ON TWO CONJECTURAL SUPERCONGRUENCES OF Z.-W. SUN

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Abstract. In this paper, we mainly prove two conjectural supercongruences of Sun by using the following identity

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2=16^n\sum_{k=0}^{n}\frac{\binom{n+k}{k}\binom{n}{k}\binom{2k}{k}^2}{(-16)^k}$$

which arises from a ${}_4F_3$ hypergeometric transformation. For any prime $p>3$, we prove that

$$\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2\equiv(-1)^{(p-1)/2}p+5p^3E_{p-3}\pmod{p^4},$$

$$\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2\equiv(-1)^{(p-1)/2}p+3p^3E_{p-3}\pmod{p^4},$$

where E_{p-3} is the $(p-3)$th Euler number.

1. Introduction

The truncated hypergeometric series are defined by

$${}_nF_{n-1}\left[\begin{matrix}x_1 & x_2 & \cdots & x_n\\ & y_1 & \cdots & y_{n-1}\end{matrix}\middle|\, z\right]_m=\sum_{k=0}^{m}\frac{(x_1)_k(x_2)_k\cdots(x_n)_k}{(y_1)_k(y_2)_k\cdots(y_{n-1})_k}\frac{z^k}{k!},$$

where

$$(x)_k=\begin{cases}1, & k=0,\\ x(x+1)\cdots(x+k-1), & k>0.\end{cases}$$

denotes the so-called Pochhammer symbol (or rising factorial). Clearly, they are truncations of the classical hypergeometric series. Since $(-x)_k/(1)_k=(-1)^k\binom{x}{k}$, sometimes we may write the truncated hypergeometric series as sums involving products of binomial coefficients. In recent years, there is a rising interest in studying supercongruences involving truncated hypergeometric series (cf., for example, [11, 13, 17, 20, 21, 22, 27]).

In 2003, Rodriguez-Villegas [17] studied hypergeometric families of Calabi-Yau manifolds and discovered (numerically) 22 supercongruences concerning truncated hypergeometric series.

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For example, he conjectured that for any odd prime p,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2}\pmod{p^2},\tag{1.1}$$

which was later confirmed by Mortenson [14] using Gaussian hypergeometric series and Gross-Koblitz formula (see [16] for details about Gross-Koblitz formula). Quite recently, Barman and Saikia [2] obtained a parametric generalization of (1.1) without using Gaussian hypergeometric series. Note that $\binom{2k}{k}\equiv 0\pmod p$ for $k\in\{(p+1)/2,\ldots,p-1\}$. Thus

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\pmod{p^2}.$$

For more parametric generalizations of (1.1), the reader may consult [7, 8, 9, 10, 13, 15, 20].

Recall that the Euler numbers E_n ($n\in\mathbb{N}$) are defined by

$$E_0=1,\ \text{and}\ \sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\text{ for }n=1,2,\ldots.$$

In 2011, Sun [21] investigated some congruences related to the Euler numbers. Especially, for any prime $p>3$ he proved the following two congruences as extensions of (1.1):

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2}+p^2E_{p-3}\pmod{p^3}\tag{1.2}$$

and

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv-2p^2E_{p-3}\pmod{p^3}.\tag{1.3}$$

In [21], Sun also conjectured many congruences most of which have been confirmed. One of them is as follows: for any prime $p>3$,

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv(-1)^{(p-1)/2}p+p^3E_{p-3}\pmod{p^4}.\tag{1.4}$$

This was confirmed by Chen, Xie and He [4] in 2016. We also note that for any prime $p>3$ Mao [11] showed that

$$\sum_{k=0}^{(p-1)/2}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv(-1)^{(p-1)/2}p+\frac{(-1)^{(p^2-1)/8}p^3}{4}E_{p-3}\left(\frac{1}{4}\right)\pmod{p^4},$$

where the Euler polynomials $E_n(x)$ ($n\in\mathbb{N}$) are given by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac12\right)^{n-k}.$$

In 2012, Sun [22] studied congruences for sums involving products of three binomial coefficients systematically. Recall that for any prime $p \equiv 1 \pmod 4$, we may write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. In [22], Sun determined $x \pmod{p^2}$ as follows:

$$(-1)^{(p-1)/4}x \equiv \sum_{k=0}^{(p-1)/2} \frac{k+1}{8^k}\binom{2k}{k}^2 \equiv \sum_{k=0}^{(p-1)/2} \frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \pmod{p^2}. \tag{1.5}$$

In the proof of (1.5), Sun also obtained the following congruences:

$$\sum_{n=0}^{p-1} \frac{n+1}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \equiv (-1)^{(p-1)/2}p \pmod{p^3} \tag{1.6}$$

and

$$\sum_{n=0}^{p-1} \frac{2n+1}{(-16)^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \equiv (-1)^{(p-1)/2}p \pmod{p^3} \tag{1.7}$$

for any odd prime p.

The main goal of this paper is to establish the following generalizations of (1.6) and (1.7) which ware conjectured by Sun (see both [24, Conjecture 4.1] and [26, Conjecture 33(ii)]).

Theorem 1.1. *For any prime $p > 3$, we have*

$$\sum_{n=0}^{p-1} \frac{n+1}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \equiv (-1)^{(p-1)/2}p + 5p^3E_{p-3} \pmod{p^4}, \tag{1.8}$$

$$\sum_{n=0}^{p-1} \frac{2n+1}{(-16)^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \equiv (-1)^{(p-1)/2}p + 3p^3E_{p-3} \pmod{p^4}. \tag{1.9}$$

Remark 1.1. In [24, Conjecture 4.1], Sun also conjectured that

$$\sum_{n=0}^{p-1} \frac{n}{32^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \equiv -2p^3E_{p-3} \pmod{p^4}.$$

This was confirmed by Mao and Cao [12, Theorem 2.1] recently.

In next section, we shall first prove (1.9) by establishing a new transformation for the summation $\sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2$. Then via a hypergeometric transformation due to Chaundy and Bullard we will show that (1.8) is actually a corollary of (1.9).

2. Proof of Theorem 1.1

In order to show (1.9) we need the following lemmas.

Lemma 2.1. *Let n be a nonnegative integer. Then we have*

$$\sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 = 16^n \sum_{k=0}^{n} \frac{\binom{n+k}{k}\binom{n}{k}\binom{2k}{k}^2}{(-16)^k}. \tag{2.1}$$

Proof. It is easy to check that

$$\binom{2k}{k}\binom{2n-2k}{n-k}=4^n\frac{(\frac12)_k(\frac12)_{n-k}}{(1)_k(1)_{n-k}}$$

and

$$\frac{(\frac12)_{n-k}}{(1)_{n-k}}=\frac{(\frac12)_n(-n)_k}{(1)_n(\frac12-n)_k}=\frac{\binom{2n}{n}(-n)_k}{4^n(\frac12-n)_k}.$$

Hence we obtain

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2=\binom{2n}{n}^2{}_4F_3\left[\begin{matrix}\frac12 & \frac12 & -n & -n\\ & 1 & \frac12-n & \frac12-n\end{matrix}\,\middle|\,1\right], \tag{2.2}$$

here we note that the hypergeometric series in the right-hand side is actually a finite sum since $(-n)_k=0$ for all $k>n$.

It is known from [1, Theorem 3.3.3] that

$${}_4F_3\left[\begin{matrix}-n & a & b & c\\ & d & e & f\end{matrix}\,\middle|\,1\right]=\frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}{}_4F_3\left[\begin{matrix}-n & a & d-b & d-c\\ & d & a+1-n-e & a+1-n-f\end{matrix}\,\middle|\,1\right] \tag{2.3}$$

provided that $a+b+c-n+1=d+e+f$. Letting $a=c=1/2,\ b=-n,\ d=1,\ e=f=1/2-n$ in (2.3) we arrive at

$$\begin{aligned}{}_4F_3\left[\begin{matrix}\frac12 & \frac12 & -n & -n\\ & 1 & \frac12-n & \frac12-n\end{matrix}\,\middle|\,1\right]=&\frac{(-n)_n^2}{(\frac12-n)_n^2}{}_4F_3\left[\begin{matrix}-n & n+1 & \frac12 & \frac12\\ & 1 & 1 & 1\end{matrix}\,\middle|\,1\right]\\ =&\frac{16^n}{\binom{2n}{n}^2}\sum_{k=0}^{n}\frac{\binom{n+k}{k}\binom{n}{k}\binom{2k}{k}^2}{(-16)^k}.\end{aligned} \tag{2.4}$$

Now substituting (2.4) into (2.2) we immediately obtain the desired (2.1). □

Remark 2.1. Note that in [22, Lemma 3.1] Sun obtained another transformation of the summation $\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2$ as follows:

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2=\sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}, \tag{2.5}$$

and he used (2.5) to prove (1.6) and (1.7). We attempted to prove (1.9) by (2.5) but failed. However, this transformation is useful for proving a congruence relation between (1.8) and (1.9).

Lemma 2.2. *For nonnegative integers k and l with $l\geq k$, we have*

$$\sum_{n=k}^{l}(-1)^n(2n+1)\binom{n+k}{2k}=(-1)^l(l-k+1)\binom{l+k+1}{2k}. \tag{2.6}$$

Proof. It can be verified directly by induction on l. □

Proof of (1.9). In view of Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2=\sum_{n=0}^{p-1}(-1)^n(2n+1)\sum_{k=0}^{n}\frac{\binom{n+k}{k}\binom{n}{k}\binom{2k}{k}^2}{(-16)^k}\\
=&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-16)^k}\sum_{n=k}^{p-1}(-1)^n(2n+1)\binom{n+k}{2k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-16)^k}(p-k)\binom{p+k}{2k}\\
=&p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{p-1}{k}\binom{p+k}{k}}{(-16)^k}\\
\equiv&p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}(1-p^2H_k^{(2)})\pmod{p^5},
\end{aligned}\tag{2.7}$$

where $H_k^{(2)}=\sum_{j=1}^{k}1/j^2$ denotes the kth harmonic number of order 2 and the last step follows from the fact

$$\binom{p-1}{k}\binom{p+k}{k}=(-1)^k\prod_{j=1}^{k}\left(1-\frac{p^2}{j^2}\right)\equiv(-1)^k(1-p^2H_k^{(2)})\pmod{p^4}$$

for k among $0,1,\ldots,p-1$.

In 2015, Sun [25, Theorem 4.1] obtained that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}H_k^{(2)}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}H_k^{(2)}\equiv-4E_{p-3}\pmod{p}\tag{2.8}$$

for any prime $p>3$.

Substituting (1.2), (1.3) and (2.8) into (2.7) we finally obtain (1.9). □

To show (1.8) we need the following preliminary results.

Lemma 2.3. *For any nonnegative integer k, we have*

$$\sum_{n=0}^{k}\binom{n+k}{n}2^n=(-1)^{k+1}-(-2)^{k+1}\sum_{n=0}^{k}\binom{n+k}{n}(-1)^n.\tag{2.9}$$

Remark 2.2. This is a corollary of the following identity due to Chaundy and Bullard [3]:

$$1=(1-x)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}x^k+x^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-x)^k.$$

Lemma 2.4. *For any positive integer k we have*

$$\sum_{n=0}^{k-1}(2n+k)\binom{-k}{n}=\frac{(-1)^{k-1}k}{2}\binom{2k}{k}.\tag{2.10}$$

Proof. Clearly,

$$\begin{aligned}\sum_{n=0}^{k-1}(2n+k)\binom{-k}{n} &= \sum_{n=0}^{k-1}(n+k)\binom{-k}{n} + \sum_{n=0}^{k-1} n\binom{-k}{n} \\ =&k\sum_{n=0}^{k-1}\binom{-k-1}{n} - k\sum_{n=0}^{k-2}\binom{-k-1}{n} = k\binom{-k-1}{k-1} = \frac{(-1)^{k-1}k}{2}\binom{2k}{k}.\end{aligned}$$

This concludes the proof. □

Lemma 2.5. *For any positive integer k, we have*

$$\sum_{n=0}^{k-1}(-2)^n(n-k+1)\binom{-k}{n} = (-1)^{k+1}(3k-1) - (-2)^k\sum_{n=0}^{k-1}(2n-2k+1)\binom{-k}{n}. \tag{2.11}$$

Proof. Note that $\binom{-k}{n} = (-1)^n\binom{n+k-1}{n}$. Thus by Lemma 2.3 we have

$$\begin{aligned}&\sum_{n=0}^{k-1}(-2)^n(n-k+1)\binom{-k}{n} = -k\sum_{n=1}^{k-1}(-2)^n\binom{-k-1}{n-1} - (k-1)\sum_{n=0}^{k-1}\binom{n+k-1}{n}2^n \\ =&2k\sum_{n=0}^{k}\binom{n+k}{n}2^n - (k-1)\sum_{n=0}^{k-1}\binom{n+k-1}{n}2^n - 5k\binom{2k}{k}2^{k-1} \\ =&(-1)^{k+1}(3k-1) - 2k(-2)^{k+1}\sum_{n=0}^{k}\binom{n+k}{n}(-1)^n + (k-1)(-2)^k\sum_{n=0}^{k-1}\binom{n+k-1}{n}(-1)^n \\ &- 5k\binom{2k}{k}2^{k-1} \\ =&(-1)^{k+1}(3k-1) + (-2)^k\sum_{n=0}^{k-1}(4n+5k-1)\binom{-k}{n} + 3k\binom{2k}{k}2^{k-1}.\end{aligned}$$

Now by Lemma 2.4 we have

$$\begin{aligned}\sum_{n=0}^{k-1}(4n+5k-1)\binom{-k}{n} =& \sum_{n=0}^{k-1}(6n+3k)\binom{-k}{n} - \sum_{n=0}^{k-1}(2n-2k+1)\binom{-k}{n} \\ =&\frac{3(-1)^{k-1}k}{2}\binom{2k}{k} - \sum_{n=0}^{k-1}(2n-2k+1)\binom{-k}{n}.\end{aligned}$$

Combining the above, we finally obtain (2.11). □

Lemma 2.6. *For any nonnegative integer k, we have the following identities:*

$$\sum_{n=0}^{k}(-2)^n(n+k+1)\binom{k}{n} = (-1)^k(3k+1), \tag{2.12}$$

$$\sum_{n=0}^{k}(2n+2k+1)\binom{k}{n}=2^k(3k+1). \tag{2.13}$$

Proof. These two identities can be easily deduced by binomial theorem. Here we just prove (2.12) as an example. It is clear that

$$\begin{aligned}\sum_{n=0}^{k}(-2)^n(n+k+1)\binom{k}{n}&=(-1)^k(k+1)+k\sum_{n=1}^{k}(-2)^n\binom{k-1}{n-1}\\=&(-1)^k(k+1)-2k\sum_{n=0}^{k-1}(-2)^n\binom{k-1}{n}=(-1)^k(3k+1).\end{aligned}$$

□

Proof of (1.8). By (2.5) and Lemma 2.6, we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-16)^k}\sum_{n=k}^{p-1}(-2)^n(n+1)\binom{k}{n-k}\\&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{8^k}\sum_{n=0}^{p-1-k}(-2)^n(n+k+1)\binom{k}{n}\\&=\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}+\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^3}{8^k}\sum_{n=0}^{p-1-k}(-2)^n(n+k+1)\binom{k}{n}\\&=\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}+\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{8^{p-k}}\sum_{n=0}^{k-1}(-2)^n(n+p-k+1)\binom{p-k}{n}\\&\equiv\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}+\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{8^{p-k}}\sum_{n=0}^{k-1}(-2)^n(n-k+1)\binom{-k}{n}\pmod{p^4},\end{aligned}\tag{2.14}$$

where in the last step we noting that $\binom{2p-2k}{p-k}\equiv 0 \pmod p$ for $k\in\{1,2,\ldots,(p-1)/2\}$. Similarly, we obtain that

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2\\&\equiv\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}+\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{(-16)^{p-k}}\sum_{n=0}^{k-1}(2n-2k+1)\binom{-k}{n}\pmod{p^4}.\end{aligned}\tag{2.15}$$

Furthermore, with the help of Lemma 2.5 we have

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{8^{p-k}}\sum_{n=0}^{k-1}(-2)^n(n-k+1)\binom{-k}{n}\\ &=\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{8^{p-k}}\left((-1)^{k+1}(3k-1)-(-2)^k\sum_{n=0}^{k-1}(2n-2k+1)\binom{-k}{n}\right)\\ &\equiv-\sum_{k=(p+1)/2}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}+2\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{(-16)^{p-k}}\sum_{n=0}^{k-1}(2n-2k+1)\binom{-k}{n}\pmod{p^4}.\end{aligned}\tag{2.16}$$

Combining (2.14)–(2.16) we arrive at

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2\\ \equiv&2\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2-\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}\pmod{p^4}.\end{aligned}\tag{2.17}$$

Substituting (1.4) and (1.9) into (2.17) we obtain (1.8). This completes the proof. □

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