INTEGER SEQUENCES AND MONOMIAL IDEALS

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Abstract. Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, \ldots, n\}$ and let W be the subset consisting of permutations $\sigma \in \mathfrak{S}_n$ avoiding 132 and 312-patterns. The monomial ideal $I_W = \left\langle \mathbf{x}^\sigma = \prod_{i=1}^n x_i^{\sigma(i)} : \sigma \in W \right\rangle$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k is called a *hypercubic ideal* in [6]. The Alexander dual $I_W^{[\mathbf{n}]}$ of I_W with respect to $\mathbf{n} = (n, \ldots, n)$ has the minimal cellular resolution supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1}. We show that the number of standard monomials of the Artinian quotient $\frac{R}{I_W^{[\mathbf{n}]}}$ equals the number of rooted-labelled unimodal forests on the vertex set $[n]$. In other words,

$$\dim_k\left(\frac{R}{I_W^{[\mathbf{n}]}}\right) = \sum_{r=1}^{n} r!\ s(n,r) = \operatorname{Per}\left([m_{ij}]_{n\times n}\right),$$

where $s(n,r)$ is the (signless) Stirling number of the first kind and $\operatorname{Per}([m_{ij}]_{n\times n})$ is the permanent of the matrix $[m_{ij}]$ with $m_{ii} = i$ and $m_{ij} = 1$ for $i \neq j$. For various subsets S of $\mathfrak{S}_n$ consisting of permutations avoiding patterns, the corresponding integer sequences $\left\{\dim_k\left(\frac{R}{I_S^{[\mathbf{n}]}}\right)\right\}_{n=1}^{\infty}$ are identified.

Key words: Permutations avoiding patterns, standard monomials, parking functions.

1. Introduction

Let G be an oriented graph on the vertex set $\{0, 1, \ldots, n\}$ rooted at 0. A nonoriented graph on $\{0, 1, \ldots, n\}$ has the symmetric adjacency matrix and it is identified with a unique rooted oriented graph on $\{0, 1, \ldots, n\}$ having the same (symmetric) adjacency matrix. Let $R = k[x_1, \ldots, x_n]$ be the standard polynomial ring in n variables over a field k. Postnikov and Shapiro [12] associated a monomial ideal $\mathcal{M}_G$ in R such that the number of standard monomials of the Artinian quotient $\frac{R}{\mathcal{M}_G}$ is precisely the number of oriented-spanning trees of G. A sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a *G-parking function* if $\mathbf{x}^{\mathbf{p}} = \prod_{i=1}^n x_i^{p_i}$ is a standard monomial of $\frac{R}{\mathcal{M}_G}$ (i.e., $\mathbf{x}^{\mathbf{p}} \notin \mathcal{M}_G$). Let $\operatorname{SPT}(G)$ be the set of (oriented) spanning trees of G rooted at 0 and $\operatorname{PF}(G)$ be the set of G-parking functions of G. Then $|\operatorname{PF}(G)| = |\operatorname{SPT}(G)|$ (see [12]).

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If G is the complete graph K_{n+1} on the vertex set $\{0, 1, \ldots, n\}$, then

$$\mathcal{M}_{K_{n+1}} = \left\langle \left(\prod_{i \in I} x_i \right)^{n-|I|+1} : \emptyset \neq I \subseteq [n] \right\rangle$$

is called a *tree ideal.* Cayley's formula for enumeration of labelled trees states that $|\mathrm{SPT}(K_{n+1})| = (n+1)^{n-1}$. Also the set $\mathrm{PF}(K_{n+1})$ of K_{n+1}-parking functions is the set PF_n of (ordinary) parking functions of length n. A finite sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ with $0 \leq p_i < n$ is called a *parking function* of length n if a nondecreasing rearrangement $p_{i_1} \leq p_{i_2} \leq \ldots \leq p_{i_n}$ of $\mathbf{p}$ satisfies $p_{i_j} < j$ for $1 \leq j \leq n$. A recursively defined bijection $\phi : \mathrm{PF}_n \longrightarrow \mathrm{SPT}(K_{n+1})$ has been constructed by Kreweras [5]. Parking functions or more generally, vector parking functions have appeared in many areas of mathematics. For more on parking functions, we refer to [11, 16]. An algorithmic bijection $\phi : \mathrm{PF}(G) \longrightarrow \mathrm{SPT}(G)$, called *DFS-burning algorithm*, is given by Perkinsons et. al. [10] for a simple graph G and by Gaydarov and Hopkins [4] for multigraph G.

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$. For $r \leq n$, consider a $\tau \in \mathfrak{S}_r$, called a *pattern*. A permutation $\sigma \in \mathfrak{S}_n$ is said to *avoid a pattern* τ if there is no subsequence in $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ that is in the same relative order as τ. Let $\mathfrak{S}_n(\tau)$ be the subset consisting of permutations $\sigma \in \mathfrak{S}_n$ that avoid pattern τ. If $r > n$, then $\mathfrak{S}_n(\tau) = \mathfrak{S}_n$. Also, if $\tau^{(i)} \in \mathfrak{S}_{r_i}$ for $1 \leq i \leq s$, then $\mathfrak{S}_n(\tau^{(1)}, \ldots, \tau^{(s)}) = \bigcap_{j=1}^{s} \mathfrak{S}_n(\tau^{(j)})$. Enumeration and combinatorial properties of the set of permutations avoiding patterns are obtained in [13].

For a nonempty subset $S \subseteq \mathfrak{S}_n$, consider the monomial ideal $I_S = \langle \mathbf{x}^\sigma = \prod_{i=1}^{n} x_i^{\sigma(i)} : \sigma \in S \rangle$ in $R = k[x_1, \ldots, x_n]$ induced by S. The monomial ideal $I_{\mathfrak{S}_n}$ is called a *permotuhedron ideal* and the Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is the tree ideal $\mathcal{M}_{K_{n+1}}$. The i^{th} Betti number $\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]})$ of $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is given by

$$\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]}) = \beta_{i+1}\left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}} \right) = (i!)S(n+1, i+1); \quad (0 \leq i \leq n-1),$$

where $S(n, r)$ is the Stirling number of the second kind, i.e., the number of set-partitions of $[n]$ into r blocks (see [12]). Further, we have already observed that the standard monomials of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$ is given by $\dim_k \left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}} \right) = |\mathrm{PF}_n| = (n+1)^{n-1}$.

For various subsets $S \subseteq \mathfrak{S}_n$, the Alexander dual $I_S^{[\mathbf{n}]}$ of I_S with respect to $\mathbf{n} = (n, \ldots, n)$ has many interesting properties similar to the Alexander dual of permutohedron ideal. The Betti numbers and enumeration of standard monomials of the Alexander dual $I_S^{[\mathbf{n}]}$ for subsets $S = \mathfrak{S}_n(132, 231)$, $\mathfrak{S}_n(123, 132)$ and $\mathfrak{S}_n(123, 132, 213)$ are obtained in [7, 8]

Let $W = \mathfrak{S}_n(132, 312)$. The monomial ideal I_W of R is called a *hypercubic ideal* in [6]. The standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$ correspond bijectively to a subset $\widetilde{\mathrm{PF}}_n$ of PF_n. An element $\mathbf{p} \in \widetilde{\mathrm{PF}}_n$

is called a *restricted parking function* of length n. We show that the number of restricted parking functions of length n is given by

$$\dim_k\left(\frac{R}{I_W^{[\mathbf{n}]}}\right) = |\widetilde{\mathrm{PF}}_n| = \sum_{r=1}^{n}(r!)\ s(n,r),$$

where $s(n,r)$ is the (signless) Stirling number of the first kind, i.e., the number of permutations of $[n]$ having exactly r cycles in its cyclic decomposition. Thus the nth term of integer sequence (A007840) in OEIS [14] can be interpreted as the number of restricted parking functions of length n, or equivalently, as the number of standard monomials of the Artinian quotient $\frac{R}{I_W^{[\mathbf{n}]}}$.

The concept of pattern avoiding permutations has been generalized to many combinatorial objects. A notion of rooted forests that avoids a set of permutations is introduced and many classes of such objects are enumerated in [1]. Let F_n be the set of rooted-labelled forests on $[n]$. Let $F_n(\tau)$ (or more generally, $F_n(\tau^{(1)},\ldots,\tau^{(r)})$) be the subset of F_n consisting of rooted-labelled forests avoiding a pattern τ (or a set of patterns $\{\tau^{(1)},\ldots,\tau^{(r)}\}$). We have

$$|F_n(213,312)| = \sum_{r=1}^{n}(r!)\ s(n,r) = |\widetilde{\mathrm{PF}}_n|.$$

It is surprising that enumeration of standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$ and enumeration of rooted-labelled forests $F_n(213,312)$ avoiding 213 and 312-patterns are related. It is an interesting problem to construct an algorithmic bijection $\phi : \widetilde{\mathrm{PF}}_n \longrightarrow F_n(213,312)$, analogous to DFS-burning algorithm that could explain the relationship between these objects.

The monomial ideal I_S for many other subsets $S \subseteq \mathfrak{S}_n$, consisting of permutations avoiding patterns are considered in the last section.

2. Hypercubic ideals and restricted Parking functions

Consider the subset $W = \mathfrak{S}_n(132,312)$ of permutations of $[n]$ that avoid 132 and 312-patterns. For $\sigma \in \mathfrak{S}_n$, it can be easily checked that $\sigma \in W$ if and only if $\sigma(1) \in [n]$ is arbitrary, and $\sigma(j) = \ell$ for $j > 1$ if either $\sigma(i) = \ell + 1$ or $\sigma(i) = \ell - 1$ for some $i < j$. Clearly, $|W| = 2^{n-1}$. The monomial ideal I_W appeared in [6], where it is called a *hypercubic ideal*. Many properties of I_W and its Alexander dual $I_W^{[\mathbf{n}]}$ with respect to $\mathbf{n} = (n,\ldots,n) \in \mathbb{N}^n$ have been obtained in [6]. We proceed to enumerate the standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$. For this purpose, we consider a little generalization.

Let $\mathbf{u} = (u_1,\ldots,u_n) \in \mathbb{N}^n$ with $1 \leq u_1 < u_2 < \ldots < u_n$. For $\sigma \in \mathfrak{S}_n$, let $\sigma\mathbf{u} = (u_{\sigma(1)},\ldots,u_{\sigma(n)})$ and $\mathbf{x}^{\sigma\mathbf{u}} = \prod_{i=1}^{n} x_i^{u_{\sigma(i)}}$. For any nonempty subset $S \subseteq \mathfrak{S}_n$, we consider the monomial ideal $I_S(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in S\rangle$ in the polynomial ring $R = k[x_1,\ldots,x_n]$. Clearly, $I_S((1,2,\ldots,n)) =$

I_S. The ideals $I_{\mathfrak{S}_n}(\mathbf{u})$ and $I_W(\mathbf{u})$ are also called a *permutohedron ideal* and a *hypercubic ideal*, respectively. For an integer $c \geq 1$, we consider the Alexander dual $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of the hypercubic ideal $I_W(\mathbf{u})$ with respect to $\mathbf{u_n}+\mathbf{c}-\mathbf{1} = (u_n+c-1,\ldots,u_n+c-1) \in \mathbb{N}^n$.

Proposition 2.1. *The minimal generators of $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are given by*

$$I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]} = \left\langle \prod_{j\in T} x_j^{\mu^{\mathbf{u}}_{j,T}} : \emptyset \neq T = \{j_1,\ldots,j_t\} \subseteq [n]; j_1 < \ldots < j_t \right\rangle,$$

where $\mu^{\mathbf{u}}_{j_1,T} = u_n - u_t + c$ and $\mu^{\mathbf{u}}_{j_i,T} = u_n - u_{t+j_i-i} + c$ for $2 \leq i \leq t$.

Proof. The minimal generators of $I_W(\mathbf{u})^{[\mathbf{u_n}]}$ are given in Theorem 3.3 of [6]. Just replace $[\mathbf{u_n}]$ by $[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]$. ☐

The Alexander dual $I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of the permutohedron ideal $I_{\mathfrak{S}_n}(\mathbf{u})$ is given by

$$I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]} = \left\langle \left(\prod_{j\in T} x_j\right)^{u_n-u_{|T|}+c} : T \in \Sigma_n \right\rangle,$$

where Σ_n is the poset of all nonempty subsets of $[n]$ ordered by inclusion. Postnikov and Shapiro [12] showed that the monomial ideal $I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is an order monomial ideal. Moreover, the minimal resolution of $I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is the cellular resolution supported on the order complex $\Delta(\Sigma_n)$ of Σ_n. Thus, the i^{th} Betti number

$$\beta_i(I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = (i!)S(n+1,i+1); \quad (0 \leq i \leq n-1),$$

where $S(n+1,i+1)$ is the Stirling number of the second kind. Further, standard monomials of $\frac{R}{I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}}$ are given in terms of λ-parking functions. Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\lambda_i = u_n - u_i + c$. A sequence $\mathbf{p} = (p_1,\ldots,p_n) \in \mathbb{N}^n$ is called a *λ-parking function* of length n, if non-decreasing rearrangement $p_{i_1} \leq p_{i_2} \leq \ldots \leq p_{i_n}$ of $\mathbf{p}$ satisfies $p_{i_j} < \lambda_{n-j+1}$ for $1 \leq j \leq n$. Let $\mathrm{PF}_n(\lambda)$ be the set of λ-parking functions of length n. Then $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $\frac{R}{I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}}$ if and only if $\mathbf{p} \in \mathrm{PF}_n(\lambda)$. Also, λ-parking functions for $\lambda = (n,n-1,\ldots,1)$ are precisely (ordinary) parking functions of length n, that is, $\mathrm{PF}_n((n,n-1,\ldots,1)) = \mathrm{PF}_n$.

The Alexander dual $I_S^{[\mathbf{n}]}$ of I_S is an order monomial ideal for $S = \mathfrak{S}_n(132,231)$, $\mathfrak{S}_n(123,132)$ and $\mathfrak{S}_n(123,132,213)$ (see [7, 8]). The minimal generators of $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ correspond to elements of poset Σ_n. The monomial ideal $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is also an order monomial ideal and its minimal resolution is the cellular resolution supported on the order complex $\Delta(\Sigma_n)$ of Σ_n. Thus, the i^{th} Betti number $\beta_i(I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = (i!)\ S(n+1,i+1)$ for $0 \leq i \leq n-1$.

We now describe standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}}$. Since $I_W(\mathbf{u}) \subseteq I_{\mathfrak{S}_n}(\mathbf{u})$, we have $I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]} \subseteq I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. Hence, standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}}$ are of the form $\mathbf{x}^{\mathbf{p}}$ for some $\mathbf{p} \in \mathrm{PF}_n(\lambda)$.

Definition 2.2. A λ-parking function $\mathbf{p} = (p_1, \ldots, p_n) \in \mathrm{PF}_n(\lambda)$ is said to be a *restricted λ-parking function* of length n if there exists a permutation $\alpha \in \mathfrak{S}_n$ such that $p_{\alpha_i} < \mu^{\mathbf{u}}_{\alpha_i, T_i}$ for all $1 \leq i \leq n$, where $\alpha_i = \alpha(i),\ T_1 = [n],\ T_i = [n] \setminus \{\alpha_1, \ldots, \alpha_{i-1}\};\ (i \geq 2)$ and $\mu^{\mathbf{u}}_{j,T}$ is as in Proposition 2.1.

Let $\widetilde{\mathrm{PF}}_n(\lambda)$ be the set of restricted λ-parking functions of length n. For $\mathbf{u} = (1, 2, \ldots, n)$ and $c = 1$, we have $\lambda = (n, n-1, \ldots, 1)$. In this case, a restricted λ-parking function is called a *restricted parking function* of length n and we simply write $\widetilde{\mathrm{PF}}_n$ for $\widetilde{\mathrm{PF}}_n(\lambda)$. Also, $\mu_{j,T} = \mu^{\mathbf{u}}_{j,T}$ is given by $\mu_{j_1,T} = n - t + 1$ and $\mu_{j_i,T} = (n-t+1) - (j_i - i);\ i \geq 2$, where $\emptyset \neq T = \{j_1, \ldots, j_t\} \subseteq [n]$ with $j_1 < \ldots < j_t$.

Proposition 2.3. *A monomial $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ if and only if $\mathbf{p} \in \widetilde{\mathrm{PF}}_n(\lambda)$ is a restricted λ-parking function of length n, with $\lambda_i = u_n - u_i + c; (1 \leq i \leq n)$. In particular, a monomial $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $\frac{R}{I_W^{[\mathbf{n}]}}$ if and only if $\mathbf{p} \in \widetilde{\mathrm{PF}}_n$ is a restricted parking function of length n.*

Proof. Standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n}]}}$ are characterized in Theorem 4.3 of [6]. Proceeding on similar lines, we get the desired result. $\square$

Using the cellular resolution of $I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ supported on the order complex $\Delta(\Sigma_n)$, we obtain the multigraded Hilbert series $H\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right)$ of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$. Proceeding as in the proof of Proposition 4.5 of [6], we get a combinatorial formula

$$
\begin{aligned}
(2.1) \qquad |\widetilde{\mathrm{PF}}_n(\lambda)| &= \dim_k\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) \\
&= \sum_{i=1}^{n} (-1)^{n-i} \sum_{\emptyset = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_i = [n]} \prod_{q=1}^{i} \left(\prod_{j \in A_q \setminus A_{q-1}} \mu^{\mathbf{u}}_{j,A_q} \right)
\end{aligned}
$$

for enumeration of standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$, where $\mu^{\mathbf{u}}_{j,A_q}$ is as in Proposition 2.1. Let $\mathcal{C}$ be a chain in Σ_n of the form

$$\mathcal{C} : A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_i = [n]$$

of length $\ell(\mathcal{C}) = i - 1$ and let $\mu^{\mathbf{u}}(\mathcal{C}) = \prod_{q=1}^{i}\left(\prod_{j \in A_q \setminus A_{q-1}} \mu^{\mathbf{u}}_{j,A_q}\right)$, where $A_0 = \emptyset$. Suppose $\mathfrak{Ch}([n])$ is the set of such chains $\mathcal{C}$ in Σ_n. Then formula (2.1) can be expressed compactly as

$$
(2.2) \qquad |\widetilde{\mathrm{PF}}_n(\lambda)| = \dim_k\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \sum_{\mathcal{C} \in \mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1} \mu^{\mathbf{u}}(\mathcal{C}).
$$

We now take $u_i = i$ in (2.2). For $c \geq 1$, let $\dim_k\left(\frac{R}{I_W^{[\mathbf{n+c-1}]}}\right) = a_n(c)$. Then we see that $a_n(c)$ is a polynomial expression in c of degree n for $n \geq 1$. In fact, $a_1(c) = c$ and $a_2(c) = c^2 + 2c$.

Lemma 2.4. *Let $n \geq 3, \mathbf{u} = (1, 2, \ldots, n)$ and $c \geq 1$. For a chain $\mathcal{C} \in \mathfrak{Ch}[n]$ of length $i-1$ of the form $A_1 \subsetneq \ldots A_r \subsetneq A_{r+1} \subsetneq \ldots \subsetneq A_i = [n]$ with $n \in A_{r+1} \setminus A_r$ and $|A_{r+1} \setminus A_r| \geq 2$, there exists a unique chain, namely $\widetilde{\mathcal{C}} : A_1 \subsetneq \ldots A_r \subsetneq A_r \cup \{n\} \subsetneq A_{r+1} \subsetneq \ldots \subsetneq A_i = [n]$ in $\mathfrak{Ch}[n]$ of length i such that $\mu^{\mathbf{u}}(\mathcal{C}) = \mu^{\mathbf{u}}(\widetilde{\mathcal{C}})$.*

Proof. Since $\mu^{\mathbf{u}}(\mathcal{C}) = \prod_{q=1}^{i} \left(\prod_{j \in A_q \setminus A_{q-1}} \mu^{\mathbf{u}}_{j,A_q} \right)$, the equality $\mu^{\mathbf{u}}(\mathcal{C}) = \mu^{\mathbf{u}}(\widetilde{\mathcal{C}})$ holds if $\mu^{\mathbf{u}}_{n,A_r \cup \{n\}} = \mu^{\mathbf{u}}_{n,A_{r+1}}$. Clearly, $\mu^{\mathbf{u}}_{n,A_r \cup \{n\}} = n - (|A_r| + 1 + n - (|A_r| + 1)) + c = c$ and $\mu^{\mathbf{u}}_{n,A_{r+1}} = n - (|A_{r+1}| + n - |A_{r+1}|) + c = c$. □

Let $\mathfrak{Ch}'[n]$ be the set of chains in Σ_n obtained from $\mathfrak{Ch}[n]$ on deleting chains $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ appearing in Lemma 2.4. Then

$$a_n(c) = \sum_{\mathcal{C} \in \mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1} \mu^{\mathbf{u}}(\mathcal{C}) = \sum_{\mathcal{C} \in \mathfrak{Ch}'([n])} (-1)^{n-\ell(\mathcal{C})-1} \mu^{\mathbf{u}}(\mathcal{C}).$$

For $\mathbf{u} = (1, 2, \ldots, n)$ and $c \geq 1$, the value $\mu^{\mathbf{u}}(\mathcal{C})$ depends on the chain $\mathcal{C}$ and c. Thus, we write $\mu^c(\mathcal{C})$ for $\mu^{\mathbf{u}}(\mathcal{C})$. Hence, $a_n(c) = \sum_{\mathcal{C} \in \mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1} \mu^c(\mathcal{C}) = \sum_{\mathcal{C} \in \mathfrak{Ch}'([n])} (-1)^{n-\ell(\mathcal{C})-1} \mu^c(\mathcal{C})$.

For $n \geq 3$, the chains in $\mathfrak{Ch}'[n]$ can be divided into three types.

- A chain $\mathcal{C} : A_1 \subsetneq \ldots \subsetneq A_i = [n]$ in $\mathfrak{Ch}'[n]$ is called a *Type-I* chain if $A_1 = \{n\}$. The Type-I chains in $\mathfrak{Ch}'[n]$ are in one-to-one correspondence with chains in $\mathfrak{Ch}[n-1]$. This correspondence is given by
$$\mathcal{C} \mapsto \mathcal{C} \setminus A_1 : A_2 \setminus \{n\} \subsetneq \ldots \subsetneq A_i \setminus \{n\} = [n-1].$$
As $\ell(\mathcal{C}) - 1 = \ell(\mathcal{C} \setminus A_1)$ and $\mu^c(\mathcal{C}) = (n-1+c)\ \mu^c(\mathcal{C} \setminus A_1)$, we have
$$\sum_{\substack{\mathcal{C} \in \mathfrak{Ch}'[n]; \\ \text{Type-I}}} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C}) = (n-1+c)\ a_{n-1}(c).$$
- A chain $\mathcal{C} : A_1 \subsetneq \ldots \subsetneq A_i = [n]$ in $\mathfrak{Ch}'[n]$ is called a *Type-II* chain if $A_{i-1} = [n-1]$. The Type-II chains in $\mathfrak{Ch}'[n]$ are in one-to-one correspondence with chains in $\mathfrak{Ch}[n-1]$. This correspondence is given by
$$\mathcal{C} \mapsto \mathcal{C}|_{[n-1]} : A_1 \subsetneq \ldots \subsetneq A_{i-1} = [n-1].$$
As $\ell(\mathcal{C}) - 1 = \ell(\mathcal{C}|_{[n-1]})$ and $\mu^c(\mathcal{C}) = (c)\ \mu^{c+1}(\mathcal{C}|_{[n-1]})$, we have
$$\sum_{\substack{\mathcal{C} \in \mathfrak{Ch}'[n]; \\ \text{Type-II}}} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C}) = (c)\ a_{n-1}(c+1).$$
- A chain $\mathcal{C} : A_1 \subsetneq \ldots \subsetneq A_i = [n]$ in $\mathfrak{Ch}'[n]$ is called a *Type-III* chain if $n \in A_1$ and $|A_1| \geq 2$. The Type-III chains in $\mathfrak{Ch}'[n]$ are in one-to-one correspondence with chains in $\mathfrak{Ch}[n-1]$.

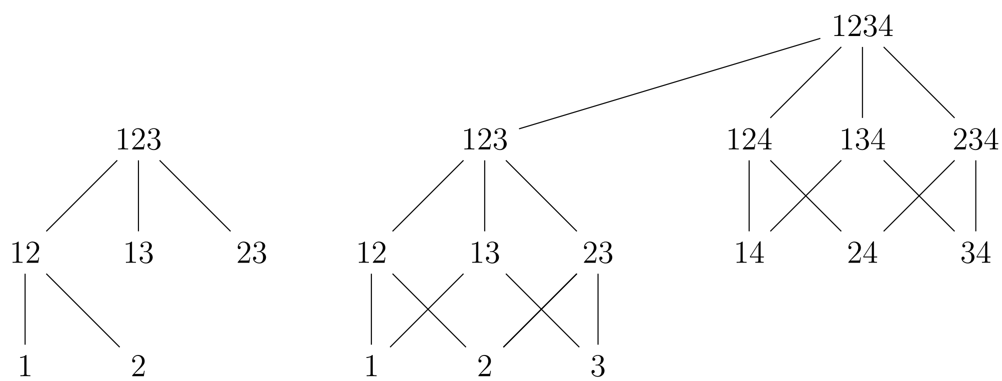
This correspondence is given by

$$\mathcal{C} \mapsto \mathcal{C} \setminus \{n\} : A_1 \setminus \{n\} \subsetneq \ldots \subsetneq A_i \setminus \{n\} = [n-1].$$

As $\ell(\mathcal{C}) = \ell(\mathcal{C} \setminus \{n\})$ and $\mu^c(\mathcal{C}) = (c)\ \mu^c(\mathcal{C} \setminus \{n\})$, we have

$$\sum_{\substack{\mathcal{C} \in \mathfrak{Ch}'[n]; \\ \text{Type-III}}} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C}) = (-c)\ a_{n-1}(c).$$

Consider the poset Σ_n and form a poset $\Lambda_n = \Sigma_{n-1} \coprod (\Sigma_{n-1} * \{n\})$; for $n \geq 2$, where $\Sigma_{n-1} * \{n\} = \{A \cup \{n\} : A \in \Sigma_{n-1}\}$ is a subposet of Σ_n. Two elements $A, B \in \Lambda_n$ are comparable if either $A, B \in \Sigma_{n-1}$ are comparable or $A, B \in \Sigma_{n-1} * \{n\}$ are comparable or $\{A, B\} = \{[n-1], [n]\}$. The Hasse diagram of Λ_n for $n = 3, 4$ are given in FIGURE-1.



Λ_3 Λ_4

FIGURE 1

Clearly, Type-II chains in $\mathfrak{Ch}'[n]$ are chains in Λ_n with an edge $[n-1] \subsetneq [n]$, while Type-III chains in $\mathfrak{Ch}'[n]$ are chains in Λ_n containing $[n]$ but not $[n-1]$.

Proposition 2.5. *For $n \geq 3$ and $c \geq 1$, $a_n(c) = \dim_k \left(\frac{R}{I_W^{[\mathbf{n+c-1}]}} \right)$ satisfies the recurrence relation*

$$a_n(c) = (n-1)a_{n-1}(c) + c\ a_{n-1}(c+1).$$

Proof. As $a_n(c) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^c(\mathcal{C}) = \sum_{\mathcal{C}\in\mathfrak{Ch}'([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^c(\mathcal{C})$, we have

$$\begin{aligned} a_n(c) &= \left[\sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-I}}} + \sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-II}}} + \sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-III}}}\right] (-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C}) \\ &= (n-1+c)\ a_{n-1}(c) + (c)\ a_{n-1}(c+1) + (-c)\ a_{n-1}(c) \\ &= (n-1)\ a_{n-1}(c) + (c)\ a_{n-1}(c+1). \end{aligned}$$

□

Replacing c by an indeterminate x, we consider polynomial $a_n(x)$. The recurrence relation in Proposition 2.5 holds for all $c \geq 1$, thus there exists a polynomial identity

$$a_n(x) = (n-1)\ a_{n-1}(x) + x\ a_{n-1}(x+1) \quad \text{for } n \geq 3. \tag{2.3}$$

Since $a_1(x) = x$ and $a_2(x) = x^2 + 2x$, on setting $a_0(x) = 1$, the recurrence relation (2.3) is valid for $n \geq 1$. Note that $a_n(0) = 0$ for $n \geq 1$.

Proposition 2.6. *For $n \geq 1$, $a_n(x) = \sum_{r=1}^{n} s(n,r)\ x(x+1)\cdots(x+r-1)$.*

Proof. Let $x^{\bar{r}} = x(x+1)\cdots(x+r-1)$ be the r^{th} *rising power* of x. Then $\{x^{\bar{r}} : r = 0,1,\ldots\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$, where $x^{\bar{0}} = 1$. As $a_n(0) = 0$ for $n \geq 1$, we can express $a_n(x) = \sum_{r=1}^{n}\alpha_n(r)x^{\bar{r}}$. As $a_n(x)$ satisfy recurrence relation (2.3) for $n \geq 1$, it follows that $\alpha_n(r)$ and the (signless) Stirling number $s(n,r)$ of the first kind satisfy the same recurrence relation with the same initial conditions (see [15]). Thus $\alpha_n(r) = s(n,r)$. □

Theorem 2.7. *For $n \geq 1$, $\dim_k\left(\frac{R}{I_W^{[\mathbf{n}]}}\right) = a_n = \sum_{r=1}^{n}(r!)\ s(n,r)$.*

Proof. Since $a_n = a_n(1)$, theorem follows from Proposition 2.6. □

Consider the integer sequence (A007840) in OEIS [14]. The nth term b_n of this sequence is the number of factorization of permutations of $[n]$ into ordered cycles and $b_n = \sum_{r=1}^{n}(r!)\ s(n,r)$. It can be verified that

$$b_n = \mathrm{Per}([m_{ij}]_{n\times n}) = \mathrm{Per}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 3 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & n \end{bmatrix},$$

where $m_{ii} = i$ and $m_{ij} = 1$ for $i \neq j$. We recall that *permanent* $\mathrm{Per}([m_{ij}]_{n\times n})$ of the matrix $[m_{ij}]_{n\times n}$ is given by $\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n} m_{i\sigma(i)}$. There are many combinatorial interpretation of the integer sequence (A007840). Theorem 2.7 gives a description of the integer sequence (A007840) in terms of enumeration of standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$, or equivalently, in terms of the number $|\widetilde{\mathrm{PF}}_n|$ of restricted parking functions of length n.

We now show that enumeration of standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$ is related to enumeration of rooted-labelled unimodal forests on $[n]$. The concept of permutations avoiding patterns has been extended to many combinatorial objects, such as, trees, graphs and posets. Let F_n be the set of (unordered) rooted-labelled forests on the vertex set $[n]$. Then $|F_n| = (n+1)^{n-1}$. A rooted-labelled forest on $[n]$ is said to *avoid a pattern* $\tau \in \mathfrak{S}_r$ if along each path from a root to a vertex, the sequence of labels do not contain a subsequence with the same relative order as in the patterns $\tau = \tau(1)\tau(2)\ldots\tau(r)$. Let $F_n(\tau)$ be the set of rooted-labelled forests on $[n]$ that avoid pattern τ. For example, if $\tau = 21$ is a transposition, then $F_n(21)$ is the set of rooted-labelled increasing forests on $[n]$. In other words, labels on any path from a root to a vertex for a forest in $F_n(21)$ form an increasing sequence. Let $F_n(\tau^{(1)}, \ldots, \tau^{(s)})$ be the set of rooted-labelled forests on $[n]$ that avoid a set $\{\tau^{(1)}, \ldots, \tau^{(s)}\}$ of patterns. The enumeration of rooted-labelled forests on $[n]$ that avoid various patterns are obtained in [1]. In particular, it is shown that $|F_n(213, 312)| = \sum_{r=1}^{n} (r!)\ s(n,r)$ for $n \geq 1$. The rooted-labelled forests on $[n]$ avoiding 213 and 312-patterns are precisely the *unimodal* forests. Since $|\widetilde{\mathrm{PF}}_n| = |F_n(213, 312)|$, an explicit or algorithmic bijection $\phi : \widetilde{\mathrm{PF}}_n \longrightarrow F_n(213, 312)$ is desired.

Before we end this section, we describe an easy extension of Theorem 2.7.

Let $b, c \geq 1$ and $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ with $u_i = u_1 + (i-1)b$. We have seen that the standard monomials of $\frac{R}{I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ are of the form $\mathbf{x}^{\mathbf{p}}$, where $\mathbf{p} \in \mathrm{PF}_n(\lambda)$ is a λ-parking function of length n and $\lambda_i = u_n - u_i + c = (n-i)b + c$. Then $|\mathrm{PF}_n(\lambda)| = c(c+nb)^{n-1}$ (see [11, 12]). Let $|\widetilde{\mathrm{PF}}_n(\lambda)| = \dim_k \left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}} \right) = \widetilde{a_n}(c)$. Actually, $\widetilde{a_n}(c)$ depends on b also, but we are treating b to be a fixed constant. Also, $\widetilde{a_n}(c)$ is a polynomial expression in c.

Proposition 2.8. *For $n \geq 3$, $b, c \geq 1$, $\widetilde{a_n}(c)$ satisfies a recurrence relation*

$$\widetilde{a_n}(c) = ((n-1)b)\ \widetilde{a_{n-1}}(c) + (c)\ \widetilde{a_{n-1}}(c+b).$$

Proof. From equation (2.2), we have

$$\widetilde{a_n}(c) = \dim_k \left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}} \right) = \sum_{\mathcal{C} \in \mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{\mathbf{u}}(\mathcal{C}),$$

where $u_i = u_1 + (i-1)b$. For such $\mathbf{u}$, Lemma 2.4 holds. Thus

$$\widetilde{a_n}(c) = \sum_{\mathcal{C} \in \mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{\mathbf{u}}(\mathcal{C}) = \sum_{\mathcal{C} \in \mathfrak{Ch}'([n])} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{\mathbf{u}}(\mathcal{C}).$$

Now proceed as in the proof of Proposition 2.5. $\square$

Replacing c with an indeterminate x, we consider polynomial $\widetilde{a_n}(x)$. Thus there is a polynomial identity

$$\widetilde{a_n}(x) = ((n-1)b)\ \widetilde{a_{n-1}}(x) + x\ \widetilde{a_{n-1}}(x+b) \quad \text{for } n \geq 3. \tag{2.4}$$

Since $\widetilde{a_1}(x) = x$ and $\widetilde{a_2}(x) = x^2 + 2bx$, on setting $\widetilde{a_0}(x) = 1$, the recurrence relation (2.4) is valid for $n \geq 1$. Again, we have $\widetilde{a_n}(0) = 0$ for $n \geq 1$.

Theorem 2.9. *For $n \geq 1$, $\widetilde{a_n}(x) = \sum_{r=1}^{n}(b^{n-r}\ s(n,r))\ x(x+b)\cdots(x+(r-1)b)$. In particular, for $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = (n-i)b + c$*

$$|\widetilde{\mathrm{PF}}_n(\lambda)| = \widetilde{a_n}(c) = b^n \sum_{r=1}^{n} s(n,r)\ \frac{\Gamma(\frac{c}{b}+r)}{\Gamma(\frac{c}{b})},$$

where Γ is the gamma function, i.e., $\Gamma(x+1) = x\ \Gamma(x)$ for $x > 0$ and $\Gamma(1) = 1$.

Proof. As in the proof of Theorem 2.7, let

$$\widetilde{a_n}(x) = \sum_{r=1}^{n} \widetilde{\alpha_n}(r)\ x(x+b)\cdots(x+(r-1)b).$$

Then from recurrence relation (2.4), $\widetilde{\alpha_n}(r)$ satisfies the recurrence relation

$$\widetilde{\alpha_n}(r) = (n-1)b\ \widetilde{\alpha_{n-1}}(r) + \widetilde{\alpha_{n-1}}(r-1); \quad \text{for } 1 \leq r \leq n,$$

with initial conditions $\widetilde{\alpha_0}(1) = 0$ and $\widetilde{\alpha_1}(1) = 1$. It is straight forward to see that $\widetilde{\alpha_n}(r) = b^{n-r}\ s(n,r)$. ☐

3. Some other cases

The Betti numbers and enumeration of standard monomials of the Artinian quotient $\frac{R}{I_S^{[\mathbf{n}]}}$ for $S = \mathfrak{S}_n(132,231), \mathfrak{S}_n(123,132)$ and $\mathfrak{S}_n(123,132,213)$ are give in [7, 8]. In this section, the monomial ideal I_S and its Alexander dual $I_S^{[\mathbf{n}]}$ are studied for various other subsets $S \subseteq \mathfrak{S}_n$ consisting of permutations avoiding patterns. For clarity of presentation, we divide these subsets into three cases.

Case 1. $S_1 = \mathfrak{S}_n(123,132,312), S_2 = \mathfrak{S}_n(123,213,231), S_3 = \mathfrak{S}_n(132,213,231)$.
Case 2. $T_1 = \mathfrak{S}_n(123,132,231), T_2 = \mathfrak{S}_n(213,312,321)$.
Case 3. $U = \mathfrak{S}_n(123,231,312)$.

We have, $|S_a| = |T_b| = |U| = n$ for $1 \leq a \leq 3$ and $1 \leq b \leq 2$ (see [13]).

Lemma 3.1. *The minimal generators of the Alexander dual $I_S^{[\mathbf{n}]}$ for $S = S_a, T_b$ or U are given as follows.*

(i) $I_{S_1}^{[\mathbf{n}]} = \left\langle x_\ell^{\ell+1},\ x_i^i \left(\prod_{j>i}^{n} x_j\right) : 1 \leq \ell \leq n-1;\ 1 \leq i \leq n \right\rangle$.

(ii) $I_{S_2}^{[\mathbf{n}]} = \left\langle x_\ell^n,\ x_i^i x_j^{j-1} : 1 \leq \ell \leq n;\ 1 \leq i < j \leq n \right\rangle$.

(iii) $I_{S_3}^{[\mathbf{n}]} = \left\langle x_\ell^n,\ x_i^i x_j^{n-(j-i)} : 1 \leq \ell \leq n;\ 1 \leq i < j \leq n \right\rangle$.

(iv) $I_{T_1}^{[\mathbf{n}]} = \left\langle x_\ell^{\ell+1},\ x_n^n,\ x_i^i x_n^i : 1 \leq \ell \leq n-1;\ 1 \leq i < n \right\rangle$.

(v) $I_{T_2}^{[\mathbf{n}]} = \left\langle x_\ell^{n-\ell+1},\ x_n^n,\ x_i^{n-i} x_n^{n-i} : 1 \leq \ell \leq n-1;\ 1 \leq i < n \right\rangle$.

(vi) $I_U^{[\mathbf{n}]} = \left\langle \prod_{j\in A} x_j^{\nu_{j,A}} : A = \{j_1, \ldots, j_t\} \in \Sigma_n \right\rangle$, *where* $\nu_{j_1,A} = n - (j_{|A|} - j_1)$ *and* $\nu_{j_i,A} = j_i - j_{i-1}$ *for* $i \geq 2$, *provided* $j_1 < j_2 < \ldots < j_t$.

Proof. We recall that a vector $\mathbf{b} \in \mathbb{N}^n$ satisfying $\mathbf{b} \leq \mathbf{n}$ (i.e., $b_i \leq n$) is maximal with $\mathbf{x}^{\mathbf{b}} \notin I_S$ if and only if $\mathbf{x}^{\mathbf{n}-\mathbf{b}}$ is a minimal generator of $I_S^{[\mathbf{n}]}$ (see Proposition 5.23 of [9]). Now proceeding as in the proof of Lemma 2.1 and 2.2 of [8], it is easy to get the minimal generators of the Alexander duals. We sketch a proof of part (i) and (vi) as proof of other parts are on similar lines.

For $\ell \in [n-1]$, let $\mathbf{b}_\ell = (n, \ldots, n-\ell-1, \ldots, n)$ (ℓ^{th} coordinate $n-\ell-1$, elsewhere n). Then $\mathbf{x}^{\mathbf{b}_\ell} \notin I_{S_1}$ and this gives the minimal generator $x_\ell^{\ell+1} \in I_{S_1}^{[\mathbf{n}]}$. For $i \in [n]$, let $\mathbf{b}_{i,n} = (n, \ldots, n, n-i, n-1, \ldots, n-1) \in \mathbb{N}^n$ (i.e., i^{th} coordinate $n-i$, first $i-1$ coordinates n, and the last $n-i$ coordinates $n-1$). Again, $\mathbf{x}^{\mathbf{b}_{i,n}} \notin I_{S_1}$ and this gives the minimal generator $x_i^i(x_{i+1}\ldots x_n) \in I_{S_1}^{[\mathbf{n}]}$. This proves part (i).

If $A = \{\ell\} \in \Sigma_n$, then taking $\widehat{\mathbf{b}}_\ell = (n, \ldots, 0, \ldots, n)$ (i.e., 0 at ℓ^{th} place and elsewhere n), we get the minimal generator $x_\ell^n \in I_U^{[\mathbf{n}]}$. For $A = \{j_1, \ldots, j_t\} \in \Sigma_n$ with $t \geq 2$ and $j_1 < \ldots < j_t$, let $\widehat{\mathbf{b}}_A = (b_1, \ldots, b_n)$, where $b_{j_1} = j_t - j_1,\ b_{j_i} = n - (j_i - j_{i-1})$ (for $i \geq 2$) and $b_r = n$ (for $r \notin A$).
Claim : $\quad \mathbf{x}^{\widehat{\mathbf{b}}_A} \notin I_U$.

Otherwise, there exists a $\sigma \in U$ such that $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\widehat{\mathbf{b}}_A}$. Thus $\sigma(j_1) \leq j_t - j_1$ and $\sigma(j_i) \leq n - (j_i - j_{i-1})$ for $2 \leq i \leq t$. We see that

$$\sigma(j_1) > \sigma(j_2) > \ldots > \sigma(j_t).$$

If $\sigma(j_{i-1}) < \sigma(j_i)$ for $1 < i \leq t$, then $\sigma(j_{i-1}), \sigma(j_i) \in [n - (j_i - j_{i-1})]$. But $|[n - (j_i - j_{i-1})]| = n - (j_i - j_{i-1})$ and $|[j_{i-1}] \coprod [j_i, n]| = n - (j_i - j_{i-1}) + 1$, where $[a, b] = \{m \in \mathbb{Z} : a \leq m \leq b\}$ denotes an integer interval for $a, b \in \mathbb{Z}$. Thus there exists $\ell \in [n] \setminus [j_{i-1}, j_i]$ such that $\sigma(\ell) \notin [n - (j_i - j_{i-1})]$. This shows that $\sigma(j_{i-1}) < \sigma(j_i) < \sigma(\ell)$. Hence, σ has a 123 or a 312-pattern, a contradiction to $\sigma \in U$. Now $\sigma(j_t) < \sigma(j_1) \leq j_t - j_1$ implies that $j_t - j_1 \geq 2$. Again, $\sigma(j_1), \sigma(j_t) \in [j_t - j_1]$, but $|[j_t - j_1]| = j_t - j_1 < |[j_1, j_t]| = j_t - j_1 + 1$. Thus there exists $\ell \in [j_1+1, j_t-1]$ such that $\sigma(\ell) > j_t - j_1$. This shows that, $\sigma(j_t) < \sigma(j_1) < \sigma(\ell)$ with $j_1 < \ell < j_t$ demonstrating that σ has a 231-pattern, a contradiction. This proves our claim. It can be shown that $\widehat{\mathbf{b}}_A$ has the desired maximality property and hence $\mathbf{x}^{\mathbf{n}-\widehat{\mathbf{b}}_A}$ is a minimal generator of $I_U^{[\mathbf{n}]}$. $\square$

We shall show that all monomial ideals in Lemma 3.1 are order monomial ideals. Let $(P, \preceq)$ be a finite poset and let $\{\omega_u : u \in P\}$ be a set of monomials in R. The monomial ideal $I = \langle \omega_u : u \in P \rangle$

is said to be an *order monomial ideal* if for any pair $u, v \in P$, there is an upper bound $w \in P$ of u and v such that ω_w divides the least common multiple $\mathrm{LCM}(\omega_u, \omega_v)$ of ω_u and ω_v. The order complex $\Delta(P)$ of a finite poset P is a simplicial complex, whose r-dimensional faces are chains $u_1 \prec u_2 \prec \ldots \prec u_{r+1}$ of length r in P. If F is a face of $\Delta(P)$, then monomial label $\mathbf{x}^{\alpha(F)}$ (say) on F is the $\mathrm{LCM}(\omega_u : u \in F)$. Let

$$\mathbb{F}_*(\Delta(P)) : \cdots \to \mathbb{F}_i \to \mathbb{F}_{i-1} \to \cdots \to \mathbb{F}_1 \to \mathbb{F}_0 \to 0$$

be the free R-complex associated to the (labelled) simplicial complex $\Delta(P)$. If $\mathbb{F}_*(\Delta(P))$ is exact at $\mathbb{F}_i$ for $i \geq 1$, then we say that $\mathbb{F}_*(\Delta(P))$ is a *cellular resolution* of I supported on $\Delta(P)$ (see [2, 3, 9]).

It is convenient to study the monomial ideal I_S in Lemma 3.1 according to the three cases already described.

Case-1. To each monomial ideal $I_{S_a}^{[\mathbf{n}]}$, we associate a poset $\Sigma_n(S_a)$ (for $1 \leq a \leq 3$) as follows.

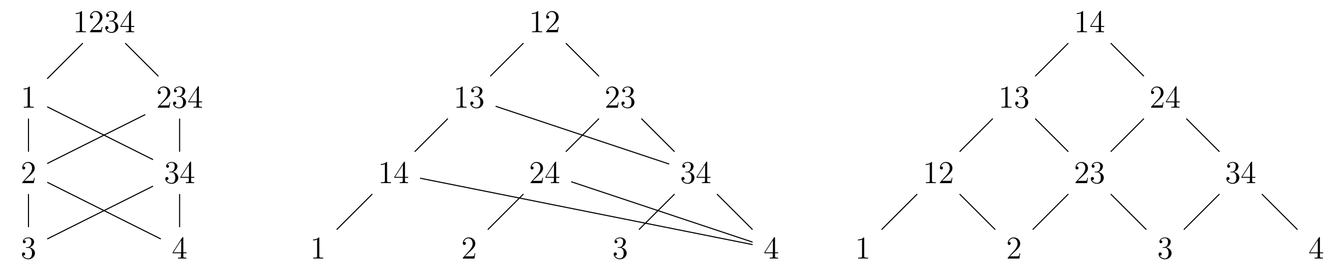
(i) Let $\Sigma_n(S_1) = \{\{\ell\} : 1 \leq \ell \leq n-1\} \cup \{[i,n] : 1 \leq i \leq n\}$, where $[i,n] = \{a \in \mathbb{N} : i \leq a \leq n\}$ and $[n,n] = \{n\}$. We define a poset structure on $\Sigma_n(S_1)$ by describing cover relations. For $\ell, \ell' \in [n-1]$ and $i, i' \in [n]$, $\{\ell\}$ covers $\{\ell'\}$ (or $[i', n]$), if $\ell' = \ell + 1$ (respectively, $i' = \ell + 2$). Also, $[i,n]$ covers $\{\ell'\}$ (or $[i',n]$) if $i = \ell'$ (respectively, $i' = i+1$). The monomial labels $\omega_{\{\ell\}} = x_\ell^{\ell+1}$ and $\omega_{[i,n]} = x_i^i x_{i+1} \ldots x_n$. Set $\mu^1_{j,C}$ for $C \in \Sigma_n(S_1)$ so that $\omega_C = \prod_{j \in C} x_j^{\mu^1_{j,C}}$. The finite poset $\Sigma_n(S_1)$ appeared in [8].

(ii) Let $\Sigma_n(S_2) = \{\{\ell\} : 1 \leq \ell \leq n\} \cup \{\{i,j\} : 1 \leq i < j \leq n\}$. A poset structure on $\Sigma_n(S_2)$ is given by the following cover relations. For $i, j, i', j' \in [n]$ with $i < j$ and $i' < j'$, $\{i,j\}$ covers $\{i', j'\}$, if either ($i = i'$ and $j' = j+1$) or ($j = i'$ and $j' = j+1$). Also, $\{i,j\}$ covers $\{i'\}$ if either ($i = i'$ and $j = n$) or ($i' = j = n$). In this case, the monomial labels $\omega_{\{\ell\}} = x_\ell^n$ and $\omega_{\{i,j\}} = x_i^i x_j^{j-1}$. Set $\mu^2_{j,C}$ for $C \in \Sigma_n(S_2)$ so that $\omega_C = \prod_{j \in C} x_j^{\mu^2_{j,C}}$.

(iii) Let $\Sigma_n(S_3) = \{\{\ell\} : 1 \leq \ell \leq n\} \cup \{\{i,j\} : 1 \leq i < j \leq n\}$. Again, a poset structure on $\Sigma_n(S_3)$ is given by the following cover relations. For $i, j, i', j' \in [n]$ with $i < j$ and $i' < j'$, $\{i,j\}$ covers $\{i', j'\}$, if either ($i = i'$ and $j = j'+1$) or ($i = i'-1$ and $j' = j$). Also, $\{i,j\}$ covers $\{i'\}$ if either ($i = i'$ and $j = i+1$) or ($j = i'$ and $j = i+1$). Again, the monomial labels $\omega_{\{\ell\}} = x_\ell^n$ and $\omega_{\{i,j\}} = x_i^i x_j^{n-(j-i)}$. Set $\mu^3_{j,C}$ for $C \in \Sigma_n(S_3)$ so that $\omega_C = \prod_{j \in C} x_j^{\mu^3_{j,C}}$.

The Hasse diagrams of $\Sigma_4(S_1)$, $\Sigma_4(S_2)$, $\Sigma_4(S_3)$ are given in Figure-2.

Proposition 3.2. (i). *The ideal $I_{S_a}^{[\mathbf{n}]}$ is an order monomial ideal for $1 \leq a \leq 3$.*
(ii). *The free complex $\mathbb{F}_*(\Delta(\Sigma_n(S_a)))$ is the cellular resolution of $I_{S_a}^{[\mathbf{n}]}$ supported on the order complex $\Delta(\Sigma_n(S_a))$ for $1 \leq a \leq 3$.*



$\Sigma_4(S_1)$ $\Sigma_4(S_2)$ $\Sigma_4(S_3)$

FIGURE 2

Proof. Given the poset structure on $\Sigma_n(S_a)$, it is a straight forward verification that $I_{S_a}^{[\mathbf{n}]}$ is an order monomial ideal. Postnikov and Shapiro [12] showed that the free complex $\mathbb{F}_*(\Delta(P))$ is a cellular resolution of the order monomial ideal $I = \langle \omega_u : u \in P \rangle$ (see Theorem 2.4 of [8]). $\square$

Remark 3.3. The cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n(S_a)))$ is minimal for $a = 1$, but nonminimal for $a = 2, 3$. Also, the r^{th} Betti number $\beta_r(I_{S_1}^{[\mathbf{n}]})$ is given by (see Theorem 2.7 of [8])

$$\beta_r(I_{S_1}^{[\mathbf{n}]}) = \sum_{s=0}^{r+1} \binom{n-1}{s} \binom{n-s}{r+1-s}; \quad (0 \leq r \leq n-1).$$

We now identify standard monomials of $\frac{R}{I_{S_a}^{[\mathbf{n}]}}$. Consider the following subsets of the set PF_n of parking functions $\mathbf{p} = (p_1, \ldots, p_n)$ of length n.

(i) $\mathrm{PF}_n^1 = \{\mathbf{p} \in \mathrm{PF}_n : p_t \leq t, \ \forall t \text{ and if } p_i = i, \text{ then } p_j = 0 \text{ for some } j \in [i, n]\}$.
(ii) $\mathrm{PF}_n^2 = \{\mathbf{p} \in \mathrm{PF}_n : \text{if } p_i \geq i, \text{ then } p_j < j - 1 \text{ for all } j \in [i+1, n]\}$.
(iii) $\mathrm{PF}_n^3 = \{\mathbf{p} \in \mathrm{PF}_n : \text{if } p_i \geq i, \text{ then } p_j < n - (j - i) \text{ for all } j \in [i+1, n]\}$.

In view of Lemma 3.1, $\mathbf{x}^{\mathbf{p}} \notin I_{S_a}^{[\mathbf{n}]}$ if and only if $\mathbf{p} \in \mathrm{PF}_n^a$ for $1 \leq a \leq 3$. Thus (fine) Hilbert series $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{x}\right)$ of $\frac{R}{I_{S_a}^{[\mathbf{n}]}}$ is given by $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{x}\right) = \sum_{\mathbf{p} \in \mathrm{PF}_n^a} \mathbf{x}^{\mathbf{p}}$. In particular, $|\mathrm{PF}_n^a| = \dim_k \left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right) = H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{1}\right)$, where $\mathbf{1} = (1, \ldots, 1)$. Using the cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n(S_a)))$ supported on the order complex $\Delta(\Sigma_n(S_a))$, the (fine) Hilber series $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{x}\right)$ is given by

$$H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{x}\right) = \frac{\sum_{i=0}^{n} (-1)^i \sum_{(C_1, \ldots, C_i) \in \mathcal{F}_{i-1}^a} \prod_{q=1}^{i} \left(\prod_{j \in C_q \setminus C_{q-1}} x_j^{\mu_{j,C_q}^a} \right)}{(1 - x_1) \cdots (1 - x_n)}, \tag{3.1}$$

where $\mathcal{F}^a_{i-1}$ is the set of $i-1$-dimensional faces of $\Delta(\Sigma_n(S_a))$, $(C_1,\ldots,C_i) \in \mathcal{F}^a_{i-1}$ is a (strict) chain $C_1 \prec \ldots \prec C_i$ of length $i-1$, $C_0 = \emptyset$ and $\mu^a_{j,C}$ is as in the definition of poset $\Sigma_n(S_a)$.

Proposition 3.4. *The number of standard monomials of $\frac{R}{I^{[\mathbf{n}]}_{S_a}}$ is given by*

$$\dim_k\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = \sum_{i=1}^{n}(-1)^{n-i}\sum_{\substack{(C_1,\ldots,C_i)\in\mathcal{F}^a_{i-1}\\ C_1\cup\ldots\cup C_i=[n]}}\prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}}\mu^a_{j,C_q}\right),$$

where summation is carried over all $i-1$-dimensional faces $(C_1,\ldots,C_i) \in \mathcal{F}^a_{i-1}$ of $\Delta(\Sigma_n(S_a))$ with $C_1 \cup \ldots \cup C_i = [n]$ and $C_0 = \emptyset$. Also,

$$\dim_k\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = \sum_{\substack{0\le i\le n;\\ (C_1,\ldots,C_i)\in\mathcal{F}^a_{i-1}}}(-1)^i\left(\prod_{q=1}^{i}\Big(\prod_{j\in C_q\setminus C_{q-1}}(\mu^a_{j,\{j\}}-\mu^a_{j,C_q})\Big)\right)\left(\prod_{l\notin C_i}\mu^a_{l,\{l\}}\right),$$

where summation is carried over all faces $(C_1,\ldots,C_i) \in \mathcal{F}^a_{i-1}$ including the empty face $C_0 = \emptyset$.

Proof. As $|\mathrm{PF}^a_n| = \dim_k\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = H\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}, \mathbf{1}\right)$, letting $\mathbf{x} \to \mathbf{1}$ in the rational function expression 3.1 of $H\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}, \mathbf{x}\right)$, and applying L'Hospital's rule, we get the first formula. For more detail, see the proof of Proposition 4.5 of [6]. In order to get the second formula, put $y_j = \frac{1}{x_j}$ in (3.1) to get a rational function, say $\tilde{H}\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}, \mathbf{y}\right)$. Now letting $\mathbf{y} \to \mathbf{1}$ in the product $\left(\prod_{j=1}^n y_j^{\mu^a_{j,\{j\}}-1}\right)\tilde{H}\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}, \mathbf{y}\right)$, we get the second formula, which is due to Postnikov and Shapiro [12].

Theorem 3.5. *The number of standard monomials of $\frac{R}{I^{[\mathbf{n}]}_{S_a}}$ is given by*

$$\dim_k\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = |\mathrm{PF}^a_n| = \frac{(n+1)!}{2}, \quad (1 \le a \le 3).$$

Proof. As $\dim_k\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = 1$ for $n = 1$, we assume that $n > 1$.
(i) Let $a = 1$. Using the second formula

$$\dim_k\left(\frac{R}{I^{[\mathbf{n}]}_{S_1}}\right) = \sum_{\substack{0\le i\le n;\\ (C_1,\ldots,C_i)\in\mathcal{F}^1_{i-1}}}(-1)^i\left(\prod_{q=1}^{i}\Big(\prod_{j\in C_q\setminus C_{q-1}}(\mu^1_{j,\{j\}}-\mu^1_{j,C_q})\Big)\right)\left(\prod_{l\notin C_i}\mu^1_{l,\{l\}}\right)$$

in Proposition 3.4, we shall show that

$$\dim_k\left(\frac{R}{I_{S_1}^{[\mathbf{n}]}}\right) = n(n!) + (n-1)((n-1)!) \sum_{\substack{1\le i\le n;\\ 0=j_0<j_1<\ldots<j_i<n}} (-1)^i \frac{1}{\prod_{q=2}^{i} j_q}. \tag{3.2}$$

The term corresponding to the empty chain is $n(n!)$. Also, for a (strict) chain $C_1 \prec \ldots \prec C_i$ in $\mathcal{F}_{i-1}^1$, the corresponding term in the second formula is zero if the chain has a singleton member. Thus surviving terms are of the form $C_l = [j_{i-l+1}, n]$ for some sequence $0 = j_0 < j_1 < \ldots < j_i < n$. Note that the term corresponding to such a chain is precisely, $(-1)^i \frac{(n-1)((n-1)!)}{j_2 j_3 \cdots j_i}$. This proves (3.2). Let $\alpha_n = \sum_{i\ge1}(-1)^{i+1}\sum_{0=j_0<j_1<\ldots<j_i<n}\frac{1}{\prod_{q=2}^{i} j_q}$. Clearly, $\alpha_1 = 0$. For $n > 1$, we claim that $\alpha_n = \frac{n}{2}$. We have,

$$\begin{aligned}
\alpha_n &= \sum_{i\ge1}(-1)^{i+1}\sum_{0=j_0<j_1<\ldots<j_i<n-1}\frac{1}{\prod_{q=2}^{i} j_q} + \sum_{i\ge1}(-1)^{i+1}\sum_{0=j_0<j_1<\ldots<j_i=n-1}\frac{1}{\prod_{q=2}^{i} j_q}\\
&= \alpha_{n-1} + \frac{1}{n-1}\sum_{i\ge2}(-1)^{i+1}\sum_{0=j_0<j_1<\ldots<j_{i-1}<n-1}\frac{1}{\prod_{q=2}^{i-1} j_q} + 1\\
&= \alpha_{n-1} - \frac{1}{n-1}\alpha_{n-1} + 1 = \frac{n-2}{n-1}\alpha_{n-1} + 1.
\end{aligned}$$

On solving this recurrence relation, we get $\alpha_n = \frac{n}{2}$ for $n > 1$. Now in view of (3.2),

$$\dim_k\left(\frac{R}{I_{S_1}^{[\mathbf{n}]}}\right) = n(n!) + (n-1)((n-1)!)\left(\frac{-n}{2}\right) = \frac{(n+1)!}{2}.$$

(ii) Let $a = 2$. As $\dim_k\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right) = 1$ or 3 for $n = 1$ or 2, respectively, we assume that $n > 2$. Suppose $\mathcal{F}^2[n] = \cup_{i=1}^{n}\{(C_1,\ldots,C_i) \in \mathcal{F}_{i-1}^2 : \cup_{j=1}^{i} C_j = [n]\}$. For $\mathcal{C} = (C_1,\ldots,C_i) \in \mathcal{F}^2[n]$, we write $\mu^2(\mathcal{C}) = \prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}} \mu_{j,C_q}^2\right)$. In view of the first formula in Proposition 3.4, we have

$$\tilde{\alpha}_n = \dim_k\left(\frac{R}{I_{S_2}^{[\mathbf{n}]}}\right) = \sum_{\mathcal{C}\in\mathcal{F}^2[n]}(-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C}).$$

Now decompose $\mathcal{F}^2[n] = \mathcal{F}^2[n]' \coprod \mathcal{F}^2[n]''$, where $\mathcal{C} = (C_1,\ldots,C_i) \in \mathcal{F}^2[n]'$ if $|C_1| = 1$ and $\mathcal{C} \in \mathcal{F}^2[n]''$ if $|C_1| = 2$. Then $\tilde{\alpha}_n = \tilde{\alpha}'_n + \tilde{\alpha}''_n$, where

$$\tilde{\alpha}'_n = \sum_{\mathcal{C}\in\mathcal{F}^2[n]'}(-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C}) \quad\text{and}\quad \tilde{\alpha}''_n = \sum_{\mathcal{C}\in\mathcal{F}^2[n]''}(-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C}).$$

A chain $\mathcal{C} = (C_1,\ldots,C_i) \in \mathcal{F}^2[n]'$ is called a *Type-I, Type-II or Type-III* chain, if $(C_1, C_2) = (\{i\},\{i,n\})$ for $i < n$, $(C_1, C_2) = (\{n\},\{i,n\})$ for $i < n$ or $(C_1, C_2) = (\{n\},\{i,n-1\})$ for $i < n-1$, respectively. Now

$$\begin{aligned}\tilde{\alpha}'_n &= \left[\sum_{\substack{\mathcal{C}\in\mathcal{F}^2[n]';\\ \text{Type-I}}} + \sum_{\substack{\mathcal{C}\in\mathcal{F}^2[n]';\\ \text{Type-II}}} + \sum_{\substack{\mathcal{C}\in\mathcal{F}^2[n]';\\ \text{Type-III}}}\right] (-1)^{n-\ell(\mathcal{C})-1}\ \mu^2(\mathcal{C})\\ &= n\tilde{\alpha}'_{n-1} - \frac{n}{n-1}\tilde{\alpha}''_n + n\tilde{\alpha}''_{n-1} = n\tilde{\alpha}_{n-1} - \frac{n}{n-1}\tilde{\alpha}''_n.\end{aligned}$$

Claim : $\tilde{\alpha}''_n = -\frac{(n-1)(n!)}{2}$.

For $1 \le t \le n-1$, consider saturated chains $\mathcal{C}^{(t)}$ in $\mathcal{F}^2[n]''$ of the form

$$\mathcal{C}^{(t)} : \{t,n\} \prec \{t,n-1\} \prec \ldots \prec \{t,t+1\} \prec \{t-1,t\} \prec \ldots \prec \{1,2\}.$$

Then $\mu^2(\mathcal{C}^{(t)}) = t((n-1)!)$. Any other chain in $\mathcal{F}^2[n]''$ is either of the form

$$\mathcal{C} : \{r,n\} \prec \ldots \prec \{r,r+1\} \prec \ldots \prec \{s-1,s\} \prec \{l,s-1\} \prec \{l,s-2\} \prec \ldots \prec \ldots$$

or

$$\mathcal{C}' : \{r,n\} \prec \ldots \prec \{r,r+1\} \prec \ldots \prec \{s'-1,s'\} \prec \{l',s'-2\} \prec \ldots \prec \ldots, \quad (\text{for } 3 \le r \le n-1),$$

where s (or s') is the largest integer such that $\{l,s-1\}$ covers $\{s-1,s\}$ in $\mathcal{C}$ (or $\{l',s'-1\}$ is not in $\mathcal{C}'$) for some $l < s-2$ (or $l' < s'-2$). Let $\tilde{\mathcal{C}} = \mathcal{C} \setminus \{\{l,s-1\}\}$ be the chain obtained from $\mathcal{C}$ on deleting $\{l,s-1\}$ and $\tilde{\mathcal{C}'} = \mathcal{C}' \cup \{\{l',s'-1\}\}$ be the chain obtained from $\mathcal{C}'$ on adjoining $\{l',s'-1\}$. Clearly, $\mu^2(\mathcal{C}) = \mu^2(\tilde{\mathcal{C}})$ and $\mu^2(\mathcal{C}') = \mu^2(\tilde{\mathcal{C}'})$. As length $\ell(\mathcal{C}) = \ell(\tilde{\mathcal{C}})+1$ and $\ell(\mathcal{C}') = \ell(\tilde{\mathcal{C}'})-1$, the terms in $\tilde{\alpha}''_n = \sum_{\mathcal{C}\in\mathcal{F}^2[n]''}(-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C})$ corresponding to chains $\mathcal{C} \in \mathcal{F}^2[n]''$ different from $\mathcal{C}^{(t)}$ cancel out. Thus

$$\tilde{\alpha}''_n = \sum_{t=1}^{n-1}(-1)^{n-\ell(\mathcal{C}^{(t)})-1}\ \mu^2(\mathcal{C}^{(t)}) = \sum_{t=1}^{n-1}(-1)^{n-(n-2)-1}\ t((n-1)!) = -\frac{(n-1)(n!)}{2}.$$

Now $\tilde{\alpha}_n = \tilde{\alpha}'_n + \tilde{\alpha}''_n = n\tilde{\alpha}_{n-1} - \frac{n}{n-1}\tilde{\alpha}''_n + \tilde{\alpha}''_n = n\tilde{\alpha}_{n-1} + \frac{n!}{2}$. On solving this recurrence, we get $\tilde{\alpha}_n = \frac{(n+1)!}{2}$, as desired.

(iii) Let $a = 3$ and assume $n > 2$. Proceeding as in part(ii), we write

$$\dim_k\left(\frac{R}{I_{S_3}^{[\mathbf{n}]}}\right) = \sum_{\mathcal{C}\in\mathcal{F}^3[n]}(-1)^{n-\ell(\mathcal{C})-1}\mu^3(\mathcal{C}),$$

where $\mathcal{F}^3[n]$ is the collection of all chains $\bar{\mathcal{C}} = (C_1,\ldots,C_i)$ in $\mathcal{F}^3_{i-1}$ (for some i) with $\cup_{j=1}^i C_j = [n]$ and $\mu^3(\bar{\mathcal{C}}) = \prod_{q=1}^i \left(\prod_{j\in C_q\setminus C_{q-1}} \mu^3_{j,C_q}\right)$. For $1 \le t \le n-1$, let $\bar{\mathcal{C}}^{(t)}$ be the chain in $\mathcal{F}^3[n]$ of the form

$$\bar{\mathcal{C}}^{(t)} : \{t\} \prec \{t,t+1\} \prec \ldots \prec \{t,n-1\} \prec \{t,n\} \prec \{t-1,n\} \prec \ldots \prec \{1,n\}$$

and $\bar{\mathcal{C}}^{(t)} \setminus \{\{t\}\}$ is the chain obtained from $\bar{\mathcal{C}}^{(t)}$ by deleting the first element $\{t\}$. Now $\mu^3(\bar{\mathcal{C}}^{(t)}) = n!$ and $\mu^3(\bar{\mathcal{C}}^{(t)} \setminus \{\{t\}\}) = t((n-1)!)$. There is one more chain $\bar{\mathcal{C}} : \{n\} \prec \{n-1,n\} \prec \ldots \prec \{1,n\}$ in $\mathcal{F}^3[n]$, with $\mu^3(\bar{\mathcal{C}}) = n!$. As in part (ii), it can be shown that the terms corresponding to remaining chains cancel out. Thus

$$\dim_k \left(\frac{R}{I_{S_3}^{[\mathbf{n}]}} \right) = n(n!) - (1+2+\ldots+(n-1))((n-1)!) = \frac{(n+1)!}{2}.$$

□

Theorem 3.5 shows that the integer sequence $\left\{ \dim_k \left(\frac{R}{I_{S_a}^{[\mathbf{n}]}} \right) = \frac{(n+1)!}{2} \right\}_{n=1}^{\infty}$ for $1 \leq a \leq 3$ is the integer sequence (A001710) in OEIS [14]. As $|\mathrm{PF}_n^a| = \frac{(n+1)!}{2}$, it is expected that the set PF_n^a could be easily enumerated. Let $\mathbf{p} \in \mathrm{PF}_n^1$. Then $p_t \leq t;\ \forall t$ and $p_i = i$ implies that $p_j = 0$ for some $j \in [i+1, n]$. We count $\mathbf{p} \in \mathrm{PF}_n^1$ according to the value s of the largest $t \in [n]$ with $p_t = t$. If $p_t < t; \forall t \in [n]$, then we take $s = 0$. As $p_n < n$, we have $0 \leq s \leq n-1$. For $s = 0$, any $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{N}^n$ such that $p_t < t;\ \forall t$ is a parking function and number of such $\mathbf{p} \in \mathrm{PF}_n^1$ is precisely $\prod_{t=1}^{n}(t) = n!$. Now let $s \geq 1$. Any sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ satisfying conditions

$$p_t \leq t\ \forall t < s,\ p_s = s,\ \text{and}\ p_j < j\ \forall j > s, \text{with at least one}\ p_j = 0, \tag{3.3}$$

is always a parking function. The number of $\mathbf{p}$ satisfying conditions (3.3) is

$$\prod_{t=1}^{s-1}(t+1) \left[\prod_{j=s+1}^{n} j - \prod_{j'=s+1}^{n} (j'-1) \right] = (n-s)((n-1)!).$$

This shows that $|\mathrm{PF}_n^1| = \sum_{s=0}^{n-1}(n-s)((n-1)!) = \frac{(n+1)!}{2}$. Similarly, PF_n^a for $a = 2, 3$ can also be enumerated. However, it is still an interesting problem to construct an (explicit) bijection $\phi : \mathrm{PF}_n^a \longrightarrow F_{n+1}(21)$, where $F_{n+1}(21)$ is the set of rooted-labelled increasing forests on $[n+1]$.

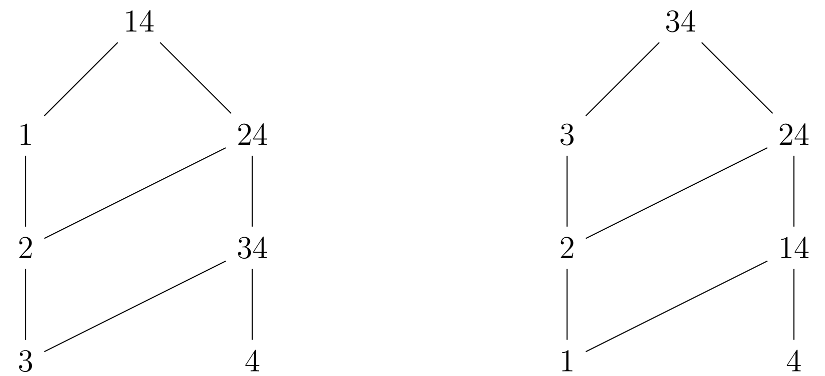
Case-2 : To monomial ideals $I_{T_1}^{[\mathbf{n}]}$ and $I_{T_2}^{[\mathbf{n}]}$, we associate finite posets $\Sigma_n(T_1)$ and $\Sigma_n(T_2)$ respectively, as below.

(i) Let $\Sigma_n(T_1) = \{\{\ell\}, \{i,n\} : 1 \leq \ell \leq n-1;\ 1 \leq i \leq n\}$, where $\{n,n\} = \{n\}$. We define a poset structure on $\Sigma_n(T_1)$ by describing cover relations. For $\ell, \ell' \in [n-1]$ and $i, i' \in [n]$, $\{\ell\}$ covers $\{\ell'\}$, if $\ell' = \ell + 1$. Also, $\{i,n\}$ covers $\{\ell'\}$ (or $\{i',n\}$) if $i = \ell'$ (respectively, $i' = i+1$). The monomial labels $\omega_{\{\ell\}} = x_\ell^{\ell+1}$, $\omega_{\{n\}} = x_n^n$ and $\omega_{\{i,n\}} = x_i^i x_n^i$ for $1 \leq \ell, i < n$. Set $\hat{\mu}_{j,C}^1$ for $C \in \Sigma_n(T_1)$ so that $\omega_C = \prod_{j \in C} x_j^{\hat{\mu}_{j,C}^1}$.

(ii) Let $\Sigma_n(T_2) = \Sigma_n(T_1)$. But the poset structure on $\Sigma_n(T_2)$ is obtained by interchanging $\{i\}$ with $\{n-i\}$ (and also, $\{i,n\}$ with $\{n-i,n\}$)(for $1 \leq i < n$) in the poset $\Sigma_n(T_1)$. The cover relations of the poset $\Sigma_n(T_2)$ are given as follows. For $\ell, \ell', i, i' \in [n-1]$, $\{\ell\}$ covers $\{\ell'\}$, if $\ell' = \ell - 1$ and $\{i,n\}$ covers $\{\ell'\}$ (or $\{i',n\}$) if $i = \ell'$ (respectively, $i' = i-1$). In addition,

$\{1, n\}$ covers $\{n\}$. The monomial labels $\omega_{\{\ell\}} = x_\ell^{n-\ell+1}$, $\omega_{\{n\}} = x_n^n$ and $\omega_{\{i,n\}} = x_i^{n-i} x_n^{n-i}$ for $1 \leq \ell, i < n$. Set $\hat{\mu}^2_{j,C}$ for $C \in \Sigma_n(T_1)$ so that $\omega_C = \prod_{j \in C} x_j^{\hat{\mu}^2_{j,C}}$.

The Hasse diagram of $\Sigma_4(T_1)$ and $\Sigma_4(T_2)$ are given in FIGURE-3.



$\Sigma_4(T_1)$ $\Sigma_4(T_2)$

FIGURE 3

Proposition 3.6. (i). *The ideals $I_{T_1}^{[\mathbf{n}]}$ and $I_{T_2}^{[\mathbf{n}]}$ are order monomial ideals.*
(ii). *The free complex $\mathbb{F}_*(\Delta(\Sigma_n(T_b)))$ is the minimal cellular resolution of $I_{S_a}^{[\mathbf{n}]}$ supported on the order complex $\Delta(\Sigma_n(T_b))$ for $1 \leq b \leq 2$. Thus the r^{th} Betti number $\beta_r(I_{T_b}^{[\mathbf{n}]})$ is given by*

$$\beta_r(I_{T_b}^{[\mathbf{n}]}) = \binom{n}{r+1} + (r+1)\binom{n-1}{r+1} + r\binom{n-1}{r}, \quad (1 \leq r \leq n-1).$$

Proof. From the definitions of the poset $\Sigma_n(T_b)$, it is clear that the ideal $I_{T_b}^{[\mathbf{n}]}$ is an order monomial ideal. Further, the cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n(T_b)))$ is the minimal resolution of $I_{S_a}^{[\mathbf{n}]}$ supported on the order complex $\Delta(\Sigma_n(T_b))$ because monomial label on any face of $\Delta(\Sigma_n(T_b))$ is different from the monomial label on subfaces. Thus the r^{th} Betti number $\beta_r(I_{T_b}^{[\mathbf{n}]})$ equals the number (strict) chains of length r in the poset $\Sigma_n(T_b)$. Since $\Sigma_n(T_2)$ is obtained from $\Sigma_n(T_1)$ by changing i to $n-i$ for $i \in [n]$, number of chains of length r in both the posets are same. We count chains of length r in $\Sigma_n(T_1)$ for $0 \leq r \leq n-1$. Consider a (strict) chain

$$\mathcal{C} : C_1 \prec C_2 \prec \ldots \prec C_s \prec C_{s+1} \prec \ldots \prec C_{r+1}.$$

If all C_j are of the form $\{t_j, n\}$ for $t_j \in [n]$, then the chain $\mathcal{C}$ can be identified with a $r+1$-subset $\{t_1, \ldots, t_{r+1}\}$ of $[n]$. Thus number of such chains is $\binom{n}{r+1}$. If $C_s = \{t_s\}$ and $C_{s+1} = \{t_{s+1}, n\}$ for some s with $t_{s+1} < t_s$, then the chain $\mathcal{C}$ can be identified with a $r+1$-subset $\{t_1, \ldots, t_{r+1}\}$ of $[n-1]$ with a chosen element t_s. Any $j \in \{t_1, \ldots, t_{r+1}\}$ represent singleton $\{j\}$ if $j \geq t_s$, while it represent $\{j, n\}$ for $j < t_s$. The number of such chains is precisely $(r+1)\binom{n-1}{r+1}$. Now we count chains $\mathcal{C}$ with $C_s = \{t_s\}$ and $C_{s+1} = \{t_s, n\}$ (i.e., $t_s = t_{s+1}$). In this case, chain $\mathcal{C}$ can be identified with a r-subset $\{t_1, \ldots, t_s = t_{s+1}, \ldots, t_{r+1}\}$ of $[n-1]$ with a chosen element t_s. Thus number of such chains is $r\binom{n-1}{r}$. Since any r-chain $\mathcal{C}$ in $\Sigma_n(T_1)$ is a chain of one of the three types, we get the desired result. $\square$

Consider the following subsets of PF_n of parking function $\mathbf{p} = (p_1, \ldots, p_n)$.

(i) $\widehat{\mathrm{PF}}_n^1 = \{\mathbf{p} \in \mathrm{PF}_n : p_t \leq t, \ \forall t \text{ and if } p_i = i, \text{ then } p_n < i\}$.
(ii) $\widehat{\mathrm{PF}}_n^2 = \{\mathbf{p} \in \mathrm{PF}_n : p_{n-t} \leq t, \ \forall t \text{ and if } p_{n-i} = i, \text{ then } p_n < i\}$.

In view of Lemma 2.4, $\mathbf{x}^{\mathbf{p}} \notin I_{T_b}^{[\mathbf{n}]}$ if and only if $\mathbf{p} \in \widehat{\mathrm{PF}}_n^b$ for $b = 1, 2$. Thus, $|\widehat{\mathrm{PF}}_n^b| = \dim_k \left(\frac{R}{I_{T_b}^{[\mathbf{n}]}}\right)$. Also, the mapping $(p_1, p_2, \ldots, p_{n-1}, p_n) \mapsto (p_{n-1}, p_{n-2}, \ldots, p_1, p_n)$ induces a bijection between $\widehat{\mathrm{PF}}_n^1$ and $\widehat{\mathrm{PF}}_n^2$.

Theorem 3.7. *The number of standard monomials of* $\frac{R}{I_{T_b}^{[\mathbf{n}]}}$ *is given by*

$$|\widehat{\mathrm{PF}}_n^b| = \dim_k \left(\frac{R}{I_{T_b}^{[\mathbf{n}]}}\right) = s(n+1, 2); \quad (b = 1, 2),$$

where $s(n+1, 2)$ *is the (signless) Stirling number of the first kind.*

Proof. We take $b = 1$. Proceeding as in Proposition 3.4, we get

$$\dim_k \left(\frac{R}{I_{T_1}^{[\mathbf{n}]}}\right) = \sum_{\mathcal{C} \in \widehat{\mathcal{F}}^1[n]} (-1)^{n-\ell(\mathcal{C})-1} \, \widehat{\mu}^1(\mathcal{C}),$$

where $\widehat{\mathcal{F}}^1[n]$ is the collection of all chains $\mathcal{C} = (C_1, \ldots, C_i)$ in $\Sigma_n(T_1)$ such that $C_1 \cup \ldots \cup C_i = [n]$ and $\widehat{\mu}^1(\mathcal{C}) = \prod_{q=1}^{i} \left(\prod_{j \in C_q \setminus C_{q-1}} \widehat{\mu}^1_{j, C_q}\right)$. For $1 \leq t \leq n$, let $\widehat{\mathcal{C}}^{(n)} : \{n\} \prec \{n-1, n\} \prec \ldots \prec \{1, n\}$,

$$\widehat{\mathcal{C}}^{(t)} : \{n-1\} \prec \ldots \prec \{t\} \prec \{t, n\} \prec \{t-1, n\} \prec \ldots \prec \{1, n\}; \quad (1 \leq t \leq n-1)$$

and $\widehat{\mathcal{C}}'^{(t)}$ be the chain obtained from $\widehat{\mathcal{C}}^{(t)}$ on deleting $\{t, n\}$. For $t = n$, we have $\{n, n\} = \{n\}$. It is clear that $\widehat{\mathcal{F}}^1[n] = \{\widehat{\mathcal{C}}^{(t)}, \widehat{\mathcal{C}}'^{(t)} : 1 \leq t \leq n\}$. Also, $\widehat{\mu}^1(\widehat{\mathcal{C}}^{(t)}) = n!$ and $\widehat{\mu}^1(\widehat{\mathcal{C}}'^{(t)}) = \frac{t-1}{t}(n!)$ for $1 \leq t \leq n$.

As $\ell(\widehat{\mathcal{C}}^{(t)}) = \ell(\widehat{\mathcal{C}}'^{(t)}) + 1 = n-1$, we see that

$$\begin{aligned}\dim_k\left(\frac{R}{I_{T_1}^{[\mathbf{n}]}}\right) &= \sum_{t=1}^{n}\left(\widehat{\mu}^1(\widehat{\mathcal{C}}^{(t)}) - \widehat{\mu}^1(\widehat{\mathcal{C}}'^{(t)})\right) = \sum_{t=1}^{n}\left(n! - \frac{t-1}{t}n!\right) \\ &= \sum_{t=1}^{n}\frac{n!}{t} = \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)n! = s(n+1,2).\end{aligned}$$

$\square$

A nice formula $|\widehat{\mathrm{PF}}_n^1| = |\widehat{\mathrm{PF}}_n^2| = s(n+1,2)$, deserves a combinatorial proof. We count parking functions $\mathbf{p} = (p_1,\ldots,p_n)$ in $\widehat{\mathrm{PF}}_n^1$ according to the value of p_n. Clearly, $0 \le p_n \le n-1$. For any $0 \le t \le n-1$, we see that $p_n = t$ implies that $p_i < i$ for all $i \le t$ and $p_j \le j$ for $j > t$. Also, any $(p_1,\ldots,p_n)$ with $p_n = t$ and $p_i < i$ for all $i \le t$, while $p_j \le j$ for all $t < j \le n-1$ is always a parking function of length n. Thus number of $\mathbf{p} = (p_1,\ldots,p_n) \in \widehat{\mathrm{PF}}_n^1$ with $p_n = t$ is $\left(\prod_{i=1}^{t} i\right)\left(\prod_{j=t+1}^{n-1}(j+1)\right) = \frac{n!}{t+1}$. Hence, $|\widehat{\mathrm{PF}}_n^1| = \sum_{t=0}^{n-1}\frac{n!}{t+1}$.

Theorem 3.7 shows that the integer sequence $\left\{\dim_k\left(\frac{R}{I_{T_b}^{[\mathbf{n}]}}\right) = s(n+1,2)\right\}_{n=1}^{\infty}$ for $b = 1,2$ is the integer sequence (A000254) in OEIS [14].

Case-3 : We finally consider the monomial ideal $I_U^{[\mathbf{n}]}$. The minimal generators $\prod_{j\in A} x_j^{\nu_{j,A}}$ of $I_U^{[\mathbf{n}]}$ are parametrized by the poset Σ_n. Again, it is straight forward to verify that the ideal $I_U^{[\mathbf{n}]}$ is an order monomial ideal and the cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n))$ supported on the order complex $\Delta(\Sigma_n)$ is the minimal free resolution of $I_U^{[\mathbf{n}]}$. Thus r^{th} Betti number $\beta_r(I_U^{[\mathbf{n}]}) = (r!)S(n+1,r+1)$ for $0 \le r \le n-1$.

Now we describe standard monomials of $\frac{R}{I_U^{[\mathbf{n}]}}$. Let $\overline{\mathrm{PF}}_n = \{\mathbf{p} \in \mathrm{PF}_n : \mathbf{x}^{\mathbf{p}} \notin I_U^{[\mathbf{n}]}\}$.

Lemma 3.8. *Let $\mathbf{p} = (p_1,\ldots,p_n) \in \mathrm{PF}_n$. Then $\mathbf{p} \in \overline{\mathrm{PF}}_n$ if and only if, there exists a permutation $\alpha \in \mathfrak{S}_n$ such that $p_{\alpha_i} < \nu_{\alpha_i,T_i}$ for all i, where $\alpha_i = \alpha(i)$, $T_1 = [n]$ and $T_j = [n] \setminus \{\alpha_1,\ldots,\alpha_{j-1}\}$ for $j \ge 2$. Also, $\nu_{j,T}$ is in the Lemma 2.4.*

Proof. Proof is similar to the proof of Theorem 4.3 of [6]. $\square$

Proceeding as in Proposition 3.4, we get a combinatorial formula for the number of standard monomials of $\frac{R}{I_U^{[\mathbf{n}]}}$.

Proposition 3.9. *The number of standard monomials of $\frac{R}{I_U^{[\mathbf{n}]}}$ is given by*

$$|\overline{\mathrm{PF}}_n| = \dim_k\left(\frac{R}{I_U^{[\mathbf{n}]}}\right) = \sum_{i=1}^{n}(-1)^{n-i}\sum_{\emptyset=C_0\subsetneq C_1\subsetneq\ldots\subsetneq C_i=[n]}\prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}}\nu_{j,C_q}\right),$$

where summation is carried over all strict chains $\emptyset = C_0 \subsetneq C_1 \subsetneq \ldots \subsetneq C_i = [n]$.

Neither using Proposition 3.9, nor by any combinatorial tricks, we could determine $|\overline{\mathrm{PF}}_n| = \dim_k \left(\frac{R}{I_U^{[\mathbf{n}]}} \right)$. Thus, we ask the following question.

Question : Is it possible to identify the sequence $\left\{ \dim_k \left(\frac{R}{I_U^{[\mathbf{n}]}} \right) \right\}_{n=1}^{\infty}$ with some well known combinatorially interesting integer sequence?

Computations for smaller values of n suggest that this integer sequence could be (A003319) in OEIS [14].

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