum_{i\geq 1} a_i x^i$ and $y(x) = \sum_{i\geq 1} y_i x^i$ be power series related by (2.1), with $a_i \geq 0$ for all $i \geq 1$. Then $y_i \geq 0$ for all i. Let R denote the radius of convergence of A. If we assume further that $y(x)$ has a finite radius of convergence, denoted by r, then*

$$y(z) = zA(y(z)) \tag{2.2}$$

for $z \in \Omega = \{z \in \mathbb{C} : |z| < r\}$, and $y(r) \leq R$, including when $y(r) = \infty$. If in addition, $a_n \geq 1$ for all large enough n, then $R < 1$, so that in particular $y(r) < 1$.

Proof. The Lagrange Inversion Theorem [18, Ch. 5.4] implies that

$$y_i = \frac{1}{i}[x^{i-1}](A(x)^i),$$

for all $i \geq 1$, where $[x^i]f(x)$ denotes the i'th coefficient of a power series $f(x)$, and it is clear that the right-hand side is non-negative.

To prove the second part of the lemma, we assume that $r > 0$, otherwise there is nothing to check. Let z be a non-negative number in Ω. Then

$$zA(y(z)) = z\sum_{i=0}^{\infty}\left[a_i\left(\sum_{j=1}^{\infty} y_j z^j\right)^i\right] = \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}(c_{ij}z^j) = \sum_{j=1}^{\infty}\left[\left(\sum_{i=1}^{j} c_{ij}\right)z^j\right]$$

where $c_{ij} = a_i[x^{j-1}]\left(\sum_{k=1}^{\infty} y_k x^k\right)^i$, and the interchange of the order of summation is permitted since each c_{ij} is non-negative. The fact that $y(z) = \sum_{j=1}^{\infty}\left[\left(\sum_{i=1}^{j} c_{ij}\right)z^j\right]$ is just a reformulation of (2.1). So $zA(y(z)) = y(z) < \infty$. Thus, the double sum above is absolutely convergent on Ω, and (2.2) is valid on Ω. Next, since the coefficients of $y(x)$ are non-negative and $y(0) = 0$, it follows that y maps $[0, r)$ bijectively to $[0, y(r))$. Therefore, if $0 < |z| < y(r)$, then $|z| = y(w)$ for some $w \in (0, r)$, and $A(|z|) = \frac{y(w)}{w} < \infty$. So $R \geq y(r)$. Finally, in the event that $a_n \geq 1$ eventually, the non-negativity of the coefficients of $A(x)$ guarantees that $A(z) = \infty$ for $z > 1$. Hence $R < 1$. □

The lemma will help us to study the radius of convergence of $A(x)$, and it provides a sufficient condition to ensure that $y(r) < \infty$, which is a hypothesis in several statements below.

When $A(x)$ satsifies the hypotheses of the lemma, for some $y(x)$, we associate to A a new function ψ, defined in the following way:

$$\psi(z) := \frac{z}{A(z)}, \tag{2.3}$$

wherever $A(z)$ is defined and non-zero. The importance of ψ is that it is inverse to y on the disk of convergence of y.

Lemma 3. *Assume the hypotheses of Lemma 2. With $\Omega = \{z \in \mathbb{C} : |z| < r\}$, the function $y : \Omega \to y(\Omega)$ is a biholomorphism, and $y^{-1} = \psi$ on $y(\Omega)$. In particular, and ψ is well-defined and analytic on $y(\Omega)$.*

Proof. Note that $y(0) = 0$, $A(0) = 1$, and A is analytic on Ω by Lemma 2. If $y(z) = 0$ for $z \in \Omega$, then by (2.2) we see that

$$z = z \cdot 1 = zA(y(z)) = y(z) = 0.$$

So y vanishes at the origin and nowhere else in Ω. Therefore, for $z \in \Omega \setminus \{0\}$ we have $A(y(z)) = y(z)/z \neq 0$, and A doesn't vanish on $y(\Omega)$. As a consequence ψ is defined there, and analytic, and it follows from (2.2) that $\psi(y(z)) = z$ for $z \in \Omega$. The injectivity of y is now an easy consequence of (2.2). Indeed, if $y(z) = y(w) \neq 0$ for $z, w \in \Omega$, then $z \neq 0$, $w \neq 0$, and

$$zA(y(z)) = y(z) = y(w) = wA(y(w)) = wA(y(z)),$$

so $w = z$. □

2.1. **Simply generated trees.** The functional equation (2.1) has a well-known interpretation in the theory of rooted trees. Suppose that $A(x) = 1 + \sum_{n\geq 1} a_n x^n$, where the a_n's are non-negative integers, and $y(x) = \sum_{i\geq 1} y_n x^n$ is related to A via (2.1). Then y_n counts the number of rooted trees with n nodes (including the root), such that for each $i \geq 1$, each internal node having i children can be colored with one of a_i colors. A family of rooted trees is commonly called *simply generated* (see [15], where this nomenclature appears to have been introduced, and also [4]) if the number of trees in the family is counted by a generating function $y(x)$ that satisfies (2.1) for some $A(x)$ of the above form.

The article [4] contains classical examples of (2.1) and some fundamental asymptotic results, including that the Catalan numbers occur as the sequence $(y_n)_{n=1}^{\infty}$ when $A(x) = \frac{1}{1-x}$, and that $y_n \sim \pi^{-\frac{1}{2}} 4^n n^{-\frac{3}{2}}$, as $n \to \infty$ (see Example 8 below). The combinatorics of trees is discussed in the textbooks [18, Ch. 5] and [6], while the latter also contains a broad treatment of the analytic framework for (2.2).

2.2. **A criterion for sharpness.** One of our main results, Theorem 5, provides a criterion that ensures that the radius of convergence of the function A in (2.2) is as small as possible. A slight variation of the theorem is converse to a known fact (see Theorem 6 below). At one point in the proof of Theorem 5 we will require the following extension lemma.

Lemma 4. *Let f be analytic on the open disk of radius $r > 0$, with power series expansion $f(z) = \sum_{n=0}^{\infty} f_n z^n$. If $f_n \geq 0$ for all $n \geq 0$, and furthermore $f(r) < \infty$, then f extends to be continuous on the closed disk $\overline{\Omega} = \{z : |z| \leq r\}$.*

Proof. The hypothesis that $f(r) < \infty$ implies that $\sum_{n=0}^{\infty} f_n z^n$ converges on $\overline{\Omega}$, and further that this convergence is uniform, by the Weierstrass M-test. □

Theorem 5. *Suppose that the generating functions $A(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ and $y(x) = \sum_{n=1}^{\infty} y_n x^n$, with radii of convergence R and r respectively, satisfy the following conditions:*

1. *$a_n \geq 0$ for $n \geq 0$.*
2. *$A(x)$ and $y(x)$ are related as formal power series by (1.1).*
3. *$r < \infty$.*

If $y(r) < R$, then $A(z) - zA'(z)$ vanishes at $z = y(r)$.

Proof. Assume that $y(r) < R$. To prove the theorem, we will show that the further assumption that $A(z) - zA'(z)$ doesn't vanish at $z = y(r)$ leads to a contradiction of the fact that r is the radius of convergence of $B(x)$. From this contradiction we obtain the desired conclusion.

Assume that $A(y(r)) - y(r)A'(y(r)) \neq 0$. Recall the function ψ, defined by $\psi(z) = \frac{z}{A(z)}$. By Lemma 3, ψ is analytic and equal to y^{-1} on $y(\Omega)$, where $\Omega = \{z \in \mathbb{C} : |z| < r\}$. We claim that there exists a small open disk E centered at $y(r)$, such that ψ extends to be analytic on $y(\Omega) \cup E$. Indeed, since $y(r) < R$, it follows that A is analytic on an open disk, which we call E, that is centered at $y(r)$ and contained in the disk of radius R. Also, since A is continuous and $A(y(r)) = y(r)/r > 0$, we may assume, by replacing E with a smaller open disk centered at $y(r)$, if necessary, that A doesn't vanish on E. It follows that ψ is analytic on $y(\Omega) \cup E$, as the reciprocal of a non-vanishing analytic function, which verifies the claim.

Now we observe, recalling from the proof of Lemma 3 that A does not vanish on $y(\Omega)$, that

$$\psi'(z) = \frac{A(z) - zA'(z)}{A(z)^2},$$

so that the assumption that $A(z) - zA'(z) \neq 0$ for $z = y(r)$ implies in particular that $\psi'(y(r)) \neq 0$. It follows that ψ is locally invertible at $y(r)$. That is, after replacing E with a smaller open disk centered at $y(r)$, if necessary, we see that the map $\psi|_E : E \to \psi(E)$ is a homeomorphism, with an analytic inverse map $\psi|_E^{-1}$. Since $\psi(y(r)) = r$ and r is a boundary point of Ω, the open set $\psi(E)$ contains r and intersects Ω. Since $y(r) < \infty$ and $y_n \geq 0$ for all $n \geq 1$, by Lemma 2, we can apply Lemma 4 to see that y extends continuously to the boundary of Ω. Therefore, if ϵ is the radius of the disk E, then there exists an open disk D centered at r and contained in $\psi(E)$, such that $|y(z) - y(r)| < \epsilon$, and hence $y(z) \in E$, if $z \in D \cap \Omega$.

For $z \in D \cap \Omega$, we have $\psi(y(z)) = z$, by Lemma 3. Since $y(z) \in E$, it follows from the injectivity of ψ on E that $y(z) = \psi|_E^{-1}(z)$. In other words, y agrees with $\psi|_E^{-1}$ on $D \cap \Omega$. Thus, y has an analytic continuation to the domain $D \cup \Omega$, which is an open set containing r. This contradicts a classical result known as Pringsheim's Theorem [6, p. 240], which asserts that an analytic function with non-negative real coefficients necessarily has a singularity at the point where its circle of convergence intersects $[0, \infty)$. This is the contradiction we sought, and the proof is complete. □

If we phrase the theorem in a slightly weaker form by replacing its conclusion with the statement that $A(z) - zA'(z) = 0$ for *some* z in $(0, R)$, then the converse is well-known to be true, and it follows from Theorem 6(1) below. Thus, we have a criterion on $A(x)$ for determining the exponential growth rate of (y_n). Theorem 6 contains even deeper asymptotic information than that, however, in particular regarding the *subexponential* (i.e. polynomial) growth rate of (y_n). This, it turns out, will be instrumental in proving Theorem 1, as it shows how information about the growth of (y_n) can certify that $A(z) - zA'(z)$ does not vanish on $(0, R)$, and hence that $y(r) = R$ (by Theorem 5).

Theorem 6. *Suppose that* $A(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ *and* $y(x) = \sum_{n=1}^{\infty} y_n x^n$ *(with radii of convergence* R *and* r*, respectively) satisfy conditions (1)-(3) of Theorem 5. Assume in addition that* $gcd\{n > 0 : a_n > 0\} = 1$*. Let* ψ *be as in* (2.3)*. If there exists* $\tau \in (0, R)$*, such that* $A(\tau) - \tau A'(\tau) = 0$*, then the following facts hold:*

1. $y(x)$ *has radius of convergence* $r = \psi(\tau)$*, which implies that* $y(r) = \tau < R$.
2. *The coefficient sequence* $(y_n)_{n=1}^{\infty}$ *satisfies the following asymptotic estimate:*
$$y_n = \frac{C}{r^n n^{\frac{3}{2}}}(1 + \mathcal{O}(n^{-1})) = \frac{C \cdot A'(\tau)^n}{n^{\frac{3}{2}}}(1 + \mathcal{O}(n^{-1})),$$
as $n \to \infty$*, where* $C = \sqrt{\frac{A(\tau)}{2\pi A''(\tau)}}$.

This theorem appears to have been first established essentially by Meir and Moon in [15, Thm. 3.1], building on techniques of Darboux [3], Pólya [14], and others, although they arrive at the conclusion of Theorem 6 under slightly different conditions on $A(x)$ than the gcd condition. In [16] the same authors generalize their analysis to a much broader class of functional equations, of which (2.1) is an example. One may also consult [4, Thm. 5] and [6, Thm. VI.6] for proofs of this more general result, and one may find in [6, pp. 467-471] a brief note about the theorem's history.

Sketch of proof: To establish (1), note that if $A(\tau) - \tau A'(\tau) = 0$ for $\tau \in (0, R)$, and we further assume that $y(r) > \tau$, then since $y(0) = 0 < \tau < y(r)$ and y is increasing on $[0, r)$ (as pointed out in the proof of Lemma 2), there is some point z in $(0, r)$ where y is analytic and $y(z) = \tau$. Since $\psi = y^{-1}$ on $(0, r)$, we see that $0 = \psi'(\tau) = (\frac{d}{dz} y(z))^{-1}$, which is absurd. Next, one can show that $z \mapsto (A(z) - zA'(z))$ is a decreasing function on $(0, R)$, by considering its power series expansion, for example, so that if $y(r) < \tau$, then $\psi'(y(r)) \neq 0$. One can then argue, as in the proof of Theorem 2, that y admits an analytic continuation to a neighborhood of r, which contradicts Pringsheim's Theorem, establishing that $y(r) = \tau$. To establish (2), the main idea is that under the hypotheses of the theorem the function ψ has a second-order zero at τ, which implies that y behaves locally like a square-root function. The Taylor coefficients of such a function are known from a generalized form of the binomial theorem to have a subexponential growth factor of $n^{-3/2}$. Finally, we note that the gcd condition in the statement of the theorem is a mild technicality, satisfied for example if (a_n) is eventually increasing, and the asymptotic expansion asserted for y_n can be easily adjusted when the gcd exceeds 1. Details can be found in the above-mentioned references.

Now we can neatly summarize, in the form of a dichotomy, the collection of statements set forth in this section.

Corollary 7. *Suppose that $A(x)$ and $y(x)$ (with radii of convergence R and r, respectively) satisfy conditions (1)-(3) of Theorem 5 and the gcd condition of Theorem 6. Then exactly one of the following is true:*

(1) *$A(z) - zA'(z)$ is non-vanishing for $z \in (0, R)$, in which case $R = y(r)$.*
(2) *$R > y(r) = \tau$, where τ is the unique solution to $A(\tau) - \tau A'(\tau) = 0$ on $(0, R)$, and $y_n = Cr^n n^{-3/2}(1 + o(1))$ as $n \to \infty$, for some constant $C > 0$.*

In particular, the absence of the $n^{-3/2}$ polynomial factor in the asymptotic expansion of y_n suffices to certify that the inequality $y(r) \le R$ is actually sharp, which is case (1). The specific function $A(x)$ from Kuperberg's conjecture as stated in the introduction exemplifies this situation, as we will show in the next section. Before doing that we close this section with two examples. The first is a classic demonstration of case (2), and the second is an illustration of the "boundary case" in case (2), namely when $A(z) - zA'(z)$ vanishes exactly at $z = R$.

2.3. **Examples.**

Example 8 (Catalan numbers)**.** Given a generating function $A(x)$ satisfying the hypotheses of Theorem 5, there exists a unique generating function $y(x)$ that satisfies (1.1). (The Lagrange Inversion formula used in the proof of Lemma 2 is one way to determine $y(x)$, but the uniqueness and existence of $y(x)$ simply amounts to equating undetermined coefficients in a power series expansion and solving a recursion.)

For the present example, take $A(x) = \frac{1}{1-x}$, with radius of convergence $R = \infty$. Let $y(x) = \sum_{n \ge 1} y_n x^n$ satisfy (1.1). Then for $n \ge 0$, $y_{n+1} = \frac{1}{n+1}\binom{2n}{n}$ is the n'th Catalan number. One way to show this (see e.g. [4, pp. 1-4]) is to solve $y(x)^2 - y(x) + x = 0$ for $y(x)$ and then develop a Taylor expansion from the solution

$$y(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

We see that the radius of convergence of $y(x)$ is $r = \frac{1}{4}$, that $y(r) = \frac{1}{2} < R$, and that

$$A(x) - xA'(x) = \frac{1-2x}{(1-x)^2}$$

vanishes at $y(r)$. Thus, the well-known asymptotic formula $y_n \sim \pi^{-1/2} 4^n n^{-3/2}$, as $n \to \infty$, is a direct application of Theorem 6 above.

Example 9 (Boundary case)**.** The question of what can happen when $A(z) - zA'(z)$ vanishes exactly at the boundary R is naturally suggested by Corollary 7 (and moreso by its use in our approach to proving Theorem 1 - see Section 3). It turns out in this case that it is also possible to see a subexponential growth factor of $n^{-3/2}$ in the asymptotic expansion of the corresponding sequence (y_n). Let

$$\begin{aligned} A(x) &= 6x + 2(1-4x)^2 - (1-4x)^{5/2} \\ &= 1 + 2x^2 + 20x^3 + 10x^4 + 12x^5 + 20x^6 + \cdots . \end{aligned}$$

The Taylor coefficients (defined in terms of the principal branch of the logarithm) of $(1-4x)^{5/2} = \sum_{n=0}^{\infty} x^n (-4)^n \binom{5/2}{n}$ are easily seen to be integers (e.g. the coefficients of $(1-4x)^{1/2}$ are integers in the previous example), and are negative except for the constant term, so considering the first few terms of $A(x)$, one finds that the hypotheses of Theorem 5 are met. Furthermore, by differentiating the expression for A, one sees that $\psi'(z)$ vanishes at $z = \frac{1}{4}$, where as usual $\psi(z) = \frac{z}{A(z)}$. (Although $A(z)$ is not analytic at $z = \frac{1}{4}$, the formal power series derivative $A'(x)$ converges at $\frac{1}{4}$ and furthermore $\lim_{z\to\frac{1}{4}}[A(z) - zA'(z)] = 0$.)

We observe that

$$\psi(z) - \frac{1}{6} = \frac{-2(1-4z)^2 + (1-4z)^{\frac{5}{2}}}{6A(z)} = (1-4z)^2 H(z),$$

where $H(z)$ is analytic in a neighborhood of $\frac{1}{4}$ and satisfies $H(z) \to -\frac{2}{9}$ as $z \to \frac{1}{4}$. Substituting $y(z)$ for z, and solving for $y(z)$, we find that

$$y(z) - \frac{1}{4} \sim \frac{-3}{4\sqrt{2}} \left(\frac{1}{6} - z\right)^{\frac{1}{2}},$$

as $z \to \frac{1}{6}$ in the slit plane $\mathbb{C} \setminus \{z \geq \frac{1}{6}\}$. The asymptotics of (y_n) are now given by the widely utilized "transfer theorem" of Flajolet and Odlyzko (see [6, Ch. VI, Cor. VI.1, p. 392] and the rest of that chapter for a detailed discussion), which implies that

$$y_n \sim \frac{K \cdot 6^n}{n^{3/2}},$$

as $n \to \infty$, where $K = \frac{\sqrt{3}}{16\sqrt{\pi}}$.

3. Proof of Theorem 1

For the rest of this section, the generating functions $A(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ and $B(x) = 1 + \sum_{n=1}^{\infty} b_n x^n$ will be those from Section 1. Recall that $\frac{1}{7}$ is the radius of convergence of $B(x)$ and $y(x)$ and that $\rho := \frac{7}{B(\frac{1}{7})} = \frac{1}{y(\frac{1}{7})}$, and that we let R denote the radius of convergence of $A(x)$. From the combinatorial definition of the sequence (a_n) and the fact that (1.1) is known to hold, it is clear that the conditions in the hypothesis of Corollary 7 are satisfied.

3.1. **Outline of the proof.** Our proof of Theorem 1 proceeds by the following steps.

(1) In Proposition 10, we will derive asymptotics for (b_n), as stated in the introduction:
$$b_n \sim K\frac{7^n}{n^7},$$
as $n \to \infty$, for an explicit constant K.

(2) The asymptotics of (b_n), specifically the presence of the n^{-7} polynomial factor as opposed to a factor of $n^{-3/2}$, will indicate by Corollary 7 that $A(z) - zA'(z)$ does not vanish on $(0, R)$, and that $R = \frac{1}{\rho}$.

(3) It will then follow from the previous step that
$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \rho\,.$$
From its combinatorial interpretation, it is clear that $(a_n)_{n\geq 2}$ is a non-decreasing sequence, since given any triangulation of an n-gon ($n \geq 2$), one can add another vertex to obtain a triangulated $(n+1)$-gon without introducing any new internal vertices. Furthermore, $a_{n+m-2} \geq a_n a_m$, for $n, m \geq 2$, since one can always obtain a triangulated $(n+m-2)$-gon by gluing together a triangulated n-gon and a triangulated m-gon along a

common edge, and this process does not introduce any new internal vertices. Thus we obtain (for $n, m \geq 2$):

$$\log a_n + \log a_m \leq \log a_{n+m-2} \leq \log a_{n+m}.$$

So $(\log a_n)_{n\geq 2}$ is superadditive sequence, which implies by a lemma generally attributed to Fekete [5] that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \sup_{n\in\mathbb{N}} \sqrt[n]{a_n}.$$

From the limsup above, this limit must be ρ. The asymptotic expression for a_n given in Theorem 1 then follows directly.

(4) The last step is to evaluate ρ exactly.

3.2. **Asymptotics of (b_n) and the exact evaluation of $\frac{1}{7}B\left(\frac{1}{7}\right)$.** In view of the outline in the previous subsection, Propositions 10 and 11 below will complete the proof of Theorem 1.

Proposition 10. *The sequence $(b_n)_{n=0}^{\infty}$ satisfies the asymptotic equivalence*

$$b_n \sim K\frac{7^n}{n^7},$$

as $n \to \infty$, where

$$K = \frac{4117715\sqrt{3}}{864\pi} \approx 2627.56\,.$$

In the introduction we described a lattice walk model in which the b_n's denote the number of n-step excursions that start and end at the origin. This interpretation suggests that a local central limit theorem will apply for the return probabilities of random paths of length n. A difficulty is that the constraints on allowable steps imply that they are not i.i.d. Luckily, there is a reflection trick due essentially to [7] which allows one to express the return probabilities as local linear combinations of the probabilities that *unconstrained* walks in the weight lattice $\cong \mathbb{Z}^2$ will end at various nearby points to the origin. Indeed, this method has been applied to derive asymptotic formulas (e.g. [20, Thm. 8]), which imply that the sequence $(\dim \operatorname{Inv}_{\mathfrak{g}}(V_{\lambda}^{\otimes n}))_{n=1}^{\infty}$, where V is any representation of an appropriate (e.g. compact, connected, semi-simple) Lie algebra $\mathfrak{g}$, is asymptotically equivalent to $C\dim(V)^n n^{-\alpha}$, where α is half the dimension of $\mathfrak{g}$ (in our case $\alpha = 14/2 = 7$), and the constant term C can also be computed, but depends on the specific representation. We none-the-less supply a direct proof of Proposition 10 based on a saddle point analysis.

Proof. Our starting point is the fact from [9] that b_n is the coefficient of $x^n y^n$ in the polynomial WM^n, where

$$M(x, y) = 1 + x + y + xy + x^2y + xy^2 + (xy)^2,$$

and

$$W(x, y) = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

We also define

$$f(x, y) = \log\left(\frac{1}{7}M(x, y)\right) - \log(x) - \log(y).$$

By Cauchy's residue formula for Taylor coefficients, we have

$$\frac{b^n}{7^n} = \frac{1}{(2\pi i)^2} \oint \oint \left[W(z_1, z_2) \cdot \frac{M(z_1, z_2)^n}{7^n} \cdot \frac{1}{(z_1 z_2)^{(n+1)}} \right] dz_1 \, dz_2.$$

We use as contours for both integrals the unit circle about the origin, i.e. $z_1 = e^{iu}$ and $z_2 = e^{iv}$, for $-\pi \leq u, v \leq \pi$. Thus,

$$\frac{b^n}{7^n} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[W(e^{iu}, e^{iv}) \cdot \exp\left(n f(e^{iu}, e^{iv}) \right) \right] du \, dv. \tag{3.1}$$

The function f satisfies $f(1,1) = f_x(1,1) = f_y(1,1) = 0$, and thus we have

$$f(x,y) = \frac{2}{7}(x-1)^2 + \frac{2}{7}(y-1)^2 + \frac{2}{7}(x-1)(y-1) + \mathcal{O}((x-1, y-1)^3),$$

as $(x,y) \to (1,1)$, where the exponent 3 in the last term signifies a multi-index that ranges over all pairs of non-negative integers whose sum is 3. It follows that

$$f(e^{iu}, e^{iv}) = -\frac{2}{7}u^2 - \frac{2}{7}v^2 - \frac{2}{7}uv + \mathcal{O}((u,v)^3),$$

as $(u,v) \to (0,0)$. Now we make an n-dependent change of variables in (3.1), namely $p = \sqrt{n}u$, and $q = \sqrt{n}v$. The integral becomes

$$\frac{4\pi^2 b^n n}{7^n} = \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left[W(e^{\frac{ip}{\sqrt{n}}}, e^{\frac{iq}{\sqrt{n}}}) \exp\left(-\frac{2}{7}p^2 - \frac{2}{7}q^2 - \frac{2}{7}pq + \mathcal{O}\left(\frac{(p,q)^3}{\sqrt{n}} \right) \right) \right] dpdq.$$

By standard estimates, which we omit (and which morally relate to the fact that there is a local central limit theorem for a lattice random walk with i.i.d. steps at work), the above integral can be approximated asymptotically by the completed integral over all of $\mathbb{R}^2$, and the contribution of the $\mathcal{O}\left(\frac{(p,q)^3}{\sqrt{n}}\right)$ term is negligible. Moreover, if for each $k \in \mathbb{N}$ we let $T_k(p,q)$ denote the order k Taylor approximation for $(p,q) \mapsto W(e^{ip}, e^{iq})$, then it suffices to consider integrals of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[T_k(pn^{-1/2}, qn^{-1/2}) \cdot \exp\left(-\frac{2}{7}p^2 - \frac{2}{7}q^2 - \frac{2}{7}pq \right) \right].$$

Computing with SAGE, we find that this integral vanishes for $k < 12$, while

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[T_{12}(pn^{-1/2}, qn^{-1/2}) \cdot \exp\left(-\frac{2}{7}p^2 - \frac{2}{7}q^2 - \frac{2}{7}pq \right) \right] dpdq \\
&= \frac{1}{n^6} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[T_{12}(p,q) \cdot \exp\left(-\frac{2}{7}p^2 - \frac{2}{7}q^2 - \frac{2}{7}pq \right) \right] dpdq \\
&= \frac{1}{n^6} \cdot \frac{4117715\sqrt{3}}{216}\pi \, .
\end{aligned}$$

In total, we have shown that $\frac{4\pi^2 b^n n}{7^n} \sim \frac{4117715\sqrt{3}}{n^6 \cdot 216}\pi$, as $n \to \infty$, which implies the asserted value of K. □

Proposition 11.

$$\rho = \frac{5\pi}{8575\pi - 15552\sqrt{3}} \, .$$

Proof. In [2, p. 8] is given the following closed formula for the generating function $B(x)$ in terms of the hypergeometric series ${}_2F_1$:

$$B(x) = \frac{1}{30x^5}\left[R_1 \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; \phi(x)\right) + R_2 \cdot {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; 3; \phi(x)\right) + 5P\right], \tag{3.2}$$

where

$$\begin{aligned} R_1(x) &= (t+1)^2(214t^3 + 45t^2 + 60t + 5)(t-1)^{-1}, \\ R_2(x) &= 6t^2(t+1)^2(101t^2 + 74t + 5)(t-1)^{-2}, \\ \phi(x) &= 27(t+1)t^2(t-1)^{-3}, \\ P(x) &= 28x^4 + 66x^3 + 46x^2 + 15x + 1. \end{aligned}$$

Evaluating these polynomials at $x = \frac{1}{7}$, the formula simplifies to

$$B\left(\frac{1}{7}\right) = \frac{7^5}{30}\left[\frac{-55296}{2401} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; 1\right) + \frac{9216}{2401} \cdot {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; 3; 1\right) + \frac{150}{7}\right].$$

It is now a matter of routine calculation to deduce the value in the proposition. One only needs standard facts about the gamma function, namely that $\Gamma(z+1) = z\Gamma(z)$ for $\mathrm{Re}(z) > 0$ and the following (see [1] and [19] respectively):

(1)
$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}(c) > \mathrm{Re}(a+b)),$$

(2)
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C}).$$

Use (1) and then (2) to simplify the above expression for $B\left(\frac{1}{7}\right)$. Then recall that $\rho = 7/B(1/7)$. $\square$

Remark: The proposition shows that $B(x)$ is not an algebraic power series, despite the fact that $B(x)$ is D-finite (i.e. solves a linear differential equation), which follows from the linear recurrence relation satisfied by the coefficients (b_n) in [2, Thm. 1.2].

4. Final remarks

This paper establishes the exact exponential growth rate of Kuperberg's sequence $(a_n)_{n=0}^{\infty} = 1, 0, 1, 1, 2, 5, 15\ldots$, which enumerates a seemingly innocuous combinatorial class and which is moreover relatively easy to describe in terms of basic geometry. It was the triumph of Kuperberg in [9] and [10] to first connect these triangulations to invariant representations in Lie theory. Using formula (3.2) we have succeeded in evaluating ρ exactly, and the analytic argument of Theorem 5 has finally shown that a_n grows like ρ^n. Thus, while combinatorics has historically provided an indispensable toolbox for studying the structure of Lie algebras and algebraic categories in general, it is a curious feature of this subject that enumeration of certain Lie representations informs the growth of the combinatorial sequence $(a_n)_{n=0}^{\infty}$. In light of this result it would be especially interesting to deduce more accurate asymptotics. Numerical evidence suggests the following.

Conjecture 12. *There exists a constant $\lambda > 0$, such that $a_n \sim \lambda \frac{\rho^n}{n^7}$, as $n \to \infty$.*

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