IDONEAL GENERA AND THE CLASSIFICATION OF K3 SURFACES COVERING AN ENRIQUES SURFACE

SIMON BRANDHORST, SERKAN SONEL, AND DAVIDE CESARE VENIANI

Abstract. We classify transcendental lattices of K3 surfaces covering an Enriques surface. In order to do so, we introduce the notion of idoneal genera, ie genera of positive definite integral lattices that only contain lattices with a vector of square 1, and classify them.

All algebraic varieties in this paper are defined over $\mathbb{C}$.
Our conventions on integral lattices are explained in §1.3.

1. Introduction

A smooth proper algebraic surface X such that $H^1(X, \mathcal{O}) = 0$ is called a *K3 surface* if its canonical bundle $\mathcal{K}$ is trivial and it is called an *Enriques surface* if $\mathcal{K}$ is not trivial, but $\mathcal{K}^{\otimes 2}$ is. The cohomology group $H^2(X, \mathbb{Z})$ of a K3 surface X, together with the Poincaré pairing, is a lattice of rank 22. It contains the *Néron–Severi lattice*, defined as the image S of the map $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ coming from the exponential sheaf sequence. The *transcendental lattice* of X is the orthogonal complement $T = S^\perp \subset H^2(X, \mathbb{Z})$.

The aim of this paper is to classify K3 surfaces covering an Enriques surface according to their transcendental lattice.

1.1. Principal results.

Our starting point is the following criterion by Keum. (Keum proved it under an additional assumption which is actually superfluous, see [15].) We denote $\Lambda^- = U \oplus U(2) \oplus E_8(-2)$.

Theorem 1.1 (Keum's criterion [7]). *A K3 surface X with transcendental lattice T covers an Enriques surface if and only if there exists a primitive embedding $T \hookrightarrow \Lambda^-$ such that the orthogonal complement of T in Λ^- contains no vector of square -2.*

For a K3 surface, the condition of covering an Enriques surface shares a lot of similarities with the condition of being a Kummer surface. Indeed, Nikulin [13] showed that a K3 surface X of transcendental lattice T is a Kummer surface if and only if there exists a primitive embedding $T \hookrightarrow U(2)^{\oplus 3}$. In [12, Corollary 4.4], Morrison restated Nikulin's criterion, characterizing in an explicit way the transcendental lattices of algebraic Kummer surfaces.

Here we restate Keum's criterion in an analogous way. Our main result is the following theorem, where $E^\dagger$ is an overlattice of $E_8(-2)$ of index 2 (or,

Date: May 26, 2020.
2010 *Mathematics Subject Classification.* 14J28, 11E12.
Key words and phrases. K3 surface, Enriques surface, lattice, genus, idoneal number.
The authors acknowledge the financial support of the following research projects: DFG SFB-TRR 45, DFG SFB-TRR 195, TÜBİTAK 118F413.

equivalently by [14, Proposition 1.4.1] and Theorem 3.6, a negative definite lattice of rank 8 and discriminant form $u_1^{\oplus 3}$).

Theorem 1.2 (see §3.5)**.** *If X is a K3 surface with transcendental lattice T of rank λ, then X covers an Enriques surface if and only if T is not isomorphic to one of the 550 (or 551, see Remark 1.4) exceptional lattices listed in Table 5 and one of the following holds:*

(i) $2 \leq \lambda \leq 6$ *and T admits a Gram matrix of the form*

$$\begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix}$$

such that a_{ij} is even for each $2 \leq i, j \leq \lambda$,

(ii) $\lambda = 7$ *and there exists an even lattice T' with* $T \cong U \oplus T'(2)$,

(iii) $\lambda = 7$ *and there exists a lattice T' with* $T \cong U(2) \oplus T'(2)$,

(iv) $\lambda = 8$ *and there exists an even lattice T' with* $T \cong U \oplus U(2) \oplus T'(2)$,

(v) $\lambda = 8$ *and there exists a lattice T' with* $T \cong U(2)^{\oplus 2} \oplus T'(2)$,

(vi) $\lambda = 9$ *and there exists an even lattice T' with* $U(2) \oplus T \cong E^\dagger \oplus T'(2)$.

(vii) $\lambda = 9$ *and there exists a lattice T' with* $U \oplus T \cong E^\dagger \oplus T'(2)$.

(viii) $\lambda = 10$ *and there exists an even lattice T' with* $T \cong E^\dagger \oplus T'(2)$,

(ix) $\lambda = 10$ *and there exists a lattice T' with* $T \cong E_8(-2) \oplus T'(2)$,

(x) $\lambda = 11$ *and there exists $n > 0$ with* $T \cong [4n] \oplus U \oplus E_8(-2)$,

(xi) $\lambda = 11$ *and there exists $n > 0$ with* $T \cong [2n] \oplus U(2) \oplus E_8(-2)$,

(xii) $\lambda = 12$ *and* $T \cong \Lambda^-$.

The case $\lambda = 12$ is trivial on account of Keum's criterion. The cases $\lambda = 2$ and $\lambda = 11$ had already been proved by Sertöz [19] and Ohashi [15], respectively. Partial attempts had been made for $\lambda = 3$ by Lee [9] and for $\lambda = 4$ by Yörük [26].

According to Ohashi [16], the number of Enriques surfaces covered by a fixed K3 surface is finite up to isomorphism. Theorem 1.2 classifies K3 surfaces for which this number is nonzero. For the related problem of computing this number explicitly, see [15, 16] and [20].

Conditions (i)–(xii) characterize even lattices T of signature $(2, \lambda - 2)$ which admit a primitive embedding $T \hookrightarrow \Lambda^-$. We say that such a lattice is *exceptional* if the orthogonal complement of T in Λ^- contains a vector of square -2 for each primitive embedding $T \hookrightarrow \Lambda^-$. Table 5 on page 19 contains the finite list of all known exceptional lattices (see Theorem 3.12 and Table 1).

An *idoneal* genus is a positive definite genus $\mathfrak{g}$ with the property that each lattice $L \in \mathfrak{g}$ contains a vector of square 1 or, equivalently, $L \cong [1] \oplus L'$ for some lattice L' (see our conventions in §1.3).

The key observation to classify exceptional lattices is contained in Proposition 3.11: the orthogonal complement in Λ^- of an exceptional lattice is always of the form $L(-2)$, where L is a lattice belonging to a uniquely determined idoneal genus. Thus, in order to compile Table 5, we are naturally led to classify all idoneal genera, which turns out to be a purely lattice-theoretical problem.

TABLE 1. The number I_r of idoneal genera of rank r and the corresponding number E_λ of exceptional lattices of signature $(2, \lambda - 2)$, under the assumption of the generalized Riemann hypothesis (see Theorem 1.5 and Theorem 3.12).

r	1	2	3	4	5	6	7	8	9	10	11	12	13	sum
I_r	1	65	110	122	107	76	47	24	13	6	4	1	1	577
λ	11	10	9	8	7	6	5	4	3	2	–	–	–	sum
E_λ	1	65	110	122	107	76	41	17	8	3	–	–	–	550

Remark 1.3. The numbers $d \in \mathbb{N}$ such that the lattice $[1] \oplus [d]$ is unique in its genus are usually called *idoneal* (sometimes also *suitable* or *convenient*) numbers. This terminology goes back to Euler [4]. If L belongs to an idoneal genus $\mathfrak{g}$ of rank 2 and determinant d, then necessarily $L \cong [1] \oplus [d]$; in particular, L is unique in $\mathfrak{g}$. Thus, a genus $\mathfrak{g}$ of rank 2 is idoneal if and only if $\mathfrak{g} = \{[1] \oplus [d]\}$, with d an idoneal number, whence the name.

Remark 1.4. There are exactly 65 idoneal numbers known, the highest one being 1848 (sequence A000926 in the OEIS [21]). Weinberger [25] proved that the sequence is complete if the generalized Riemann hypothesis holds. If it does not hold, then there could exist one more idoneal genus of rank 2 (and determinant $> 10^8$) and one more exceptional lattice of rank 10. All other statements in this paper are valid also without the assumption of the generalized Riemann hypothesis.

Theorem 1.5 (see §2.7)**.** *For each $r \in \mathbb{N}$, there exist I_r idoneal genera of rank r, with I_r given in Table 1 for $r \leq 13$ and $I_r = 0$ otherwise. There exist no other idoneal genera of rank $r \neq 2$ and there exists at most one more of rank 2 if the generalized Riemann hypothesis does not hold (see Remark 1.4). The list of all 577 known idoneal genera can be found in the ancillary file on arXiv.*

1.2. **Contents of the paper.** The paper is divided into two sections.

In §2 we classify idoneal genera. The main idea is to look at the twigs of idoneal genera, which must satisfy a certain mass condition (see Proposition 2.6). We then employ the Smith–Minkowski–Siegel mass formula in order find an explicit bound on the determinant of idoneal genera.

In §3, after recalling some results by Nikulin on the discriminant form of integral lattices, we first determine which transcendental lattices embed into Λ^- and then we list all exceptional lattices. We conclude the paper with the proof of Theorem 1.2.

1.3. **Conventions on lattices.** In this paper, an (integral) *lattice* of rank r is a finitely generated free $\mathbb{Z}$-module $L \cong \mathbb{Z}^r$ endowed with a nondegenerate symmetric bilinear pairing $b\colon L \times L \to \mathbb{Z}$. A morphism $L \to L'$ of $\mathbb{Z}$-modules is an *isomorphism* of lattices if it is an isomorphism of $\mathbb{Z}$-modules respecting the bilinear pairings. If it exists, we write $L \cong L'$. We denote the group of automorphisms of L by $\mathrm{Aut}(L)$.

If $e_1, \ldots, e_r \in L$ is a system of generators, the associated *Gram matrix* is the square matrix with entries $b_{ij} = b(e_i, e_j)$. A lattice L is denoted by any of its Gram matrices, using the following shorthand notation:

$$[b_{11}, b_{12}, b_{22}, \ldots, b_{1n}, \ldots, b_{rr}] = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{12} & b_{22} & & \\ \vdots & & \ddots & \vdots \\ b_{1r} & & \ldots & b_{rr} \end{pmatrix}.$$

The *determinant* $\det(L)$ is the determinant of any such matrix. A lattice L is called *even* if $e^2 = b(e,e) \in 2\mathbb{Z}$ for each $e \in L$, otherwise it is called *odd*.

An embedding $L \hookrightarrow L'$ is *primitive* if L'/L is free, and L' is an *overlattice* of L of index n if $\operatorname{rank} L' = \operatorname{rank} L$ and $n = |L'/L|$.

We write $L(n)$ for the lattice with the pairing defined by the composition $L \times L \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}$. The (positive definite) ADE lattices are denoted A_n, D_n, E_n and the hyperbolic plane is denoted U.

The abelian group $A = L^\vee/L$, where

$$L^\vee = \{e \in L \otimes \mathbb{Q} \mid b(e,f) \in \mathbb{Z} \text{ for all } f \in L\},$$

has order $\det(L)$. If L is even, the finite quadratic form $q(L)\colon A \to \mathbb{Q}/2\mathbb{Z}$ induced by the linear extension of b to $\mathbb{Q}$ is called the *discriminant (quadratic) form* of L.

In this paper, a *genus* is a complete set of isomorphism classes of lattices which are equivalent over $\mathbb{R}$ and over $\mathbb{Z}_p$ for each prime p to a given lattice. Each genus is a finite set ([8, Kapitel VII, Satz (21.3)]). The *parity*, *rank*, *signature* or *determinant* of a genus $\mathfrak{g}$ are by definition the parity, rank, signature or determinant of any lattice $L \in \mathfrak{g}$.

Acknowledgements. The authors would like to warmly thank Alex Degtyarev, Markus Kirschmer, Stéphane Louboutin, Rainer Schulze-Pillot, Ali Sinan Sertöz and John Voight for sharing their insights.

2. Classification of idoneal genera

Trivially, the only idoneal genus of rank 1 is $\mathfrak{g} = \{[1]\}$. We already observed that there exists a bijective correspondence between idoneal genera of rank 2 and Euler's idoneal numbers (Remark 1.3). This section is dedicated to the classification of idoneal genera of rank ≥ 3.

We first fix the notation concerning the Smith–Minkowski Siegel mass formula (§2.2–§2.4). In §2.5 the main idea is explained: on account of Proposition 2.6, we can classify idoneal genera by searching for slender genera, which are genera satisfying a certain condition on the mass (Definition 2.5). We are led to compare the mass of $\mathfrak{g}$ with the mass of a related genus $\tilde{\mathfrak{g}}$, which is done in §2.6. Finally, §2.7 contains the proof of Theorem 1.5.

We assume that the reader is familiar with Conway–Sloane's paper [2], to which we refer for further details.

2.1. **Zeta functions.**

The usual gamma function (respectively Riemann zeta function) is denoted by Γ (respectively ζ). For $D \in \mathbb{Z}$ we introduce the following Dirichlet character modulo $4D$:

$$\chi_D(m) = \begin{cases} 0 & \text{if } (m, 2D) \neq 1, \\ \left(\frac{D}{m}\right) & \text{if } (m, 2D) = 1, \end{cases}$$

where $\left(\frac{D}{m}\right)$ denotes the Jacobi symbol. In [2, §7] the zeta function $\zeta_D(s)$ is defined as the Dirichlet L-series with respect to the character χ_D.

$$\zeta_D(s) = \sum_{m=1,3,5,\ldots} \left(\frac{D}{m}\right) m^{-s} = \prod_p \frac{1}{1-\chi_D(p)p^{-s}}.$$

For a genus $\mathfrak{g}$ of rank $n = 2s$ or $n = 2s-1$ and determinant d, we put

$$D = (-1)^s d, \qquad \varepsilon_p(\mathfrak{g}) = \chi_D(p).$$

2.2. **Jordan decomposition.** Each lattice L admits a Jordan decomposition over the p-adic integers

$$L = \ldots \oplus \frac{1}{p} J_p^{-1}(L) \oplus 1 J_p^0(L) \oplus p J_p^1(L) \oplus p^2 J_p^2(L) \oplus \ldots$$

Since the pairing of L takes values in $\mathbb{Z}$, each Jordan constituent $J_p^i(L)$ with $i < 0$ is a lattice of dimension 0. Nonetheless, Conway–Sloane's formalism takes the Jordan constituent $J_2^{-1}(L)$ into account to compute the mass of L.

We write $J_p^i(\mathfrak{g})$ for the ith p-adic Jordan constituent of any $L \in \mathfrak{g}$. Denoting by ν_p the p-adic valuation, we have

$$\sum_i i \dim J_p^i(\mathfrak{g}) = \nu_p(\det \mathfrak{g}). \tag{1}$$

2.3. **Mass and p-mass.** The *mass* of a genus $\mathfrak{g}$ is defined as

$$\mathrm{m}(\mathfrak{g}) = \sum_{L\in\mathfrak{g}} \frac{1}{|\operatorname{Aut}(L)|}.$$

The *p-mass* of $\mathfrak{g}$ is defined by the formula

$$\mathrm{m}_p(\mathfrak{g}) = \Delta_p(\mathfrak{g}) \cdot \chi_p(\mathfrak{g}) \cdot \operatorname{type}_p(\mathfrak{g}), \tag{2}$$

where $\operatorname{type}_p(\mathfrak{g}) = 1$ if $p \neq 2$ and the other factors are defined as follows:

$$\begin{aligned} \Delta_p(\mathfrak{g}) &= \prod_i M_p^i(\mathfrak{g}) && \text{(diagonal product)},\\ \chi_p(\mathfrak{g}) &= \prod_{i<j} p^{\frac{1}{2}(j-i)\dim J_p^i(\mathfrak{g}) \dim J_p^j(\mathfrak{g})} && \text{(cross product)},\\ \operatorname{type}_2(\mathfrak{g}) &= 2^{n_{\mathrm{I,I}}(\mathfrak{g}) - n_{\mathrm{II}}(\mathfrak{g})} && \text{(type factor)}. \end{aligned}$$

Here, $n_{\mathrm{II}}(\mathfrak{g})$ is the sum of the dimensions of all Jordan constituents that have type II, $n_{\mathrm{I,I}}$ is the total number of pairs of adjacent constituents $J_2^i(\mathfrak{g}), J_2^{i+1}(\mathfrak{g})$ that are both of type I, and $M_p^i(\mathfrak{g})$ is the *diagonal factor* associated to $J_p^i(\mathfrak{g})$. How to compute the diagonal factor is explained in [2, §5].

Note that it is customary to write $\mathrm{m}(L)$, $\mathrm{m}_p(L)$ and so on, but we preferred to stress the dependence on the genus and not on the chosen representative.

Remark 2.1 (cf [2, §7])**.** If $p \nmid 2d$, then the Jordan decomposition of any $L \in \mathfrak{g}$ is concentrated in degree 0. Hence, $\mathrm{m}_p(\mathfrak{g}) = \Delta_p(\mathfrak{g}) = M_p^0(\mathfrak{g})$ takes on the so-called *standard value*

$$\operatorname{std}_p(\mathfrak{g}) = \frac{1}{2(1-p^{-2})(1-p^{-4})\ldots(1-p^{2-2s})\cdot(1-\varepsilon_p(\mathfrak{g})p^{-s})}.$$

2.4. **Smith–Minkowski–Siegel mass formula.** The following formula, known as the *Smith–Minkowski–Siegel mass formula*, relates the mass of $\mathfrak{g}$ to its p-masses $\mathrm{m}_p(\mathfrak{g})$

$$\mathrm{m}(\mathfrak{g}) = 2\pi^{-\frac{1}{4}n(n+1)} \cdot \prod_{j=1}^{n} \Gamma\left(\tfrac{1}{2}j\right) \cdot \prod_{p} (2\,\mathrm{m}_p(\mathfrak{g})). \tag{3}$$

2.5. **Twigs and slender genera.** If L', L'' are two lattices belonging to the same genus $\mathfrak{g}$, then also $[1] \oplus L'$ and $[1] \oplus L''$ belong to the same genus, denoted by $\tilde{\mathfrak{g}}$.

Definition 2.2. We say that $\mathfrak{g}$ is a *twig* of a genus $\mathfrak{f}$ if $\mathfrak{f} = \tilde{\mathfrak{g}}$.

Lemma 2.3 (see [2, Lemma 3])**.** *A genus $\mathfrak{f}$ can have at most two twigs. If $\mathfrak{f}$ has two twigs, then one of them is odd and the other one is even.*

Lemma 2.4. *If $\mathfrak{T}$ is the set of twigs of an idoneal genus $\mathfrak{f}$, then*

$$2\,\mathrm{m}(\mathfrak{f}) \leq \sum_{\mathfrak{g} \in \mathfrak{T}} \mathrm{m}(\mathfrak{g}). \tag{4}$$

Proof. Any lattice in $\mathfrak{f}$ is of the form $[1] \oplus L$ with $L \in \mathfrak{g}$, $\mathfrak{g} \in \mathfrak{T}$. If L and L' are not isomorphic, then $[1] \oplus L$ and $[1] \oplus L'$ are also not isomorphic, by the uniqueness of the decomposition of positive definite lattices into irreducible lattices ([8, Satz 27.2]). Therefore, since $|\operatorname{Aut}([1] \oplus L)| \geq 2|\operatorname{Aut}(L)|$, we have

$$\mathrm{m}(\mathfrak{f}) = \sum_{\mathfrak{g} \in \mathfrak{T}} \sum_{L \in \mathfrak{g}} \frac{1}{|\operatorname{Aut}([1] \oplus L)|} \leq \frac{1}{2} \sum_{\mathfrak{g} \in \mathfrak{T}} \mathrm{m}(\mathfrak{g}).$$ □

Definition 2.5. A genus $\mathfrak{g}$ is called *slender* if $\mathrm{m}(\tilde{\mathfrak{g}}) \leq \mathrm{m}(\mathfrak{g})$.

Proposition 2.6. *Any idoneal genus has at least one slender twig.*

Proof. If $\mathfrak{f}$ is an idoneal genus and $\mathfrak{T}$ is its set of twigs, then $\mathfrak{T}$ has at most 2 elements by Lemma 2.3. If each $\mathfrak{g} \in \mathfrak{T}$ is not slender, ie $\mathrm{m}(\mathfrak{f}) > \mathrm{m}(\mathfrak{g})$, then $2\,\mathrm{m}(\mathfrak{f}) > \sum_{\mathfrak{g} \in \mathfrak{T}} \mathrm{m}(\mathfrak{g})$, contradicting Lemma 2.4. □

Remark 2.7. If an idoneal genus has two twigs, not both of them need to be slender. For instance, the two twigs of the idoneal genus $\mathfrak{f} = \{[1]^{\oplus 9}, [1] \oplus E_8\}$ are $\mathfrak{g} = \{[1]^{\oplus 8}\}$, which is slender, and $\mathfrak{h} = \{E_8\}$, which is not.

2.6. **Comparison of masses.** We set out to compare the mass of $\tilde{\mathfrak{g}}$ with the mass of $\mathfrak{g}$. The Smith–Minkowski–Siegel mass formula (3) implies

$$\frac{\mathrm{m}(\tilde{\mathfrak{g}})}{\mathrm{m}(\mathfrak{g})} = \pi^{-\frac{1}{2}(n+1)} \Gamma\left(\tfrac{1}{2}(n+1)\right) \prod_{p} \frac{\mathrm{m}_p(\tilde{\mathfrak{g}})}{\mathrm{m}_p(\mathfrak{g})}. \tag{5}$$

By the definition of p-mass (2), it holds that

$$\prod_{p} \frac{\mathrm{m}_p(\tilde{\mathfrak{g}})}{\mathrm{m}_p(\mathfrak{g})} = \underbrace{\frac{\Delta_2(\tilde{\mathfrak{g}})}{\Delta_2(\mathfrak{g})}}_{A} \cdot \underbrace{\prod_{p \mid d, p \neq 2} \frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})}}_{B} \cdot \underbrace{\prod_{p \nmid 2d} \frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})}}_{C} \cdot \underbrace{\prod_{p} \frac{\chi_p(\tilde{\mathfrak{g}})}{\chi_p(\mathfrak{g})}}_{D} \cdot \underbrace{\frac{\mathrm{type}_2(\tilde{\mathfrak{g}})}{\mathrm{type}_2(\mathfrak{g})}}_{E}. \tag{6}$$

We proceed to estimate the factors $A, \dots, E$. To begin with, we observe that the Jordan constituents behave in the following way:

$$J_p^i(\tilde{\mathfrak{g}}) = \begin{cases} J_p^i(\mathfrak{g}) & \text{for } i \neq 0, \\ [1] \oplus J_p^0(\mathfrak{g}) & \text{for } i = 0. \end{cases} \tag{7}$$

Proposition 2.8 (factor A)**.** *For a genus* $\mathfrak{g}$*,*

$$\frac{\Delta_2(\tilde{\mathfrak{g}})}{\Delta_2(\mathfrak{g})} \geq \begin{cases} \frac{1}{2} & \text{if } \mathfrak{g} \text{ is odd,} \\ \frac{1}{8} & \text{if } \mathfrak{g} \text{ is even.} \end{cases}$$

Proof. We refer to [2] for an explanation of the terms used in this proof.

If $J_2^0(\mathfrak{g})$ is of type II_{2t} or I_{2t+1} or I_{2t+2}, then $J_2^0(\tilde{\mathfrak{g}})$ is of type I_{2t+1} or I_{2t+2} or $\mathrm{I}_{2(t+2)+1}$. Thus by [2, Table 1] the type factor of $J_2^0(\mathfrak{g})$ is $2t\pm$ or $2t+1$ and that of $J_2^0(\tilde{\mathfrak{g}})$ is $2t\pm$, $2(t+1)\pm$, $2t+1$ or $2(t+1)+1$. However if $J_2^0(\mathfrak{g})$ is of type $2t+$, then $J_2^0(\tilde{\mathfrak{g}})$ cannot be of type $2t-$ since the octane value of $J_2^0(\mathfrak{g})$ increases by 1 when passing to $J_2^0(\tilde{\mathfrak{g}})$.

Using [2, Table 2], for $t \neq 0$ we have the following inequalities:

$$M_2(2(t+1)-) \leq M_2(2t+1) \leq M_2(2t+),$$
$$M_2(2t\pm) \leq M_2(2(t+1)\pm),$$
$$M_2(2t+1) \leq M_2(2(t+1)+1).$$

This leads us to the two cases for $t > 0$. If $J_2^0(\mathfrak{g})$ has type $2t+$, then

$$\frac{M_2^0(\tilde{\mathfrak{g}})}{M_2^0(\mathfrak{g})} \geq \frac{M_2(2(t+1)-)}{M_2(2t+)} = \frac{1}{(1+2^{-t})(1+2^{-t-1})} \geq \frac{8}{15} \geq \frac{1}{2},$$

else

$$\frac{M_2^0(\tilde{\mathfrak{g}})}{M_2^0(\mathfrak{g})} \geq \frac{M_2(2t-)}{M_2(2t+1)} = (1-2^{-t}) \geq \frac{1}{2}.$$

For $t = 0$ we reach the lower bound $1/2$.

Suppose first that $\mathfrak{g}$ is odd, so that $J_2^0(\mathfrak{g})$ is of type I. Then both $J_2^{-1}(\mathfrak{g})$ and $J_2^1(\mathfrak{g})$ are bound, so their contributions do not vary when passing to $\tilde{\mathfrak{g}}$; using the estimates above, we obtain

$$\frac{\Delta_2(\tilde{\mathfrak{g}})}{\Delta_2(\mathfrak{g})} = \frac{M_2^0(\tilde{\mathfrak{g}})}{M_2^0(\mathfrak{g})} \geq \frac{1}{2}.$$

Assume now that $\mathfrak{g}$ is even, so that $J_2^0(\mathfrak{g})$ is of type II. As $J_2^0(\tilde{\mathfrak{g}})$ is of type I, the status of $J_2^{-1}(\mathfrak{g})$ and $J_2^1(\mathfrak{g})$ might change from free to bound, so we must take into account their contributions. Arguing as above,

$$\frac{\Delta_2(\tilde{\mathfrak{g}})}{\Delta_2(\mathfrak{g})} = \frac{M_2^{-1}(\tilde{\mathfrak{g}})}{M_2^{-1}(\mathfrak{g})} \cdot \frac{M_2^0(\tilde{\mathfrak{g}})}{M_2^0(\mathfrak{g})} \cdot \frac{M_2^1(\tilde{\mathfrak{g}})}{M_2^1(\mathfrak{g})} \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$ □

Proposition 2.9 (factor B)**.** *Define*

$$\xi(n,d) = \prod_{p \mid d,\, p \neq 2} \left(1 + p^{-\max(0, \lceil \frac{n-\nu_p(d)}{2} \rceil)}\right)^{-1}.$$

If $\mathfrak{g}$ *is a genus of rank* n *and determinant* d*, then*

$$\prod_{p \mid d, p \neq 2} \frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})} \geq \xi(n,d).$$

Proof. Indeed, since $\dim J_p^0(\mathfrak{g}) \geq n-\nu_p(d)$ by (1), and since the contribution of $J_0^i(\mathfrak{g})$ does not vary when passing to $\tilde{\mathfrak{g}}$ for $i \geq 1$, we have

$$\frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})} = \frac{M_p^0(\tilde{\mathfrak{g}})}{M_p^0(\mathfrak{g})} \geq \left(1 + p^{-\max(0,\left\lceil \frac{n-\nu_p(d)}{2}\right\rceil)}\right)^{-1}.$$

We obtain the result by multiplying over all odd prime divisors of d. □

Lemma 2.10 (cf [10, Lemma 5.1])**.** *If $s > 1$, then*

$$(1+2^{-s})\frac{\zeta(2s)}{\zeta(s)} \leq \zeta_D(s) \leq (1-2^{-s})\zeta(s).$$

Proof. On the one hand,

$$\zeta_D(s) \geq \prod_{p\neq 2} \frac{1}{1+p^{-s}} = (1+2^{-s}) \prod_p \frac{1}{1+p^{-s}} = (1+2^{-s})\frac{\zeta(2s)}{\zeta(s)}.$$

On the other hand,

$$\zeta_D(s) \leq \prod_{p\neq 2} \frac{1}{1-p^{-s}} = (1-2^{-s})\zeta(s).$$ □

Proposition 2.11 (factor C)**.** *For a genus $\mathfrak{g}$ of determinant d and rank n,*

$$\prod_{p\nmid 2d} \frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})} = \begin{cases} \zeta_D(s) & \text{if } n = 2s-1, \\ \displaystyle\prod_{p\mid 2d}(1-p^{-2s})\frac{\zeta(2s)}{\zeta_D(s)} & \text{if } n = 2s. \end{cases}$$

Proof. By Remark 2.1, for $p \nmid 2d$ it holds that

$$\frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})} = \frac{\mathrm{std}_p(\tilde{\mathfrak{g}})}{\mathrm{std}_p(\mathfrak{g})}.$$

If $\mathfrak{g}$ is of rank $n = 2s-1$, then $\tilde{\mathfrak{g}}$ is of rank $2s$, hence

$$\frac{\mathrm{std}_p(\tilde{\mathfrak{g}})}{\mathrm{std}_p(\mathfrak{g})} = \frac{1}{1-\varepsilon_p(\tilde{\mathfrak{g}})p^{-s}} = \frac{1}{1-(D/p)p^{-s}},$$

whereas if $\mathfrak{g}$ is of rank $n = 2s$, then $\tilde{\mathfrak{g}}$ is of rank $2s+1 = 2(s+1)-1$, hence

$$\frac{\mathrm{std}_p(\tilde{\mathfrak{g}})}{\mathrm{std}_p(\mathfrak{g})} = \frac{1-\varepsilon_p(\mathfrak{g})p^{-s}}{1-p^{-2s}} = \frac{1-(D/p)p^{-s}}{1-p^{-2s}}.$$

The result ist obtained by multiplying over all primes $p \nmid 2d$. □

Corollary 2.12 (factor C, $n \geq 3$)**.** *For a genus $\mathfrak{g}$ of determinant d and rank $n = 2s$ or $n = 2s-1$, with $s > 1$,*

$$\prod_{p\nmid 2d} \frac{\Delta_p(\tilde{\mathfrak{g}})}{\Delta_p(\mathfrak{g})} \geq (1+2^{-s})\frac{\zeta(2s)}{\zeta(s)}. \tag{8}$$

Proof. This follows from Lemma 2.10 and Proposition 2.11. □

Proposition 2.13 (factor D)**.** *For a genus $\mathfrak{g}$ of determinant d,*

$$\prod_p \frac{\chi_p(\tilde{\mathfrak{g}})}{\chi_p(\mathfrak{g})} = \sqrt{d}.$$

Proof. From (1) and (7) it follows that

$$\begin{aligned}\chi_p(\tilde{\mathfrak{g}}) &= \prod_{1<j} p^{\frac{1}{2}j(1+\dim J_p^0(\mathfrak{g}))\dim J_p^j(\mathfrak{g})} \prod_{0<i<j} p^{\frac{1}{2}(j-i)\dim J_p^i(\mathfrak{g})\dim J_p^j(\mathfrak{g})} \\ &= \prod_{j} p^{\frac{1}{2}j\dim J_p^j(\mathfrak{g})} \prod_{0\le i<j} p^{\frac{1}{2}(j-i)\dim J_p^i(\mathfrak{g})\dim J_p^j(\mathfrak{g})} \\ &= p^{\frac{1}{2}\nu_p(d)}\chi_p(\mathfrak{g}).\end{aligned}$$

The result is obtained by multiplying over all primes. □

Proposition 2.14 (factor E)**.** *For a genus $\mathfrak{g}$ of determinant d,*

$$\frac{\operatorname{type}_2(\tilde{\mathfrak{g}})}{\operatorname{type}_2(\mathfrak{g})} = \begin{cases} 1 & \text{if } \mathfrak{g} \text{ is odd,} \\ 2^{\max(0,n-\nu_2(d))} & \text{if } \mathfrak{g} \text{ is even.}\end{cases}$$

Proof. If $\mathfrak{g}$ is odd, then $J_2^0(\mathfrak{g})$ is of type I. In this case, both $n_{\mathrm{I,I}}$ and n_{II} do not vary when passing to $\tilde{\mathfrak{g}}$. (Equality holds.) On the other hand, if $\mathfrak{g}$ is even, then $J_2^0(\mathfrak{g})$ is even, so $n_{\mathrm{I,I}}(\tilde{\mathfrak{g}}) \ge n_{\mathrm{I,I}}(\mathfrak{g})$ and $n_{\mathrm{II}}(\tilde{\mathfrak{g}}) = n_{\mathrm{II}}(\mathfrak{g}) - \dim J_2^0(\mathfrak{g})$. We conclude observing that $\dim J_2^0(\mathfrak{g}) \ge \max(0, n-\nu_2(d))$ by (1). □

We can now draw all estimates together in order to obtain a lower bound on $\mathrm{m}(\tilde{\mathfrak{g}})/\mathrm{m}(\mathfrak{g})$ for genera $\mathfrak{g}$ of rank > 2 (Theorem 2.15). The case of rank 2 is more delicate and is treated in Theorem 2.17.

Theorem 2.15. *Define ξ as in Proposition 2.9 and*

$$\begin{aligned} F_{\mathrm{I}}(n,d) &= \frac{1}{2}\,\xi(n,d)\sqrt{d}, \\ F_{\mathrm{II}}(n,d) &= \frac{1}{8}\,\xi(n,d)\sqrt{d}\cdot 2^{\max(0,n-\nu_2(d))}.\end{aligned}$$

For $s>1$, put

$$c_n = \pi^{-\frac{1}{2}(n+1)}\Gamma\left(\tfrac{1}{2}(n+1)\right)(1+2^{-s})\frac{\zeta(2s)}{\zeta(s)}.$$

Then, for a genus $\mathfrak{g}$ of rank $n=2s$ or $n=2s-1$ and determinant d,

$$\frac{\mathrm{m}(\tilde{\mathfrak{g}})}{\mathrm{m}(\mathfrak{g})} \ge \begin{cases} c_n F_{\mathrm{I}}(n,d) & \text{if } \mathfrak{g} \text{ is odd,} \\ c_n F_{\mathrm{II}}(n,d) & \text{if } \mathfrak{g} \text{ is even.}\end{cases}$$

Proof. The statement follows from (5) and (6), by comparing the factors $A,\dots,E$ using respectively Proposition 2.8, Proposition 2.9, Corollary 2.12 (here, $s>1$ is crucial), Proposition 2.13 and Proposition 2.14. □

We turn to the case $s=1$.

Lemma 2.16. *Put $\kappa_1 = 3/2$, $\kappa_2 = 3$. Then, for every $d>0$,*

$$\zeta_{-d}(1) \le \kappa_1 \log 4d + \kappa_2.$$

Proof. Put $D=-d$ and let $e \equiv 0,1 \mod 4$, $e \neq 0$. Define $\psi_e(m)$ as the Kronecker symbol $\left(\frac{e}{m}\right)$. The associated L-series is $L(\psi_e, s) = \sum_{m=1}^{\infty}\psi_e(m)m^{-s}$. By [6, §12.14, Theorem 14.3] we have $L(\chi_e, 1) \le 2 + \log|e|$.

Now, if $D \equiv 0 \mod 4$, then $\zeta_D(s) = L(\psi_D, s)$. If $D \equiv 2, 3 \mod 4$, then $\zeta_D(s) = \zeta_{4D}(s) = L(\psi_{4D}, s)$. Finally, if $D \equiv 1 \mod 4$, then

$$\begin{aligned}(1 - \psi_D(2)2^{-s})L(\psi_D, s) &= \sum_{m=1}^{\infty} \psi_D(m)m^{-s} - \sum_{m=1}^{\infty} \psi_D(2m)(2m)^{-s} \\ &= \sum_{m=1,3,5,\dots}^{\infty} \chi_D(m)m^{-s} \\ &= \zeta_D(s).\end{aligned}$$

Note that $(1 - \psi_D(2)2^{-1}) \leq 3/2$. In any case, we conclude that

$$\zeta_D(1) \leq 3 + \tfrac{3}{2}\log(4d).$$ □

Theorem 2.17. *Put $c_2' = \pi^{-\frac{3}{2}}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2\pi}$, $\kappa_1 = 3/2$, $\kappa_2 = 3$ and*

$$\tilde{F}_{\mathrm{P}}(d) = \frac{F_{\mathrm{P}}(2, d)}{\kappa_1 \log 4d + \kappa_2} \quad \textit{for } \mathrm{P} = \mathrm{I}, \mathrm{II},$$

where $F_{\mathrm{I}}, F_{\mathrm{II}}$ are defined as in Theorem 2.15. Then, for a genus $\mathfrak{g}$ of rank 2 and determinant d,

$$\frac{\mathrm{m}(\tilde{\mathfrak{g}})}{\mathrm{m}(\mathfrak{g})} \geq \begin{cases} c_2'\tilde{F}_{\mathrm{I}}(d) & \textit{if } \mathfrak{g} \textit{ is odd} \\ c_2'\tilde{F}_{\mathrm{II}}(d) & \textit{if } \mathfrak{g} \textit{ is even.} \end{cases}$$

Proof. The statement follows from (5) and (6), by comparing the factors $A, \dots, E$ using respectively Proposition 2.8, Proposition 2.9, Proposition 2.11 together with Lemma 2.16, Proposition 2.13 and Proposition 2.14. □

2.7. **Proof of Theorem 1.5.** Let us first fix $r \geq 4$ and let $\mathfrak{f}$ be an idoneal genus of rank r and determinant d. Then, by Proposition 2.6, $\mathfrak{f}$ has at least one slender twig $\mathfrak{g}$ of rank $n = r - 1$ (Definition 2.5). On account of Theorem 2.15, since d is also the determinant of $\mathfrak{g}$, d belongs to one of the following sets, depending on the parity of $\mathfrak{g}$:

$$\begin{aligned} D_n^{\mathrm{I}} &= \{d \in \mathbb{Z}_{>0} : c_n F_{\mathrm{I}}(n, d) \leq 1\}, \\ D_n^{\mathrm{II}} &= \{d \in \mathbb{Z}_{>0} : c_n F_{\mathrm{II}}(n, d) \leq 1\}. \end{aligned}$$

Note that $\sqrt{p}/(1 + p^{-n}) > 1$ unless $(n, p) = (0, 2), (1, 2)$ or $(0, 3)$. Hence

$$F_{\mathrm{I}}(n, d) < F_{\mathrm{I}}(n, d')$$

whenever d' is a multiple of d. This property implies the following preliminary bound on the rank n.

Lemma 2.18. *For $n \geq 21$, the sets D_n^{I} and D_n^{II} are empty.*

Proof. Indeed, we have $F_{\mathrm{P}}(n, d) \geq F_{\mathrm{P}}(n, 1) \geq \frac{1}{8}$ for $n \geq 2$ and $\mathrm{P} = \mathrm{I}, \mathrm{II}$. Moreover, $1 + 2^{-s} > 1$ and the functions $\zeta(2s)/\zeta(s)$ and $\pi^{-\frac{1}{2}(n+1)}\Gamma(\frac{1}{2}(n+1))$ are increasing for $s \geq 2$ and $n \geq 6$. Therefore, for $n \geq 21$ we have

$$c_n \geq c_{21} > 8.$$ □

TABLE 2. The number S_n of slender genera of rank $n \geq 2$ and the corresponding maximal determinant d_n (see Addendum 2.19).

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
S_n	534	977	1546	2134	2396	2296	1808	1230	712	361	157	61	19	4	1
d_n	10800	6912	4096	4096	1152	1024	1024	320	128	68	36	17	8	3	1

In order to determine the set D_n^{I}, we set $D_0 = \{1\}$ and compute inductively

$$D_{i+1} = \{pd \in \mathbb{Z}_{>0} : d \in D_i, p \text{ prime}, pd \in D_n^{\mathrm{I}}\},$$

so that $D_n^{\mathrm{I}} = \bigcup_i D_i$. The set D_n^{II} can be computed analogously, by first fixing $\nu_2(d)$. Thus, the following strategy enumerates all idoneal genera of rank r.

(i) Set $n = r - 1$.
(ii) Compute the set $D_n = D_n^{\mathrm{I}} \cup D_n^{\mathrm{II}}$.
(iii) For each $d \in D_n$, enumerate all sets of twigs of genera of rank r and determinant d.
(iv) For each set of twigs, determine if it satisfies condition (4).
(v) For each set satisfying (4), determine it it is the set of twigs of an idoneal genus.

For rank $r = 3$ we argue similarly, using Theorem 2.17 instead of Theorem 2.15 and

$$\tilde{D}_2^{\mathrm{I}} = \{d \in \mathbb{Z}_{>0} : c_2 \tilde{F}_{\mathrm{I}}(d) \leq 1\},$$
$$\tilde{D}_2^{\mathrm{II}} = \{d \in \mathbb{Z}_{>0} : c_2 \tilde{F}_{\mathrm{II}}(d) \leq 1\},$$

instead of $D_n^{\mathrm{I}}, D_n^{\mathrm{II}}$. The sets $\tilde{D}_2^{\mathrm{I}}, \tilde{D}_2^{\mathrm{II}}$ can also be computed inductively, by observing that $\tilde{F}_{\mathrm{I}}(d) < \tilde{F}_{\mathrm{I}}(pd)$ for every prime $p \neq 3$, or $p = 3$ and $d > 39$. Similarly, $\tilde{F}_{\mathrm{II}}(d) < \tilde{F}_{\mathrm{II}}(pd)$ for $p > 5$, or $p = 2$ and $4|d$, or $p = 3$ and $d > 39$. Since our bound on $\zeta_D(1)$ is far from optimal, a further step (ii-b) speeds up the computation:

(ii-b) Sieve the results of step (ii) using $F_{\mathrm{P}}(2, d)/\zeta_D(1)$ instead of $\tilde{F}_{\mathrm{P}}(d)$.

As a byproduct of this strategy, using Steps (i)–(iii), one can classify all slender genera of rank $n \geq 2$.

Addendum 2.19. *For $n \in \mathbb{N}, n \geq 2$, there exist exactly S_n slender genera of rank n and maximal determinant d_n, with S_n and d_n given in Table 2 for $n \leq 16$ and $S_n = 0$ otherwise.*

Remark 2.20. The computations were carried out using `sageMath` [22] or `Magma` [1]. Step (iii) can be carried out using the `sageMath` [22] function `genera` found in `sage.quadratic_forms.genera.genus` and implemented by the first author. Step (iv) involves only computations of the mass and is not difficult for a computer to carry out. We used the code in `sageMath` written by John Hanke for this purpose [5]. Step (v) involves computing all representatives of a genus (stopping whenever a representative which does not represent 1 is found). It is the most time-consuming step. For this purpose, we used Kneser's neighbour method (see [8, §28] and [18] for algorithmic considerations). The most important subtasks are isometry testing and the computation of the orthogonal groups. We used the algorithm of [17] shipped with `PARI` [23].

Remark 2.21. In its general formulation, the Smith–Minkowski–Siegel mass formula computes the weighted average of representations of a lattice by a genus. If L is a lattice of rank l and $\mathfrak{g}$ a positive definite genus of rank m, then

$$\frac{1}{\mathrm{m}(\mathfrak{g})}\sum_{M\in\mathfrak{g}}\frac{a(L,M)}{|O(M)|}=\frac{\gamma(m-l)}{\gamma(m)}\prod\alpha_p(L,\mathfrak{g}) \tag{9}$$

where $a(L,M)$ is the number of representations of L by M, $\alpha_p(L,\mathfrak{g})$ is the local representation density of L by the genus $\mathfrak{g}$ and γ is known [8, Kapitel X, Satz (33.6)]. If $\mathfrak{g}$ is an idoneal genus, then $a([1],M)\geq 1$ so that (9) yields

$$\frac{\gamma(m-1)}{\gamma(m)}\prod\alpha_p(1,\mathfrak{g})\geq 1.$$

This gives an alternative way to bound the determinant of an idoneal genus. However, the local representation densities $\alpha_p(1,\mathfrak{g})$ need to be estimated.

3. K3 surfaces covering an Enriques surface

The main theorem of this paper, namely Theorem 1.2, is proven in §3.5. First, some elementary properties of finite quadratic forms and Nikulin's theory of discriminant forms are reviewed (§3.1 and §3.2). Then, the possible shapes of the discriminant form of lattices of signature $(2,\lambda-2)$ embedding in Λ^- are determined in §3.3 and listed in Table 4. Finally, we find all exceptional lattices in §3.4.

3.1. **Finite quadratic forms.** Let A be a finite commutative group. The bilinear form associated to a finite quadratic form $q\colon A\to\mathbb{Q}/2\mathbb{Z}$ is given by

$$q^\flat\colon A\times A\to\mathbb{Q}/\mathbb{Z},\quad q^\flat(\alpha,\beta)=(q(\alpha+\beta)-q(\alpha)-q(\beta))/2.$$

A torsion quadratic form q is *nondegenerate* if the homomorphism of groups $A\to\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ given by $\alpha\mapsto(\beta\mapsto q^\flat(\alpha,\beta))$ is an isomorphism. If $H\subset A$ is a subgroup, then $q|H$ denotes the restriction of q to H.

We adopt Miranda–Morrison's notation [11] for the elementary finite quadratic forms u_k, v_k and $w^\varepsilon_{p,k}$. Moreover, the degenerate quadratic form $q\colon A\to\mathbb{Z}/2\mathbb{Z}$ with $A\cong\mathbb{Z}/2\mathbb{Z}$ taking the values $0 \mod 2\mathbb{Z}$ (resp. $1 \mod 2\mathbb{Z}$) on the non-trivial element is denoted by $\langle 0\rangle$ (resp. $\langle 1\rangle$). If A is 2-elementary, then the short exact sequence $0\to q^\perp\to q\to q/q^\perp\to 0$ splits. Hence, q can be written as the direct sum of copies of $u_1, v_1, w^1_{2,1}, w^3_{2,1}, \langle 0\rangle, \langle 1\rangle$. A (possibly degenerate) finite quadratic form q is *odd* if $q\cong w^\varepsilon_{2,1}\oplus q'$ for some ε and some finite quadratic form q', otherwise q is *even*.

From now on, unless explicitly stated, we assume a quadratic form to be nondegenerate.

The *signature* of a finite quadratic form q is defined as $\operatorname{sign} q=s_+-s_-$ mod 8, where (s_+,s_-) is the signature of any lattice L such that $q(L)\cong q$. Explicit formulas for the signature of elementary quadratic forms were found by Wall [24] (and are also reproduced in [14, Proposition 1.11.2]). Given a prime number p, q_p denotes the p-torsion part of q. We put $\ell_p(q) = \ell(q_p)$, where ℓ denotes the length of a finite abelian group.

Lemma 3.1. *For any finite quadratic form q, $\ell_2(q) \equiv \operatorname{sign} q \mod 2$.*

Proof. This can be checked for all u_k, v_k, $w^{\varepsilon}_{p,k}$. The claim follows by linearity, as each finite quadratic form is isomorphic to the direct sum of elementary forms [3]. □

Lemma 3.2. *Any finite quadratic q form contains a 2-elementary subgroup H of length $\ell_2(q) - 1$ such that $q|H$ is even. A finite quadratic q form contains a 2-elementary subgroup H of length $\ell_2(q)$ such that $q|H$ is even if and only if q is even.*

Proof. Let G be the underlying group of q. The subgroup $H = \{\alpha \in G : 2\alpha = 0\}$ is 2-elementary of length $\ell_2(q)$. The (possibly degenerate) quadratic form $q|H$ is even if and only q is.

Write $q|H$ as the direct sum of copies of $u_1, v_1, w^1_{2,1}, w^3_{2,1}, \langle 0\rangle, \langle 1\rangle$. By [14, Proposition 1.8.2], we can suppose that there are at most two copies of $w^{\varepsilon}_{2,1}$. But $w^1_{2,1} \oplus w^1_{2,1}$ and $w^3_{2,1} \oplus w^3_{2,1}$ contain a copy of $\langle 1\rangle$, while $w^1_{2,1} \oplus w^3_{2,1}$ contains a copy of $\langle 0\rangle$, so we conclude. □

The following lemma uses similar ideas as the previous ones. We leave the details to the reader.

Lemma 3.3. *Let $q = u_1^{\oplus n}$. If H is a subgroup of q and $m = \max\{0, \ell_2(H) - n\}$, then there exists a (possibly degenerate) finite quadratic form q' such that*

$$q|H \cong u_1^{\oplus m} \oplus q'.$$

3.2. **Nikulin's theory of discriminant forms.** If L is an even lattice, $q(L)$ denotes its discriminant form.

Theorem 3.4 (Nikulin [14, Theorem 1.9.1])**.** *For each finite quadratic form q and prime number p there exists a unique p-adic lattice $K_p(q)$ of rank $\ell_p(q)$ whose discriminant form is isomorphic to q_p, except in the case when $p = 2$ and q is odd.*

We introduce the following conditions (depending on $s, s' \in \mathbb{Z}$) on a finite quadratic form q.

A(s):	$\operatorname{sign} q \equiv s \mod 8$.
B(s, s'):	for all primes $p \neq 2$, $\ell_p(q) \leq s + s'$; moreover, $\lvert q\rvert \equiv (-1)^{s'} \operatorname{discr} K_p(q) \mod (\mathbb{Z}_p^\times)^2$ if $\ell_p(q) = s + s'$.
C(s):	$\ell_2(q) \leq s$; moreover, $\lvert q\rvert \equiv \pm \operatorname{discr} K_2(q) \mod (\mathbb{Z}_2^\times)^2$ if $\ell_2(q) = s$ and q is even.

Theorem 3.5 (Nikulin's Existence Theorem [14, Theorem 1.10.1])**.** *An even lattice of signature (t_+, t_-), $t_+, t_- \in \mathbb{Z}_{\geq 0}$, and discriminant quadratic form q exists if and only if q satisfies conditions A$(t_+ - t_-)$, B(t_+, t_-) and C$(t_+ + t_-)$.*

Theorem 3.6 (Nikulin [14, Theorem 1.14.2])**.** *If T is an even, indefinite lattice satisfying the following conditions:*

(a) $\operatorname{rank} T \geq \ell(q(T)) + 2$ *for all* $p \neq 2$,
(b) if $\operatorname{rank} T = \ell_2(q(T))$, *then* $q(T) \cong u_1 \oplus q'$ *or* $q(T) \cong v_1 \oplus q'$,
then the genus of T *contains only one class.*

Given a pair of nonnegative integers (m_+, m_-) and a finite quadratic form q, Nikulin establishes a useful way to enumerate the set of primitive embeddings of a fixed lattice T into any even lattice belonging to the genus $\mathfrak{g}$ of signature (m_+, m_-) and discriminant form q ([14, Proposition 1.15.1]). The following proposition is a simplified version of Nikulin's proposition in the case that the genus $\mathfrak{g}$ contains only one class.

Proposition 3.7. *Let* T *be an even lattice of signature* (t_+, t_-) *and* Λ *be an even lattice of signature* (m_+, m_-) *which is unique in its genus. Then, for each lattice* S*, there exists a primitive embedding* $T \hookrightarrow \Lambda$ *with* $T^\perp \cong S$ *if and only if* $\operatorname{sign} S = (m_+ - t_+, m_- - t_-)$ *and there exist subgroups* $H \subset q(\Lambda)$ *and* $K \subset q(T)$*, and an isomorphism of quadratic forms* $\gamma\colon q(\Lambda)|H \to q(T)|K$*, whose graph we denote by* Γ*, such that*

$$q(S) \cong q(\Lambda) \oplus (-q(T))|\Gamma^\perp/\Gamma.$$

3.3. **Transcendental lattices embedding in** Λ^-**.** Recall that we defined $\Lambda^- = U \oplus U(2) \oplus E_8(-2)$. We have $\operatorname{sign} \Lambda^- = (2, 10)$ and $q(\Lambda^-) \cong u_1^{\oplus 5}$.

In this section the possible discriminant forms of even lattices of signature $(2, \lambda - 2)$ which embed primitively into Λ^- are determined. Necessarily, $2 \leq \lambda \leq 12$.

Lemma 3.8. *Let* $f\colon F \to \mathbb{Q}/2\mathbb{Z}$ *and* $g\colon G \to \mathbb{Q}/2\mathbb{Z}$ *be finite quadratic forms with* $f \cong u_1^{\oplus n}$*. Let* $H \subset F$ *and* $K \subset G$ *be subgroups and* $\gamma\colon f|H \to g|K$ *an isometry. Let* Γ *be the graph of* γ *in* $F \oplus G$*. Then*

$$u_1^{\oplus \ell_2(H)} \oplus \Big(f \oplus (-g)|\Gamma^\perp/\Gamma\Big) \cong f \oplus (-g).$$

In particular,

$$\ell_2(f) + \ell_2(g) = \ell_2(\Gamma^\perp/\Gamma) + 2\ell_2(H).$$

Proof. Recall that $f^\flat$ and $g^\flat$ denote the bilinear forms induced by the quadratic forms f and g, respectively. Let $C = \ker(f^\flat|H)$. The exact sequence

$$0 \to C^\perp \to F \to \operatorname{Hom}(C, \mathbb{Q}/\mathbb{Z}) \to 0$$

(induced by $f^\flat$) splits, because F is 2-elementary by assumption.

Let $s\colon \operatorname{Hom}(K, \mathbb{Q}/\mathbb{Z}) \to F$ be a section and $C_s^\vee$ its image. Then, since C is a totally isotropic subspace with respect to $f^\flat$, we infer that $f^\flat|(C \oplus C_s^\vee) \cong (u_1^\flat)^{\oplus \ell}$, with $\ell = \ell_2(C)$. By modifying the section s, we may assume that $f^\flat|C_s^\vee = 0$. Consider the subgroups $F_s' = (H \oplus C_s^\vee)^\perp \subset F$, $D_s = (C \oplus C_s^\vee)^\perp \subset H \oplus C_s^\vee$ and $G_s' = \gamma(D_s)^\perp \subset G$. Putting $f' = f|F_s'$, $g' = g|G_s'$ and $d = f|D_s$, we obtain

$$\begin{aligned} f^\flat &\cong f^\flat|D_s \oplus (C \oplus C_s^\vee) \oplus F_s' \cong d^\flat \oplus (u_1^\flat)^{\oplus \ell} \oplus (f')^\flat, \\ g^\flat &\cong g^\flat|\gamma(D_s) \oplus G_s' \cong d^\flat \oplus (g')^\flat. \end{aligned}$$

Let $\varphi\colon G_s' \to C_s^\vee$ be defined by $f^\flat(\varphi(\beta), \alpha) = g^\flat(\beta, \gamma(\alpha))$ for all $\alpha \in C \subset H$ and $\beta \in G_s'$. Define $\psi\colon G_s' \to \Gamma^\perp$ by $\psi(\beta) = \varphi(\beta) + \beta$. Since $f^\flat|C_s^\vee = 0$, we

TABLE 3. The three cases appearing in the proof of Proposition 3.9.

λ_{case}	$\ell_2(q(T))$	$\ell_2(H)$	$\ell_2(q(T^\perp))$	parity of $q(T)$
λ_{a}	$\lambda-2$	$\lambda-2$	$12-\lambda$	even
λ_{b}	λ	$\lambda-1$	$12-\lambda$	even or odd
λ_{c}	λ	λ	$10-\lambda$	even

have

$$f^\flat \oplus (-g^\flat)(\psi(\beta),\psi(\beta')) = f^\flat(\varphi(\beta),\varphi(\beta')) - g^\flat(\beta,\beta') = -g^\flat(\beta,\beta'). \tag{10}$$

It follows that $f^\flat \oplus (-g^\flat)$ restricted to $F'_s \oplus \psi(G'_s) \subset \Gamma^\perp$ is nondegenerate. Since the orders of $\Gamma^\perp/\Gamma$ and $F'_s \oplus \psi(G'_s)$ coincide, this shows that

$$(f \oplus (-g))^\flat|\Gamma^\perp/\Gamma \cong (f' \oplus (-g'))^\flat.$$

We claim that we can choose the section s in such a way that the quadratic forms coincide. Indeed, if $f|C_s^\vee = 0$, then (10) also holds at the level of quadratic forms, so we can replace $f^\flat, g^\flat$ by f, g, respectively, and we are done (note that $d^{\oplus 2} \cong u_1^{\oplus \ell_2(H)-\ell}$).

If $C \oplus C_s^\vee = F$, the section s can be modified so that $f|C_s^\vee = 0$, because $f \cong u_1^{\oplus n}$. Otherwise, as $f|(C \oplus C_s^\vee)^\perp$ is then even, nondegenerate and nonzero, there exists $\alpha \in (C \oplus C_s^\vee)^\perp \subset C^\perp$ with $f(\alpha) = 1$. Let $\delta_1, \dots, \delta_\ell$ be a basis of $\operatorname{Hom}(C, \mathbb{Q}/\mathbb{Z})$, so that $C_s^\vee = \langle s(\delta_1), \dots, s(\delta_\ell)\rangle$. Define s' by $s'(\delta_i) = s(\delta_i)$ if $x(s(\delta_i)) = 0$ and $s'(\delta_i) = s(\delta_i) + \alpha$ else. By replacing s with s' we are again in the situation where $x|C_s^\vee = 0$ and we conclude. □

Proposition 3.9. *An even lattice T of signature $(2, \lambda-2)$ embeds primitively into Λ^- if and only if $q(T)$ is of the form given in Table 4 for some nondegenerate finite quadratic form q satisfying the given conditions. In that case, $q(T^\perp)$ is isomorphic to the form given in the corresponding column of the table.*

Proof. If $\lambda = 12$, then $\operatorname{rank} T = \operatorname{rank} \Lambda^-$, so the claim is trivial. For the rest of the proof we will suppose that $2 \le \lambda \le 11$.

Assume first that T embeds in Λ^- and let $H \subset q(T)$ be the subset given by Proposition 3.7. By the last equation in Lemma 3.8 we see that

$$\ell_2(q(T)) + 10 = \ell_2(q(T^\perp)) + 2\ell_2(H).$$

Using $\ell_2(q(T^\perp)) \le \operatorname{rank}(T^\perp) = 12 - \lambda$ and $\ell_2(H) \le \ell_2(q(T))$, we see that

$$\lambda - 2 = 10 - \operatorname{rank}(T^\perp) \le 2\ell_2(H) - \ell_2(q(T)) \le \ell_2(q(T)) \le \operatorname{rank}(T) = \lambda.$$

By Lemma 3.1, $\ell_2(q(T))$ can only assume two values, namely $\lambda - 2$ or λ .

We also infer that if $\ell_2(q(T)) = \lambda - 2$, then $\ell_2(q(H)) = \lambda - 2$, whereas if $\ell_2(q(T)) = \lambda$, then either $\ell_2(H) = \lambda - 1$ or $\ell_2(H) = \lambda$ (if $\lambda = 11$ only $\ell_2(H) = \lambda - 1$ is possible, as $\ell_2(H) \le \ell_2(q(\Lambda^-)) = 10$). Moreover, by Lemma 3.2, q must be even whenever $\ell_2(H) = \ell_2(q(T))$.

Summarizing, for each $\lambda \in \{2, \dots, 11\}$ we have three cases, described in Table 3, except for $\lambda = 11$ where the last case does not occur.

Let $m = \max\{0, \ell_2(H) - 5\}$. Lemma 3.3 ensures the existence of a subgroup of H (hence of $q(T)$) isometric to $u_1^{\oplus m}$; therefore, $q(T) \cong u_1^{\oplus m} \oplus q$. The form of $q(T^\perp)$ is then given by Lemma 3.8 and we can apply Theorem 3.5 to find all necessary conditions on q. We only write those that are also sufficient: for instance, condition A$(12-\lambda)$ for $q(T^\perp)$ is always equivalent to A(λ) for $q(T)$; if $2 \le \lambda \le 6$, condition B$(2, \lambda - 2)$ for $q(T)$ automatically implies B$(0, 12-\lambda)$ for $q(T^\perp)$.

Conversely, if T is given as in one of the rows of Table 4, then the conditions on q, together with Proposition 3.7 and Lemma 3.2, ensure the existence of a primitive embedding $T \hookrightarrow \Lambda^-$ with $q(T^\perp)$ of the given form. □

3.4. **Exceptional lattices.** Recall that we defined an (even) lattice T of signature $(2, \lambda - 2)$ to be exceptional if the following holds: T embeds primitively into Λ^- and, for every primitive embedding $T \hookrightarrow \Lambda^-$, $T^\perp$ contains a vector of square -2.

Here we list all exceptional lattices. We start with an elementary but useful lemma, of which we omit the proof. The following proposition explains the connection between exceptional lattices and idoneal genera.

Lemma 3.10. *A lattice L satisfies $\ell_2(q(L)) = \operatorname{rank} L$ if and only if there exists a lattice L' with $L = L'(2)$. In this case, L' is even if and only if $q(L)$ is even.*

Proposition 3.11. *For each exceptional lattice T there exists a unique idoneal genus $\mathfrak{g}$ with the following property: for each primitive embedding $T \hookrightarrow \Lambda^-$, there exists $L \in \mathfrak{g}$ with $T^\perp \cong L(-2)$.*

Proof. Consider a primitive embedding $T \hookrightarrow \Lambda^-$ of an exceptional lattice T of rank $\lambda \in \{2, \dots 11\}$ and let $H \subset q(T)$ be the subset given by Proposition 3.7. The orthogonal complement $T^\perp$ has rank $12 - \lambda$.

We claim that we are in case λ_{b} of Table 3. Indeed, by inspection of Table 4, one sees that $q(T)$ is even if and only if $q(T^\perp)$ is even. In case λ_{a}, $\ell_2(q(T^\perp)) = \operatorname{rank}(q(T^\perp))$ and $q(T)$ is even, so $T^\perp \cong T'(2)$ for some even lattice T', by Lemma 3.10. Thus, $T^\perp$ does not contain a vector of square -2 and T cannot be exceptional. If an embedding as in case λ_{c} exists, then an embedding as in case λ_{b} exists: it suffices to choose a smaller H. Moreover, $q(T)$ is even in case λ_{c}, so we can argue as before and T cannot be exceptional.

In the second case of Table 3 we have $\ell_2(q(T^\perp)) = \operatorname{rank}(q(T^\perp))$. By Lemma 3.10, there exists a positive definite lattice L of rank $12 - \lambda$ with $T^\perp = L(-2)$. Let $\mathfrak{g}$ be the genus of L.

The discriminant form of $T^\perp$ is determined by the discriminant form of T according to Table 4. According to [14, Corollary 1.16.3], the genus of a lattice is determined by its signature, parity and discriminant bilinear form. Hence, each lattice L such that $T^\perp \cong L(-2)$ belongs to the same genus $\mathfrak{g}$.

Conversely, as Λ^- is unique in its genus, each lattice $L \in \mathfrak{g}$ satisfies $L(-2) \cong T^\perp$ for some embedding $T \hookrightarrow \Lambda^-$, by Proposition 3.7. Since $T^\perp$ always contains a vector of square -2, each lattice $L \in \mathfrak{g}$ contains a vector of square 1, ie $\mathfrak{g}$ is idoneal. □

TABLE 4. Discriminant forms of even lattices T of signature $(2, \lambda - 2)$ embedding primitively into Λ^- (see Proposition 3.9).

λ_{case}	$q(T)$	$q(T^\perp)$	$\ell_2(q)$	conditions on q
2_{a}	q	$u_1^{\oplus 5} \oplus (-q)$	0	C(0)
2_{b}	q	$u_1^{\oplus 4} \oplus (-q)$	2	–
2_{c}	q	$u_1^{\oplus 3} \oplus (-q)$	2	even
3_{a}	q	$u_1^{\oplus 4} \oplus (-q)$	1	C(1), even
3_{b}	q	$u_1^{\oplus 3} \oplus (-q)$	3	–
3_{c}	q	$u_1^{\oplus 2} \oplus (-q)$	3	even
4_{a}	q	$u_1^{\oplus 3} \oplus (-q)$	2	C(2), even
4_{b}	q	$u_1^{\oplus 2} \oplus (-q)$	4	–
4_{c}	q	$u_1 \oplus (-q)$	4	even
5_{a}	q	$u_1^{\oplus 2} \oplus (-q)$	3	C(3), even
5_{b}	q	$u_1 \oplus (-q)$	5	–
5_{c}	q	$-q$	5	even
6_{a}	q	$u_1 \oplus (-q)$	4	C(4), even
6_{b}	q	$-q$	6	–
6_{c}	$u_1 \oplus q$	$-q$	4	even
7_{a}	q	$-q$	5	B(5, 0), C(5), even
7_{b}	$u_1 \oplus q$	$-q$	5	B(5, 0)
7_{c}	$u_1^{\oplus 2} \oplus q$	$-q$	3	B(5, 0), even
8_{a}	$u_1 \oplus q$	$-q$	4	B(4, 0), C(4), even
8_{b}	$u_1^{\oplus 2} \oplus q$	$-q$	4	B(4, 0)
8_{c}	$u_1^{\oplus 3} \oplus q$	$-q$	2	B(4, 0), even
9_{a}	$u_1^{\oplus 2} \oplus q$	$-q$	3	B(3, 0), C(3), even
9_{b}	$u_1^{\oplus 3} \oplus q$	$-q$	3	B(3, 0)
9_{c}	$u_1^{\oplus 4} \oplus q$	$-q$	1	B(3, 0), even
10_{a}	$u_1^{\oplus 3} \oplus q$	$-q$	2	B(2, 0), C(2), even
10_{b}	$u_1^{\oplus 4} \oplus q$	$-q$	2	B(2, 0)
10_{c}	$u_1^{\oplus 5} \oplus q$	$-q$	0	B(2, 0)
11_{a}	$u_1^4 \oplus q$	$-q$	1	B(1, 0), C(1), even
11_{b}	$u_1^5 \oplus q$	$-q$	1	B(1, 0)
12	u_1^5	–	–	–

Theorem 3.12. *For each $\lambda \in \mathbb{N}$ there exist E_λ exceptional lattices of signature $(2, \lambda - 2)$, with E_λ given in Table 1 if $2 \le \lambda \le 11$ and $E_\lambda = 0$ otherwise. There exist no other exceptional lattices of rank $\lambda \neq 10$, and there exists at most one more of rank 10 if the generalized Riemann hypothesis does not hold (see Remark 1.4). The list of all known 550 exceptional lattices is contained in Table 5.*

Proof. Thanks to Proposition 3.11 and Theorem 1.5, one can proceed in the following way.

(i) For each $\mathfrak{g}$ of rank $r = 12 - \lambda$ in the list we pick any $L \in \mathfrak{g}$ and compute $x = q(L(-2))$.
(ii) If x is one of the suitable forms $q(T^\perp)$ in Table 4, then we compute the corresponding $y = q(T)$.
(iii) We add all lattices T of signature $(2, \lambda - 2)$ and discriminant form y to Table 5.

In this way we enumerate all exceptional lattices. □

The following turns out to be true.

Addendum 3.13. *All exceptional lattices are unique in their genus.*

3.5. **Proof of Theorem 1.2.** Let X be a K3 surface with transcendental lattice T of rank λ. By Keum's criterion (Theorem 1.1), if T is not an exceptional lattice, then X covers an Enriques surface if and only if there exists a primitive embedding $T \hookrightarrow \Lambda^-$. Theorem 1.2 follows from Theorem 3.12 once we prove that the conditions given by Proposition 3.9 are equivalent to conditions (i)–(xii). Therefore, we need to analyze all cases of Table 4.

Let us first consider the case $2 \leq \lambda \leq 6$. We want to prove that (i) holds if and only if one of the following holds

(a) $\ell_2(q(T)) = \lambda - 2$, $q(T)$ is even and satisfies condition $\mathrm{C}(\lambda - 2)$ (case λ_{a});
(b) $\ell_2(q(T)) = \lambda$ (case λ_{b} or λ_{c}).

Let $e_1, \ldots, e_\lambda$ be a system of generators of T. Suppose first that the corresponding Gram matrix satisfies (i).

If a_{1j} is even for $2 \leq j \leq \lambda$, then $\ell_2(q(T)) = \operatorname{rank} T = \lambda$ by Lemma 3.10, hence (b) holds.

If this is not true, we can suppose a_{12} to be odd and a_{1j} to be even for $3 \leq j \leq \lambda$, up to relabelling and substituting e_j with $e_j + e_2$. Let T' be the sublattice generated by $e_1' = 2e_1, e_2, \ldots, e_\lambda$. Then $q(T') \cong u_1 \oplus q(T)$, where the copy of u_1 is generated by $e_1'/2$ and $e_2/2$. Since $T' \cong T''(2)$ with T'' even, $q(T')$ is even and $\ell_2(q(T')) = \lambda$, by Lemma 3.10. Moreover, $q(T')$ satisfies condition $\mathrm{C}(\lambda)$ by Theorem 3.5. This implies (a).

Conversely, if (b) holds, then Lemma 3.10 implies that $T \cong T'(2)$ for some lattice T'. If T' is even, then (i) holds. If T' is odd, then up to relabelling we can suppose that $e_1^2 \equiv 2 \mod 4$. Then, up to substituting e_j with $e_j + e_1$, we can suppose that $e_j^2 \equiv 0 \mod 4$. Hence, (i) holds.

Finally, suppose that (a) holds. Since T exists, q satisfies also $\mathrm{A}(4 - \lambda)$ and condition $\mathrm{B}(2, \lambda - 2)$. Therefore, $u_1 \oplus q$ satisfies conditions $\mathrm{A}(4 - \lambda)$, $\mathrm{B}(2, \lambda - 2)$ and $\mathrm{C}(\lambda)$, so the genus $\mathfrak{g}$ of even lattices of signature $(2, \lambda - 2)$ and discriminant form $u_1 \oplus q$ is nonempty. All lattices T' in $\mathfrak{g}$ are of the form $T' \cong T''(2)$ for some even lattice T''. By Proposition 1.4.1 in [14], T must be an overlattice of such a lattice T'. The fact that $\det T' = 4 \det T$ implies T' has index 2 in T. Therefore we can find a basis $e_1, \ldots e_\lambda$ with $e_1 \notin T'$ and $e_j \in T'$ for $j = 2, \ldots, \lambda$. Then, the corresponding Gram matrix satisfies (i).

We now turn to $\lambda \geq 7$. The arguments for $\lambda \in \{7, \ldots, 11\}$ are very similar, so we illustrate here only the case $\lambda = 10$.

Suppose case λ_{a} holds, ie $\lambda = 10$ and $q(T) \cong u_1^{\oplus 3} \oplus q$, with $\ell_2(q) = 2$, q even and satisfying conditions B$(2,0)$, C(2). Since T exists, q satisfies also A(2), by Theorem 3.5. Hence, using Theorem 3.5 again and Lemma 3.10, we infer that there exists an even lattice T' of signature $(2,0)$ such that $T'(2)$ has discriminant form q. Since T is unique in its genus (Theorem 3.6), $T \cong E^\dagger \oplus T'(2)$, so (viii) holds.

Suppose case λ_{b} or λ_{c} holds, ie $\lambda = 10$ and $q(T) \cong u_1^{\oplus 4} \oplus q$, with $\ell_2(q) = 2$, and q satisfying condition B$(2,0)$. Since T exists, q satisfies also A(2) and C(2). Hence, there exists a lattice T' of signature $(2,0)$ such that $T'(2)$ has discriminant form q. Again by uniqueness, $T \cong E_8(-2) \oplus T'(2)$, so (ix) holds.

Conversely, if (viii) holds, then $q(T) \cong u_1^{\oplus 3} \oplus q$, with $q = q(T')$ being an even finite quadratic form satisfying conditions A(2), B$(2,0)$ and C(2) by Theorem 3.5. Hence, by Theorem 3.5 and Proposition 3.7, there exists a primitive embedding $T \hookrightarrow \Lambda^-$. An analogous argument works for (ix).

Remark 3.14. For the equivalence between (vi) and case 9_{a}, one uses the following fact: if q, q' are two torsion quadratic forms, then $u_1 \oplus q \cong u_1 \oplus q'$ if and only if $q \cong q'$ (see [8, Kapitel I, Satz (4.3)]).

Table 5. Exceptional lattices (see Theorem 3.12).

λ	No.	$\det(T)$	T
2	1	4	$[2]^{\oplus 2}$
2	2	8	$[2] \oplus [4]$
2	3	16	$[2] \oplus [8]$
3	1	-8	$[2] \oplus U(2)$
3	2	-16	$[4] \oplus [0,1,1](2)$
3	3	-24	$[2]^{\oplus 2} \oplus [-6]$
3	4	-32	$[2] \oplus [4] \oplus [-4]$
3	5	-32	$[2] \oplus U(4)$
3	6	-40	$[2] \oplus [2,1,-2](2)$
3	7	-64	$[4] \oplus [0,2,-1](2)$
3	8	-72	$[-2] \oplus [6]^{\oplus 2}$
4	1	16	$[0,1,1](2)^{\oplus 2}$
4	2	32	$[2] \oplus [-4] \oplus U(2)$
4	3	48	$[-2] \oplus [6] \oplus U(2)$
4	4	64	$[2] \oplus [-1,1,1,-1,1,1](-2)$
4	5	64	$[0,1,1](2) \oplus U(4)$
4	6	64	$[0,1,1](2) \oplus [0,2,-1](2)$
4	7	64	$[0,1,1](2) \oplus [0,2,1](2)$
4	8	80	$[-2] \oplus [2] \oplus [2,1,-2](2)$
4	9	96	$[2] \oplus [1,2,-1,0,1,1](-2)$
4	10	128	$[2] \oplus [-4] \oplus [0,1,1](4)$
4	11	128	$[2] \oplus [-2,1,2,0,1,2](-2)$
4	12	144	$[6] \oplus [-6] \oplus U(2)$
4	13	144	$[0,3,2,0,2,0,0,-5,1,1](2)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
4	14	192	$[6] \oplus [0,2,-2,0,1,1](-2)$
4	15	256	$[-8] \oplus [0,2,1,0,1,1](2)$
4	16	256	$[8] \oplus [0,2,1,0,1,1](-2)$
4	17	256	$[0,1,1](2) \oplus U(8)$
5	1	-32	$[-2] \oplus U(2)^{\oplus 2}$
5	2	-64	$[2] \oplus [-2] \oplus [-4] \oplus U(2)$
5	3	-96	$[-2]^{\oplus 2} \oplus [6] \oplus U(2)$
5	4	-96	$[2] \oplus [-2] \oplus [-6] \oplus U(2)$
5	5	-128	$[-2] \oplus [4] \oplus [-4] \oplus [0,1,1](2)$
5	6	-128	$[-2] \oplus U(2) \oplus U(4)$
5	7	-128	$[2] \oplus [-2] \oplus [-8] \oplus U(2)$
5	8	-128	$[-2] \oplus U(2) \oplus [-1,1,3](2)$
5	9	-160	$[-2]^{\oplus 2} \oplus [10] \oplus U(2)$
5	10	-192	$[-12] \oplus [0,1,1](2) \oplus U(2)$
5	11	-224	$[-2] \oplus U(2) \oplus [2,1,-3](2)$
5	12	-224	$[-2]^{\oplus 2} \oplus [14] \oplus U(2)$
5	13	-256	$[2] \oplus [-2] \oplus [-4] \oplus U(4)$
5	14	-256	$[-2] \oplus [-4] \oplus [8] \oplus U(2)$
5	15	-256	$[-2]^{\oplus 2} \oplus [16] \oplus U(2)$
5	16	-256	$[-2] \oplus [2] \oplus [0,2,-2,0,1,2](-2)$
5	17	-256	$[2] \oplus [-2] \oplus [-16] \oplus U(2)$
5	18	-288	$U(2) \oplus [0,3,-2,0,1,1](-2)$
5	19	-288	$[0,3,2,0,1,0,0,1,0,1,0,1,1,0,0](-2)$
5	20	-320	$[2] \oplus [-2] \oplus [-20] \oplus U(2)$
5	21	-384	$[-6] \oplus [0,1,1](2) \oplus U(4)$
5	22	-384	$[-2]^{\oplus 2} \oplus [6] \oplus U(4)$
5	23	-384	$[-2] \oplus [0,1,1](2) \oplus [2,1,-1](4)$
5	24	-384	$[-2] \oplus [-6] \oplus [8] \oplus U(2)$
5	25	-416	$[-2]^{\oplus 2} \oplus [26] \oplus U(2)$
5	26	-480	$[6] \oplus U(2) \oplus [2,1,3](-2)$
5	27	-512	$[-2] \oplus U(4)^{\oplus 2}$
5	28	-512	$[-8] \oplus [0,1,1](2) \oplus U(4)$
5	29	-512	$[2] \oplus [-8]^{\oplus 2} \oplus U(2)$
5	30	-512	$[0,2,-2,0,1,1,0,1,0,-2,0,1,0,0,2](-2)$
5	31	-512	$[-8] \oplus U(2) \oplus [-1,1,3](2)$
5	32	-512	$[2] \oplus [-2] \oplus [-32] \oplus U(2)$
5	33	-576	$[-2] \oplus [-4] \oplus [18] \oplus U(2)$
5	34	-640	$[-2]^{\oplus 2} \oplus [10] \oplus U(4)$
5	35	-672	$[-2] \oplus [-6] \oplus [14] \oplus U(2)$
5	36	-768	$[-48] \oplus [0,1,1](2) \oplus U(2)$
5	37	-864	$[-6] \oplus [0,1,1](2) \oplus [0,1,1](6)$
5	38	-1024	$[2] \oplus [-2] \oplus [-4] \oplus U(8)$
5	39	-1024	$[2] \oplus [-2] \oplus [-16] \oplus U(4)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
5	40	-1536	$U(4) \oplus [0,2,-1,0,1,3](-2)$
5	41	-2048	$[2] \oplus [-8]^{\oplus 2} \oplus U(4)$
6	1	64	$[-2]^{\oplus 2} \oplus U(2)^{\oplus 2}$
6	2	128	$[-2]^{\oplus 2} \oplus [2] \oplus [-4] \oplus U(2)$
6	3	192	$[-2] \oplus [-6] \oplus U(2)^{\oplus 2}$
6	4	192	$[2] \oplus [-2] \oplus U(2) \oplus A_2(-2)$
6	5	256	$[2] \oplus [-2] \oplus [-4]^{\oplus 2} \oplus U(2)$
6	6	256	$[-2]^{\oplus 2} \oplus U(2) \oplus U(4)$
6	7	256	$[-2]^{\oplus 2} \oplus U(2) \oplus [0,2,-1](2)$
6	8	256	$[-2] \oplus [-8] \oplus U(2)^{\oplus 2}$
6	9	320	$[-2]^{\oplus 3} \oplus [10] \oplus U(2)$
6	10	320	$U(2)^{\oplus 2} \oplus [2,1,3](-2)$
6	11	384	$[-2]^{\oplus 2} \oplus [-4] \oplus [6] \oplus U(2)$
6	12	384	$[2] \oplus [-2] \oplus [-4] \oplus [-6] \oplus U(2)$
6	13	448	$[-2] \oplus [-14] \oplus U(2) \oplus [0,1,1](2)$
6	14	512	$[-2] \oplus [-4] \oplus U(4) \oplus U(2)$
6	15	512	$[-2]^{\oplus 2} \oplus [-4] \oplus [8] \oplus U(2)$
6	16	512	$[-2]^{\oplus 3} \oplus [16] \oplus U(2)$
6	17	512	$[-2] \oplus [2] \oplus U(2) \oplus [3,1,3](-2)$
6	18	512	$[-2] \oplus [-16] \oplus U(2)^{\oplus 2}$
6	19	576	$[2] \oplus [-2] \oplus [-6]^{\oplus 2} \oplus U(2)$
6	20	576	$[-2]^{\oplus 2} \oplus U(2) \oplus [-2,1,4](2)$
6	21	576	$[-2]^{\oplus 3} \oplus [18] \oplus U(2)$
6	22	640	$[-2]^{\oplus 2} \oplus [-4] \oplus [10] \oplus U(2)$
6	23	704	$[-2] \oplus [2] \oplus U(2) \oplus [3,1,4](-2)$
6	24	704	$[2] \oplus [-2] \oplus [-2,1,2,0,1,2,0,1,1,2](-2)$
6	25	768	$[2] \oplus [-2] \oplus U(4) \oplus A_2(-2)$
6	26	768	$[2] \oplus [-2] \oplus [-4] \oplus [-12] \oplus U(2)$
6	27	768	$[-2] \oplus [-6] \oplus U(2) \oplus U(4)$
6	28	768	$[-6] \oplus [-8] \oplus [0,1,1](2) \oplus U(2)$
6	29	768	$[2] \oplus [-8] \oplus U(2) \oplus A_2(-2)$
6	30	768	$[-2] \oplus [-24] \oplus [0,1,1](2) \oplus U(2)$
6	31	896	$[-2] \oplus [-4] \oplus U(2) \oplus [2,1,-3](2)$
6	32	896	$[-2]^{\oplus 2} \oplus [-4] \oplus [14] \oplus U(2)$
6	33	960	$[-2] \oplus [-30] \oplus [0,1,1](2) \oplus U(2)$
6	34	960	$[-2] \oplus [-6] \oplus U(2) \oplus [1,2,-1](2)$
6	35	1024	$[2] \oplus [-2] \oplus [-4]^{\oplus 2} \oplus U(4)$
6	36	1024	$[-8] \oplus U(2) \oplus [0,2,-1,0,1,1](-2)$
6	37	1024	$[-2] \oplus [-8] \oplus U(2) \oplus U(4)$
6	38	1024	$[-2] \oplus U(4) \oplus [0,2,-2,0,1,1](-2)$
6	39	1024	$[-2]^{\oplus 2} \oplus [4] \oplus [-16] \oplus U(2)$
6	40	1024	$[2] \oplus U(2) \oplus [3,1,3,-1,1,3](-2)$
6	41	1024	$[2] \oplus [-2] \oplus [-8]^{\oplus 2} \oplus U(2)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
6	42	1024	$[-2]^{\oplus 2} \oplus U(2) \oplus U(8)$
6	43	1024	$U(2)^{\oplus 2} \oplus [4,2,5](-2)$
6	44	1088	$[-2]^{\oplus 2} \oplus U(2) \oplus [-2,1,8](2)$
6	45	1152	$[-2]^{\oplus 2} \oplus [2] \oplus [-4] \oplus [6] \oplus [-6]$
6	46	1280	$[-2]^{\oplus 2} \oplus [4] \oplus [-20] \oplus U(2)$
6	47	1280	$[0,1,1](2) \oplus U(2) \oplus [3,1,7](-2)$
6	48	1280	$[-2]^{\oplus 2} \oplus [-8] \oplus [10] \oplus U(2)$
6	49	1472	$[-2] \oplus [-46] \oplus U(2) \oplus [0,1,1](2)$
6	50	1536	$[-2] \oplus [-4] \oplus [0,1,1](2) \oplus [2,1,-1](4)$
6	51	1536	$[0,1,1](2) \oplus U(2) \oplus [4,2,7](-2)$
6	52	1536	$[-2]^{\oplus 2} \oplus [-6] \oplus [16] \oplus U(2)$
6	53	1664	$[-2]^{\oplus 2} \oplus [-4] \oplus [26] \oplus U(2)$
6	54	1728	$[-2] \oplus [-6] \oplus U(2) \oplus [-2,1,4](2)$
6	55	1728	$[2] \oplus [-6]^{\oplus 3} \oplus [0,1,1](2)$
6	56	1728	$[2] \oplus [-2] \oplus [0,3,2](2) \oplus A_2(-2)$
6	57	1792	$[-2] \oplus U(2) \oplus [-1,3,-1,2,0,4](-2)$
6	58	2048	$[-2] \oplus [-4] \oplus [-8] \oplus [8] \oplus U(2)$
6	59	2048	$U(4) \oplus U(2) \oplus [3,1,3](-2)$
6	60	2048	$[-2] \oplus [8] \oplus U(2) \oplus [3,1,3](-2)$
6	61	2048	$[2] \oplus [-2] \oplus [-4] \oplus [-32] \oplus U(2)$
6	62	2048	$[-2] \oplus [-4] \oplus U(4)^{\oplus 2}$
6	63	2304	$[2] \oplus U(2) \oplus [4,2,4,1,-1,4](-2)$
6	64	3072	$[-2] \oplus U(4) \oplus [0,2,-1,0,1,3](-2)$
6	65	3072	$[-6] \oplus [-8] \oplus [0,1,1](2) \oplus U(4)$
6	66	3072	$[2] \oplus [-2] \oplus U(2) \oplus A_2(-8)$
6	67	4096	$[2] \oplus U(4) \oplus [3,1,3,-1,1,3](-2)$
6	68	4096	$[2] \oplus [-2] \oplus [-4]^{\oplus 2} \oplus U(8)$
6	69	4096	$[2] \oplus [-2] \oplus [-8]^{\oplus 2} \oplus U(4)$
6	70	4096	$[2] \oplus [-8]^{\oplus 3} \oplus U(2)$
6	71	4096	$[-8] \oplus U(4) \oplus [0,2,-1,0,1,1](-2)$
6	72	8192	$U(4)^{\oplus 2} \oplus [3,1,3](-2)$
6	73	12288	$[-8] \oplus U(4) \oplus [0,2,-1,0,1,3](-2)$
6	74	15552	$[2] \oplus [6] \oplus [-6]^{\oplus 4}$
6	75	16384	$[2] \oplus U(8) \oplus [3,1,3,-1,1,3](-2)$
6	76	16384	$[2] \oplus [-8]^{\oplus 3} \oplus U(4)$
7	1	-128	$[-2]^{\oplus 3} \oplus U(2)^{\oplus 2}$
7	2	-256	$[-2]^{\oplus 2} \oplus [-4] \oplus U(2)^{\oplus 2}$
7	3	-384	$[-2]^{\oplus 2} \oplus [-6] \oplus U(2)^{\oplus 2}$
7	4	-384	$[-2]^{\oplus 2} \oplus [2] \oplus A_2(-2) \oplus U(2)$
7	5	-512	$[-2]^{\oplus 3} \oplus [0,1,1](4) \oplus U(2)$
7	6	-512	$[-2]^{\oplus 3} \oplus U(4) \oplus U(2)$
7	7	-512	$[2] \oplus [-2] \oplus U(2) \oplus A_3(-2)$
7	8	-512	$[-2]^{\oplus 2} \oplus [-8] \oplus U(2)^{\oplus 2}$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
7	9	-640	$[-2]^{\oplus 2} \oplus [-10] \oplus U(2)^{\oplus 2}$
7	10	-640	$[-2] \oplus U(2)^{\oplus 2} \oplus [2,1,3](-2)$
7	11	-768	$[-2] \oplus [-4] \oplus [-6] \oplus U(2)^{\oplus 2}$
7	12	-768	$[2] \oplus [-2] \oplus [-4] \oplus A_2(-2) \oplus U(2)$
7	13	-896	$[-2]^{\oplus 2} \oplus U(2) \oplus [1,1,-2,0,1,2](-2)$
7	14	-896	$[-2] \oplus U(2)^{\oplus 2} \oplus [2,1,4](-2)$
7	15	-1024	$[-2]^{\oplus 2} \oplus [-4] \oplus U(4) \oplus U(2)$
7	16	-1024	$[-2] \oplus [-4] \oplus [-8] \oplus U(2)^{\oplus 2}$
7	17	-1024	$U(2)^{\oplus 2} \oplus [2,1,3,0,1,2](-2)$
7	18	-1024	$[-2] \oplus U(2)^{\oplus 2} \oplus [3,1,3](-2)$
7	19	-1024	$[-2]^{\oplus 2} \oplus [-16] \oplus U(2)^{\oplus 2}$
7	20	-1152	$[2] \oplus A_2(-2)^{\oplus 2} \oplus U(2)$
7	21	-1152	$[-2]^{\oplus 3} \oplus U(6) \oplus U(2)$
7	22	-1152	$[-2]^{\oplus 3} \oplus [2,1,-4](2) \oplus U(2)$
7	23	-1280	$[-4] \oplus U(2)^{\oplus 2} \oplus [2,1,3](-2)$
7	24	-1280	$[-2] \oplus [-4] \oplus [-10] \oplus U(2)^{\oplus 2}$
7	25	-1408	$[-2]^{\oplus 2} \oplus [-22] \oplus U(2)^{\oplus 2}$
7	26	-1536	$[-2]^{\oplus 2} \oplus [-6] \oplus U(4) \oplus U(2)$
7	27	-1536	$[2] \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-6] \oplus U(2)$
7	28	-1536	$[-2]^{\oplus 2} \oplus [2] \oplus [-4] \oplus [-12] \oplus U(2)$
7	29	-1536	$[-2] \oplus U(2) \oplus [1,1,-3,0,2,1,0,0,1,2](-2)$
7	30	-1536	$[0,1,1](2) \oplus U(2) \oplus [2,1,4,0,1,2](-2)$
7	31	-1536	$[-2] \oplus [2] \oplus U(2) \oplus [2,1,3,1,1,3](-2)$
7	32	-1664	$[-2]^{\oplus 2} \oplus [-26] \oplus U(2)^{\oplus 2}$
7	33	-1664	$[-2]^{\oplus 3} \oplus [-2,1,6](2) \oplus U(2)$
7	34	-1792	$[2] \oplus [-2] \oplus [-4] \oplus [2,1,4](-2) \oplus U(2)$
7	35	-1920	$[-6] \oplus U(2)^{\oplus 2} \oplus [2,1,3](-2)$
7	36	-1920	$[-2] \oplus [-6] \oplus [-10] \oplus U(2)^{\oplus 2}$
7	37	-2048	$[-2] \oplus [-4]^{\oplus 2} \oplus [0,2,1](2) \oplus U(2)$
7	38	-2048	$[-4] \oplus [0,1,1](2) \oplus [3,1,3](-2) \oplus U(2)$
7	39	-2048	$U(2) \oplus [0,2,-1,0,1,2,0,1,1,2,0,0,1,1,2](-2)$
7	40	-2048	$[-2] \oplus [-8]^{\oplus 2} \oplus U(2)^{\oplus 2}$
7	41	-2048	$[-2] \oplus [-8] \oplus U(2) \oplus [0,2,-2,0,1,1](-2)$
7	42	-2048	$U(2)^{\oplus 2} \oplus [3,1,3,-1,1,3](-2)$
7	43	-2048	$[2] \oplus [-2] \oplus U(2) \oplus [2,1,2,1,1,6](-2)$
7	44	-2048	$[-2]^{\oplus 2} \oplus [-32] \oplus U(2)^{\oplus 2}$
7	45	-2048	$[-2] \oplus [-4]^{\oplus 2} \oplus U(4) \oplus U(2)$
7	46	-2176	$[-2] \oplus U(2)^{\oplus 2} \oplus [3,1,6](-2)$
7	47	-2304	$[-4] \oplus U(2) \oplus [0,3,1,0,1,1,0,1,0,1](-2)$
7	48	-2304	$[-4] \oplus [-6]^{\oplus 2} \oplus U(2)^{\oplus 2}$
7	49	-2432	$[-2]^{\oplus 2} \oplus [-38] \oplus U(2)^{\oplus 2}$
7	50	-2560	$[-2] \oplus U(2)^{\oplus 2} \oplus [2,1,3](-4)$
7	51	-2560	$U(2)^{\oplus 2} \oplus [3,1,3,1,1,3](-2)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
7	52	-2560	$[2,1,3](-2) \oplus U(2) \oplus [-2,1,1,1,0,1](-2)$
7	53	-2688	$[-2]^{\oplus 2} \oplus [6] \oplus [2,1,4](-2) \oplus U(2)$
7	54	-2688	$[-2] \oplus [2] \oplus U(2) \oplus [3,1,3,0,1,3](-2)$
7	55	-2816	$[2] \oplus [-2] \oplus U(2) \oplus [2,1,2,1,0,8](-2)$
7	56	-3072	$[-2]^{\oplus 2} \oplus [-12] \oplus U(4) \oplus U(2)$
7	57	-3072	$[-2] \oplus [-4] \oplus [-24] \oplus U(2)^{\oplus 2}$
7	58	-3072	$[2] \oplus [-2] \oplus U(2) \oplus [2,1,4,1,0,4](-2)$
7	59	-3072	$U(2) \oplus [-1,1,2,1,0,2,0,1,0,2,0,0,1,1,3](-2)$
7	60	-3072	$[-2] \oplus [-6] \oplus [-16] \oplus U(2)^{\oplus 2}$
7	61	-3200	$U(2) \oplus [1,1,-1,0,1,1,0,1,0,3,0,-1,0,2,3](-2)$
7	62	-3456	$[-2]^{\oplus 2} \oplus [-6] \oplus [2,1,-4](2) \oplus U(2)$
7	63	-3456	$[-2] \oplus [2] \oplus [-6]^{\oplus 3} \oplus U(2)$
7	64	-3456	$[-18] \oplus U(2)^{\oplus 2} \oplus A_2(-2)$
7	65	-3584	$[-2]^{\oplus 2} \oplus U(2) \oplus [3,1,3,0,2,-2](-2)$
7	66	-3584	$[2] \oplus [-2] \oplus [-8] \oplus [2,1,4](-2) \oplus U(2)$
7	67	-3840	$[-4] \oplus [2,1,-1](2) \oplus [2,1,3](-2) \oplus U(2)$
7	68	-4096	$U(4) \oplus U(2) \oplus [2,1,3,0,1,2](-2)$
7	69	-4096	$[-2]^{\oplus 2} \oplus [-4] \oplus U(8) \oplus U(2)$
7	70	-4096	$[-2] \oplus U(2) \oplus [0,4,1,0,1,1,0,2,0,2](-2)$
7	71	-4096	$[-2] \oplus U(4) \oplus [3,1,3](-2) \oplus U(2)$
7	72	-4096	$[-4] \oplus U(2)^{\oplus 2} \oplus [4,2,5](-2)$
7	73	-4096	$[-8] \oplus U(2)^{\oplus 2} \oplus [3,1,3](-2)$
7	74	-4096	$[-2] \oplus [-4] \oplus [-8] \oplus U(4) \oplus U(2)$
7	75	-4224	$[-2] \oplus [-6] \oplus [-22] \oplus U(2)^{\oplus 2}$
7	76	-4608	$[2] \oplus U(2) \oplus [2,1,4,1,0,4,0,-1,1,2](-2)$
7	77	-4608	$[2] \oplus [-2] \oplus [-24] \oplus A_2(-2) \oplus U(2)$
7	78	-5120	$[-2] \oplus U(2)^{\oplus 2} \oplus [7,3,7](-2)$
7	79	-5120	$[-16] \oplus U(2)^{\oplus 2} \oplus [2,1,3](-2)$
7	80	-5760	$[-30] \oplus U(2)^{\oplus 2} \oplus A_2(-2)$
7	81	-6144	$[-8] \oplus [0,1,1](2) \oplus A_2(-4) \oplus U(2)$
7	82	-6144	$[-2] \oplus [2] \oplus U(2) \oplus [3,1,3,1,-1,7](-2)$
7	83	-6144	$U(2) \oplus [0,2,-1,0,1,2,0,1,0,2,0,1,-1,1,4](-2)$
7	84	-6144	$[-2] \oplus [0,1,1](4) \oplus A_2(-4) \oplus U(2)$
7	85	-6272	$[-14] \oplus U(2)^{\oplus 2} \oplus [2,1,4](-2)$
7	86	-6912	$[-2] \oplus [-4] \oplus [18] \oplus A_2(-2) \oplus U(2)$
7	87	-8192	$[0,2,1](2) \oplus U(2) \oplus [3,1,3,-1,1,3](-2)$
7	88	-8192	$[2] \oplus U(2) \oplus [3,1,3,1,-1,3,1,1,-1,5](-2)$
7	89	-8192	$[2] \oplus [-2] \oplus [-8]^{\oplus 3} \oplus U(2)$
7	90	-8192	$U(2) \oplus [2,1,2,1,1,2,1,1,0,3,0,-1,1,3,2](-2)$
7	91	-8192	$[-2]^{\oplus 2} \oplus U(2) \oplus [-3,1,5,1,-3,5](-2)$
7	92	-8192	$[-2] \oplus [-8]^{\oplus 2} \oplus [0,1,1](4) \oplus U(2)$
7	93	-8192	$U(4) \oplus U(2) \oplus [3,1,3,-1,1,3](-2)$
7	94	-10240	$U(2)^{\oplus 2} \oplus [4,2,7,0,2,4](-2)$

Continued on next page

Table 5, continued from previous page

λ	No.	det(T)	T
7	95	-10240	$U(4) \oplus U(2) \oplus [3,1,3,1,1,3](-2)$
7	96	-10368	$[-6] \oplus U(6) \oplus A_2(-2) \oplus U(2)$
7	97	-12288	$U(4) \oplus U(2) \oplus [2,1,7,0,1,2](-2)$
7	98	-16384	$U(8) \oplus U(2) \oplus [2,1,3,0,1,2](-2)$
7	99	-16384	$[-8] \oplus U(4) \oplus [3,1,3](-2) \oplus U(2)$
7	100	-18432	$[2] \oplus A_2(-4)^{\oplus 2} \oplus U(2)$
7	101	-20736	$[-4] \oplus [-6]^{\oplus 2} \oplus U(6) \oplus U(2)$
7	102	-24576	$[-8] \oplus [0,2,1](2) \oplus A_2(-4) \oplus U(2)$
7	103	-31104	$[-18] \oplus U(2) \oplus [-1,1,2,-1,1,2,1,-1,1,2](-2)$
7	104	-32768	$[2] \oplus [-32] \oplus U(2) \oplus [3,1,3,-1,1,3](-2)$
7	105	-32768	$U(2) \oplus [1,1,-1,0,2,4,0,2,0,4,0,2,2,2,6](-2)$
7	106	-32768	$[-8] \oplus U(2) \oplus [3,1,3,-1,1,-1,1,-1,1,3](-2)$
7	107	-131072	$U(2) \oplus [4,2,-1,-2,1,7,0,2,2,4,0,2,2,0,4](-2)$
8	1	256	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4}$
8	2	512	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-4]$
8	3	768	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-6]$
8	4	768	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus A_2(-2)$
8	5	1024	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]^{\oplus 2}$
8	6	1024	$U(2)^{\oplus 2} \oplus [-2] \oplus A_3(-2)$
8	7	1024	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-8]$
8	8	1280	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,3](-2)$
8	9	1280	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-10]$
8	10	1536	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-6]$
8	11	1536	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus A_2(-2)$
8	12	1792	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,2,1,0,3](-2)$
8	13	1792	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,4](-2)$
8	14	2048	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-8]$
8	15	2048	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,2,1,1,0,3](-2)$
8	16	2048	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,0,1,2](-2)$
8	17	2048	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [3,1,3](-2)$
8	18	2048	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 3}$
8	19	2304	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6] \oplus A_2(-2)$
8	20	2304	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-18]$
8	21	2304	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6]^{\oplus 2}$
8	22	2560	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus [2,1,3](-2)$
8	23	2560	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-10]$
8	24	2816	$U(2)^{\oplus 2} \oplus [2,1,2,1,0,3,0,0,1,2](-2)$
8	25	3072	$U(2)^{\oplus 2} \oplus [2,1,3,1,2,3,0,1,1,2](-2)$
8	26	3072	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-12]$
8	27	3072	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-6]$
8	28	3072	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6] \oplus [-8]$
8	29	3072	$U(2)^{\oplus 2} \oplus [-6] \oplus A_3(-2)$
8	30	3072	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,1,1,3](-2)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
8	31	3328	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,7](-2)$
8	32	3328	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-26]$
8	33	3584	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,2,1,1,3](-2)$
8	34	3840	$U(2)^{\oplus 2} \oplus [2,1,2,1,0,3,1,0,0,3](-2)$
8	35	3840	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [4,1,4](-2)$
8	36	4096	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,3,0,1,1,2](-2)$
8	37	4096	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-16]$
8	38	4096	$U(2)^{\oplus 2} \oplus [-2] \oplus [3,1,3,1,-1,3](-2)$
8	39	4096	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,3,1,1,1,3](-2)$
8	40	4096	$U(2)^{\oplus 2} \oplus [2,1,2,1,0,2,1,1,1,5](-2)$
8	41	4096	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-8]$
8	42	4352	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,1,0,4](-2)$
8	43	4608	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus [2,1,5](-2)$
8	44	4608	$U(2)^{\oplus 2} \oplus [-2] \oplus [-12] \oplus A_2(-2)$
8	45	4864	$U(2)^{\oplus 2} \oplus [2,1,3,1,0,3,0,1,0,2](-2)$
8	46	5120	$U(2)^{\oplus 2} \oplus [-2] \oplus [3,1,3,1,1,3](-2)$
8	47	5120	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-10]$
8	48	5120	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,6,0,1,2](-2)$
8	49	5120	$U(2)^{\oplus 2} \oplus [-2] \oplus [-8] \oplus [2,1,3](-2)$
8	50	5376	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6] \oplus [2,1,4](-2)$
8	51	5376	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-42]$
8	52	5632	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-22]$
8	53	6144	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,3,1,1,3](-2)$
8	54	6144	$U(2)^{\oplus 2} \oplus [2,1,3,1,0,3,0,1,1,3](-2)$
8	55	6144	$U(2)^{\oplus 2} \oplus [-2] \oplus [-16] \oplus A_2(-2)$
8	56	6144	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6] \oplus [3,1,3](-2)$
8	57	6144	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus A_2(-4)$
8	58	6400	$U(2)^{\oplus 2} \oplus [-2] \oplus [-10] \oplus [2,1,3](-2)$
8	59	6912	$U(2)^{\oplus 2} \oplus [-6]^{\oplus 2} \oplus A_2(-2)$
8	60	6912	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,3,1,0,1,3](-2)$
8	61	6912	$U(2)^{\oplus 2} \oplus [2,1,5](-2) \oplus A_2(-2)$
8	62	7168	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-14]$
8	63	7168	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,0,1,6](-2)$
8	64	7680	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6] \oplus [-20]$
8	65	8192	$U(2)^{\oplus 2} \oplus [-8] \oplus [2,1,3,0,1,2](-2)$
8	66	8192	$U(2)^{\oplus 2} \oplus [-2] \oplus [-8] \oplus [3,1,3](-2)$
8	67	8192	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [3,1,3](-4)$
8	68	8192	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-32]$
8	69	8192	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus [-8]^{\oplus 2}$
8	70	8192	$U(2)^{\oplus 2} \oplus [-4]^{\oplus 2} \oplus [3,1,3](-2)$
8	71	8192	$U(2)^{\oplus 2} \oplus [3,1,3,-1,1,3,-1,1,1,3](-2)$
8	72	8448	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,2,1,1,1,9](-2)$
8	73	8448	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,4,0,1,5](-2)$

Continued on next page

Table 5, continued from previous page

λ	No.	det(T)	T
8	74	8960	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,7,0,0,1,2](-2)$
8	75	9216	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6] \oplus A_2(-4)$
8	76	9216	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,10,0,1,2](-2)$
8	77	9216	$U(2)^{\oplus 2} \oplus [-6] \oplus [2,1,4,0,1,2](-2)$
8	78	10240	$U(2)^{\oplus 2} \oplus [2,1,3](-2) \oplus [3,1,3](-2)$
8	79	10240	$U(2)^{\oplus 2} \oplus [-2] \oplus [3,1,3,-1,1,6](-2)$
8	80	10752	$U(2)^{\oplus 2} \oplus [-4] \oplus [-14] \oplus A_2(-2)$
8	81	11264	$U(2)^{\oplus 2} \oplus [2,1,3,0,1,2,0,1,0,6](-2)$
8	82	11520	$U(2)^{\oplus 2} \oplus [-2] \oplus [-10] \oplus [2,1,5](-2)$
8	83	12288	$U(2)^{\oplus 2} \oplus [-6] \oplus [3,1,3,1,-1,3](-2)$
8	84	12288	$U(2)^{\oplus 2} \oplus [3,1,3,1,-1,3,1,1,-1,4](-2)$
8	85	12288	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,7,0,1,1,2](-2)$
8	86	12288	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,5,1,1,-1,5](-2)$
8	87	12288	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-24]$
8	88	12288	$U(2)^{\oplus 2} \oplus [3,1,3,1,1,3,1,1,1,3](-2)$
8	89	13056	$U(2)^{\oplus 2} \oplus [2,1,3,0,1,6,0,0,1,2](-2)$
8	90	13824	$U(2)^{\oplus 2} \oplus [-4] \oplus [-6]^{\oplus 3}$
8	91	14336	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,4,1,0,1,4](-2)$
8	92	15360	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-30]$
8	93	16384	$U(2)^{\oplus 2} \oplus [-16] \oplus [2,1,3,0,1,2](-2)$
8	94	16384	$U(2)^{\oplus 2} \oplus [-4]^{\oplus 2} \oplus [4,2,5](-2)$
8	95	16384	$U(2)^{\oplus 2} \oplus [-8] \oplus [3,1,3,1,-1,3](-2)$
8	96	18432	$U(2)^{\oplus 2} \oplus [2,1,6,0,1,3,0,1,-1,3](-2)$
8	97	20480	$U(2)^{\oplus 2} \oplus [2,1,7,0,1,2,0,2,0,4](-2)$
8	98	20480	$U(2)^{\oplus 2} \oplus [-8] \oplus [3,1,3,1,1,3](-2)$
8	99	20736	$U(2)^{\oplus 2} \oplus [-6] \oplus [-18] \oplus A_2(-2)$
8	100	20736	$U(2)^{\oplus 2} \oplus [-6]^{\oplus 2} \oplus [2,1,5](-2)$
8	101	24576	$U(2)^{\oplus 2} \oplus [-8] \oplus [2,1,7,0,1,2](-2)$
8	102	24576	$U(2)^{\oplus 2} \oplus [3,1,3](-2) \oplus A_2(-4)$
8	103	27648	$U(2)^{\oplus 2} \oplus [2,1,7,0,1,2,0,3,0,6](-2)$
8	104	27648	$U(2)^{\oplus 2} \oplus [-4] \oplus [-6]^{\oplus 2} \oplus [-12]$
8	105	28672	$U(2)^{\oplus 2} \oplus [-14] \oplus [3,1,3,-1,1,3](-2)$
8	106	32000	$U(2)^{\oplus 2} \oplus [-2] \oplus [-10]^{\oplus 3}$
8	107	32768	$U(2)^{\oplus 2} \oplus [2,1,7,-1,2,7,0,1,1,2](-2)$
8	108	32768	$U(2)^{\oplus 2} \oplus [-4] \oplus [4,2,5,2,1,5](-2)$
8	109	32768	$U(2)^{\oplus 2} \oplus [-2] \oplus [3,1,3,1,-1,3](-4)$
8	110	32768	$U(2)^{\oplus 2} \oplus [-8]^{\oplus 2} \oplus [3,1,3](-2)$
8	111	34560	$U(2)^{\oplus 2} \oplus A_2(-2) \oplus [2,1,3](-6)$
8	112	36864	$U(2)^{\oplus 2} \oplus [-6] \oplus [-8] \oplus A_2(-4)$
8	113	41472	$U(2)^{\oplus 2} \oplus [-12] \oplus [-18] \oplus A_2(-2)$
8	114	49152	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,7,0,1,2](-4)$
8	115	49152	$U(2)^{\oplus 2} \oplus [-6] \oplus [-8]^{\oplus 3}$
8	116	55296	$U(2)^{\oplus 2} \oplus A_2(-2) \oplus [3,1,3](-6)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
8	117	61440	$U(2)^{\oplus 2} \oplus [-30] \oplus [3,1,3,-1,1,3](-2)$
8	118	62208	$U(2)^{\oplus 2} \oplus [2,1,5](-2) \oplus A_2(-6)$
8	119	65536	$U(2)^{\oplus 2} \oplus [4,2,7,0,2,4,0,2,0,4](-2)$
8	120	110592	$U(2)^{\oplus 2} \oplus [-6]^{\oplus 2} \oplus A_2(-8)$
8	121	131072	$U(2)^{\oplus 2} \oplus [4,2,7,2,3,7,2,3,-1,7](-2)$
8	122	196608	$U(2)^{\oplus 2} \oplus [4,2,15,0,2,4,0,2,0,4](-2)$
9	1	−512	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 5}$
9	2	−1024	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-4]$
9	3	−1536	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-6]$
9	4	−1536	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus A_2(-2)$
9	5	−2048	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus A_3(-2)$
9	6	−2048	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-4]^{\oplus 2}$
9	7	−2560	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [2,1,3](-2)$
9	8	−2560	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-10]$
9	9	−3072	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-4] \oplus [-6]$
9	10	−3072	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-12]$
9	11	−3584	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,2,1,0,3](-2)$
9	12	−4096	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,2,1,0,2,1,0,0,3](-2)$
9	13	−4096	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,3,0,1,2](-2)$
9	14	−4096	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus A_3(-2)$
9	15	−4608	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-18]$
9	16	−4608	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,2,1,1,1,2,1,1,0,0,3](-2)$
9	17	−4608	$U(2)^{\oplus 2} \oplus [-2] \oplus A_2(-2)^{\oplus 2}$
9	18	−5120	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,2,1,0,4](-2)$
9	19	−5632	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-22]$
9	20	−5632	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,2,1,0,3,0,0,1,2](-2)$
9	21	−6144	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,2,1,1,0,3,0,0,0,1,2](-2)$
9	22	−6144	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,3,1,1,3](-2)$
9	23	−6144	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,1,1,3,0,1,-1,2](-2)$
9	24	−6144	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]^{\oplus 2} \oplus [-6]$
9	25	−7168	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [2,1,4](-2)$
9	26	−7168	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus [2,1,2,1,0,3](-2)$
9	27	−7680	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-10] \oplus A_2(-2)$
9	28	−7680	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6] \oplus [2,1,3](-2)$
9	29	−8192	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,3,1,1,1,3,0,0,1,-1,2](-2)$
9	30	−8192	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,2,1,0,2,1,1,1,3](-2)$
9	31	−8192	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,1,1,3,0,1,1,2](-2)$
9	32	−8704	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-34]$
9	33	−9216	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-6] \oplus [-12]$
9	34	−10240	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,6,0,1,2](-2)$
9	35	−10240	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-8] \oplus [2,1,3](-2)$
9	36	−10240	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [2,1,3](-2)$
9	37	−10752	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-14] \oplus A_2(-2)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
9	38	-11776	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,3,0,1,5](-2)$
9	39	-12288	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,2,1,1,0,3,1,0,1,0,4](-2)$
9	40	-12288	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,2,1,0,2,1,0,0,7](-2)$
9	41	-12288	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6] \oplus [3,1,3](-2)$
9	42	-12288	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4] \oplus [-6] \oplus [-8]$
9	43	-12800	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-10]^{\oplus 2}$
9	44	-13312	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,4,0,1,4](-2)$
9	45	-13824	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-6] \oplus [-18]$
9	46	-13824	$U(2)^{\oplus 2} \oplus [-2] \oplus A_2(-2) \oplus [2,1,5](-2)$
9	47	-14336	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3,1,1,3,1,0,0,3](-2)$
9	48	-14848	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-58]$
9	49	-15360	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-12] \oplus [2,1,3](-2)$
9	50	-16384	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,2,1,0,3,1,0,1,3](-2)$
9	51	-16384	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,2,1,0,2,1,1,1,5](-2)$
9	52	-16384	$U(2)^{\oplus 2} \oplus [2,1,3,1,0,3,1,2,1,4,0,1,-1,1,2](-2)$
9	53	-17920	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-10] \oplus [2,1,4](-2)$
9	54	-18432	$U(2)^{\oplus 2} \oplus [2,1,2,1,0,3,1,0,1,4,0,0,1,0,2](-2)$
9	55	-18432	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6]^{\oplus 2} \oplus [-8]$
9	56	-18432	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4]^{\oplus 2} \oplus [-6]^{\oplus 2}$
9	57	-19968	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-26] \oplus A_2(-2)$
9	58	-20480	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2,1,2,1,0,14](-2)$
9	59	-20992	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-82]$
9	60	-22528	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,2,1,1,3,1,0,0,7](-2)$
9	61	-22528	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]^{\oplus 2} \oplus [-22]$
9	62	-23040	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [2,1,3](-6)$
9	63	-24576	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,3,0,0,1,2,0,0,0,1,5](-2)$
9	64	-24576	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6] \oplus [3,1,3,-1,1,3](-2)$
9	65	-24576	$U(2)^{\oplus 2} \oplus [-2] \oplus [-8] \oplus [2,1,3,1,1,3](-2)$
9	66	-25600	$U(2)^{\oplus 2} \oplus [2,1,3](-2) \oplus [2,1,2,1,0,4](-2)$
9	67	-27648	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus [-6] \oplus [2,1,5](-2)$
9	68	-27648	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-12] \oplus [-18]$
9	69	-28672	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,3,1,0,0,3,0,-1,1,0,3](-2)$
9	70	-28672	$U(2)^{\oplus 2} \oplus [2,1,4](-2) \oplus [2,1,3,0,1,2](-2)$
9	71	-30720	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3](-2) \oplus A_2(-4)$
9	72	-32256	$U(2)^{\oplus 2} \oplus [-14] \oplus A_2(-2)^{\oplus 2}$
9	73	-32768	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,4,1,1,2,4,0,1,1,1,2](-2)$
9	74	-33280	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [2,1,33](-2)$
9	75	-36864	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,3,1,0,1,3,1,1,0,1,6](-2)$
9	76	-38400	$U(2)^{\oplus 2} \oplus [-10] \oplus [2,1,3](-2) \oplus A_2(-2)$
9	77	-40960	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,4,0,1,3,0,-1,1,3](-2)$
9	78	-40960	$U(2)^{\oplus 2} \oplus [-2] \oplus [-8]^{\oplus 2} \oplus [2,1,3](-2)$
9	79	-41472	$U(2)^{\oplus 2} \oplus [-18] \oplus [2,1,2,1,1,2,1,1,1,3](-2)$
9	80	-43008	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,8,0,1,0,2,0,1,1,0,2](-2)$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
9	81	-46080	$U(2)^{\oplus 2} \oplus [-2] \oplus [-4] \oplus [-6] \oplus [2,1,8](-2)$
9	82	-49152	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6] \oplus [3,1,3,1,-1,5](-2)$
9	83	-49152	$U(2)^{\oplus 2} \oplus [-4] \oplus [2,1,4,0,1,3,0,1,1,3](-2)$
9	84	-49152	$U(2)^{\oplus 2} \oplus [3,1,3](-2) \oplus [2,1,4,0,1,2](-2)$
9	85	-53248	$U(2)^{\oplus 2} \oplus [2,1,4,0,1,5,0,0,1,2,0,0,1,0,2](-2)$
9	86	-55296	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,2,1,1,0,7,1,1,0,-2,7](-2)$
9	87	-55296	$U(2)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6] \oplus [2,1,5](-4)$
9	88	-59904	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,20](-2) \oplus A_2(-2)$
9	89	-61440	$U(2)^{\oplus 2} \oplus [-2] \oplus [2,1,3](-2) \oplus [4,2,7](-2)$
9	90	-64000	$U(2)^{\oplus 2} \oplus [-2] \oplus [-10] \oplus [2,1,5,0,1,3](-2)$
9	91	-65536	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,3,1,1,1,4,1,0,0,1,4](-2)$
9	92	-69120	$U(2)^{\oplus 2} \oplus A_2(-2) \oplus [2,1,5,1,2,6](-2)$
9	93	-71680	$U(2)^{\oplus 2} \oplus [-4]^{\oplus 2} \oplus [-10] \oplus [2,1,4](-2)$
9	94	-73728	$U(2)^{\oplus 2} \oplus [-2] \oplus [-12] \oplus [-16] \oplus A_2(-2)$
9	95	-73728	$U(2)^{\oplus 2} \oplus [-2] \oplus [-6]^{\oplus 2} \oplus [-8]^{\oplus 2}$
9	96	-90112	$U(2)^{\oplus 2} \oplus [-2] \oplus [-22] \oplus [3,1,3,-1,1,3](-2)$
9	97	-96768	$U(2)^{\oplus 2} \oplus [-14] \oplus [2,1,5](-2) \oplus A_2(-2)$
9	98	-98304	$U(2)^{\oplus 2} \oplus [2,1,3,1,1,4,-1,0,1,4,0,0,1,-1,5](-2)$
9	99	-102400	$U(2)^{\oplus 2} \oplus [2,1,2,1,1,7,1,1,2,7,0,0,1,1,2](-2)$
9	100	-110592	$U(2)^{\oplus 2} \oplus [2,1,6,0,1,3,0,1,-1,3,0,1,1,1,4](-2)$
9	101	-110592	$U(2)^{\oplus 2} \oplus [2,1,2,1,0,3,1,0,0,6,1,0,1,-2,8](-2)$
9	102	-153600	$U(2)^{\oplus 2} \oplus [-4] \oplus [-10] \oplus [-12] \oplus [2,1,3](-2)$
9	103	-163840	$U(2)^{\oplus 2} \oplus [2,1,4,1,2,4,0,-1,1,5,0,-1,1,1,5](-2)$
9	104	-184320	$U(2)^{\oplus 2} \oplus [-6] \oplus [2,1,8](-2) \oplus [3,1,3](-2)$
9	105	-196608	$U(2)^{\oplus 2} \oplus [3,1,3,-1,1,6,0,0,1,4,0,0,2,2,4](-2)$
9	106	-221184	$U(2)^{\oplus 2} \oplus [-6] \oplus [3,1,4,1,0,4,1,0,0,4](-2)$
9	107	-225792	$U(2)^{\oplus 2} \oplus [-42] \oplus [2,1,2,1,1,2,1,0,0,6](-2)$
9	108	-286720	$U(2)^{\oplus 2} \oplus [2,1,4](-2) \oplus [4,2,7,0,2,4](-2)$
9	109	-294912	$U(2)^{\oplus 2} \oplus [-2] \oplus [6,3,7,-3,1,7,3,0,0,7](-2)$
9	110	-614400	$U(2)^{\oplus 2} \oplus [3,1,4,1,0,4,1,2,2,7,1,2,2,2,8](-2)$
10	1	1024	$E_8(-2) \oplus [2]^{\oplus 2}$
10	2	2048	$E_8(-2) \oplus [2] \oplus [4]$
10	3	3072	$E_8(-2) \oplus [2] \oplus [6]$
10	4	4096	$E_8(-2) \oplus [2] \oplus [8]$
10	5	5120	$E_8(-2) \oplus [2] \oplus [10]$
10	6	6144	$E_8(-2) \oplus [2] \oplus [12]$
10	7	7168	$E_8(-2) \oplus [2] \oplus [14]$
10	8	8192	$E_8(-2) \oplus [2] \oplus [16]$
10	9	9216	$E_8(-2) \oplus [2] \oplus [18]$
10	10	10240	$E_8(-2) \oplus [2] \oplus [20]$
10	11	12288	$E_8(-2) \oplus [2] \oplus [24]$
10	12	13312	$E_8(-2) \oplus [2] \oplus [26]$
10	13	15360	$E_8(-2) \oplus [2] \oplus [30]$

Continued on next page

Table 5, continued from previous page

λ	No.	det(T)	T
10	14	16384	$E_8(-2) \oplus [2] \oplus [32]$
10	15	18432	$E_8(-2) \oplus [2] \oplus [36]$
10	16	21504	$E_8(-2) \oplus [2] \oplus [42]$
10	17	22528	$E_8(-2) \oplus [2] \oplus [44]$
10	18	24576	$E_8(-2) \oplus [2] \oplus [48]$
10	19	25600	$E_8(-2) \oplus [2] \oplus [50]$
10	20	28672	$E_8(-2) \oplus [2] \oplus [56]$
10	21	30720	$E_8(-2) \oplus [2] \oplus [60]$
10	22	33792	$E_8(-2) \oplus [2] \oplus [66]$
10	23	37888	$E_8(-2) \oplus [2] \oplus [74]$
10	24	40960	$E_8(-2) \oplus [2] \oplus [80]$
10	25	43008	$E_8(-2) \oplus [2] \oplus [84]$
10	26	46080	$E_8(-2) \oplus [2] \oplus [90]$
10	27	49152	$E_8(-2) \oplus [2] \oplus [96]$
10	28	58368	$E_8(-2) \oplus [2] \oplus [114]$
10	29	59392	$E_8(-2) \oplus [2] \oplus [116]$
10	30	61440	$E_8(-2) \oplus [2] \oplus [120]$
10	31	71680	$E_8(-2) \oplus [2] \oplus [140]$
10	32	73728	$E_8(-2) \oplus [2] \oplus [144]$
10	33	79872	$E_8(-2) \oplus [2] \oplus [156]$
10	34	87040	$E_8(-2) \oplus [2] \oplus [170]$
10	35	90112	$E_8(-2) \oplus [2] \oplus [176]$
10	36	95232	$E_8(-2) \oplus [2] \oplus [186]$
10	37	104448	$E_8(-2) \oplus [2] \oplus [204]$
10	38	107520	$E_8(-2) \oplus [2] \oplus [210]$
10	39	114688	$E_8(-2) \oplus [2] \oplus [224]$
10	40	122880	$E_8(-2) \oplus [2] \oplus [240]$
10	41	133120	$E_8(-2) \oplus [2] \oplus [260]$
10	42	136192	$E_8(-2) \oplus [2] \oplus [266]$
10	43	168960	$E_8(-2) \oplus [2] \oplus [330]$
10	44	172032	$E_8(-2) \oplus [2] \oplus [336]$
10	45	181248	$E_8(-2) \oplus [2] \oplus [354]$
10	46	194560	$E_8(-2) \oplus [2] \oplus [380]$
10	47	215040	$E_8(-2) \oplus [2] \oplus [420]$
10	48	237568	$E_8(-2) \oplus [2] \oplus [464]$
10	49	245760	$E_8(-2) \oplus [2] \oplus [480]$
10	50	259072	$E_8(-2) \oplus [2] \oplus [506]$
10	51	279552	$E_8(-2) \oplus [2] \oplus [546]$
10	52	286720	$E_8(-2) \oplus [2] \oplus [560]$
10	53	319488	$E_8(-2) \oplus [2] \oplus [624]$
10	54	337920	$E_8(-2) \oplus [2] \oplus [660]$
10	55	353280	$E_8(-2) \oplus [2] \oplus [690]$
10	56	365568	$E_8(-2) \oplus [2] \oplus [714]$

Continued on next page

Table 5, continued from previous page

λ	No.	$\det(T)$	T
10	57	394240	$E_8(-2) \oplus [2] \oplus [770]$
10	58	417792	$E_8(-2) \oplus [2] \oplus [816]$
10	59	473088	$E_8(-2) \oplus [2] \oplus [924]$
10	60	532480	$E_8(-2) \oplus [2] \oplus [1040]$
10	61	778240	$E_8(-2) \oplus [2] \oplus [1520]$
10	62	860160	$E_8(-2) \oplus [2] \oplus [1680]$
10	63	1351680	$E_8(-2) \oplus [2] \oplus [2640]$
10	64	1397760	$E_8(-2) \oplus [2] \oplus [2730]$
10	65	1892352	$E_8(-2) \oplus [2] \oplus [3696]$
11	1	-2048	$U(2) \oplus E_8(-2) \oplus [2]$

References

1. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).
2. John H. Conway and Neil J. A. Sloane, *Low-dimensional lattices. IV. The mass formula*, Proc. R. Soc. Lond. A **419** (1988), no. 1857, 259–286.
3. Martin Eichler, *Quadratische Formen und orthogonale Gruppen*, Springer-Verlag, Berlin, 1952.
4. Leonhard Euler, *Illustratio paradoxi circa progressionem numerorum idoneorum sive congruorum*, Nova Acta Academiae Scientarum Imperialis Petropolitinae **15** (1806), 29–32 (Latin), delivered to the St. Petersburg Academy April 20, 1778. English translation by Jordan Bell: arXiv:math/0507352 (2005).
5. Jonathan Hanke, *Quadratic forms library for SageMath. Tickets #4470, #5418 and #5954 at trac.sagemath.org*, 2007.
6. Loo Keng Hua, *Introduction to number theory*, Springer-Verlag, Berlin, 1982.
7. JongHae Keum, *Every algebraic Kummer surface is the K3-cover of an Enriques surface*, Nagoya Math. J. **118** (1990), 99–110.
8. Martin Kneser, *Quadratische Formen*, Springer-Verlag, Berlin, 2002 (German), revised and edited in collaboration with Rudolf Scharlau.
9. Kwangwoo Lee, *Which K3 surfaces with Picard number 19 cover an Enriques surface*, Bull. Korean Math. Soc. **49** (2012), no. 1, 213–222.
10. David Lorch and Markus Kirschmer, *Single-class genera of positive integral lattices*, LMS J. Comput. Math. **16** (2013), 172–186.
11. Rick Miranda and David R. Morrison, *Embeddings of integral quadratic forms*, preliminary draft, `web.math.ucsb.edu/~drm/manuscripts/eiqf.pdf`, 2009.
12. David R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984), 105–121.
13. Viacheslav V. Nikulin, *On Kummer surfaces*, Trudy Mosk. Mat. Ob. **39** (1975), 278–293 (Russian), English translation: Math USSR-Izv. **9**, 261–275 (1975).
14. ———, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238 (Russian), English translation: Math USSR-Izv. **14** (1979), no. 1, 103–167 (1980). MR 525944 (80j:10031)
15. Hisanori Ohashi, *Counting Enriques quotients of a K3 surface*, RIMS preprint (2007), no. 1609, 1–6, `www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1609.pdf`.
16. ———, *On the number of Enriques quotients of a K3 surface*, Publ. RIMS, Kyoto Univ. **43** (2007), 181–200.

17. Wilhelm Plesken and Bernd Souvignier, *Computing isometries of lattices*, J. Symbolic Comput. **24** (1997), no. 3-4, 327–334, Computational algebra and number theory (London, 1993). MR 1484483
18. Rudolf Scharlau and Boris Hemkemeier, *Classification of integral lattices with large class number*, Math. Comp. **67** (1998), no. 222, 737–749. MR 1458224
19. Ali Sinan Sertöz, *Which singular K3 surfaces cover an Enriques surface*, Proc. Am. Math. Soc. (2005), no. 133, 43–50.
20. Ichiro Shimada and Davide Cesare Veniani, *Enriques quotients of singular K3 surfaces with small discriminants*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) (2020), to appear.
21. Neil J. A. Sloane and OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*. *oeis.org/A000926*, 2020.
22. William A. Stein et al., *Sage Mathematics Software (Version 9.0)*, The Sage Development Team, 2020, www.sagemath.org.
23. The PARI Group, Univ. Bordeaux, *PARI/GP version* 2.11.1, 2018, pari.math.u-bordeaux.fr/.
24. Charles T. C. Wall, *Quadratic forms on finite groups, and related topics*, Topology **2** (1963), 281–298.
25. Peter J. Weinberger, *Exponents of the class groups of complex quadratic fields*, Acta Arith. **22** (1973), 117–124.
26. Oğuzhan Yörük, *Which algebraic K3 surfaces doubly cover an Enriques surface: a computational approach*, 2019, master thesis, Bilkent University.

(Simon Brandhorst) Fachrichtung Mathematik, Universität des Saarlandes, Campus E24, 66123 Saarbrücken, Germany
E-mail address: brandhorst@math.uni-sb.de

(Serkan Sonel) Matematik Bolümü, Bilkent Üniversitesi, 06800 Ankara, Turkey
E-mail address: serkan.sonel@bilkent.edu.tr

(Davide Cesare Veniani) Institut für Geometrie und Topologie, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail address: davide.veniani@mathematik.uni-stutttgart.de