Proof of a conjecture of Adamchuk

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Abstract. In this paper, we prove a congruence which confirms a conjecture of Adamchuk. For any prime $p\equiv 1 \pmod 3$ and $a\in\mathbb{Z}^+$, we have

$$\sum_{k=1}^{\frac{2}{3}(p^a-1)}\binom{2k}{k}\equiv 0 \pmod{p^2}.$$

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1. Introduction

In the past decades, many people studied congruences for sums of binomial coefficients (see, for instance, [2,4,5,9–11,13,22,23]). In 2011, Sun [23] proved that for any odd prime p and $a\in\mathbb{Z}^+$,

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right) \pmod{p^2}, \tag{1.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol. Liu and Petrov [7] showed some congruences on sums of q-binomial coefficients.

In 2006, Adamchuk [1] conjectured that for any prime $p\equiv 1 \pmod 3$,

$$\sum_{k=1}^{\frac{2}{3}(p-1)}\binom{2k}{k}\equiv 0 \pmod{p^2}.$$

Pan and Sun [19] used a combinatorial identity to deduce that if p is prime then

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right) \pmod p \quad\text{for}\quad d=0,1,\ldots,p.$$

Sun told me he posed the following conjecture which generalizes Adamchuk's conjecture:

Conjecture 1.1. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod 3$ or $2|a$, then*

$$\sum_{k=1}^{\frac{2}{3}(p^a-1)} \binom{2k}{k} \equiv 0 \pmod{p^2}.$$

Recall that the Bernoulli numbers $\{B_n\}$ and the Bernoulli polynomials $\{B_n(x)\}$ are defined as follows:

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi) \text{ and } B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n \in \mathbb{N}).$$

Mattarei and Tauraso [14] proved that for any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) - \frac{p^2}{3} B_{p-2}\left(\frac{1}{2}\right) \pmod{p^2}.$$

The main objective of this paper is to obtain the following result.

Theorem 1.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod 3$ and $a \in \mathbb{Z}^+$, then*

$$\sum_{k=1}^{\frac{2}{3}(p^a-1)} \binom{2k}{k} \equiv 0 \pmod{p^2}.$$

In order to prove Theorem 1.2, we fist show the following interesting congruence.

Theorem 1.3. *For any prime $p \equiv 1 \pmod 3$, we have*

$$\sum_{\substack{k=0 \\ k \neq (p-1)/3}}^{(p-1)/2} \frac{\binom{2k}{k}}{3k+1} \equiv 0 \pmod p.$$

We shall prove Theorem 1.3 in Section 2, Section 3 is devoted to prove Theorem 1.2.

2. Proof of Theorem 1.3

Define the *hypergeometric series*

$$ {}_{m+1}F_m\left[\begin{matrix} \alpha_0 & \alpha_1 & \ldots & \alpha_m \\ & \beta_1 & \ldots & \beta_m \end{matrix} \middle| z \right] := \sum_{k=0}^{\infty} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_m)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot \frac{z^k}{k!}, \tag{2.1}$$

where $\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m, z \in \mathbb{C}$ and

$$(\alpha)_k = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1), & \text{if } k \geq 1, \\ 1, & \text{if } k = 0. \end{cases}$$

For a prime p, let $\mathbb{Z}_p$ denote the ring of all p-adic integers and let

$$\mathbb{Z}_p^\times := \{a \in \mathbb{Z}_p : a \text{ is prime to } p\}.$$

For each $\alpha \in \mathbb{Z}_p$, define the p-adic order $\nu_p(\alpha) := \max\{n \in \mathbb{N} : p^n \mid \alpha\}$ and the p-adic norm $|\alpha|_p := p^{-\nu_p(\alpha)}$. Define the p-adic gamma function $\Gamma_p(\cdot)$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1\le j<n\\(k,p)=1}} k, \qquad n = 1, 2, 3, \dots,$$

and

$$\Gamma_p(\alpha) = \lim_{\substack{|\alpha-n|_p\to 0\\ n\in\mathbb{N}}} \Gamma_p(n), \qquad \alpha \in \mathbb{Z}_p.$$

In particular, we set $\Gamma_p(0) = 1$. Throughout the whole paper, we only need to use the most basic properties of Γ_p, and all of them can be found in [15, 17]. For example, we know that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } |x|_p = 1, \\ -1, & \text{if } |x|_p > 1. \end{cases} \tag{2.2}$$

Lemma 2.1. *For any nonnegative integer n, we have*

$${}_2F_1\left[\begin{matrix}-3n & -3n+\frac{1}{2}\\ & -4n+\frac{2}{3}\end{matrix}\middle|\, \frac{4}{3}\right] = \frac{1}{4^n}\,{}_2F_1\left[\begin{matrix}-n & -n+\frac{1}{2}\\ & -2n+\frac{5}{6}\end{matrix}\middle|\, 1\right]. \tag{2.3}$$

Proof. By using package `Sigma` due to Schneider [18], we find that both sides of (2.3) satisfy the same recurrence:

$$(3n+2)(6n+1)S[n] - 2(12n+1)(12n+7)S(n+1) = 0,$$

and it is easy to check that both sides of (2.3) are equal for $n = 0, 1, 2$. □

Lemma 2.2. ([6]). *For any prime $p > 3$, we have the following congruences modulo p*

$$H_{\lfloor p/2\rfloor} \equiv -2q_p(2), \ H_{\lfloor p/3\rfloor} \equiv -\frac{3}{2}q_p(3), \ H_{\lfloor p/6\rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3).$$

Proof of Theorem 1.3. First for any $\alpha, s \in \mathbb{Z}_p$, we have

$$\frac{\binom{2k}{k}}{4^k} = \frac{\left(\frac{1}{2}\right)_k}{(1)_k}, \quad \frac{\left(\frac{1}{3}\right)_k}{\left(\frac{4}{3}\right)_k} = \frac{1}{3k+1} \text{ and } (\alpha+sp)_k \equiv (\alpha)_k \pmod{p}.$$

For each $(p+2)/3 \le k \le (p-1)/2$, we have

$$\begin{aligned}\frac{\left(\frac{1}{3}-\frac{p}{6}\right)_k}{\left(\frac{4}{3}-\frac{2p}{3}\right)_k} &= \frac{\frac{p}{6}\left(\frac{1}{3}-\frac{p}{6}\right)_{(p-1)/3}\left(\frac{p}{6}+1\right)_{k-(p+2)/3}}{\frac{-p}{3}\left(\frac{4}{3}-\frac{2p}{3}\right)_{(p-4)/3}\left(-\frac{p}{3}+1\right)_{k-(p-1)/3}} \equiv -\frac{1}{2}\frac{\left(\frac{1}{3}\right)_{(p-1)/3}(1)_{k-(p+2)/3}}{\left(\frac{4}{3}\right)_{(p-4)/3}(1)_{k-(p-1)/3}} \\ &= -\frac{1}{2}\frac{\left(\frac{1}{3}\right)_{(p-1)/3}}{\left(\frac{4}{3}\right)_{(p-4)/3}}\frac{1}{k-(p-1)/3} \equiv -\frac{3}{2}\frac{\left(\frac{1}{3}\right)_{(p-1)/3}}{\left(\frac{4}{3}\right)_{(p-4)/3}}\frac{1}{3k+1} \pmod{p}.\end{aligned}$$

And

$$\frac{\left(\frac{1}{3}\right)_{(p-1)/3}}{\left(\frac{4}{3}\right)_{(p-4)/3}}=\frac{p-1}{3}\frac{\left(\frac{1}{3}\right)_{(p-1)/3}(p-4)/3!}{\left(\frac{4}{3}\right)_{(p-4)/3}(p-1)/3!}\equiv-\frac{1}{3}(-1)^{(p-1)/3}(-1)^{(p-4)/3}=\frac{1}{3}\pmod p.$$

Hence for each $(p+2)/3\le k\le(p-1)/2$,

$$\frac{\left(\frac{1}{3}-\frac{p}{6}\right)_k}{\left(\frac{4}{3}-\frac{2p}{3}\right)_k}\equiv-\frac{1}{2}\frac{1}{3k+1}\pmod p.$$

That means that

$$\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}4^k\equiv-\frac{1}{2}\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}4^k\pmod p.$$

So

$$\begin{aligned}\sum_{\substack{k=0\\k\neq(p-1)/3}}^{(p-1)/2}\frac{\binom{2k}{k}}{3k+1}&=\sum_{\substack{k=0\\k\neq(p-1)/3}}^{(p-1)/2}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}4^k\\&\equiv\sum_{\substack{k=0\\k\neq(p-1)/3}}^{(p-1)/2}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}4^k-3\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}4^k\pmod p.\end{aligned}$$

Thus, we only need to prove that

$$\sum_{\substack{k=0\\k\neq(p-1)/3}}^{(p-1)/2}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}4^k\equiv-\frac{3}{2}\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}4^k\pmod p.\tag{2.4}$$

Set

$$\sum_{\substack{k=0\\k\neq(p-1)/3}}^{(p-1)/2}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}4^k=\mathfrak{A}-\mathfrak{F},$$

where

$$\mathfrak{A}={}_2F_1\left[\begin{matrix}\frac{1-p}{2}&\frac{1}{3}-\frac{p}{6}\\&\frac{4}{3}-\frac{2p}{3}\end{matrix}\middle|\,4\right]$$

$$\mathfrak{F}=\frac{\left(\frac{1-p}{2}\right)_{(p-1)/3}\left(\frac{1}{3}-\frac{p}{6}\right)_{(p-1)/3}}{(1)_{(p-1)/3}\left(\frac{4}{3}-\frac{2p}{3}\right)_{(p-1)/3}}4^{(p-1)/3}.$$

In view of [16, 15.8.1], we have

$${}_2F_1\left[\begin{matrix}a&b\\&c\end{matrix}\middle|\,z\right]=(1-z)^{-a}{}_2F_1\left[\begin{matrix}a&c-b\\&c\end{matrix}\middle|\,\frac{z}{z-1}\right].$$

Setting $a=\frac{1-p}{2}, b=\frac{1}{3}-\frac{p}{6}, c=\frac{4}{3}-\frac{2p}{3}$, we have

$$\mathfrak{A}=(-3)^{(p-1)/2}{}_2F_1\left[\begin{matrix}\frac{1-p}{2} & 1-\frac{p}{2}\\ & \frac{4}{3}-\frac{2p}{3}\end{matrix}\middle|\,\frac{4}{3}\right]$$

Set $n=\frac{p-1}{6}$ in Lemma 2.1, n is a nonnegative integer because of $p\equiv 1\pmod 3$, so we have

$${}_2F_1\left[\begin{matrix}\frac{1-p}{2} & 1-\frac{p}{2}\\ & \frac{4}{3}-\frac{2p}{3}\end{matrix}\middle|\,\frac{4}{3}\right]=\frac{1}{2^{(p-1)/3}}{}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{2}{3}-\frac{p}{6}\\ & \frac{7}{6}-\frac{p}{3}\end{matrix}\middle|\,1\right].$$

Substituting $m=\frac{p-1}{6}, b=\frac{2}{3}-\frac{p}{6}, c=\frac{7}{6}-\frac{p}{3}$ into [16, 15.8.6], we have

$${}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{2}{3}-\frac{p}{6}\\ & \frac{7}{6}-\frac{p}{3}\end{matrix}\middle|\,1\right]=\frac{\left(\frac{2}{3}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{7}{6}-\frac{p}{3}\right)_{(p-1)/6}}(-1)^{(p-1)/6}{}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{p}{6}\\ & \frac{1}{2}\end{matrix}\middle|\,1\right].$$

Hence

$$\mathfrak{A}=(-3)^{(p-1)/2}\frac{1}{2^{(p-1)/3}}\frac{\left(\frac{2}{3}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{7}{6}-\frac{p}{3}\right)_{(p-1)/6}}(-1)^{(p-1)/6}{}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{p}{6}\\ & \frac{1}{2}\end{matrix}\middle|\,1\right].$$

Setting $n=\frac{p-1}{6}, b=\frac{p}{6}, c=\frac{1}{2}$ in [16, 15.4.24], we have

$${}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{p}{6}\\ & \frac{1}{2}\end{matrix}\middle|\,1\right]=\frac{\left(\frac{1}{2}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{1}{2}\right)_{(p-1)/6}}.$$

Notice that $(p-1)/2+(p-1)/6=2(p-1)/3$ is even, so

$$\mathfrak{A}=3^{(p-1)/2}\frac{1}{2^{(p-1)/3}}\frac{\left(\frac{2}{3}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{7}{6}-\frac{p}{3}\right)_{(p-1)/6}}\frac{\left(\frac{1}{2}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{1}{2}\right)_{(p-1)/6}}.$$

Now we calculate the right-side of (2.4),

$$\begin{aligned}
&\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}4^k=\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\binom{2k}{k}}{3k+1}\equiv\sum_{k=(p+2)/3}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}(-4)^k}{3k+1}\\
&=\sum_{k=0}^{(p-7)/6}\frac{\binom{(p-1)/2}{k}(-4)^{(p-1)/2-k}}{3((p-1)/2-k)+1}\equiv-2(-1)^{p-1)/2}\sum_{k=0}^{(p-7)/6}\frac{\binom{2k}{k}}{(6k+1)(16)^k}\\
&=-2(-1)^{p-1)/2}\sum_{k=0}^{(p-7)/6}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{6}\right)_k}{(1)_k\left(\frac{7}{6}\right)_k4^k}\equiv-2(-1)^{p-1)/2}\sum_{k=0}^{(p-7)/6}\frac{\left(\frac{1+p}{2}\right)_k\left(\frac{1-p}{6}\right)_k}{(1)_k\left(\frac{7}{6}+\frac{p}{3}\right)_k4^k}\\
&=-2(-1)^{p-1)/2}(\mathfrak{L}-\mathfrak{Q})\pmod p,
\end{aligned}\tag{2.5}$$

where

$$\mathfrak{L}={}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{1}{2}+\frac{p}{2}\\ & \frac{7}{6}+\frac{p}{3}\end{matrix}\middle|\,\frac{1}{4}\right],\qquad\mathfrak{Q}=\frac{\left(\frac{1+p}{2}\right)_{(p-1)/6}\left(\frac{1-p}{6}\right)_{(p-1)/6}}{(1)_{(p-1)/6}\left(\frac{7}{6}+\frac{p}{3}\right)_{(p-1)/6}}\left(\frac{1}{4}\right)^{(p-1)/6}.$$

Substituting $a=\frac{1-p}{6}, b=\frac{1+p}{2}, c=\frac{7}{6}+\frac{p}{3}$ in [16, 15.8.1], and then by using [16, 15.4.31] with $a=\frac{1-p}{6}$ we have

$$\mathfrak{L}=\left(\frac{3}{4}\right)^{(p-1)/6}{}_2F_1\left[\begin{matrix}\frac{1-p}{6} & \frac{2}{3}-\frac{p}{6}\\ & \frac{7}{6}+\frac{p}{3}\end{matrix}\middle|-\frac{1}{3}\right]=\left(\frac{3}{4}\right)^{(p-1)/6}\left(\frac{8}{9}\right)^{(p-1)/3}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{7}{6}+\frac{p}{3}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(1+\frac{p}{3}\right)}.$$

In view of [8, Lemma 17,(3)], we have

$$\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{7}{6}+\frac{p}{3}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(1+\frac{p}{3}\right)}=\frac{3}{p}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{7}{6}+\frac{p}{3}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{p}{3}\right)}=-\frac{3}{p}\frac{\Gamma_p\left(\frac{4}{3}\right)\Gamma_p\left(\frac{7}{6}+\frac{p}{3}\right)}{\Gamma_p\left(\frac{3}{2}\right)\Gamma_p\left(\frac{p}{3}\right)}.$$

So

$$\mathfrak{L}=-\frac{34^{(p-1)/3}}{p3^{(p-1)/2}}\frac{\Gamma_p\left(\frac{4}{3}\right)\Gamma_p\left(\frac{7}{6}+\frac{p}{3}\right)}{\Gamma_p\left(\frac{3}{2}\right)\Gamma_p\left(\frac{p}{3}\right)}. \tag{2.6}$$

Thus, by (2.4, (2.5) and (2.6), we just need to prove that

$$\mathfrak{A}-\mathfrak{F}\equiv 3(-1)^{(p-1)/2}(\mathfrak{L}-\mathfrak{Q}) \pmod p. \tag{2.7}$$

By [8, Lemma 17, (3)] we know that

$$\mathfrak{A}=\frac{3^{\frac{p-1}{2}}}{2^{\frac{p-1}{3}}}\frac{\left(\frac{2}{3}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{7}{6}-\frac{p}{3}\right)_{(p-1)/6}}\frac{\left(\frac{1}{2}-\frac{p}{6}\right)_{(p-1)/6}}{\left(\frac{1}{2}\right)_{(p-1)/6}}=\frac{6}{p}\frac{3^{\frac{p-1}{2}}}{2^{\frac{p-1}{3}}}\frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{7}{6}-\frac{p}{3}\right)\Gamma_p\left(\frac{1}{2}\right)}{\Gamma_p\left(\frac{2}{3}-\frac{p}{6}\right)\Gamma_p\left(\frac{1}{2}+\frac{p}{6}\right)\Gamma_p\left(-\frac{p}{6}\right)\Gamma_p\left(\frac{1}{3}+\frac{p}{6}\right)}.$$

We know that for any $\alpha\in\mathbb{Z}_p$,

$$\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)}\equiv\Gamma_p'(0)+H_{p-\langle-\alpha\rangle_p-1} \pmod p, \tag{2.8}$$

where $H_n=\sum_{k=1}^n\frac{1}{k}$ is the nth classic harmonic number.
So we have

$$p2^{\frac{p-1}{3}}\mathfrak{A}\equiv 6\cdot 3^{\frac{p-1}{2}}\frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{7}{6}\right)\Gamma_p\left(\frac{1}{2}\right)}{\Gamma_p\left(\frac{2}{3}\right)\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(0\right)\Gamma_p\left(\frac{1}{3}\right)}\left(1-\frac{p}{3}H_{\frac{p-7}{6}}+\frac{p}{6}H_{\frac{p-1}{2}}\right) \pmod{p^2}.$$

So by [8, Definition 4], we have

$$p2^{(p-1)/3}\mathfrak{A}\equiv-(-3)^{(p-1)/2}\frac{\Gamma_p\left(\frac{1}{6}\right)\Gamma_p\left(\frac{1}{3}\right)}{\Gamma_p\left(\frac{1}{2}\right)}\left(1-\frac{p}{3}H_{(p-1)/6}-2p+\frac{p}{6}H_{(p-1)/2}\right) \pmod{p^2},$$

Similarly, we have

$$p2^{\frac{p-1}{3}}\mathfrak{F}\equiv-2^{p-1}\frac{\Gamma_p\left(\frac{1}{6}\right)\Gamma_p\left(\frac{1}{3}\right)}{\Gamma_p\left(\frac{1}{2}\right)}\left(1-\frac{p}{6}H_{\frac{p-1}{6}}-2p-\frac{5p}{6}H_{\frac{p-1}{3}}+\frac{p}{2}H_{\frac{p-1}{2}}\right) \pmod{p^2},$$

$$3p2^{\frac{p-1}{3}}(-1)^{\frac{p-1}{2}}\mathfrak{L} \equiv \frac{2^{p-1}}{(-3)^{\frac{p-1}{2}}}\frac{\Gamma_p\left(\frac{1}{6}\right)\Gamma_p\left(\frac{1}{3}\right)}{\Gamma_p\left(\frac{1}{2}\right)}\left(1+\frac{p}{3}H_{\frac{p-1}{6}}+2p\right) \pmod{p^2},$$

$$3p2^{\frac{p-1}{3}}(-1)^{\frac{p-1}{2}}\mathfrak{Q} \equiv \frac{\Gamma_p\left(\frac{1}{6}\right)\Gamma_p\left(\frac{1}{3}\right)}{\Gamma_p\left(\frac{1}{2}\right)}\left(1+\frac{2p}{3}H_{\frac{p-1}{3}}+\frac{p}{3}H_{\frac{p-1}{6}}-\frac{p}{2}H_{\frac{p-1}{2}}+2p\right) \pmod{p^2}.$$

Therefore (2.7) is equivalent to

$$\begin{aligned}&-(-3)^{\frac{p-1}{2}}\left(1-\frac{p}{3}H_{\frac{p-1}{6}}-2p+\frac{p}{6}H_{\frac{p-1}{2}}\right)+2^{p-1}\left(1-\frac{p}{6}H_{\frac{p-1}{6}}-2p-\frac{5p}{6}H_{\frac{p-1}{3}}+\frac{p}{2}H_{\frac{p-1}{2}}\right)\\ &\equiv \frac{2^{p-1}}{(-3)^{\frac{p-1}{2}}}\left(1+\frac{p}{3}+2p\right)-\left(1+\frac{2p}{3}H_{\frac{p-1}{3}}+\frac{p}{3}H_{\frac{p-1}{6}}-\frac{p}{2}H_{\frac{p-1}{2}}+2p\right) \pmod{p^2}.\end{aligned}$$

By Lemma 2.2, we just need to prove that

$$2^{p-1}-(-3)^{(p-1)/2}-\frac{2^{p-1}}{(-3)^{(p-1)/2}}+1 \equiv 0 \pmod{p^2}.$$

By using Fermat little theorem and $\left(\frac{-3}{p}\right)=\left(\frac{p}{3}\right)=1$, we immediately get that

$$2^{p-1}-(-3)^{\frac{p-1}{2}}-\frac{2^{p-1}}{(-3)^{\frac{p-1}{2}}}+1=\left(2^{p-1}-(-3)^{\frac{p-1}{2}}\right)\left(1-\frac{1}{(-3)^{\frac{p-1}{2}}}\right)\equiv 0 \pmod{p^2}.$$

Therefore the proof of Theorem 1.3 is complete. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Now $p\equiv 1 \pmod 3$, so $\left(\frac{p^a}{3}\right)=1$, by (1.1) we have

$$\sum_{k=1}^{p^a-1}\binom{2k}{k}\equiv 0 \pmod{p^2}.$$

Thus we only need to prove that

$$\sum_{k=(2p^a+1)/3}^{p^a-1}\binom{2k}{k}\equiv 0 \pmod{p^2}.$$

Let k and l be positive integers with $k+l=p^a$ and $0<l<p^a/2$. In view of [20], we have

$$\frac{l}{2}\binom{2l}{l}=\frac{(2l-1)!}{(l-1)!^2}\not\equiv 0 \pmod{p^a} \tag{3.1}$$

and

$$\binom{2k}{k} \equiv -p^a \frac{(l-1)!^2}{(2l-1)!} = -\frac{2p^a}{l\binom{2l}{l}} \pmod{p^2}. \tag{3.2}$$

So we have

$$\sum_{k=(2p^a+1)/3}^{p-1} \binom{2k}{k} = \sum_{k=1}^{(p^a-1)/3} \binom{2p^a-2k}{p^a-k} \equiv -2p^a \sum_{k=1}^{(p^a-1)/3} \frac{1}{k\binom{2k}{k}} \pmod{p^2}.$$

Hence we only need to show that

$$p^{a-1} \sum_{k=1}^{(p^a-1)/3} \frac{1}{k\binom{2k}{k}} \equiv 0 \pmod{p}. \tag{3.3}$$

It is easy to see that for $k = 1, 2, \ldots, (p^a-1)/2$,

$$\frac{\binom{(p^a-1)/2}{k}}{\binom{2k}{k}/(-4)^k} = \frac{\binom{(p^a-1)/2}{k}}{\binom{1/2}{k}} = \prod_{j=0}^{k-1} \frac{(p^a-1)/2-j}{-1/2-j} = \prod_{j=0}^{k-1}\left(1-\frac{p^a}{2j+1}\right) \equiv 1 \pmod{p}. \tag{3.4}$$

This, with Fermat little theorem yields that

$$\begin{aligned} p^{a-1} \sum_{k=1}^{(p^a-1)/3} \frac{1}{k\binom{2k}{k}} &\equiv p^{a-1} \sum_{k=1}^{(p^a-1)/3} \frac{1}{k\binom{(p^a-1)/2}{k}(-4)^k} \equiv -2p^{a-1} \sum_{k=1}^{(p^a-1)/3} \frac{1}{\binom{(p^a-3)/2}{k-1}(-4)^k} \\ &= \frac{1}{2} p^{a-1} \sum_{k=0}^{(p^a-4)/3} \frac{1}{\binom{(p^a-3)/2}{k}(-4)^k} \pmod{p}. \end{aligned}$$

Now we set $n = (p^a-1)/2, m = (p^a-1)/3, \lambda = -\frac{1}{4}$, then

$$\sum_{k=0}^{m-1} \frac{\lambda^k}{\binom{n-1}{k}} = \sum_{k=0}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} - \sum_{k=m}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}}.$$

So we only need to prove that

$$p^{a-1} \sum_{k=0}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} \equiv p^{a-1} \sum_{k=m}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} \pmod{p}. \tag{3.5}$$

In view of [24], we have

$$\sum_{k=0}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} = n \sum_{k=0}^{n-1} \frac{\lambda^k}{(\lambda+1)^{k+1}} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \frac{(-1)^i}{i+1} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=0}^{n-1} \frac{(\lambda+1)^{k+1}}{k+1}.$$

It is easy to show that for each $0 \le k \le n-1$

$$\sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \frac{(-1)^i}{i+1} = \int_0^1 \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} (-x)^i dx = \int_0^1 (1-x)^{n-1-k} dx = \frac{1}{n-k}.$$

Hence

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} &= n \sum_{k=0}^{n-1} \frac{\lambda^k}{(\lambda+1)^{k+1}(n-k)} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=0}^{n-1} \frac{(\lambda+1)^{k+1}}{k+1} \\ &= n \sum_{k=1}^{n} \frac{\lambda^{n-k}}{(\lambda+1)^{n-k+1} k} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=0}^{n-1} \frac{(\lambda+1)^{k+1}}{k+1} \\ &= \frac{n\lambda^n}{(\lambda+1)^{n+1}} \left(\sum_{k=1}^{n} \frac{(\lambda+1)^k}{k\lambda^k} + \sum_{k=1}^{n} \frac{(\lambda+1)^k}{k} \right).\end{aligned}$$

In the same way, we have

$$\sum_{k=m}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} = n \sum_{k=0}^{n-1-m} \frac{\lambda^{m+k}}{(\lambda+1)^{k+1}} \sum_{i=0}^{n-1-m-k} \frac{(-1)^i \binom{n-1-m-k}{i}}{m+i+1} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=m}^{n-1} \frac{(\lambda+1)^{k+1}}{k+1}.$$

It is easy to check that for each $0 \le k \le n-1-m$

$$\begin{aligned}\sum_{i=0}^{n-1-m-k} \binom{n-1-m-k}{i} \frac{(-1)^i}{m+i+1} &= \int_0^1 \sum_{i=0}^{n-1-m-k} \binom{n-1-m-k}{i} (-x)^i x^m dx \\ &= \int_0^1 x^m (1-x)^{n-1-m-k} dx = B(m+1, n-m-k),\end{aligned}$$

where $B(P,Q)$ stands for the beta function. It is well known that the beta function relate to gamma function:

$$B(P,Q) = \frac{\Gamma(P)\Gamma(Q)}{\Gamma(P+Q)}.$$

So

$$B(m+1, n-m-k) = \frac{\Gamma(m+1)\Gamma(n-m-k)}{\Gamma(n-k+1)} = \frac{m!(n-m-k-1)!}{(n-k)!} = \frac{1}{(m+1)\binom{n-k}{m+1}}.$$

Therefore

$$\begin{aligned}\sum_{k=m}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} &= \frac{n}{m+1} \sum_{k=0}^{n-1-m} \frac{\lambda^{m+k}}{(\lambda+1)^{k+1}\binom{n-k}{m+1}} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=m}^{n-1} \frac{(\lambda+1)^{k+1}}{k+1} \\ &= \frac{n}{m+1} \sum_{k=m+1}^{n} \frac{\lambda^{m+n-k}}{(\lambda+1)^{n-k+1}\binom{k}{m+1}} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=m+1}^{n} \frac{(\lambda+1)^k}{k} \\ &= \frac{n\lambda^n}{(\lambda+1)^{n+1}} \left(\frac{\lambda^m}{m+1} \sum_{k=m+1}^{n} \frac{(\lambda+1)^k}{\lambda^k \binom{k}{m+1}} + \sum_{k=m+1}^{n} \frac{(\lambda+1)^k}{k} \right).\end{aligned}$$

By (3.5), we just need to show that

$$p^{a-1}\frac{\lambda^m}{m+1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}}\equiv p^{a-1}\sum_{k=1}^{n}\frac{(\lambda+1)^k}{k\lambda^k}+p^{a-1}\sum_{k=1}^{m}\frac{(\lambda+1)^k}{k}\pmod p. \tag{3.6}$$

It is obvious that

$$\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}}=\sum_{k=m+1}^{n}\frac{(-3)^k}{\binom{k}{m+1}}=\sum_{k=m+1}^{n}\frac{1}{\binom{k}{m+1}}\sum_{j=0}^{k}\binom{k}{j}(-4)^j=\mathfrak{B}+\mathfrak{C},$$

where

$$\mathfrak{B}=\sum_{j=m+1}^{n}(-4)^j\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{k}{m+1}},\qquad \mathfrak{C}=\sum_{j=0}^{m}(-4)^j\sum_{k=m+1}^{n}\frac{\binom{k}{j}}{\binom{k}{m+1}}.$$

By the following transformation

$$\frac{\binom{k}{j}}{\binom{k}{m+1}}=\frac{k!(m+1)!(k-m-1)!}{j!(k-j)!k!}=\frac{(m+1)!(k-m-1)!(j-m-1)!}{j!(k-j)!(j-m-1)!}=\frac{\binom{k-m-1}{j-m-1}}{\binom{j}{m+1}}.$$

We have

$$\mathfrak{B}=\sum_{j=m+1}^{n}(-4)^j\sum_{k=j}^{n}\frac{\binom{k-m-1}{j-m-1}}{\binom{j}{m+1}}=\sum_{j=m+1}^{n}\frac{(-4)^j}{\binom{j}{m+1}}\sum_{k=0}^{n-j}\binom{k+j-m-1}{j-m-1}.$$

By [3, (1.48)], we have

$$\mathfrak{B}=\sum_{j=m+1}^{n}\frac{(-4)^j}{\binom{j}{m+1}}\binom{n-m}{j-m}.$$

It is easy to show that

$$\frac{\binom{n-m}{j-m}}{\binom{j}{m+1}}=\frac{(n-m)!(m+1)!(j-m-1)!}{j!(n-j)!(j-m)!}=\frac{n+1}{j-m}\frac{\binom{n}{j}}{\binom{n+1}{m+1}}.$$

Thus,

$$\mathfrak{B}=\frac{n+1}{\binom{n+1}{m+1}}\sum_{j=m+1}^{n}\frac{(-4)^j}{j-m}\binom{n}{j}.$$

Now we calculate $\mathfrak{C}$. First we have the following transformation

$$\frac{\binom{k}{j}}{\binom{k}{m+1}}=\frac{k!(m+1)!(k-m-1)!}{j!(k-j)!k!}=\frac{(m+1)!(k-m-1)!(m-j+1)!}{j!(k-j)!(m-j+1)!}=\frac{\binom{m+1}{j}}{\binom{k-j}{m-j+1}}.$$

Thus,

$$\mathfrak{C}=\sum_{j=0}^{m}\binom{m+1}{j}(-4)^j\sum_{k=m+1}^{n}\frac{1}{\binom{k-j}{m-j+1}}=\sum_{j=0}^{m}\binom{m+1}{j}(-4)^j\sum_{k=0}^{n-m-1}\frac{1}{\binom{k+m+1-j}{m-j+1}}.$$

By using package `Sigma`, we find the following identity,

$$\sum_{k=0}^{N} \frac{1}{\binom{k+i}{i}} = \frac{i}{i-1} - \frac{N+1}{(i-1)\binom{N+i}{N}}.$$

Substituting $N = n-m-1, i = m+1-j$ into the above identity, we have

$$\mathfrak{C} = \sum_{j=0}^{m-1} \binom{m+1}{j} (-4)^j \left(\frac{m+1-j}{m-j} - \frac{n-m}{(m-j)\binom{n-j}{n-m-1}} \right) + (m+1)(-4)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

It is easy to check that

$$\frac{(n-m)\binom{m+1}{j}}{\binom{n-j}{n-m-1}} = \frac{(m+1)!((n-m)!(m+1-j)!}{j!(n-j)!(m+1-j)!} = \frac{(m+1)!((n-m)!}{j!(n-j)!} = \frac{(n+1)\binom{n}{j}}{\binom{n+1}{m+1}}.$$

Therefore

$$\mathfrak{C} = (m+1) \sum_{j=0}^{m-1} \binom{m}{j} \frac{(-4)^j}{m-j} - \frac{n+1}{\binom{n+1}{m+1}} \sum_{j=0}^{m-1} \binom{n}{j} \frac{(-4)^j}{m-j} + (m+1)(-4)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

Hence

$$\mathfrak{B} + \mathfrak{C} = (m+1) \sum_{j=0}^{m-1} \binom{m}{j} \frac{(-4)^j}{m-j} + \frac{n+1}{\binom{n+1}{m+1}} \sum_{\substack{j=0 \\ j\neq m}}^{n} \binom{n}{j} \frac{(-4)^j}{j-m} + (m+1)(-4)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

That is

$$\frac{\lambda^m}{m+1}(\mathfrak{B} + \mathfrak{C}) = \lambda^m \sum_{j=0}^{m-1} \binom{m}{j} \frac{(-4)^j}{m-j} + \frac{\lambda^m}{\binom{n}{m}} \sum_{\substack{j=0 \\ j\neq m}}^{n} \binom{n}{j} \frac{(-4)^j}{j-m} + H_{n-m}. \tag{3.7}$$

In view of (3.4), we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(-3)^k}{k} &= \int_0^1 \sum_{k=1}^{n} (-3)^k x^{k-1} dx = -3 \int_0^1 \sum_{k=0}^{n-1} (-3x)^k dx = -3 \int_0^1 \frac{1-(-3x)^n}{1+3x} dx \\
&= 3 \int_0^1 \sum_{k=1}^{n} \binom{n}{k} (-1)^k (1+3x)^{k-1} dx = \int_1^4 \sum_{k=1}^{n} (-1)^k y^{k-1} dy \\
&= \sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{4^k-1}{k} \equiv \sum_{k=1}^{n} \frac{\binom{2k}{k}}{k} - \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} \pmod p
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} &= \int_0^1 \sum_{k=1}^{n} \binom{n}{k} (-1)^k x^{k-1} dx = \int_0^1 \frac{(1-x)^n - 1}{x} dx = \int_0^1 \frac{y^n - 1}{1-y} dy \\
&= -\int_0^1 \sum_{k=0}^{n-1} y^k dy = -\sum_{k=0}^{n-1} \frac{1}{k+1} = -\sum_{k=1}^{n} \frac{1}{k}.
\end{aligned}$$

In view of [22, (1.20)], and by (3.1), (3.2) we have

$$p^{a-1}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\equiv p^{a-1}\sum_{k=1}^{p^a-1}\frac{\binom{2k}{k}}{k}\equiv 0 \pmod p. \tag{3.8}$$

This, with [22, (1.20)] yields that

$$p^{a-1}\sum_{k=1}^{n}\frac{(\lambda+1)^k}{k\lambda^k}=p^{a-1}\sum_{k=1}^{n}\frac{(-3)^k}{k}\equiv p^{a-1}H_n \pmod p.$$

On the other hand, by [3, (1.48)] we have

$$\sum_{k=1}^{m}\frac{(\lambda+1)^k-1}{k}=\sum_{k=1}^{m}\frac{3^k-1}{k4^k}=\sum_{k=1}^{m}\frac{1}{k}\sum_{j=1}^{k}\binom{k}{j}\frac{1}{(-4)^j}=\sum_{j=1}^{m}\frac{1}{j(-4)^j}\sum_{k=j}^{m}\binom{k-1}{j-1}$$
$$=\sum_{j=1}^{m}\frac{1}{j(-4)^j}\binom{m}{j}=\frac{1}{(-4)^m}\sum_{j=0}^{m-1}\frac{(-4)^j}{m-j}\binom{m}{j}.$$

Hence

$$\sum_{k=1}^{m}\frac{(\lambda+1)^k}{k}=\frac{1}{(-4)^m}\sum_{j=0}^{m-1}\frac{(-4)^j}{m-j}\binom{m}{j}+H_m.$$

So modulo p we have

$$p^{a-1}\sum_{k=1}^{m}\frac{(\lambda+1)^k}{k}+p^{a-1}\sum_{k=1}^{n}\frac{(\lambda+1)^k}{k\lambda^k}\equiv p^{a-1}\lambda^m\sum_{j=0}^{m-1}\frac{(-4)^j}{m-j}\binom{m}{j}+p^{a-1}(H_m+H_n).$$

It is obvious that

$$p^{a-1}H_n=p^{a-1}\sum_{k=1}^{n}\frac{1}{k}\equiv p^{a-1}\sum_{j=1}^{(p-1)/2}\frac{1}{jp^{a-1}}=H_{(p-1)/2} \pmod p$$

and $p^{a-1}H_m\equiv H_{\lfloor p/3\rfloor} \pmod p$, $p^{a-1}H_{n-m}\equiv H_{\lfloor p/6\rfloor} \pmod p$.
This, with (3.6), (3.7) and Lemma 2.2 yields that we only need to prove that

$$\frac{p^{a-1}}{\binom{n}{m}}\sum_{\substack{j=0\\ j\neq m}}^{n}\binom{n}{j}\frac{(-4)^j}{j-m}\equiv 0 \pmod p.$$

Now $p\equiv 1 \pmod 3$, so by [8, Lemma 17, (2)], we can deduce that $p\nmid\binom{n}{m}$. So we only need to prove that

$$p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n}\binom{n}{j}\frac{(-4)^j}{j-m}\equiv 0 \pmod p.$$

It is obvious that

$$p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n}\binom{n}{j}\frac{(-4)^j}{j-m}\equiv 3p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n}\binom{n}{j}\frac{(-4)^j}{3j+1}\pmod p. \tag{3.9}$$

In view of (3.9), we know that There are only the items $3j+1=p^{a-1}(3k+1)$ with $k=0,1,\ldots,(p-1)/2$ and $k\neq(p-1)/3$, so by Fermat little theorem and Lucas congruence, we have

$$\begin{aligned}p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n}\frac{\binom{n}{j}(-4)^j}{3j+1}&\equiv\sum_{\substack{k=0\\ k\neq(p-1)/3}}^{(p-1)/2}\frac{\binom{n}{kp^{a-1}+\frac{p^{a-1}-1}{3}}(-4)^{kp^{a-1}+\frac{p^{a-1}-1}{3}}}{3k+1}\\ &\equiv(-4)^{\frac{p^{a-1}-1}{3}}\binom{\frac{p^{a-1}-1}{2}}{\frac{p^{a-1}-1}{3}}\sum_{\substack{k=0\\ k\neq(p-1)/3}}^{(p-1)/2}\frac{\binom{n}{k}(-4)^k}{3k+1}\pmod p.\end{aligned}$$

By Theorem 1.3, we immediately get the desired reslut.
Therefore the proof of Theorem 1.2 is complete. □

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