NEW SIGN UNCERTAINTY PRINCIPLES

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Abstract. We prove new sign uncertainty principles which vastly generalize the recent developments of Bourgain, Clozel & Kahane and Cohn & Gonçalves, and apply our results to a variety of spaces and operators. In particular, we establish new sign uncertainty principles for Fourier and Dini series, the Hilbert transform, the discrete Fourier and Hankel transforms, spherical harmonics, and Jacobi polynomials, among others. We present numerical evidence highlighting the relationship between the discrete and continuous sign uncertainty principles for the Fourier and Hankel transforms, which in turn are connected with the sphere packing problem via linear programming. Finally, we explore some connections between the sign uncertainty principle on the sphere and spherical designs.

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1. Introduction

The uncertainty principle, discovered by W. Heisenberg in 1927, is one of the cornerstones of quantum mechanics. It can be expressed via Heisenberg's inequality:

$$\inf_{a,b\in\mathbb{R}} \int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 \,\mathrm{d}x \int_{-\infty}^{\infty} (\xi-b)^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \geqslant \frac{\|f\|_{L^2(\mathbb{R})}^4}{16\pi^2},$$

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where $\widehat{f}$ denotes the Fourier transform of f. This estimate reflects the fact that the Fourier transform of a highly localized function must necessarily be widely dispersed in frequency space. Six years later, G. H. Hardy developed a more refined theory in this respect, and in particular established the following result: If there exist $a, b > 0$, such that the estimates $f(x) = O(e^{-a\pi x^2})$, $\widehat{f}(\xi) = O(e^{-b\pi\xi^2})$ hold, then $f \equiv 0$ whenever $ab > 1$, and f must coincide with a polynomial multiple of the Gaussian function $e^{-a\pi x^2}$ if $ab = 1$. Thus the uncertainty inequalities of Heisenberg and Hardy respectively explore, in a quantitative way, the notions of *concentration* around the origin and *decay* at infinity; see [14] for further details.

In 2010, motivated by applications to number theory, Bourgain, Clozel & Kahane [4] investigated an analogue of the uncertainty principle, where the notions of concentration and decay are replaced by that of *nonnegativity*. To describe it precisely, consider the following setting. Given $d \geqslant 1$, a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *eventually nonnegative* if $f(x) \geqslant 0$ for all sufficiently large $|x|$. In this case, consider the quantity

$$r(f) := \inf\{r > 0 : f(x) \geqslant 0 \text{ if } |x| \geqslant r\},$$

which corresponds to the radius of the last sign change of f. Normalize the Fourier transform,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} \, \mathrm{d}x, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ represents the usual inner product in $\mathbb{R}^d$. Let $\mathcal{A}_+(d)$ denote the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ which satisfy the following conditions:

- $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \in L^1(\mathbb{R}^d)$, and $\widehat{f}$ is real-valued (i.e. f is even);
- f is eventually nonnegative while $\widehat{f}(0) \leqslant 0$;
- $\widehat{f}$ is eventually nonnegative while $f(0) \leqslant 0$.

The product $r(f)r(\widehat{f})$ is invariant under rescaling, and becomes a natural quantity to consider. In this setting, the authors of [4] estimated the quantity

$$\mathbb{A}_+(d) := \inf_{f \in \mathcal{A}_+(d) \setminus \{\mathbf{0}\}} \sqrt{r(f) r(\widehat{f})}. \tag{1.2}$$

In particular, it is shown in [4, Théorème 3.1] that $\mathbb{A}_+(d)$ is bounded from below, and that in fact it grows linearly with the square root of the dimension.

Very recently, Cohn & Gonçalves [7] discovered a complementary uncertainty principle which is connected with the linear programming bounds of Cohn & Elkies [6] for the sphere packing problem. To describe it precisely, let $\mathcal{A}_-(d)$ denote the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ which satisfy the following conditions:

- $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \in L^1(\mathbb{R}^d)$, and $\widehat{f}$ is real-valued (i.e. f is even);
- f is eventually nonnegative while $\widehat{f}(0) \leqslant 0$;
- $-\widehat{f}$ is eventually nonnegative while $f(0) \geqslant 0$.

In a similar spirit to [4], the authors of [7] showed that the quantity

$$\mathbb{A}_-(d) := \inf_{f\in\mathcal{A}_-(d)\setminus\{\mathbf{0}\}} \sqrt{r(f)r(-\widehat{f})} \tag{1.3}$$

is bounded from below, and that in fact it grows linearly with $\sqrt{d}$. We shall refer to the boundedness of the quantities defined in (1.2), (1.3) as the ± 1 *uncertainty principles*; see §1.1 below (in particular, the statement of Theorem 1.8) for further information. Our first main result consists in the following generalization of the ± 1 uncertainty principles.

Theorem 1.1 (Operator Sign Uncertainty Principle)**.** *Let*[1] $s \in \{+,-\}$. *Let* X, Y *be two arbitrary measure spaces, equipped with positive measures* μ, ν, *respectively. Let* $\mathcal{F} \subseteq L^1(X,\mu) \times L^1(Y,\nu)$ *be a given family of pairs of functions. Assume that there exist real numbers* $p, q > 1$ *and* $a, b, c > 0$, *such that, for every* $(f,g) \in \mathcal{F}$,

- $\|g\|_{L^\infty(Y,\nu)} \leqslant a\|f\|_{L^1(X,\mu)}$;
- $\|g\|_{L^q(Y,\nu)} \leqslant b\|f\|_{L^p(X,\mu)}$;
- $\|f\|_{L^p(X,\mu)} \leqslant c\|g\|_{L^q(Y,\nu)}$;
- $\int_X f \,\mathrm{d}\mu \leqslant 0, \;\; s\int_Y g \,\mathrm{d}\nu \leqslant 0$.

Then, for every nonzero $(f,g) \in \mathcal{F}$, *the following inequality holds:*

$$\mu(\{x \in X : f(x) < 0\})^{\frac{1}{p'}} \nu(\{y \in Y : s\, g(y) < 0\})^{\frac{1}{q}} \geqslant a^{-1} b^{-\frac{q'}{q}} (2c)^{-q'}, \tag{1.4}$$

where $p' = p/(p-1)$ *denotes the exponent conjugate to* p, *and similarly for* q'.

The designation *Operator Sign Uncertainty Principle* derives from the fact that the family $\mathcal{F}$ is usually defined in terms of a given invertible operator $T : L^p(X,\mu) \to L^q(Y,\nu)$, i.e., it is often the case that $\mathcal{F} = \{(f, T(f)) : f \in \mathcal{S}\}$, for some $\mathcal{S} \subseteq L^p(X,\mu)$. For instance, if

$$\mathcal{F} = \{(f,\widehat{f}) : f, \widehat{f} \in L^1(\mathbb{R}^d) \text{ and both eventually nonnegative}\},$$

then the hypotheses of Theorem 1.1 are satisfied with $p = q = 2$ and $a = b = c = 1$. Since $f(x), \widehat{f}(\xi) \geqslant 0$ for $|x| \geqslant r(f), |\xi| \geqslant r(\widehat{f})$, respectively, it follows that

$$\frac{1}{16} \leqslant |\{x \in \mathbb{R}^d : f(x) < 0\}||\{\xi \in \mathbb{R}^d : \widehat{f}(\xi) < 0\}| \leqslant |B_1^d|^2 r(f)^d r(\widehat{f})^d. \tag{1.5}$$

Here, $|E|$ represents the Lebesgue measure of a given set $E \subseteq \mathbb{R}^d$, and $B_1^d \subseteq \mathbb{R}^d$ denotes the unit ball centered at the origin. In turn, estimate (1.5) immediately implies the aforementioned ± 1 uncertainty principles of Bourgain, Clozel & Kahane and Cohn & Gonçalves.

Theorem 1.1 opens the door to a variety of novel sign uncertainty principles of interest, as evidenced by the many examples explored in §2, §3, §4 below, which we shall introduce as

[1] Henceforth we shall use the letter s to denote a sign from $\{+,-\}$ and, by a slight but convenient abuse of notation, we will sometimes identify the signs $\{+,-\}$ with the integers $\{+1,-1\}$.

further main results of the present article. For instance, in §2 we establish a sign uncertainty principle for Fourier series. In §3, we describe some discrete sign uncertainty principles, which in the limit seem to converge back to the continuous ± 1 uncertainty principles. In §4, we discuss sign uncertainty principles for certain convolution operators on spaces of bandlimited functions, including the Hilbert transform. These connections are entirely new, and can potentially find many applications in several different branches of mathematics.

Motivation for our second main result comes from letting $Y = \mathbb{N} := \{0, 1, 2, 3, \ldots\}$ in Theorem 1.1, and taking $\mathcal{F}$ to be the family of pairs $(f, \widehat{f})$, where $\widehat{f} : \mathbb{N} \to \mathbb{R}$ is the coefficient sequence obtained by expanding f in some orthonormal basis. We shall derive a result that applies to a wide class of metric measure spaces, which we proceed to describe. Let $X = (X, d, \lambda)$ be a metric measure space, with a distance function $d : X \times X \to [0, \infty)$, and a probability measure λ. Further consider the space $L^2(X, \lambda)$ of square-integrable, *real-valued* functions $f : X \to \mathbb{R}$, which we will simply denote by $L^2(X)$ if no confusion arises. Given $x \in X$ and $r > 0$, let $B(x, r) := \{y \in X : d(x, y) \leqslant r\}$.

Definition 1.2 (Admissible space)**.** *The space (X, d, λ) is admissible if there exists an orthonormal basis $\{\varphi_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ of $L^2(X)$ and a fixed point*[2] *$\mathfrak{o} \in X$, such that $\varphi_0 \equiv 1$, and, for every $n \in \mathbb{N}$,*

$$\varphi_n(\mathfrak{o}) := \lim_{r\to 0^+} \frac{1}{\lambda(B(\mathfrak{o}, r))} \int_{B(\mathfrak{o},r)} \varphi_n \, \mathrm{d}\lambda = \|\varphi_n\|_{L^\infty(X)} < \infty. \tag{1.6}$$

Definition 1.3 (The $\mathcal{A}_s(X)$-cone)**.** *Let $s \in \{+, -\}$. Let (X, d, λ) be an admissible space, for which $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(X)$ satisfying* (1.6) *for some $\mathfrak{o} \in X$. Then $\mathcal{A}_s(X)$ consists of all square-integrable functions $f : X \to \mathbb{R}$, such that:*

- *If $f = \sum_{n=0}^{\infty} \widehat{f}(n)\varphi_n$ then*
$$\sum_{n=0}^{\infty} |\widehat{f}(n)| \|\varphi_n\|_{L^\infty(X)} < \infty; \tag{1.7}$$
- $\widehat{f}(0) \leqslant 0$;
- $\{s\widehat{f}(n)\}_{n\in\mathbb{N}}$ *is eventually nonnegative while $sf(\mathfrak{o}) \leqslant 0$.*

Here $\widehat{f}(n) = \langle f, \varphi_n\rangle_{L^2(X)} = \int_X f\varphi_n \, \mathrm{d}\lambda$. Note that $\mathcal{A}_s(X) \subseteq L^1(X)$ since $L^2(X) \subseteq L^1(X)$. From (1.7), it also follows that $\widehat{f} \in \ell^1(\mathbb{N})$ if $f \in \mathcal{A}_s(X)$. Indeed, for each n, it holds that $\|\varphi_n\|_{L^\infty(X)} \geqslant 1$, since

$$\|\varphi_n\|_{L^1(X)} \leqslant \|\varphi_n\|_{L^2(X)} = \|\varphi_n\|^2_{L^2(X)} \leqslant \|\varphi_n\|_{L^1(X)} \|\varphi_n\|_{L^\infty(X)}.$$

Since the series $\sum_{n=0}^{\infty} \widehat{f}(n)\varphi_n$ converges absolutely and uniformly, the function f would coincide λ-almost everywhere with a continuous function *if* each φ_n were continuous. While

[2]It may be useful to think of $\mathfrak{o}$ as the *origin of* X with respect to the basis $\{\varphi_n\}_{n\in\mathbb{N}}$.

this is the case for most of our applications, the latter continuity property is not strictly necessary to make sense of the value of a given $f \in \mathcal{A}_s(X)$ at $\mathfrak{o}$. Indeed, in the current setting, one can easily show that $\mathfrak{o}$ is a Lebesgue point of f, and invoke (1.7) to define $f(\mathfrak{o})$ as follows:

$$f(\mathfrak{o}) := \lim_{r\to 0^+} \frac{1}{\lambda(B(\mathfrak{o},r))} \int_{B(\mathfrak{o},r)} f \, \mathrm{d}\lambda = \sum_{n=0}^{\infty} \widehat{f}(n) \|\varphi_n\|_{L^\infty(X)}.$$

Given $r_1, r_2 \in [0,\infty)$, we write $r_1 \sim r_2$ if $\lambda(B(\mathfrak{o},r_1)) = \lambda(B(\mathfrak{o},r_2))$, or equivalently if $B(\mathfrak{o},r_1) = B(\mathfrak{o},r_2)$ up to λ-null sets. One easily checks that $\sim$ defines an equivalence relation on $[0,\infty)$, and that each equivalence class is an interval which contains its infimum. Let $\mathcal{R} := \{\inf I : I \in [0,\infty)/\sim\}$. Given $f \in \mathcal{A}_s(X)$, we define[3] the following quantities:

$$r(f;X) := \inf\{r \in \mathcal{R} : f(x) \geqslant 0 \text{ for } \lambda\text{-a.e. } x \in X \text{ such that } d(x,\mathfrak{o}) \geqslant r\}; \tag{1.8}$$

$$k_s(\widehat{f}) := \min\{k \geqslant 1 : s\widehat{f}(n) \geqslant 0 \text{ if } n \geqslant k\}. \tag{1.9}$$

Note that $r(f;X)$ can be $+\infty$, or equal to the smallest $r_0 > 0$ for which $X \subseteq B(\mathfrak{o},r_0)$. On the other hand, if f is nonzero, then $r(f;X) > 0$ as long as $\lambda(\{\mathfrak{o}\}) = 0$, for otherwise $f \geqslant 0$ (λ-a.e.), which contradicts $\widehat{f}(0) \leqslant 0$. Moreover, $s\widehat{f}(n)$ cannot be nonnegative for all $n \geqslant 0$, for otherwise

$$0 \leqslant \sum_{n=0}^{\infty} s\widehat{f}(n)\varphi_n(\mathfrak{o}) = sf(\mathfrak{o}) \leqslant 0,$$

and therefore $\widehat{f}(n) = 0$, for all $n \geqslant 0$, which is absurd because f is nonzero. We also have that $k_-(\widehat{f}) \geqslant 2$, for otherwise

$$f(x) - \widehat{f}(0) = \sum_{n=1}^{\infty} \widehat{f}(n)\varphi_n(x) \geqslant \sum_{n=1}^{\infty} \widehat{f}(n)\varphi_n(\mathfrak{o}) = f(\mathfrak{o}) - \widehat{f}(0),$$

whence $f(x) \geqslant f(\mathfrak{o}) \geqslant 0$ for all $x \in X$, which is absurd because $\widehat{f}(0) \leqslant 0$ and f is nonzero. On the other hand, it might be the case that $k_+(\widehat{f}) = 1$ (e.g. take $f \equiv -1$); but if $\widehat{f}(0) = 0$, then it is easy to see that $k_+(\widehat{f}) \geqslant 2$ as well.

We are now ready to state our second main result.

Theorem 1.4 (Orthonormal Sign Uncertainty Principle)**.** *Let $s \in \{+,-\}$. Let (X,d,λ) be an admissible space, for which $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(X)$ satisfying* (1.6) *for some $\mathfrak{o} \in X$. Then, for every nonzero $f \in \mathcal{A}_s(X)$, the following inequality holds:*

$$\lambda(B(\mathfrak{o},r(f;X))) \sum_{n=1}^{k_s(\widehat{f})} \|\varphi_{n-1}\|^2_{L^\infty(X)} \geqslant \frac{1}{16}. \tag{1.10}$$

[3]Definition (1.8) turns out to be more adequate than merely taking the infimum over all $r \geqslant 0$. Indeed, let $X = \mathbb{N}$, with $d(n,m) := |n-m|$ and counting measure λ. Then $\mathcal{R} = \mathbb{N}$, and $r(f;X)$ coincides with the unique integer $m \geqslant 1$, for which $f(m-1) < 0$ but $f(n) \geqslant 0$ for all $n \geqslant m$.

Theorems 1.1 and 1.4 are not entirely unrelated: for instance, the latter easily follows from the former in the special case when the orthonormal basis satisfies $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{L^\infty(X)} < \infty$ (with a lower bound which possibly differs from $\frac{1}{16}$). If the space $L^2(X)$ is finite dimensional, then a corresponding version of Theorem 1.4 holds; we omit the obvious statement, but note that the proof is exactly the same. Consequences of Theorem 1.4 to a variety of settings will be explored in §2. In particular, we establish a sign uncertainty principle for spherical harmonics in §2.1. It turns out that, in the case of the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$, the zero set of a minimizer to the restricted problem on a finite dimensional subspace $V = \text{span}\{\varphi_n\}_{n=0}^N$ exhibits natural geometric structure. In particular, we shall see how to relate this zero set to the set of cosine distances of certain spherical designs.

1.1. **Further Background.** We briefly expand on the history of previous work which inspired the present paper, and its connections to our main results. The initial lower and upper bounds for $\mathbb{A}_+(d)$ of Bourgain, Clozel & Kahane [4] were subsequently sharpened by Gonçalves, Oliveira e Silva & Steinerberger [17]. Cohn & Gonçalves [7] then discovered that the sign uncertainty principle is connected with the linear programming bounds for the sphere packing problem, and exploited this connection to prove that $\mathbb{A}_+(12) = \sqrt{2}$. Crucially, they realized the applicability of the powerful machinery devised by Viazovska [34] in her solution to the eight-dimensional sphere packing problem to construct eigenfunctions of the Fourier transform via certain Laplace transforms of modular forms. To understand this connection in greater depth, we shall briefly discuss the upper bounds on sphere packings via linear programming from the groundbreaking work of Cohn & Elkies [6]. Let $\mathcal{A}_{LP}(d)$ denote the set of functions $f : \mathbb{R}^d \to \mathbb{R}$, which satisfy the following conditions:

- $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \in L^1(\mathbb{R}^d)$, and $\widehat{f}$ is real-valued (i.e. f is even);
- $-f$ is eventually nonnegative while $\widehat{f}(0) = 1$;
- $\widehat{f}$ is nonnegative and $f(0) = 1$.

In [6, Theorem 3.2] it is shown that, given any sphere packing $\mathcal{P} \subseteq \mathbb{R}^d$ of congruent balls, its upper density $\bar{\delta}(\mathcal{P})$ satifies

$$\bar{\delta}(\mathcal{P}) \leqslant r(-f)^d |B^d_{\frac{1}{2}}|, \tag{1.11}$$

for any $f \in \mathcal{A}_{LP}(d)$. Therefore the quantity

$$\mathbb{A}_{LP}(d) := \inf_{f\in\mathcal{A}_{LP}(d)} r(-f)$$

becomes of interest. High precision numerical data indicated that the upper bound (1.11) agrees with the packing density of the honeycomb, E_8, and Leech lattices in dimensions $2, 8$, and 24, respectively. In a celebrated breakthrough, Viazovska [34] found the magical function f realizing equality in (1.11) when $d = 8$, thereby proving optimality of the E_8-lattice packing and showing that $\mathbb{A}_{LP}(8) = \sqrt{2}$. Shortly thereafter, Cohn, Kumar, Miller, Radchenko &

Viazovska [9] used similar methods to prove the optimality of the Leech lattice when $d = 24$, thereby showing that $\mathbb{A}_{LP}(24) = 2$. An elementary geometric argument reveals that the honeycomb packing is optimal if $d = 2$ (see e.g. [21]), but the corresponding magical function is yet to be discovered. Cohn & Gonçalves [7] later noticed that the -1 uncertainty principle described in the previous section underpins the construction in dimensions $d \in \{8, 24\}$. The connection is simple to describe: If $f \in \mathcal{A}_{LP}(d)$, then $\widehat{f} - f \in \mathcal{A}_{-}(d)$ and $r(\widehat{f} - f) \leqslant r(-f)$, and therefore $\mathbb{A}_{-}(d) \leqslant \mathbb{A}_{LP}(d)$. In [7], the authors performed extensive numerical calculations, producing compelling evidence towards the following conjecture, which if proved would establish a precise mathematical link between the sign uncertainty principle and the sphere packing problem, and clarify the constructions in [9, 34].

Conjecture 1.5. $\mathbb{A}_{LP}(d) = \mathbb{A}_{-}(d)$, *for every* $d \geqslant 1$.

Indeed, one can extract the -1 eigenfunctions from [9, 34], and then use Poisson-type summation formulae for the E_8 and Leech lattices (in the same way as the Eisenstein series E_6 was used to prove optimality in [7]) in order to conclude that $\mathbb{A}_{LP}(8) = \mathbb{A}_{-}(8) = \sqrt{2}$ and $\mathbb{A}_{LP}(24) = \mathbb{A}_{-}(24) = 2$. Cohn & Elkies [6] further showed that $\mathbb{A}_{LP}(1) = 1$, and that the function $f(x) = (1 - |x|)_+$ is optimal; from their proof, one can easily derive that $\mathbb{A}_{-}(1) = 1$, and that a corresponding minimizer is given by the function $x \mapsto (\widehat{f} - f)(x) = \frac{\sin^2(\pi x)}{(\pi x)^2} - (1 - |x|)_+$. Together with $\mathbb{A}_{+}(12) = \sqrt{2}$ (recall [7]), these constitute a complete list of dimensions d for which $\mathbb{A}_{\pm}(d), \mathbb{A}_{LP}(d)$ are known. From the possible equality in (1.11) for the honeycomb packing when $d = 2$, Cohn & Elkies [6] further conjectured that $\mathbb{A}_{LP}(2) = (\frac{4}{3})^{\frac{1}{4}}$. Therefore one should also expect that $\mathbb{A}_{-}(2) = (\frac{4}{3})^{\frac{1}{4}}$.

Conjecture 1.6. $\mathbb{A}_{LP}(2) = \mathbb{A}_{-}(2) = (\frac{4}{3})^{\frac{1}{4}}$.

As a consequence of our new sign uncertainty principle for the discrete Fourier transform (see §3.1 and §6.1 below), we now have compelling numerical evidence pointing towards the solution of the one-dimensional $+1$ uncertainty principle.

Conjecture 1.7. $\mathbb{A}_{+}(1) = (2\varphi)^{-\frac{1}{2}}$, *where* $\varphi := \frac{1+\sqrt{5}}{2}$ *denotes the golden ratio.*

To the best of our knowledge, these are the only dimensions for which even a guess of the actual solution exists, all other dimensions remaining for the most part entirely mysterious. We believe that solving Conjectures 1.6 or 1.7 would require brand new techniques, which could potentially be applied to other dimensions, and open windows of possibilities. Even though the exact answer is not known, or even conjectured, in any other dimension $d \notin \{1, 2, 8, 12, 24\}$, it has been established that radial minimizers exist in all dimensions, and that such minimizers must necessarily vanish at infinitely many radii greater than $\mathbb{A}_{+}(d)$. This was shown in [17, Theorem 4] for the $+1$ uncertainty principle, and the technique was later [7] adapted to handle the -1 uncertainty principle. The following result summarizes the state-of-the-art knowledge of minimizers for the ± 1 uncertainty principles.

Theorem 1.8 ([4, 7, 17])**.** *Let $d \geqslant 1$. Then the following two-sided inequalities hold:*

$$\frac{1}{\sqrt{2\pi e}} \leqslant \frac{\mathbb{A}_+(d)}{\sqrt{d}} \leqslant \frac{1}{\sqrt{2\pi}} + o_d(1); \tag{1.12}$$

$$\frac{1}{\sqrt{2\pi e}} \leqslant \frac{\mathbb{A}_-(d)}{\sqrt{d}} \leqslant 0.3194... + o_d(1). \tag{1.13}$$

Moreover, for each $s \in \{+,-\}$ and $d \geqslant 1$, there exists a radial function $f \in \mathcal{A}_s(d) \setminus \{\mathbf{0}\}$, such that $\widehat{f} = sf$, $f(0) = 0$, $r(f) = \mathbb{A}_s(d)$. Any such function must vanish at infinitely many radii greater than $\mathbb{A}_s(d)$.

The number 0.3194... in (1.13) is derived from the classical upper bounds of Kabatiansky & Levenshtein [24] for the sphere packing problem. Indeed, the construction in [10] reveals how the same bound can be obtained via linear programming, whence $\mathbb{A}_{LP}(d) \leqslant (0.3194... + o_d(1))\sqrt{d}$. The upper bound in (1.13) then follows from the aforementioned estimate $\mathbb{A}_-(d) \leqslant \mathbb{A}_{LP}(d)$. In spite of the distinct upper bounds in (1.12), (1.13), it is conjectured in [7] (with strong numerical evidence) that there exists a constant $c > 0$, for which $\mathbb{A}_+(d) \sim \mathbb{A}_-(d) \sim c\sqrt{d}$, as $d \to \infty$. Moreover, there are reasons to believe that c might not be too far from 0.3194. The structural statement in Theorem 1.8 (concerning the double roots of the minimizers) stem from a seemingly new observation concerning Hermite polynomials, which relates their pointwise values to linear flows on the torus $\mathbb{T}^d$, and extends to other families of orthogonal polynomials; see [18] for further applications of this idea. The proof of [17, Theorem 4] can easily be adapted to show that minimizers for $\mathbb{A}_{LP}(d)$ exist, and must also have infinitely many double roots. Finally, some equivalent formulations of the ± 1 uncertainty principles, and mass concentration phenomena exhibited by the corresponding minimizing sequences, were the subject of very recent explorations in [16]. Further related recent results can be found in [5, 19].

1.2. **Outline.** In §2, we establish sign uncertainty principles for spherical harmonics (§2.1), Jacobi polynomials (§2.2), Fourier series (§2.3), and Dini series (§2.4). In §3, we establish sign uncertainty principles for the discrete Fourier transform (§3.1), the discrete Hankel transform (§3.2), and the Hamming cube (§3.3). In §4, we establish sign uncertainty principles for convolution kernels in bandlimited function spaces (§4.1), the Hilbert transform of bandlimited functions (§4.2), and the Hankel transform (§4.3). The main results are proved in §5. Finally, in §6, we present our numerical findings related to the discrete Fourier transform (§6.1), and the discrete Hankel transform (§6.2).

2. Sign Uncertainty for Classical Orthogonal Systems

2.1. **Spherical Harmonics.** Let $\mathbb{S}^{d-1} = \{\omega \in \mathbb{R}^d : |\omega| = 1\}$ denote the unit sphere, equipped with the geodesic distance $d_g : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to [0, \pi]$, $d_g(\omega, \nu) := \arccos(\langle \omega, \nu \rangle)$, and normalized surface measure $\bar{\sigma}$, induced from the ambient space $\mathbb{R}^d$ in the natural way and satisfying $\bar{\sigma}(\mathbb{S}^{d-1}) = 1$. The special orthogonal group $\mathrm{SO}(d)$ consists of all $d \times d$ orthogonal matrices of unit determinant, and acts transitively on the unit sphere $\mathbb{S}^{d-1}$. The vector space of spherical harmonics on $\mathbb{S}^{d-1}$ of degree n, denoted $\mathcal{H}_n^d$, consists of restrictions to $\mathbb{S}^{d-1}$ of real-valued harmonic polynomials on $\mathbb{R}^d$ which are homogeneous of degree n. The spaces $\mathcal{H}_n^d$ are mutually orthogonal and span $L^2(\mathbb{S}^{d-1}) = L^2(\mathbb{S}^{d-1}, \bar{\sigma})$,

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^d.$$

Let $h_n := \dim \mathcal{H}_n^d$, and denote the north pole by $\eta = (0, \ldots, 0, 1) \in \mathbb{S}^{d-1}$.

Definition 2.1 (Signed basis)**.** *An orthonormal basis* $\{Y_{n,j} \in \mathcal{H}_n^d : n \in \mathbb{N}, j = 1, \ldots, h_n\}$ *of* $L^2(\mathbb{S}^{d-1})$ *is signed if:*

- $Y_{n,j}(\eta) \geqslant 0$, *for every* $n \in \mathbb{N}, j = 1, 2, \ldots, h_n$;
- $Y_{n,j}(\eta) > 0$, *for every* $j = 1, 2, \ldots, h_n$, *provided* n *is sufficiently large.*

A signed basis for $L^2(\mathbb{S}^{d-1})$ can be constructed as follows. Given a continuous function $f : \mathbb{S}^{d-1} \to \mathbb{R}$, let $Z(f) := \{\omega \in \mathbb{S}^{d-1} : f(\omega) = 0\}$ denote its zero set. Start with an arbitrary basis $\mathcal{Y} = \{Y_{n,j} \in \mathcal{H}_n^d : n \in \mathbb{N}, j = 1, 2, \ldots, h_n\}$ of $L^2(\mathbb{S}^{d-1})$, and consider the corresponding zero set,

$$\mathcal{Z}(\mathcal{Y}) := \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{h_n} Z(Y_{n,j}).$$

Since $\bar{\sigma}(\mathcal{Z}(\mathcal{Y})) = 0$, we can, for each $\varepsilon > 0$, find a rotation $\rho \in \mathrm{SO}(d)$, such that $|\rho(\eta) - \eta| < \varepsilon$ and $\rho(\eta) \notin \mathcal{Z}(\mathcal{Y})$. Therefore there exists a sequence of signs $\{s_{n,j}\} \subseteq \{+, -\}^{\mathbb{N}}$, for which $\{s_{n,j} Y_{n,j} \circ \rho : n \in \mathbb{N}, j = 1, 2, \ldots, h_n\}$ is a signed basis for $L^2(\mathbb{S}^{d-1})$.

Henceforth, we fix a signed orthonormal basis $\{Y_{n,j} : n \in \mathbb{N}, j = 1, 2, \ldots, h_n\}$ of $L^2(\mathbb{S}^{d-1})$. Any real-valued, square-integrable function $f : \mathbb{S}^{d-1} \to \mathbb{R}$ can be expanded as follows:

$$f = \sum_{n=0}^{\infty} \sum_{j=1}^{h_n} \widehat{f}(n, j) Y_{n,j}, \tag{2.1}$$

where $\widehat{f}(n, j) = \int_{\mathbb{S}^{d-1}} f(\omega) Y_{n,j}(\omega) \, \mathrm{d}\bar{\sigma}(\omega)$.

Definition 2.2 (The $\mathcal{B}_s(\mathbb{S}^{d-1})$-cone)**.** *Let* $s \in \{+, -\}$. *Then* $\mathcal{B}_s(\mathbb{S}^{d-1})$ *consists of all continuous functions* $f : \mathbb{S}^{d-1} \to \mathbb{R}$, *such that:*

- $\widehat{f}(0, 1) \leqslant 0$;

- $\{s\widehat{f}(n,j) : n \in \mathbb{N}, j = 1, 2, \ldots, h_n\}$ *is eventually nonnegative while* $sf(\eta) \leqslant 0$.

Given $f \in \mathcal{B}_s(\mathbb{S}^{d-1})$, set

$$\theta(f) := \inf\{\theta \in (0,\pi] : f(\omega) \geqslant 0 \text{ if } d_g(\omega,\eta) \geqslant \theta\};$$
$$k_s(\widehat{f}) := \min\{k \geqslant 1 : s\widehat{f}(n,j) \geqslant 0 \text{ if } n \geqslant k\},$$

and define the quantity

$$\mathbb{B}_s(\mathbb{S}^{d-1}) := \inf_{f \in \mathcal{B}_s(\mathbb{S}^{d-1}) \setminus \{\mathbf{0}\}} (1 - \cos(\theta(f)))^{\frac{1}{2}} k_s(\widehat{f}), \tag{2.2}$$

which is estimated by our next result.

Theorem 2.3. *Let* $s \in \{+,-\}$ *and* $d \geqslant 2$. *Then the following estimates hold:*

$$\mathbb{B}_s(\mathbb{S}^{d-1}) \geqslant \frac{2\Gamma(\frac{d+1}{2})^{\frac{2}{d-1}}}{(4e^{\frac{1}{12}})^{\frac{2}{d-1}}(d^2-1)^{\frac{1}{2}}}, \tag{2.3}$$

$$\mathbb{B}_+(\mathbb{S}^{d-1}) \leqslant \sqrt{2}, \quad \textit{and} \quad \mathbb{B}_-(\mathbb{S}^{d-1}) \leqslant 2\sqrt{2}. \tag{2.4}$$

Remark. Since $(1-\cos\theta)^{\frac{1}{2}} = \sqrt{2}\sin\frac{\theta}{2} \approx \theta$ if $0 \leqslant \theta \leqslant \pi$, a similar uncertainty principle would be obtained if $(1-\cos(\theta(f)))^{\frac{1}{2}}$ were replaced by $\theta(f)$ in (2.2). We made this choice with a view towards identity (2.5) below, which would otherwise be merely a two-sided inequality instead of an equality. Further note that, by Stirling's formula, the lower bound in (2.3) is $e^{-1} + O(d^{-1})$; here, e is the base of the natural logarithm, and $O(d^{-1})$ denotes a quantity which is bounded in absolute value by Cd^{-1}, for some absolute constant $C > 0$.

The proof of Theorem 2.3 involves Gegenbauer polynomials, which are particular instances of Jacobi polynomials, discussed in §2.2 below. As with most results in this section, Theorem 2.3 ultimately boils down to a special case of a more general result from §2.2. More precisely, the proof of the lower bound (2.3) proceeds in two steps. Firstly, via a zonal symmetrization procedure, we may assume the existence of an eventually nonnegative sequence of coefficients $\{a_n\}_{n\in\mathbb{N}}$, for which

$$f(\omega) = \sum_{n=0}^{\infty} a_n C_n^{d/2-1}(\langle\omega,\eta\rangle).$$

Here, $C_n^{d/2-1}$ denotes the Gegenbauer polynomial of degree n and order $\frac{d}{2}-1$; see (2.10) below. Secondly, the map $g(x) \mapsto g(\langle\omega,\eta\rangle)$ defines a bijection between the set $\mathcal{B}_s(I; \frac{d-3}{2}, \frac{d-3}{2})$ from Definition 2.12 below and the set of functions in $\mathcal{B}_s(\mathbb{S}^{d-1})$ which are invariant under rotations that fix the north pole. Consequently, the following identity holds:

$$\mathbb{B}_s(\mathbb{S}^{d-1})^2 = \mathbb{B}_s\left([-1,1]; \tfrac{d-3}{2}, \tfrac{d-3}{2}\right), \tag{2.5}$$

where the right-hand side is defined in (2.13) below. Therefore Theorem 2.3 will ultimately follow from Theorem 2.13; see §5.3 for details.

Definition 2.4 (The class $\mathcal{B}_s^0(\mathbb{S}^{d-1})$)**.** *Let $s \in \{+,-\}$ and $d \geqslant 2$. Then $\mathcal{B}_s^0(\mathbb{S}^{d-1})$ consists of all functions $f \in \mathcal{B}_s(\mathbb{S}^{d-1})$ which are invariant under rotations that fix the north pole η, and satisfy $f(\eta) = 0$.*

Further define the quantity

$$\mathbb{B}_s^0(\mathbb{S}^{d-1}) := \inf_{f \in \mathcal{B}_s^0(\mathbb{S}^{d-1}) \setminus \{\mathbf{0}\}} (1 - \cos(\theta(f)))^{\frac{1}{2}} k_s(\widehat{f}).$$

The following result is a direct consequence of (2.5) and Proposition 2.14 below.

Proposition 2.5. *Let $s \in \{+,-\}$ and $d \geqslant 2$. Then $\mathbb{B}_s^0(\mathbb{S}^{d-1}) = \mathbb{B}_s(\mathbb{S}^{d-1})$.*

For the remainder of this section, we investigate polynomials in $\mathcal{B}_s^0(\mathbb{S}^{d-1})$ which are *optimal* in the following sense.

Definition 2.6 (s-optimal polynomial in $\mathcal{B}_s^0(\mathbb{S}^{d-1})$)**.** *Let $s \in \{+,-\}$ and $d \geqslant 2$. A polynomial $f \in \mathcal{B}_s^0(\mathbb{S}^{d-1})$ is locally s-optimal if there exists $\delta > 0$, such that*

$$(1 - \cos(\theta(f)))^{\frac{1}{2}} k_s(\widehat{f}) < (1 - \cos(\theta(h)))^{\frac{1}{2}} k_s(\widehat{h}),$$

for any polynomial $h \in \mathcal{B}_s^0(\mathbb{S}^{d-1})$ satisfying $\deg(h) \leqslant \deg(f)$ and $0 < \inf_{c>0} \|f - ch\|_{L^\infty(\mathbb{S}^{d-1})} < \delta$. The polynomial f is said to be globally s-optimal if one can take $\delta = +\infty$.

2.1.1. *Connections with Spherical Designs.* A fundamental tool employed in the solutions of the sphere packing problem in 8 and 24 dimensions [34, 9] and of the +1-uncertainty principle in 12 dimensions [7] is the Poisson summation formula associated with certain modular forms; recall the discussion in §1.1. Poisson summation is often used to extract sharp lower bounds, and to access information about the root location of the conjectural minimizer. On the sphere $\mathbb{S}^{d-1}$, the role of Poisson summation seems to be played by *spherical designs*; see [1] for an excellent introduction to this topic.

Let us introduce some terminology. A finite subset $\Omega \subseteq \mathbb{S}^{d-1}$ is called a *spherical t-design* if, for every polynomial $f : \mathbb{S}^{d-1} \to \mathbb{R}$ of degree at most t,

$$\int_{\mathbb{S}^{d-1}} f(\omega)\, \mathrm{d}\bar{\sigma}(\omega) = \frac{1}{\#\Omega} \sum_{\omega \in \Omega} f(\omega).$$

We say that Ω has m distances if the *set of cosine distances*,

$$\alpha(\Omega) := \{\langle \omega, \omega' \rangle : \omega, \omega' \in \Omega,\, \omega \neq \omega'\},$$

is such that $\#\alpha(\Omega) = m$; in this case, we write $\alpha(\Omega) = \{\alpha_m < \alpha_{m-1} < \ldots < \alpha_1\}$. Note that necessarily $t \leqslant 2m$, for otherwise the nonnegative, nonzero function

$$f(\omega) = (1 - \langle \omega, \omega_1 \rangle) \prod_{j=1}^{m} (\langle \omega, \omega_1 \rangle - \alpha_j)^2, \ \ (\omega_1 \in \Omega)$$

would have zero average on $\mathbb{S}^{d-1}$. Moreover, if $t = 2m$, then Ω cannot contain a pair of antipodal points, for otherwise $\alpha_m = -1$, and the function

$$g(\omega) = (1 - \langle \omega, \omega_1 \rangle^2) \prod_{j=1}^{m-1} (\langle \omega, \omega_1 \rangle - \alpha_j)^2$$

would have zero average on $\mathbb{S}^{d-1}$, which is again impossible.

Delsarte, Goethals & Seidel [12] showed that, if $\Omega \subseteq \mathbb{S}^{d-1}$ is a spherical t-design, then

$$\#\Omega \geqslant \binom{d + \lfloor t/2 \rfloor - 1}{\lfloor t/2 \rfloor} + \binom{d + \lceil t/2 \rceil - 2}{\lceil t/2 \rceil - 1}. \tag{2.6}$$

A spherical t-design $\Omega \subseteq \mathbb{S}^{d-1}$ is said to be *tight* if equality holds in (2.6). It is also shown in [12] that, if Ω is a spherical t-design, then Ω is tight if and only if $\#\alpha(\Omega) = \lceil t/2 \rceil$ and Ω is antipodal if t is odd.

The regular $(t+1)$-gon is a tight t-design on $\mathbb{S}^1 \subseteq \mathbb{R}^2$, for any $t \geqslant 1$. By contrast, tight t-designs on $\mathbb{S}^{d-1}$ with $d \geqslant 3$ are rare. In particular, Bannai & Damerell [2, 3] established the following: if $d \geqslant 3$, then tight t-spherical designs can only exist when $t \in \{1, 2, 3, 4, 5, 7, 11\}$. Moreover, modulo isometries: if $t = 1$, then Ω consists of a pair of antipodal points; if $t = 2$, then Ω is the regular simplex with $d + 1$ vertices; if $t = 3$, then $\Omega = \{\pm e_j\}_{j=1}^{d}$ is the cross-polytope with $2d$ vertices; and if $t = 11$, then $d = 24$ and Ω is the set of 196 560 minimal vectors of the Leech Lattice. The complete classification of spherical t-designs is open for $t \in \{4, 5, 7\}$, although several examples are known; see [1, p. 1401] and [8, Table 1].

Definition 2.7 (s-optimal spherical design)**.** *Let $s \in \{+, -\}$ and $d \geqslant 2$. Let $\Omega \subseteq \mathbb{S}^{d-1}$ be a tight spherical t-design with $\alpha(\Omega) = \{\alpha_m < \alpha_{m-1} < \ldots < \alpha_1\}$, where $m = \lceil t/2 \rceil$. For $m \geqslant 2$, let $a = 1$ if $\alpha_m = -1$, $a = 2$ if $\alpha_m > -1$, and define the polynomial*

$$P(\omega) := (x - 1)(x - \alpha_m)^a (x - \alpha_1) \prod_{j=2}^{m-1} (x - \alpha_j)^2, \ \ \text{where } x = \langle \omega, \eta \rangle. \tag{2.7}$$

If $m = 1$, set $P(\omega) := (x - 1)(x - \alpha_1)$. We say that Ω is locally (resp. globally) s-optimal if the polynomial P is locally (resp. globally) s-optimal in $\mathcal{B}_s^0(\mathbb{S}^{d-1})$.

Since every tight spherical design generates a quadrature rule for the measure associated to Gegenbauer polynomials (see §2.2.2), the zonal symmetrization argument from the proof of Theorem 2.3 leads to the following result.

Proposition 2.8. *Let $s \in \{+,-\}$ and $d \geqslant 2$. Let $\Omega \subseteq \mathbb{S}^{d-1}$ be a spherical t-design with $\alpha(\Omega) = \{\alpha_m < \alpha_{m-1} < \ldots < \alpha_1\}$. Let $f \in \mathcal{B}_s(\mathbb{S}^{d-1}) \setminus \{\mathbf{0}\}$ be a polynomial satisfying $\deg(f) \leqslant t$, and further assume $f(\eta) = 0$ if $s = +1$. Then $\theta(f) \geqslant \arccos(\alpha_1)$. Moreover, if $\theta(f) = \arccos(\alpha_1)$ and f is invariant under rotations that fix the north pole η, then f coincides with a positive multiple of the polynomial P defined in* (2.7)*.*

The discussion preceding Corollary 2.17 below implies that every tight spherical t-design is in fact locally s-optimal. Moreover, in light of Proposition 2.8, a tight spherical t-design is globally s-optimal if the corresponding polynomial P defined via (2.7) satisfies[4] $k_s(\widehat{P}) = 2$. In the following examples, given a certain set of nodes $X = (x_m, x_{m-1}, \ldots, x_0)$, $W = (w_m, w_{m-1}, \ldots, w_0)$ will be such that $\left\{\frac{w_j}{\sum_{i=0}^m w_i}\right\}_{j=0}^m$ is the set of weights of the quadrature rule associated with the nodes in X.

Example 2.9 (Simplex)**.** *The regular simplex on $\mathbb{S}^{d-1}$ is a tight 2-spherical design with $d+1$ vertices and one cosine distance, $-\frac{1}{d}$. It induces a quadrature rule of degree $t = 2$ for the Gegenbauer measure $w_{\nu-\frac{1}{2},\nu-\frac{1}{2}}$ (see* (2.8) *below), $\nu = \frac{d}{2} - 1$, with $X = \left(\frac{-1}{2\nu+2}, 1\right)$ and $W = (2\nu+2, 1)$. One easily checks that this quadrature rule integrates all polynomials of degree at most 2 exactly, for all $\nu \geqslant 0$. Moreover, letting*[5]

$$P(x) = (x-1)\left(x + \frac{1}{2\nu+2}\right) = \frac{-(2\nu+1)}{4\nu+4} G_1^\nu(x) + \frac{1}{2\nu+2} G_2^\nu(x),$$

we have that $k_+(\widehat{P}) = 2$. Hence P is a globally $+1$-optimal polynomial in $\mathcal{B}_+^0(I; \nu-\frac{1}{2}, \nu-\frac{1}{2})$, and the regular simplex is a globally $+1$-optimal tight 2-design on $\mathbb{S}^{d-1}$.

Example 2.10 (Cross-polytope)**.** *The cross-polytope $\{\pm e_j\}_{j=1}^d$ on $\mathbb{S}^{d-1}$ is a tight 3-spherical design with $2d$ vertices and two cosine distances, $\{-1, 0\}$. It induces a quadrature rule of degree $t = 3$ for $w_{\nu-\frac{1}{2},\nu-\frac{1}{2}}$, $\nu = \frac{d}{2} - 1$, with $X = (-1, 0, 1)$ and $W = (1, 4\nu+2, 1)$. One easily checks that this quadrature rule integrates all polynomials of degree at most 3 exactly, for all $\nu \geqslant 0$. Moreover, letting*

$$P(x) = (x^2-1)x = \frac{-(2\nu+1)}{4(\nu+2)} G_1^\nu(x) + \frac{3}{4(\nu+1)(\nu+2)} G_3^\nu(x),$$

we have that $k_+(\widehat{P}) = 2$. Hence P is a globally $+1$-optimal polynomial in $\mathcal{B}_+^0(I; \nu-\frac{1}{2}, \nu-\frac{1}{2})$, and the cross-polytope is a globally $+1$-optimal tight 3-design on $\mathbb{S}^{d-1}$.

We summarize the preceding discussion in the following result.

[4]Recall that $k_s(\widehat{P}) \geqslant 2$ since $P \in \mathcal{B}_s^0(\mathbb{S}^{d-1})$.

[5]The *modified Gegenbauer polynomials* are defined as $G_n^\nu(x) := \nu^{-1} C_n^\nu(x)$ for $\nu \geqslant 0$, with the understanding that $G_n^0(x) = \lim_{\nu \to 0^+} \nu^{-1} C_n^\nu(x)$.

Theorem 2.11. *Let $d \geqslant 2$. Every tight spherical t-design is locally s-optimal, for any $s \in \{+,-\}$. Furthermore:*

- *The regular simplex on $\mathbb{S}^{d-1}$ with $d+1$ vertices is a globally $+1$-optimal tight 2-design;*
- *The cross-polytope on $\mathbb{S}^{d-1}$ with $2d$ vertices is a globally $+1$-optimal tight 3-design.*

We have not been able to find any globally -1-optimal design, nor any further globally $+1$-optimal designs.

2.2. **Jacobi Polynomials.** Let $\{P_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ denote the Jacobi polynomials with parameters $\alpha, \beta > -1$. These are defined in [32, Ch. IV] as the orthogonal polynomials on the interval $I := [-1,1]$, associated with the measure

$$w_{\alpha,\beta}(x)\,\mathrm{d}x = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta \mathbb{1}_I(x)\,\mathrm{d}x, \tag{2.8}$$

and normalized in such a way that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{2.9}$$

If $\alpha = \beta = \nu - \frac{1}{2}$, then

$$P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x) = \frac{\binom{n+\nu}{n}}{\binom{n+2\nu-1}{n}} C_n^\nu(x), \tag{2.10}$$

where C_n^ν is the Gegenbauer polynomial of degree n and order ν. The constant $c_{\alpha,\beta}$ in (2.8) is chosen in such a way that $w_{\alpha,\beta}(x)\,\mathrm{d}x$ defines a probability measure,

$$c_{\alpha,\beta}^{-1} = \int_{-1}^{1} (1-x)^\alpha(1+x)^\beta\,\mathrm{d}x = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{2.11}$$

Rodrigues' formula [32, (4.3.1)] states that

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n [(1-x)^{n+\alpha}(1+x)^{n+\beta}],$$

from which it can be deduced that

$$\begin{aligned} h_n^{(\alpha,\beta)} &:= \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)^2 w_{\alpha,\beta}(x)\,\mathrm{d}x \\ &= \frac{1}{2n+\alpha+\beta+1}\frac{\Gamma(\alpha+\beta+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \end{aligned}$$

Here, $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ has to be replaced by $\Gamma(n+\alpha+\beta+2)$ if $n=0$; see [32, (4.3.3)]. Setting

$$p_n^{(\alpha,\beta)} := (h_n^{(\alpha,\beta)})^{-\frac{1}{2}} P_n^{(\alpha,\beta)},$$

we then have that $\{p_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ constitutes an orthonormal basis of $L^2(I) = L^2(I, w_{\alpha,\beta})$. Any real-valued function $f : [-1,1] \to \mathbb{R}$ in $L^2(I)$ can be decomposed as

$$f(x) = \sum_{n=0}^{\infty} \widehat{f}(n) p_n^{(\alpha,\beta)}(x), \tag{2.12}$$

where $\widehat{f}(n)$ denotes the n-th coefficient of f with respect to the orthonormal basis $\{p_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$.

Definition 2.12 (The $\mathcal{B}_s(I;\alpha,\beta)$-cone)**.** *Let $s \in \{+,-\}$, and let $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. Then $\mathcal{B}_s(I;\alpha,\beta)$ consists of all continuous functions $f : [-1,1] \to \mathbb{R}$, such that:*

- $\widehat{f}(0) \leqslant 0$;
- $\{s\widehat{f}(n)\}_{n\in\mathbb{N}}$ *is eventually nonnegative while* $sf(1) \leqslant 0$.

The proof of Theorem 2.13 below will reveal that the space[6] $(I, d, w_{\alpha,\beta}(x)\,\mathrm{d}x)$ is admissible in the sense of Definition 1.2, with respect to the basis $\{p_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ and $\mathfrak{o} = 1$. Moreover, $\mathcal{B}_s(I;\alpha,\beta) = \mathcal{A}_s(I)$ (recall Definition 1.3). Specializing (1.8), (1.9) to the present case, we are led to consider

$$r(f; I) = \inf\{r \in (0,2] : f(x) \geqslant 0 \text{ if } x \in [-1, 1-r)\};$$
$$k_s(\widehat{f}) = \min\{k \geqslant 1 : s\widehat{f}(n) \geqslant 0 \text{ if } n \geqslant k\},$$

together with the quantity

$$\mathbb{B}_s(I;\alpha,\beta) := \inf_{f\in\mathcal{B}_s(I;\alpha,\beta)\setminus\{\mathbf{0}\}} r(f;I) k_s(\widehat{f})^2, \tag{2.13}$$

which is estimated by our next result.

Theorem 2.13. *Let $s \in \{+,-\}$ and $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. Then the following estimate holds:*

$$\mathbb{B}_s(I;\alpha,\beta) \geqslant \frac{2\Gamma(\alpha+2)^{\frac{2}{\alpha+1}}}{(4e^{\frac{1}{12}})^{\frac{2}{\alpha+1}}(\alpha+\beta+2)(\alpha+2)}. \tag{2.14}$$

Moreover, $\mathbb{B}_+(I;\alpha,\beta) \leqslant 2$ and $\mathbb{B}_-(I;\alpha,\beta) \leqslant 8$.

Remark. By Stirling's formula, the right-hand side of (2.14) satisfies

$$\frac{2\Gamma(\alpha+2)^{\frac{2}{\alpha+1}}}{(4e^{\frac{1}{12}})^{\frac{2}{\alpha+1}}(\alpha+\beta+2)(\alpha+2)} = \frac{2e^{-2}}{1+\frac{\beta}{\alpha}}\left(1 + O\left(\frac{1}{\alpha+1}\right)\right).$$

The upper bounds $\mathbb{B}_+(I;\alpha,\beta) \leqslant 2$ and $\mathbb{B}_-(I;\alpha,\beta) \leqslant 8$ are produced by the polynomials

$$f_+(x) = -1 + \frac{P_1^{(\alpha,\beta)}(x)}{P_1^{(\alpha,\beta)}(1)} \quad \text{and} \quad f_-(x) = -\frac{P_1^{(\alpha,\beta)}(x)}{P_1^{(\alpha,\beta)}(1)} + \frac{P_2^{(\alpha,\beta)}(x)}{P_2^{(\alpha,\beta)}(1)}, \tag{2.15}$$

[6] Here, $d : I \times I \to [0,2]$ denotes the restriction of the usual Euclidean distance.

respectively. We have performed extensive numerical searches in order to find polynomials up to degree 30 which lead to better upper bounds, but were unable to find any. Nevertheless, we would be extremely surprised if the polynomials $f_\pm$ from (2.15) turned out to be extremal.

We are interested in the following restricted optimum:

$$\mathbb{B}_s^0\,(I;\alpha,\beta) := \inf\left\{r(f;I)k_s(\widehat{f})^2 :\ f \in \mathcal{B}_s\,(I;\alpha,\beta)\setminus\{\mathbf{0}\}, f(1)=0\right\},$$

which according to the next result coincides with (2.13).

Proposition 2.14. *Let $s \in \{+,-\}$, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, and $f \in \mathcal{B}_s(I;\alpha,\beta)\setminus\{\mathbf{0}\}$. Then there exists a polynomial g such that $(f+g) \in \mathcal{B}_s(I;\alpha,\beta)\setminus\{\mathbf{0}\}$, $(f+g)(1)=0$, $k_s(\widehat{f}+\widehat{g}) = k_s(\widehat{f})$, and $r(f+g;I) < r(f;I)$. In particular, $\mathbb{B}_s^0\,(I;\alpha,\beta) = \mathbb{B}_s\,(I;\alpha,\beta)$.*

2.2.1. *Connections with Quadrature.* A finite set $\{(x_j,\lambda_j)\}_{j=0}^m$ with $-1 \leqslant x_m < x_{m-1} < \ldots < x_0 \leqslant 1$ and $\lambda_j > 0$ for $j = 0,\ldots,m$ is said to generate a quadrature rule of degree t for the measure $w_{\alpha,\beta}$ if, for every polynomial f of degree at most t,

$$\int_{-1}^1 f(x)w_{\alpha,\beta}(x)\,\mathrm{d}x = \sum_{j=0}^m \lambda_j f(x_j).$$

$X := \{x_j\}_{j=0}^m$ is the set of *nodes* and $\Lambda := \{\lambda_j\}_{j=0}^m$ is the set of *weights*. Note that necessarily $t \leqslant 2m+1$, for otherwise the integral of the polynomial $\prod_{j=0}^m (x-x_j)^2$ against the measure $w_{\alpha,\beta}$ would be zero, which is absurd. Similarly, if $x_m = -1 < -x_0$ or $x_m > -1 = -x_0$, then $t \leqslant 2m$, and if $x_0 = -x_m = 1$, then $t \leqslant 2m-1$.

Quadrature rules where t is as large as possible can be completely classified via the Gauss–Jacobi quadrature [32, Theorem 3.4.1], with nodes given by the zeros of Jacobi polynomials, and weights given by the Christoffel numbers; see [12]. A quick review follows.

- Assume that $-1 < x_m < x_0 < 1$ and $t = 2m+1$. Then $q(x) = \prod_{j=0}^m (x-x_j)$ is orthogonal to all polynomials of degree $\leqslant m$ with respect to the measure $w_{\alpha,\beta}$, and therefore $q = c\,p_{m+1}^{(\alpha,\beta)}$, for some $c > 0$.
- Assume that $-1 = x_m < x_0 < 1$ (resp. $-1 < x_m < x_0 = 1$) and $t = 2m$. Then $q(x) = \prod_{j=0}^{m-1}(x-x_j)$ (resp. $q(x) = \prod_{j=1}^m (x-x_j)$) is orthogonal to all polynomials of degree $\leqslant m-1$ with respect to $w_{\alpha,\beta+1}$ (resp. $w_{\alpha+1,\beta}$), and therefore $q = c\,p_m^{(\alpha,\beta+1)}$ (resp. $q = c\,p_m^{(\alpha+1,\beta)}$), for some $c > 0$.
- Assume that $-1 = x_m < x_0 = 1$ and $t = 2m-1$. Then $q(x) = \prod_{j=1}^{m-1}(x-x_j)$ is orthogonal to all polynomials of degree $\leqslant m-2$ with respect to $w_{\alpha+1,\beta+1}$, and therefore $q = c\,p_{m-1}^{(\alpha+1,\beta+1)}$, for some $c > 0$.

Definition 2.15 (s-optimal polynomial in $\mathcal{B}_s^0(I;\alpha,\beta)$)**.** *Let $s \in \{+,-\}$ and $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. A polynomial $f \in \mathcal{B}_s^0(I;\alpha,\beta)$ is locally s-optimal if there exists $\delta > 0$, such that*

$$r(f;I)k_s(\widehat{f})^2 < r(h;I)k_s(\widehat{h})^2,$$

for any polynomial $h \in \mathcal{B}_s^0(I;\alpha,\beta)$ satisfying $\deg(h) \leqslant \deg(f)$ and $0 < \inf_{c>0}\|f - ch\|_{L^\infty(I)} < \delta$. The polynomial f is said to be globally s-optimal if one can take $\delta = +\infty$.

In what follows, we let $x_{1,m}^{(\alpha,\beta)}$ denote the largest zero of the polynomial $p_m^{(\alpha,\beta)}$.

Theorem 2.16. *Let $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. Define the polynomials*

$$\begin{aligned} P(x) &:= (1-x)\frac{p_m^{(\alpha+1,\beta)}(x)^2}{x_{1,m}^{(\alpha+1,\beta)} - x}, \quad (m \geqslant 1); \\ Q(x) &:= (1-x^2)\frac{p_{m-1}^{(\alpha+1,\beta+1)}(x)^2}{x_{1,m-1}^{(\alpha+1,\beta+1)} - x}, \quad (m \geqslant 2). \end{aligned} \tag{2.16}$$

Then P and Q are locally s-optimal in $\mathcal{B}_s^0(I;\alpha,\beta)$, for any $s \in \{+,-\}$.

2.2.2. Quadrature and Spherical Designs. Aiming to establish a connection between spherical designs and the sign uncertainty principle for spherical harmonics, we now restrict attention to Gegenbauer polynomials. For notational simplicity, set $\mu_\nu := w_{\nu-\frac{1}{2},\nu-\frac{1}{2}}$. Let $\Omega \subseteq \mathbb{S}^{d-1}$ be a tight spherical t-design with set of cosine distances $\{\alpha_m < \alpha_{m-1} < \ldots < \alpha_1\} \subseteq [-1,1)$, where $t = 2m$ if $\alpha_m > -1$, and $t = 2m-1$ if $\alpha_m = -1$. Define

$$\ell_j := \#\{(\omega,\omega') \in \Omega^2 : \langle \omega,\omega'\rangle = \alpha_j\},$$

and further set $\ell_0 = 1$, $x_0 = 1$, and $\{x_j = \alpha_j\}_{j=1}^m$. We note that $\{(x_j, \frac{\ell_j}{\#\Omega^2})\}_{j=0}^m$ generates a quadrature rule of degree t for μ_ν. Indeed, if f is a polynomial of degree at most t, and $\bar\sigma$ denotes the normalized surface measure on $\mathbb{S}^{d-1}$, then

$$\int_{(\mathbb{S}^{d-1})^2} f(\langle \zeta,\nu\rangle)\,\mathrm{d}\bar\sigma(\zeta)\,\mathrm{d}\bar\sigma(\nu) = \frac{1}{\#\Omega^2}\sum_{\omega,\omega'\in\Omega} f(\langle\omega,\omega'\rangle) = \sum_{j=0}^m \frac{\ell_j}{\#\Omega^2} f(x_j).$$

In particular, $\{\alpha_j\}_{j=1}^m \setminus \{-1\}$ coincide with the zeros of the polynomial $p_m^{(\nu+1/2,\nu-1/2)}$ or $p_{m-1}^{(\nu+1/2,\nu+1/2)}$, depending on whether $\alpha_m > -1$ or $\alpha_m = -1$, respectively. On the other hand, if $\eta \in \mathbb{S}^{d-1}$ denotes the north pole as usual, then

$$\int_{(\mathbb{S}^{d-1})^2} f(\langle \zeta,\nu\rangle)\,\mathrm{d}\bar\sigma(\zeta)\,\mathrm{d}\bar\sigma(\nu) = \int_{\mathbb{S}^{d-1}} f(\langle\zeta,\eta\rangle)\,\mathrm{d}\bar\sigma(\zeta) = \int_{-1}^1 f(x)\mu_\nu(x)\,\mathrm{d}x.$$

Moreover, one easily checks that the map $f(x) \mapsto F(\omega) := f(\langle\omega,\eta\rangle)$ defines a bijection between the sets $\mathcal{B}_s^0(I;\nu-\frac{1}{2},\nu-\frac{1}{2})$ and $\mathcal{B}_s^0(\mathbb{S}^{d-1})$, and that $k_s(\widehat{f}) = k_s(\widehat{F})$ and $r(f;I) =$

$1 - \cos(\theta(F))$. With these considerations in place, one easily checks that Theorem 2.16 specializes to the following result.

Corollary 2.17. *Let $d \geqslant 2$, and set $\alpha = \beta = \frac{d-3}{2}$ in Theorem 2.16. Then, for any $s \in \{+, -\}$, the polynomials $f := P(\langle \cdot, \eta \rangle)$ and $g := Q(\langle \cdot, \eta \rangle)$ (where P, Q were defined in* (2.16)*) are locally s-optimal in $\mathcal{B}^0_s(\mathbb{S}^{d-1})$.*

2.3. **Fourier Series.** Given $d \geqslant 1$, the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ can be defined as the set of equivalence classes under the equivalence relation $x \sim y$ if $x - y \in \mathbb{Z}^d$. Equivalently, we will think of $\mathbb{T}^d$ as the following subset of $\mathbb{C}^d$:

$$\mathbb{T}^d = \{(e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) \in \mathbb{C}^d : (x_1, \ldots, x_d) \in [-\tfrac{1}{2}, \tfrac{1}{2}]^d\}$$

Functions on $\mathbb{T}^d$ are in one-to-one correspondence with functions on $\mathbb{R}^d$ which are 1-periodic in each coordinate. The Haar probability measure on $\mathbb{T}^d$, denoted λ, is simply the restriction of d-dimensional Lebesgue measure to the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Translation invariance of the Lebesgue measure, and periodicity of functions on $\mathbb{T}^d$, imply that

$$\int_{\mathbb{T}^d} f \, \mathrm{d}\lambda = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} f(x) \, \mathrm{d}x.$$

Given a real-valued function $f \in L^1(\mathbb{T}^d) = L^1(\mathbb{T}^d, \lambda)$, and $m \in \mathbb{Z}^d$, define the Fourier coefficient

$$\widehat{f}(m) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i \langle x, m \rangle} \, \mathrm{d}\lambda(x).$$

An immediate consequence is the estimate $\|\widehat{f}\|_{\ell^\infty(\mathbb{Z}^d)} \leqslant \|f\|_{L^1(\mathbb{T}^d)}$. If $f \in L^1(\mathbb{T}^d)$ and $\widehat{f} \in \ell^1(\mathbb{Z}^d)$, then Fourier inversion applies, and implies that, for λ-almost every $x \in \mathbb{T}^d$,

$$f(x) = \sum_{m \in \mathbb{Z}^d} \widehat{f}(m) e^{2\pi i \langle x, m \rangle}.$$

In particular, f is almost everywhere equal to a continuous function on $\mathbb{T}^d$; see [20, Prop. 3.1.14]. If moreover $f \in L^2(\mathbb{T}^d)$, then Plancherel's identity states that

$$\|f\|^2_{L^2(\mathbb{T}^d)} = \sum_{m \in \mathbb{Z}^d} |\widehat{f}(m)|^2.$$

As an immediate consequence of Theorem 1.1, we obtain the following result.

Theorem 2.18. *Let $s \in \{+, -\}$, $d \geqslant 1$. Let $f \in L^1(\mathbb{T}^d)$ be nonzero and such that $\widehat{f} \in \ell^1(\mathbb{Z}^d)$,*

$$\int_{\mathbb{T}^d} f \, \mathrm{d}\lambda \leqslant 0, \ \text{and} \ \sum_{m \in \mathbb{Z}^d} s\widehat{f}(m) \leqslant 0.$$

Then the following inequality holds:

$$\lambda(\{x \in \mathbb{T}^d : f(x) < 0\}) \cdot \#\{m \in \mathbb{Z}^d : s\widehat{f}(m) < 0\} \geqslant \frac{1}{16}.$$

The space $(\mathbb{T}^d, d_\infty, \lambda)$ is admissible for $\mathfrak{o} = (0, \ldots, 0) \in \mathbb{T}^d$ in the sense of Definition 1.2. Here, $d_\infty : \mathbb{T}^d \times \mathbb{T}^d \to [0, 1]$ is defined via

$$d_\infty(x, y) := \max_{1 \leqslant j \leqslant d} |x_j - y_j|,$$

where $|x|$ denotes the distance from x to 0 in $\mathbb{T}^1$. The following result then follows from Theorem 1.4, or more directly from Theorem 2.18.

Theorem 2.19. *Let $s \in \{+, -\}$, $d \geqslant 1$. Let $f \in \mathcal{A}_s(\mathbb{T}^d)$ be a nonzero, even function, for which there exist $r_f \in (0, 1], k_{f,s} \geqslant 1$ with the following properties: $f(x) \geqslant 0$ if $r_f \leqslant d_\infty(x, \mathfrak{o})$ while $\widehat{f}(0) \leqslant 0$, and $s\widehat{f}(m) \geqslant 0$ if $|m| \geqslant k_{f,s}$ while $sf(\mathfrak{o}) \leqslant 0$. Then the following inequality holds:*

$$r_f(2k_{f,s} - 1) \geqslant 2^{-(1+\frac{4}{d})}.$$

In the companion paper [16], we established the following estimate:

$$\inf_{f \in \mathcal{A}_+(\mathbb{T}^1) \setminus \{\mathbf{0}\}} \sqrt{r_f k_{f,+}} \leqslant \mathbb{A}_+(1); \tag{2.17}$$

see [16, Prop. 4]. We do not know whether an analogous result holds for $s = -1$. Another open problem is to determine whether equality holds in (2.17), in which case the statement could be regarded as a transference principle between the continuous and discrete settings. It would also be interesting to prove a similar result for Dini series, which should relate to the higher dimensional ± 1 uncertainty principles $\mathbb{A}_s(d)$, $d \geqslant 2$, and are the subject of the next section.

2.4. Dini Series.

The Dini series of a function $f : [0, 1] \to \mathbb{R}$ is given by

$$f(x) = \mathfrak{B}_0(x) + \sum_{n=1}^{\infty} c_n J_\nu(\lambda_n x), \tag{2.18}$$

where $0 < \lambda_1 < \lambda_2 < \ldots$ denote the positive zeros of the function

$$zJ_\nu'(z) + HJ_\nu(z) = (H + \nu)J_\nu(z) - zJ_{\nu+1}(z). \tag{2.19}$$

Here, J_ν is the Bessel function of the first kind of order $\nu \geqslant -\frac{1}{2}$, and $H \in \mathbb{R}$. The initial term in (2.18), $\mathfrak{B}_0(x)$, depends on the sign of $H + \nu$. If $H + \nu > 0$, then $\mathfrak{B}_0 \equiv 0$; if $H + \nu < 0$, then the function (2.19) has two purely imaginary zeros $\pm i\lambda_0$, whose contributions are manifested by taking $\mathfrak{B}_0(x)$ to be an appropriate multiple of $J_\nu(i\lambda_0 x)$; if $H + \nu = 0$, then the imaginary zeros coalesce at the origin, and $\mathfrak{B}_0(x) = 2(\nu + 1)x^\nu \int_0^1 t^{\nu+1} f(t)\, \mathrm{d}t$. Note that the functions $x \mapsto J_\nu(\lambda_n x), n \in \mathbb{N}$, are orthogonal in $[0, 1]$ with respect to the measure $x\, \mathrm{d}x$. Indeed, [35,

§5.11-(8)] implies that, for all real numbers $k \neq \ell$,

$$\int_0^1 J_\nu(kx) J_\nu(\ell x) x \, \mathrm{d}x = \frac{k J_{\nu+1}(k) J_\nu(\ell) - \ell J_\nu(k) J_{\nu+1}(\ell)}{k^2 - \ell^2}. \tag{2.20}$$

If k, ℓ are distinct zeros of (2.19), then one can invoke the usual recurrence relations for Bessel functions in order to deduce that the integral in (2.20) vanishes.

If $H + \nu = 0$, then the elements of the sequence $\{\lambda_n\}_{n \geqslant 1}$ coincide with the positive zeros of the function $J_{\nu+1}$. In this case, if $\nu = -\frac{1}{2}$, then $J_{\nu+1}(x) = (\frac{2}{\pi x})^{\frac{1}{2}} \sin(x)$ and $\lambda_n = \pi n$; hence the Dini series (2.18) specializes to the Fourier series from §2.3. In this way, Dini series for $H + \nu = 0$ are seen to generalize one-dimensional Fourier series to the higher dimensional radial case.

In order to properly place Dini series within the scope of Theorem 1.4, we need to normalize the functions $J_\nu(\lambda_n x)$, in such a way as to ensure that their maximum is attained at the origin. This is most easily done by introducing the even, entire function $A_\nu(z) := \Gamma(\nu + 1)(\frac{1}{2}z)^{-\nu} J_\nu(z)$, since $|A_\nu(z)| \leqslant A_\nu(0) = 1$. One can then rescale the results from [35, §18.33], and invoke the identity [35, §5.11-(11)],

$$\int_0^1 A_\nu^2(\lambda_n x) x^{2\nu+1} \, \mathrm{d}x = \frac{A_\nu^2(\lambda_n)}{2},$$

in order to derive the following proposition.

Proposition 2.20. *Let $\nu \geqslant -\frac{1}{2}$. For every $f \in L^2\left([0,1], \frac{x^{2\nu+1}}{2(\nu+1)} \mathrm{d}x\right)$, we have that*

$$f(x) = \widehat{f}(0) + 2\sqrt{\nu+1} \sum_{n=1}^\infty \widehat{f}(n) \frac{A_\nu(\lambda_n x)}{A_\nu(\lambda_n)} \tag{2.21}$$

in the L^2-sense, where $\{\lambda_n\}_{n \geqslant 1}$ denote the positive zeros of the Bessel function $J_{\nu+1}$,

$$\widehat{f}(n) = \frac{2\sqrt{\nu+1}}{A_\nu(\lambda_n)} \int_0^1 f(x) A_\nu(\lambda_n x) \frac{x^{2\nu+1} \, \mathrm{d}x}{2(\nu+1)}, \tag{2.22}$$

for all $n \geqslant 1$, and

$$\widehat{f}(0) = \int_0^1 f(x) \frac{x^{2\nu+1} \, \mathrm{d}x}{2(\nu+1)}.$$

Moreover, if f is continuous and of bounded variation in $[0,1]$, then the Dini series (2.21) of f converges absolutely and uniformly in $[0,1]$.

Identity [35, §12.11-(1)] translates into

$$\int_0^1 A_\nu(kx) x^{2\nu+1} \, \mathrm{d}x = \frac{A_{\nu+1}(k)}{2(k+1)},$$

and reveals that the functions $\{A_\nu(\lambda_n x)\}_{n\geqslant 1}$ are orthogonal to the constant function $\mathbf{1}$. Consequently, the orthonormal basis

$$\{\mathbf{1}\} \cup \left\{ \frac{2\sqrt{\nu+1}}{A_\nu(\lambda_n)} A_\nu(\lambda_n x) \right\}_{n\geqslant 1}$$

satisfies all the hypotheses of Theorem 1.4 with $\mathfrak{o} = 0$. We can then use the well-known asymptotic formulae $\lambda_n \sim \pi n$ and

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos(z - \nu\pi/2 - \pi/4) + O(|z|^{-3/2}),$$

see [35, §7.1], in order to deduce that $A_\nu(\lambda_n)^{-2} \sim \lambda_n^{2\nu+1}$, where the implied constant depends only on ν. The following result can then be derived from Theorem 1.4 at once.

Theorem 2.21. *Let $s \in \{+,-\}$, $\nu \geqslant -\frac{1}{2}$. Let $f : [0,1] \to \mathbb{R}$ be a nonzero continuous function of bounded variation, whose coefficients $\{\widehat{f}(n)\}_{n\geqslant 1}$ defined in (2.22) satisfy*

$$\sum_{n=1}^{\infty} n^{\nu+\frac{1}{2}} |\widehat{f}(n)| < \infty.$$

Suppose that there exist $r_f \in (0,1]$, $k_{f,s} \geqslant 1$, such that $f(x) \geqslant 0$ if $x \in [r_f, 1]$ while $\widehat{f}(0) \leqslant 0$, and $s\widehat{f}(n) \geqslant 0$ if $n \geqslant k_{f,s}$ while $sf(0) \leqslant 0$. Then there exists $c_\nu > 0$, such that

$$r_f\, k_{f,s}^{2\nu+2} \geqslant c_\nu. \tag{2.23}$$

The constant c_ν in (2.23) depends only on ν and can be made explicit, e.g. by appealing to [27, Lemma 2.5]. However, the number of terms in the required asymptotic expansion grows linearly with the parameter ν, and as such we have omitted the precise formulation of the corresponding (somewhat cumbersome) statement.

3. Sign Uncertainty in Discrete Spaces

3.1. **Discrete Fourier Transform.** Let $q \geqslant 1$ be an integer, and let $\mathbb{Z}_{2q+1}$ denote the set of equivalence classes of integers modulo $2q+1$. The choice of a residue class of odd size is convenient[7] for numerical purposes, since we can then place the origin (in the sense of Definition 1.2) at $n = 0$.

If $f : \mathbb{Z}_{2q+1} \to \mathbb{R}$ is real-valued and even, then its discrete Fourier transform $\widehat{f}$, defined via

$$\widehat{f}(k) = \frac{1}{\sqrt{2q+1}} \sum_{n=-q}^{q} f(n) e^{-2\pi i \frac{kn}{2q+1}} = \frac{1}{\sqrt{2q+1}} \left(f(0) + 2\sum_{n=1}^{q} f(n) \cos\left(2\pi \frac{kn}{2q+1}\right)\right)$$

[7]However, everything that follows can be easily adapted to residue classes of arbitrary size.

is likewise real-valued and even. Since the discrete Fourier transform defines an isometry from $L^2(\mathbb{Z}_{2q+1}) \simeq \mathbb{R}^{2q+1}$ onto itself, and $\max_{-q\leqslant k\leqslant q} |\widehat{f}(k)| \leqslant (2q+1)^{-\frac{1}{2}} \sum_{n=-q}^{q} |f(n)|$, the following result is a direct consequence of Theorem 1.1.

Theorem 3.1. *Let $s \in \{+,-\}$ and $q \geqslant 1$ be an integer. Let $f : \mathbb{Z}_{2q+1} \to \mathbb{R}$ be nonzero and even. Assume that $sf(0) \leqslant 0$ and $\widehat{f}(0) \leqslant 0$. Then the following inequality holds:*

$$\#\{n \in \mathbb{Z}_{2q+1} : f(n) < 0\} \cdot \#\{k \in \mathbb{Z}_{2q+1} : s\widehat{f}(k) < 0\} \geqslant \frac{2q+1}{16}.$$

The following problem will be of interest.

Problem 3.1 (Feasibility Linear Programming Problem for the discrete Fourier transform)**.** *Let*

$$\mathbb{A}_s^{\mathrm{disc}}(q) := \min\{k_{f,s} \geqslant 0 : f \in \mathcal{A}_s^{\mathrm{disc}}(q)\},$$

where $\mathcal{A}_s^{\mathrm{disc}}(q)$ denotes the set of even functions $f : \mathbb{Z}_{2q+1} \to \mathbb{R}$, such that $sf(0), \widehat{f}(0) \leqslant 0$ and $f(\pm q), s\widehat{f}(\pm q) \geqslant 1$, and $k_{f,s}$ is the smallest nonnegative integer, for which $f(n), s\widehat{f}(n) \geqslant 0$ if $k_{f,s} \leqslant |n| \leqslant q$. Here, $|n|$ denotes the absolute value of the representation of n in the interval $\{-q, \dots, 0, \dots, q\}$.

Definition 3.2 (s-Feasibility)**.** *Let $s \in \{+,-\}$. A pair (k,q) is s-feasible if there exists $f \in \mathcal{A}_s^{\mathrm{disc}}(q)$, such that $k_{f,s} \leqslant k$.*

The following result is an immediate consequence of Theorem 3.1 and Definition 3.1.

Corollary 3.3. *Let $s \in \{+,-\}$ and $q \geqslant 1$ be an integer. Then*

$$\frac{\mathbb{A}_s^{\mathrm{disc}}(q)}{\sqrt{2q+1}} \geqslant \frac{1}{8}.$$

Problem 3.1 can be solved numerically with a linear programming solver, and we have done so. Numerical evidence presented in §6.1 strongly supports the following conjecture.

Conjecture 3.4. *Let $s \in \{+,-\}$. If (k,q) is s-feasible, then $(k+1,q), (k,q-1)$ are s-feasible. The function $q \mapsto \mathbb{A}_s^{\mathrm{disc}}(q)$ is non-decreasing, and its range contains all integers $k \geqslant 2$ if $s = +1$, and all integers $k \geqslant 3$ if $s = -1$. Moreover,*

$$\lim_{q\to\infty} \frac{\mathbb{A}_s^{\mathrm{disc}}(q)}{\sqrt{2q+1}} = \mathbb{A}_s(1).$$

where $\mathbb{A}_s(1)$ denotes the optimal constant for the one-dimensional continuous sign uncertainty principles defined in (1.2), (1.3).

Since the discrete Fourier transform is a proper discretization of the Fourier transform (1.1), it is natural to expect that the discrete uncertainty principles converge to their continuous counterparts, in the limit when $q \to \infty$. Indeed, this is what seems to happen numerically. Moreover, the numerical patterns in §6.1 (see Table 1) are relatively straightforward to identify, and they provide evidence towards the following conjecture. As before, we let $\varphi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio.

Conjecture 3.5. *The pair $(k, \lfloor (k-1)^2\varphi \rfloor)$ is $+1$-feasible, for every integer $k \geqslant 3$. The pair $(k, \left\lfloor \frac{k^2-2k+2}{2} \right\rfloor)$ is -1-feasible, for every integer $k \geqslant 4$. Moreover, if $\widetilde{q}_+(k) = \lfloor (k-1)^2\varphi \rfloor$ and $\widetilde{q}_-(k) = \left\lfloor \frac{k^2-2k+2}{2} \right\rfloor$, then $k = \mathbb{A}_s(\widetilde{q}_s(k)) + o(k)$.*

Conjectures 3.4 and 3.5 imply that

$$\mathbb{A}_+(1) = \lim_{k\to\infty} \frac{k}{\sqrt{2\lfloor (k-1)^2\varphi \rfloor + 1}} = (2\varphi)^{-\frac{1}{2}}, \tag{3.1}$$

which is Conjecture 1.7. There are several reasons to believe (3.1) to hold, one of them being that the companion -1 uncertainty principle yields the correct answer in the limit. Indeed, Conjectures 3.4 and 3.5 together imply that $\mathbb{A}_-(1) = 1$, which is known to hold; recall the discussion in §1.1, and see §6.1 below for further details.

3.2. **Discrete Hankel Transform.** The discrete Hankel transform was proposed by Siegman in 1977, and later on several other versions were put forward; see [13]. To the best of our knowledge, none of the proposed explicit forms defines a unitary operator; rather, they are only *asymptotically unitary.* In one way or another, they all properly discretize a given compactly supported function f, and then appeal to Bessel–Fourier series in order to further discretize the Hankel transform of f. Fisk Johnson [13] proposes several approaches, which turn out to work well in practice since they are already very close to being unitary when applied to "short" vectors. Since Theorem 1.1 only requires approximate inversion, it seems reasonable to expect that a sign uncertainty principle holds for each of the kernels defined in [13, (13), (16)–(19)]; for the sake of brevity, we chose not to fully pursue this line of investigation.

The main purpose of this section is to formulate a sign uncertainty principle for the discrete Hankel transform of Fisk Johnson, and to start discussing the numerical experiments which we conducted. Since (after normalization) the Hankel transform of order $\nu = \frac{d}{2} - 1$ coincides with the Fourier transform of a radial function in $\mathbb{R}^d$, one may expect that, in the limit, the corresponding discrete sign uncertainty principle converges to the continuous sign uncertainty principle in all dimensions. We proceed to describe the evidence we obtained in support of this possibility.

Given $\nu \geqslant -\frac{1}{2}$, let $\{j_n\}_{n\geqslant 1}$ denote the positive zeros of the Bessel function J_ν. Our starting point is formula [13, (13)], for $N = q+1$ and $T = \sqrt{j_{q+1}}$. Fisk Johnson proposes a discretization of the following version of the Hankel transform of parameter $\nu \geqslant -\frac{1}{2}$,

$$\widetilde{H}_\nu(f)(x) = \int_0^\infty f(y) J_\nu(xy) y \, \mathrm{d}y, \tag{3.2}$$

which we proceed to describe. Define the discrete Hankel transform with parameter $\nu \geqslant -\frac{1}{2}$ of a given[8] $f : [q] \to \mathbb{R}$, as follows:

$$H_\nu^{\mathrm{disc}}(f)(m) = \frac{2}{j_{q+1}} \sum_{n=1}^{q} f(n) \frac{J_\nu(j_m j_n / j_{q+1})}{J_{\nu+1}(j_n)^2}.$$

Each of the values $f(n)$ is to be interpreted as the evaluation of some continuous function at the node $j_n (j_{q+1})^{-\frac{1}{2}}$. By showing that the kernel of the composition $H_\nu^{\mathrm{disc}} H_\nu^{\mathrm{disc}}$ satisfies[9]

$$\frac{4}{J_{\nu+1}(j_\ell) j_{q+1}^2} \sum_{n=1}^{q} \frac{J_\nu(j_m j_n / j_{q+1}) J_\nu(j_n j_\ell / j_{q+1})}{J_{\nu+1}(j_n)^2} = \delta_{m,\ell} + o(1), \text{ as } q \to \infty,$$

where the term $o(1)$ is already small for small values of q, the author argues that $H_\nu^{\mathrm{disc}} H_\nu^{\mathrm{disc}} \approx \mathrm{Id}$; see [13, (11)]. We turn to the following feasibility problem.

Problem 3.2 (Feasibility Linear Programming Problem for the discrete Hankel transform)**.** *Let*

$$\mathbb{A}_s^{\mathrm{disc}}(q, \nu) := \min\{k_{f,s} : f \in \mathcal{A}_s^{\mathrm{disc}}(q, \nu)\},$$

where $\mathcal{A}_s^{\mathrm{disc}}(q,\nu)$ denotes the set of functions $f : [q] \to \mathbb{R}$, such that $sf(1), H_\nu^{\mathrm{disc}}(f)(1) \leqslant 0$ and $f(q), s\widehat{f}(q) \geqslant 1$, and $k_{f,s}$ is the smallest nonnegative integer, for which $f(n)$, $sH_\nu^{\mathrm{disc}}(f)(n) \geqslant 0$ if $k_{f,s} \leqslant n \leqslant q$.

Definition 3.6 ((s,ν)-Feasibility)**.** *Let $s \in \{+,-\}, \nu \geqslant -\frac{1}{2}$. A pair (k,q) is (s,ν)-feasible it there exists $f \in \mathcal{A}_s^{\mathrm{disc}}(q,\nu)$, such that $k_{f,s} \leqslant k$.*

In §6.2 below, we present compelling numerical evidence towards the following conjecture.

Conjecture 3.7. *Let $s \in \{+,-\}, \nu \geqslant -\frac{1}{2}$. If (k,q) is (s,ν)-feasible, then $(k+1,q), (k,q-1)$ are (s,ν)-feasible. The function $q \mapsto \mathbb{A}_s^{\mathrm{disc}}(q,\nu)$ is non-decreasing, and its range contains $\mathbb{N} \setminus [k_0]$, for some $k_0 \geqslant 1$. Moreover, if $\nu = \frac{d}{2} - 1$ and $n_q = \mathbb{A}_s^{\mathrm{disc}}(q,\nu)$, then*

$$\lim_{q\to\infty} \frac{j_{n_q}}{\sqrt{2\pi j_{q+1}}} = \mathbb{A}_s(d), \tag{3.3}$$

[8]Here, $[q] := \{1, 2, \ldots, q\}$.
[9]Here, $\delta_{m,\ell}$ denotes the usual Kronecker delta: $\delta_{m,\ell} = 1$ if $m = \ell$, and $\delta_{m,\ell} = 0$ otherwise.

where $\mathbb{A}_s(d)$ denotes the optimal constant for the continuous sign uncertainty principles defined in (1.2), (1.3), *and $\{j_n\}_{n\geqslant 1}$ are the positive zeros of the Bessel function J_ν.*

If $f:\mathbb{R}^d\to\mathbb{R}$ is radial and $\nu=\frac{d}{2}-1$, then identity (4.3) below can be rephrased as

$$|\xi|^{\frac{d}{2}-1}\widehat{f}(\xi)=c_\nu\,\widetilde{H}_\nu[y^\nu f(y)](2\pi|\xi|),$$

for some $c_\nu>0$, and therefore the factor $\sqrt{2\pi}$ in (3.3) is to be expected. The particular cases $d\in\{8,12,24\}$ are especially interesting since it is known that $\mathbb{A}_-(8)=\mathbb{A}_+(12)=\sqrt{2}$ and $\mathbb{A}_-(24)=2$. In these cases, the numerical data presented in §6.2 corroborate Conjecture 3.7. Moreover, if $d\in\{2,8,12,24\}$, then our numerics point to the following more structured version of Conjecture 3.7.

Conjecture 3.8. *The following statements hold:*

- $\left(k,\lfloor\frac{\sqrt{3}(k^2-2k+2)}{4}\rfloor\right)$ *is $(-1,\frac{2}{2}-1)$-feasible, for every integer $k\geqslant 4$;*
- $\left(k,\lfloor\frac{k^2}{4}\rfloor\right)$ *is $(-1,\frac{8}{2}-1)$-feasible, for every integer $k\geqslant 4$;*
- $\left(k,\lfloor\frac{k^2+6k-8}{8}\rfloor\right)$ *is $(-1,\frac{24}{2}-1)$-feasible, for every integer $k\geqslant 4$;*
- $\left(k,\lfloor\frac{k^2-2}{4}\rfloor\right)$ *is $(+1,\frac{12}{2}-1)$-feasible, for every integer $k\geqslant 3$.*

Moreover, if we write the pairs above as $(k,\widetilde{q}_s(k,\nu))$ for $(s,\nu)=(-,0),(-,3),(-,11),(+,5)$, respectively, then

$$k=\mathbb{A}_s^{\mathrm{disc}}(\widetilde{q}_s(k,\nu),\nu)+o(k),\ \text{ as } k\to\infty.$$

Noting that $j_n\sim\pi n$, as $n\to\infty$, Conjectures 3.7 and 3.8 would imply that $\mathbb{A}_-(8)=\mathbb{A}_+(12)=\sqrt{2}$ and $\mathbb{A}_-(24)=2$, which are known to be true, but also that $\mathbb{A}_-(2)=(\frac{4}{3})^{\frac{1}{4}}$, which is the content of Conjecture 1.6.

3.3. **Hamming Cube.** The Hamming cube $H_N:=\{-1,1\}^N$ can be equipped with normalized counting measure, $\lambda_H:=2^{-N}\#$, and the Hamming distance $d_H:H_N\times H_N\to[N]$,

$$d_H(x,y):=\#\{n\in[N]:x_n\neq y_n\}.$$

We write $x=(x_1,\ldots,x_N)\in H_N$ with $x_j=\pm1$, for each j, and let $\mathbf{1}=(1,\ldots,1)\in H_N$. An orthonormal basis of $L^2(H_N)=L^2(H_N,\lambda_H)$ is given by $\{\varphi_S:S\subseteq[N]\}$, where $\varphi_S:H_N\to\{-1,1\}$ are the monomials defined via $\varphi_S(x):=\prod_{i\in S}x_i$, with the understanding that $\varphi_\emptyset\equiv 1$. Every function $f:H_N\to\mathbb{R}$ admits an expansion of the form

$$f=\sum_{S\subseteq[N]}\widehat{f}(S)\varphi_S,$$

with (real-valued) coefficients given by

$$\widehat{f}(S) := \frac{1}{2^N} \sum_{x \in H_N} f(x)\varphi_S(x).$$

Let $\widehat{H}_N = \{c : 2^{[N]} \to \mathbb{R}\}$ denote the finite dimensional vector space of sequences of real numbers indexed by subsets of $[N]$, and define

$$\|c\|^2_{L^2(\widehat{H}_N)} := \frac{1}{2^N} \sum_{S \subseteq [N]} |c(S)|^2.$$

The operator $T : H_N \to \widehat{H}_N, f \mapsto 2^{\frac{N}{2}} \widehat{f}$, defines an isometric isomorphism, in the sense that

$$\|T(f)\|^2_{L^2(\widehat{H}_N)} = \sum_{S \subseteq [N]} |\widehat{f}(S)|^2 = \|f\|^2_{L^2(H_N)}.$$

Moreover, $\sup_{S \subseteq [N]} |T(f)(S)| \leqslant 2^{\frac{N}{2}} \|f\|_{L^1(H_N)}$. We can then apply Theorem 1.1 to the operator T, with $p = q = 2$, $a = 2^{\frac{N}{2}}$, and $b = c = 1$, and obtain the following result.

Theorem 3.9. *Let $s \in \{+, -\}$. Let $f : H_N \to \mathbb{R}$ be nonzero, and such that*

$$\sum_{x \in H_N} f(x) \leqslant 0, \qquad sf(\mathbf{1}) \leqslant 0.$$

Then the following estimate holds:

$$\#\{x \in H_N : f(x) < 0\} \cdot \#\{S \subseteq [N] : s\widehat{f}(S) < 0\} \geqslant 2^{N-4}.$$

In particular, if $f(x) \geqslant 0$ if $d_H(x, \mathbf{1}) \geqslant r$ and $s\widehat{f}(S) \geqslant 0$ if $\#S \geqslant k$, then

$$\sum_{n=1}^{r} \binom{N}{n-1} \sum_{n=1}^{k} \binom{N}{n-1} \geqslant 2^{N-4}.$$

We proceed to describe an application of Theorem 3.9 to information theory; for further context, see [25, 26], together with the recent breakthrough solution of the Sensitivity Conjecture [23]. Given a Boolean function $f : H_N \to \{0, 1\}$, let $\deg(f)$ denote the smallest possible degree of a real-valued polynomial p satisfying $f(x) = p(x)$, for all $x \in H_N$; in other words, $\deg(f) := \max\{\#S : \widehat{f}(S) \neq 0\}$.

Corollary 3.10. *Given a non-constant Boolean function $f : H_N \to \{0, 1\}$, define*

$$m(f) := \min\left(\#\{x \in H_N : f(x) = 0\}, \#\{x \in H_N : f(x) = 1\}\right).$$

Then the following estimate holds:

$$\sum_{n=0}^{\deg(f)} \binom{N}{n} \geqslant \frac{2^{N-4}}{m(f)}.$$

Given a Boolean function $f : H_N \to \{0,1\}$ for which $m(f)$ is small (i.e. in the complementary regime to that of the Sensitivity Conjecture [23]), Corollary 3.10 yields a non-trivial lower bound for $\deg(f)$ in terms of a seemingly unexplored measure of the complexity of f.

Proof of Corollary 3.10. Note that $c := 2^{-N}\sum_{x\in H_N} f(x)$ satisfies $0 < c < 1$ because f is non-constant. Set $F(x) := f(x) - c$, for each $x \in H_N$, and $s := -\operatorname{sign}(F(\mathbf{1}))$. In this way, F is not identically zero, $\widehat{F}(\emptyset) = 2^{-N}\sum_{x\in H_N} F(x) = 0$, $sF(\mathbf{1}) < 0$, and $\widehat{F}(S) = \widehat{f}(S)$ if $\#S \geqslant 1$. As a consequence of Theorem 3.9, we can estimate:

$$\begin{aligned} 2^{N-4} &\leqslant \#\{x \in H_N : F(x) < 0\} \cdot \#\{S \subseteq [N] : s\widehat{F}(S) < 0\} \\ &= \#\{x \in H_N : f(x) = 0\} \cdot \#\{S \subseteq [N] : s\widehat{f}(S) < 0\} \\ &\leqslant \#\{x \in H_N : f(x) = 0\} \sum_{n=0}^{\deg(f)} \binom{N}{n}. \end{aligned}$$

A similar reasoning applied to the function $G(x) := c - f(x)$, $x \in H_N$, yields

$$2^{N-4} \leqslant \#\{x \in H_N : f(x) = 1\} \sum_{n=0}^{\deg(f)} \binom{N}{n},$$

and the result follows. □

4. Sign Uncertainty for Convolution Operators

4.1. **Convolution Kernels in Bandlimited Function Spaces.** Let PW_d denote the L^1-*Paley–Wiener space of bandlimited functions in* $\mathbb{R}^d$, i.e. the set of all real-valued, continuous functions $f \in L^1(\mathbb{R}^d)$, whose Fourier support is contained on the unit ball, $\operatorname{supp}(\widehat{f}) \subseteq B_1^d$. Given a function $\psi : \mathbb{R}^d \to \mathbb{R}$ for which there exist $a, b, c \in (0,\infty)$, such that $\|\psi\|_{L^\infty} = a$, $\|\psi\|_{L^1} = b$, and $c|\widehat{\psi}(\xi)| \geqslant 1$ if $\xi \in B_1^d$, consider the associated convolution operator, $T_\psi(f) := f * \psi$. Young's convolution inequality and Plancherel's Theorem together imply that $\|T_\psi(f)\|_{L^\infty} \leqslant a\|f\|_{L^1}$, $\|T_\psi(f)\|_{L^1} \leqslant b\|f\|_{L^1}$, $\|T_\psi(f)\|_{L^2} \leqslant b\|f\|_{L^2}$, and $\|f\|_{L^2} \leqslant c\|T_\psi(f)\|_{L^2}$, for every $f \in PW_d$. Therefore the family $\mathcal{F} = \{(f, T_\psi(f)) : f \in PW_d\}$ satisfies the hypotheses of Theorem 1.1 with $p = q = 2$, and we obtain the following result.

Theorem 4.1. *Let $d \geqslant 1$. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be as above and $s = \operatorname{sign}(\widehat{\psi}(0))$. Let $f \in PW_d \backslash \{\mathbf{0}\}$ be such that $\int_{\mathbb{R}^d} f \leqslant 0$. Then the following inequality holds:*

$$|\{x \in \mathbb{R}^d : f(x) < 0\}||\{\xi \in \mathbb{R}^d : sT_\psi(f)(\xi) < 0\}| \geqslant (16a^2b^2c^4)^{-1}.$$

In particular, if there exist $r_1, r_{2,s} > 0$ such that $f(x) \geqslant 0$ if $|x| \geqslant r_1$, and $sT_\psi(f)(\xi) \geqslant 0$ if $|\xi| \geqslant r_{2,s}$, then

$$r_1 r_{2,s} \geqslant \left(16a^2b^2c^4|B_1^d|^2\right)^{-\frac{1}{d}}.$$

Theorem 4.1 can be extended to the more general setting of locally compact abelian groups; the reader is referred to [31] for the relevant background.

4.2. **Hilbert Transform of Bandlimited Functions.** It is of interest to consider the situation in which the kernel ψ from §4.1 above fails to be integrable. For instance, if $d=1$, then the choice $\psi(x)=\frac{1}{\pi x}$ leads to the Hilbert transform $\mathcal{H}$, as long as the convolution is taken in the principal value sense. It is well-known that $\mathcal{H}$ defines a bounded operator in $L^p(\mathbb{R})$, for all $p\in(1,\infty)$, and that the optimal constant in $\|\mathcal{H}(f)\|_{L^p}\leqslant C_p\|f\|_{L^p}$ is given by

$$C_p := \begin{cases} \tan(\frac{\pi}{2p}), & \text{if } 1<p\leqslant 2,\\ \cot(\frac{\pi}{2p}), & \text{if } 2<p<\infty; \end{cases} \tag{4.1}$$

see [29]. Moreover, since $\widehat{\mathcal{H}(f)}(\xi)=-i\,\mathrm{sign}(\xi)\widehat{f}(\xi)$, we have that $\mathcal{H}(\mathcal{H}(f))=-f$, hence the reverse inequality, $\|f\|_{L^p}\leqslant C_p\|\mathcal{H}(f)\|_{L^p}$, holds with the same optimal constant. Now, if $f\in PW_1$ (recall the definition in §4.1), then $\widehat{f}$ is supported in $[-1,1]$, and consequently

$$\|\mathcal{H}(f)\|_{L^\infty}\leqslant\|\widehat{\mathcal{H}(f)}\|_{L^1}=\|\widehat{f}\|_{L^1}\leqslant 2\|\widehat{f}\|_{L^\infty}\leqslant 2\|f\|_{L^1}.$$

Note that $\widehat{f}$ is continuous since $f\in L^1$. A necessary condition for $\mathcal{H}(f)$ to be integrable if $f\in L^1$ is that $\widehat{f}(0)=0$, in which case $\widehat{\mathcal{H}(f)}(0)=0$ as well. We then conclude that

$$\mathcal{F}=\{(f,\mathcal{H}(f))\,:\,f\in PW_1\,;\,\widehat{f}(0)=0\}$$

satisfies all the hypotheses of Theorem 1.1, with $p=q\in(1,\infty)$, $a=2$, and $b=c=C_p$. As a consequence, we obtain the following result.

Theorem 4.2. *Let $s\in\{+,-\}$ and $p\in(1,\infty)$. Let $f\in PW_1$ satisfy $\widehat{f}(0)=0$. Suppose that there exist $r_1,r_{2,s}>0$, such that $f(y)\geqslant 0$ if $|y|\geqslant r_1$, and $s\mathcal{H}(f)(x)\geqslant 0$ if $|x|\geqslant r_{2,s}$. Then the following estimate holds:*

$$r_1^{1/p'}r_{2,s}^{1/p}\geqslant 2^{-(p'+2)}C_p^{-\frac{p+1}{p-1}},$$

where C_p is given by (4.1) *above.*

Theorem 4.2 can probably be extended to a certain class of singular integral operators given by Calderón–Zygmund kernels of convolution type (see [20, Ch. 5]) which includes the higher dimensional Riesz transforms.

4.3. **Hankel Transform.** The Hankel transform with parameter $\nu>-1$ of a function $f:\mathbb{R}_+\to\mathbb{R}$ is given by

$$H_\nu(f)(x)=\int_0^\infty f(y)A_\nu(xy)y^{2\nu+1}\,\mathrm{d}y, \tag{4.2}$$

where $A_\nu(z)=\Gamma(\nu+1)(\frac{1}{2}z)^{-\nu}J_\nu(z)$, and J_ν is the Bessel function of the first kind. Alternative ways to define the Hankel Transform exist, the most common one having A_ν replaced by J_ν, and $y^{2\nu+1}\,\mathrm{d}y$ replaced by $y\,\mathrm{d}y$; recall (3.2), and see e.g. [33]. However, the choice of kernel in (4.2) suits us better since the function $A_\nu(z)$ is entire, $A_\nu(0)=1$, and routine computations show that, if $f:\mathbb{R}^d\to\mathbb{R}$ is radial, then its Fourier transform $\widehat{f}$, as defined in (1.1), is also radial, and satisfies

$$\widehat{f}(\xi)=c_d H_{\frac{d}{2}-1}(f)(2\pi|\xi|), \tag{4.3}$$

for some $c_d>0$. The analogue of (2.20) over the unbounded region of integration $(0,\infty)$ reveals the following Plancherel-type identity:

$$\int_0^\infty |H_\nu(f)(x)|^2 x^{2\nu+1}\,\mathrm{d}x = 4^\nu\Gamma^2(\nu+1)\int_0^\infty |f(y)|^2 y^{2\nu+1}\,\mathrm{d}y.$$

Moreover, since $|A_\nu(x)|\leqslant A_\nu(0)=1$, we easily obtain that

$$\sup_{x>0}|H_\nu(f)(x)|\leqslant \int_0^\infty |f(y)|y^{2\nu+1}\,\mathrm{d}y.$$

Therefore, for a given $s\in\{+,-\}$, the family

$$\mathcal{F}=\Big\{(f,H_\nu(f))\ :\ f,H_\nu(f)\in L^1(\mathbb{R}_+,y^{2\nu+1}\,\mathrm{d}y),$$
$$\int_0^\infty f(y)y^{2\nu+1}\,\mathrm{d}y,\ s\int_0^\infty H_\nu(f)(x)x^{2\nu+1}\,\mathrm{d}x\leqslant 0\Big\}$$

satisfies the hypotheses of Theorem 1.1 when $p=q=2$, $a=1$, and $b=1/c=2^\nu\Gamma(\nu+1)$. It is then easy to derive the following result.

Theorem 4.3. *Let $s\in\{+,-\}$ and $\nu>-1$. Let $f:\mathbb{R}_+\to\mathbb{R}$ be a nonzero continuous function, such that $f,H_\nu(f)\in L^1(\mathbb{R}_+,y^{2\nu+1}\,\mathrm{d}y)$. Assume that there exist $r_1,r_{2,s}>0$, such that $f(y)\geqslant 0$ if $y\geqslant r_1$ while $H_\nu(f)(0)\leqslant 0$, and $sH_\nu(f)(x)\geqslant 0$ if $x\geqslant r_{2,s}$ while $sf(0)\leqslant 0$. Then the following estimate holds:*

$$r_1 r_{2,s}\geqslant 4^{\nu-2}\Gamma^2(\nu+1).$$

5. Proofs of Main Results

5.1. **Proof of Theorem 1.1.**

Proof. Since $\int_X f\,\mathrm{d}\mu\leqslant 0$, we have that

$$\|f\|_{L^1(X,\mu)}\leqslant 2\int_{\{f<0\}}|f|\,\mathrm{d}\mu\leqslant 2\mu(\{f<0\})^{\frac{1}{p'}}\|f\|_{L^p(X,\mu)}, \tag{5.1}$$

where the last estimate follows from Hölder's inequality. On the other hand, the hypotheses, convexity of L^p-norms, the fact that $s\int_Y g\,\mathrm{d}\nu\leqslant 0$, and a second application of Hölder's inequality, together yield

$$\|f\|^q_{L^p(X,\mu)}\leqslant c^q\|g\|^q_{L^q(Y,\nu)}$$

$$
\begin{aligned}
&\leqslant c^q \|g\|_{L^\infty(Y,\nu)}^{q-1} \|g\|_{L^1(Y,\nu)} \\
&\leqslant 2c^q a^{q-1} \|f\|_{L^1(X,\mu)}^{q-1} \int_{\{sg<0\}} |g| \,\mathrm{d}\nu \\
&\leqslant 2c^q a^{q-1} \|f\|_{L^1(X,\mu)}^{q-1} \nu(\{sg<0\})^{\frac{1}{q'}} \|g\|_{L^q(Y,\nu)} \\
&\leqslant 2c^q a^{q-1} b \|f\|_{L^1(X,\mu)}^{q-1} \nu(\{sg<0\})^{\frac{1}{q'}} \|f\|_{L^p(X,\mu)}.
\end{aligned}
$$

Cancelling one power of $\|f\|_{L^p(X,\mu)}$ (which is allowed since f is nonzero), taking the $(q-1)$-th root on both sides, and plugging the resulting estimate into (5.1), we finally obtain:

$$
\|f\|_{L^1(X,\mu)} \leqslant a b^{\frac{q'}{q}} (2c)^{q'} \mu(\{f<0\})^{\frac{1}{p'}} \nu(\{sg<0\})^{\frac{1}{q}} \|f\|_{L^1(X,\mu)},
$$

from where (1.4) follows at once. □

5.2. **Proof of Theorem 1.4.**

Proof. Let $f \in \mathcal{A}_s(X) \setminus \{\mathbf{0}\}$. On the one hand,

$$
0 \geqslant \widehat{f}(0) = \int_X f \,\mathrm{d}\lambda \geqslant \int_{X \setminus B(\mathfrak{o}, r(f;X))} f \,\mathrm{d}\lambda - \int_{B(\mathfrak{o}, r(f;X))} |f| \,\mathrm{d}\lambda,
$$

and therefore

$$
\|f\|_{L^1(X)} \leqslant 2 \int_{B(\mathfrak{o}, r(f;X))} |f| \,\mathrm{d}\lambda \leqslant 2\lambda(B(\mathfrak{o}, r(f;X)))^{\frac{1}{2}} \|f\|_{L^2(X)}. \tag{5.2}
$$

On the other hand,

$$
0 \geqslant sf(\mathfrak{o}) = \sum_{n=0}^{\infty} s\widehat{f}(n)\varphi_n(\mathfrak{o}) \geqslant \sum_{n=k_s(\widehat{f})}^{\infty} s\widehat{f}(n)\|\varphi_n\|_{L^\infty(X)} - \sum_{n=0}^{k_s(\widehat{f})-1} |\widehat{f}(n)|\|\varphi_n\|_{L^\infty(X)}, \tag{5.3}
$$

where in the latter estimate we used the facts that $\varphi_n(\mathfrak{o}) = \|\varphi_n\|_{L^\infty(X)}$, for all $n \in \mathbb{N}$, and $s\widehat{f}(n) \geqslant 0$ if $n \geqslant k_s(\widehat{f})$. We also have that

$$
|\widehat{f}(n)| = \left| \int_X f\varphi_n \,\mathrm{d}\lambda \right| \leqslant \|f\|_{L^1(X)} \|\varphi_n\|_{L^\infty(X)},
$$

and therefore

$$
\begin{aligned}
\|f\|_{L^2(X)}^2 &= \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \\
&\leqslant \|f\|_{L^1(X)} \sum_{n=0}^{\infty} |\widehat{f}(n)|\|\varphi_n\|_{L^\infty(X)} \\
&\leqslant 2\|f\|_{L^1(X)} \sum_{n=0}^{k_s(\widehat{f})-1} |\widehat{f}(n)|\|\varphi_n\|_{L^\infty(X)}
\end{aligned}
$$

$$\leqslant 2\|f\|_{L^1(X)}\|f\|_{L^2(X)}\left(\sum_{n=0}^{k_s(\widehat{f})-1}\|\varphi_n\|^2_{L^\infty(X)}\right)^{\frac{1}{2}}.$$

From the second to the third lines, we appealed to (5.3). Cancelling one power of $\|f\|_{L^2(X)}$ from both sides, and plugging the resulting estimate into (5.2), yields (1.10). □

5.3. **Proof of Theorem 2.3.**

Proof. The strategy is to establish identity (2.5), and then invoke Theorem 2.13. With this purpose in mind, let $f \in \mathcal{B}_s(\mathbb{S}^{d-1}) \setminus \{\mathbf{0}\}$, and let $\mathrm{SO}_\eta(d) \subseteq \mathrm{SO}(d)$ denote the subgroup of rotations which fix the north pole $\eta \in \mathbb{S}^{d-1}$, equipped with Haar probability measure γ. Consider the *partially radialized* function $g : \mathbb{S}^{d-1} \to \mathbb{R}$, defined by

$$g(\omega) = \int_{\mathrm{SO}_\eta(d)} f(\rho\,\omega)\,\mathrm{d}\gamma(\rho). \tag{5.4}$$

One easily checks that g is continuous, $sg(\eta) = sf(\eta) \leqslant 0$, and that $\theta(g) \leqslant \theta(f)$. Note that the possibility that $g \equiv 0$ cannot be excluded, so we split the analysis into two cases.

First we consider the case when g is nonzero. Set $\nu = \frac{d}{2} - 1$, and let $Z_n(\omega) := C_n^\nu(\langle \omega, \eta\rangle)$ denote the zonal harmonic of degree n. Here, C_n^ν is the Gegenbauer polynomial of degree n; see (2.10). If $d \geqslant 3$, then $\frac{n+\nu}{\nu} C_n^\nu(\langle\cdot,\cdot\rangle)$ is the reproducing kernel of $\mathcal{H}_n^d$ with respect to the normalized surface measure on $\mathbb{S}^{d-1}$; see [11, Def. 1.2.2 and Theorem 1.2.6]. Consequently,

$$\int_{\mathrm{SO}_\eta(d)} P(\rho\,\omega)\,\mathrm{d}\gamma(\rho) = P(\eta)\frac{Z_n(\omega)}{Z_n(\eta)}, \text{ for every } P \in \mathcal{H}_n^d. \tag{5.5}$$

To verify identity (5.5), one checks that the left-hand side depends on ω only through its inner product with the north pole, invokes [11, Lemma 1.7.1], and sets $\omega = \eta$ to compute the leading constant on the right-hand side. It follows from (2.1), (5.4), (5.5) that

$$g(\omega) = \sum_{n=0}^{\infty} a_n Z_n(\omega), \text{ where } a_n := \sum_{j=1}^{h_n} \widehat{f}(n,j)\frac{Y_{n,j}(\eta)}{Z_n(\eta)}.$$

From (2.9) and (2.10), we have that $Z_n(\eta) = C_n^\nu(1) = \binom{n+2\nu-1}{n} > 0$, and since the basis $\{Y_{n,j}\}$ is signed, it follows that $sa_n \geqslant 0$, for every $n \geqslant k_s(\widehat{f})$. Set $G(x) := g(\omega)$, where $x = \langle \omega, \eta\rangle$. The function $G : [-1,1] \to \mathbb{R}$ is continuous, and satisfies $sG(1) = sg(\eta) \leqslant 0$. Moreover, for every $x \in [-1, \cos(\theta(f))]$, we have that $G(x) = \sum_{n=0}^\infty a_n C_n^\nu(x) \geqslant 0$, where $sa_n \geqslant 0$, for every $n \geqslant k_s(\widehat{f})$. As a consequence, we obtain the following lower bound:

$$(1-\cos(\theta(f)))k_s(\widehat{f})^2 \geqslant \mathbb{B}_s(I; \nu - \tfrac{1}{2}, \nu - \tfrac{1}{2}). \tag{5.6}$$

If $g \equiv 0$, then $a_n = 0$ for all $n \geqslant 0$, and since $Y_{n,j}(\eta) > 0$ for all sufficiently large n, we also have that $\widehat{f}(n,j) = 0$ for all sufficiently large n. Hence f is a polynomial. In turn, this implies $\theta(f)=\pi$, for otherwise f would have to vanish identically on the spherical cap $\{\omega\in\mathbb{S}^{d-1}:\theta(f)<d_g(\omega,\eta)\leqslant\pi\}$, which cannot happen unless f were the zero polynomial. This shows that $(1-\cos(\theta(f)))k_+(\widehat{f})^2\geqslant 2$ and[10] $(1-\cos(\theta(f)))k_-(\widehat{f})^2\geqslant 8$. On the other hand, the functions

$$f_+(\omega)=-1+\frac{C_1^\nu(x)}{C_1^\nu(1)},\quad f_-(\omega)=-\frac{C_1^\nu(x)}{C_1^\nu(1)}+\frac{C_2^\nu(x)}{C_2^\nu(1)},$$

respectively belong to $\mathcal{B}_+(\mathbb{S}^{d-1})$, $\mathcal{B}_-(\mathbb{S}^{d-1})$ as functions of ω, and respectively belong to $\mathcal{B}_+(I;\nu-\frac12,\nu-\frac12)$, $\mathcal{B}_-(I;\nu-\frac12,\nu-\frac12)$ as functions of $x=\langle\omega,\eta\rangle$. They also satisfy $(1-\cos(\theta(f_+)))k_+(\widehat{f_+})^2=2$ and $(1-\cos(\theta(f_-)))k_-(\widehat{f_-})^2=8$, hence (5.6) still holds. This also establishes the upper bounds in (2.4). We conclude that $\mathbb{B}_s(\mathbb{S}^{d-1})^2\geqslant\mathbb{B}_s(I;\nu-\frac12,\nu-\frac12)$. Conversely, given a function F in $\mathcal{B}_s(I;\nu-\frac12,\nu-\frac12)$, then $f:=F(\langle\cdot,\eta\rangle)$ belongs to $\mathcal{B}_s(\mathbb{S}^{d-1})$, and satisfies

$$(1-\cos(\theta(f)))^{\frac12}k_s(\widehat{f})=r(F;I)^{\frac12}k_s(\widehat{F}).$$

This shows that $\mathbb{B}_s(\mathbb{S}^{d-1})^2\leqslant\mathbb{B}_s(I;\nu-\frac12,\nu-\frac12)$, and therefore (2.5) holds.

Theorem 2.13 then implies the following lower bound:

$$\begin{aligned}\mathbb{B}_s(\mathbb{S}^{d-1})&=\mathbb{B}_s(I;\nu-\tfrac12,\nu-\tfrac12)^{\frac12}\\&\geqslant\left[\frac{\Gamma(\nu+\frac32)^{\frac{2}{\nu+1/2}}}{(4e^{\frac{1}{12}})^{\frac{2}{\nu+1/2}}(\nu+\frac12)(\nu+\frac32)}\right]^{\frac12}=\frac{2\Gamma(\frac{d+1}{2})^{\frac{2}{d-1}}}{(4e^{\frac{1}{12}})^{\frac{2}{d-1}}(d^2-1)^{\frac12}}.\end{aligned}$$

This concludes the proof of the theorem. ∎

5.4. **Proof of Theorem 2.13.**

Proof. Let $\alpha\geqslant\beta\geqslant-\frac12$. Consider the interval $I=[-1,1]$, equipped with the restricted Euclidean metric d and the probability measure $w_{\alpha,\beta}$. Then $(I,d,w_{\alpha,\beta})$ is an admissible space in the sense of Definition 1.2, with $\mathfrak{o}=1$. Indeed, if $\alpha=\max\{\alpha,\beta\}\geqslant-\frac12$, then from [32, Theorem 7.32.1] and (2.9) it follows that

$$\max_{-1\leqslant x\leqslant1}|P_n^{(\alpha,\beta)}(x)|=P_n^{(\alpha,\beta)}(1),\tag{5.7}$$

and therefore the orthogonal basis $\{p_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ of $L^2(I)$ satisfies (1.6) with $\mathfrak{o}=1$.

Moreover, the class $\mathcal{A}_s(I)$ from Definition 1.3 coincides with the class $\mathcal{B}_s(I;\alpha,\beta)$ from Definition 2.12. To see why this is the case, note that (5.7) and the second condition required by Definition 1.3 together imply that

$$\sum_{n=0}^\infty|\widehat{f}(n)|p_n^{(\alpha,\beta)}(1)<\infty.\tag{5.8}$$

[10] Recall that, by the discussion preceding the statement of Theorem 1.4, we must have $k_-(\widehat{f})\geqslant 2$.

Therefore the series (2.12) converges absolutely and uniformly, and the function f is continuous. This shows that $\mathcal{A}_s(I) \subseteq \mathcal{B}_s(I;\alpha,\beta)$. Conversely, the sequence $\{s\widehat{f}(n)\}_{n\in\mathbb{N}}$ being eventually nonnegative implies that (5.8) holds if and only if $\sum_{n=0}^{\infty} \widehat{f}(n)p_n^{(\alpha,\beta)}(1) < \infty$, which in turn is equivalent to the limit $\lim_{r\to 1^-} \sum_{n=0}^{\infty} \widehat{f}(n)p_n^{(\alpha,\beta)}(1)r^n$ existing and being finite. The latter limit exists and equals $f(1)$ since the power series of any real-valued, continuous function on I is Abel summable. It follows that $\mathcal{A}_s(I) = \mathcal{B}_s(I;\alpha,\beta)$, as claimed.

From Theorem 1.4, it then follows directly that

$$\left(\int_{1-r(f;I)}^{1} w_{\alpha,\beta}(x)\,\mathrm{d}x\right)\sum_{n=1}^{k_s(\widehat{f})} \frac{P_{n-1}^{(\alpha,\beta)}(1)^2}{h_{n-1}^{(\alpha,\beta)}} \geqslant \frac{1}{16}. \tag{5.9}$$

To estimate the left-hand side of (5.9), start by noting that the confluent form of the Christoffel–Darboux formula for Jacobi polynomials (see [32, (4.5.8)]) implies that

$$\sum_{n=1}^{k_s(\widehat{f})} \frac{P_{n-1}^{(\alpha,\beta)}(1)^2}{h_{n-1}^{(\alpha,\beta)}} = \frac{\Gamma(\alpha+k_s(\widehat{f})+1)\Gamma(\alpha+\beta+k_s(\widehat{f})+1)\Gamma(\beta+1)}{\Gamma(\alpha+2)\Gamma(k_s(\widehat{f}))\Gamma(\beta+k_s(\widehat{f}))\Gamma(\alpha+\beta+2)}. \tag{5.10}$$

A version of Stirling's formula for the Gamma function [30] states that

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}e^{\mu(x)}, \text{ for every } x > 0,$$

where the function μ satisfies the two-sided inequality $\frac{1}{12x+1} < \mu(x) < \frac{1}{12x}$. Moreover, it is elementary to check that

$$\left(1+\frac{a}{x}\right)^x \leqslant \exp(a), \text{ for every } a, x \geqslant 0.$$

In particular, if $x \geqslant y \geqslant -1, k \geqslant 1$, then we may estimate:

$$\begin{aligned}\frac{\Gamma(k+x+1)}{\Gamma(k+y+1)} &\leqslant e^{\frac{1}{12}}\frac{(k+x+1)^{k+x+\frac{1}{2}}e^{-k-x-1}}{(k+y+1)^{k+y+\frac{1}{2}}e^{-k-y-1}}\\ &= e^{\frac{1}{12}}e^{y-x}(k+x+1)^{x-y}\left(1+\frac{x-y}{k+y+1}\right)^{k+y+\frac{1}{2}}\\ &\leqslant e^{\frac{1}{12}}(k+x+1)^{x-y} \leqslant e^{\frac{1}{12}}k^{x-y}(x+2)^{x-y}.\end{aligned}$$

Applying the latter estimate (twice) to (5.10), with $k = k_s(\widehat{f})$, yields

$$\begin{aligned}&\frac{\Gamma(\alpha+k_s(\widehat{f})+1)\Gamma(\alpha+\beta+k_s(\widehat{f})+1)\Gamma(\beta+1)}{\Gamma(\alpha+2)\Gamma(k_s(\widehat{f}))\Gamma(\beta+k_s(\widehat{f}))\Gamma(\alpha+\beta+2)}\\ &\qquad\leqslant \frac{e^{\frac{1}{6}}(\alpha+2)^{\alpha+1}(\alpha+\beta+2)^{\alpha+1}\Gamma(\beta+1)}{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)}k_s(\widehat{f})^{2\alpha+2}.\end{aligned} \tag{5.11}$$

On the other hand, a crude estimate together with identity (2.11) yield

$$\int_{1-r(f;I)}^{1} w_{\alpha,\beta}(x)\,\mathrm{d}x \leqslant c_{\alpha,\beta}2^{\beta}\int_{1-r(f;I)}^{1}(1-x)^{\alpha}\,\mathrm{d}x = \frac{1}{2^{\alpha+1}}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+1)}r(f;I)^{\alpha+1}. \quad (5.12)$$

The lower bound in (2.14) now follows from (5.9), (5.10), (5.11), (5.12). Since the upper bounds were already established via (2.15), this concludes the proof of the theorem. □

5.5. **Proof of Proposition 2.14.**

Proof. We split the proof into the cases $s \in \{+,-\}$.

Case $s=-1$. Let $f \in \mathcal{B}_-(I;\alpha,\beta)\setminus\{\mathbf{0}\}$, and consider the auxiliary polynomial g_-,

$$g_-(x) = \frac{(1-x_{1,n})}{p_n^{(\alpha,\beta)}(1)^2}\frac{p_n^{(\alpha,\beta)}(x)^2}{(x-x_{1,n})},$$

where $x_{1,n}$ denotes the largest zero[11] of $p_n^{(\alpha,\beta)}$. Clearly, $g_-(1)=1$, $g_-(x)\leqslant 0$ if $-1\leqslant x\leqslant x_{1,n}$, and $\widehat{g}_-(0)=0$ (since $p_n^{(\alpha,\beta)}$ is orthogonal to all polynomials of degree less than n). We claim that $\widehat{g}_-(n)\geqslant 0$, for all $n\geqslant 1$. Indeed, [15, Theorem] states that, for all $m,n\geqslant 0$,

$$p_n^{(\alpha,\beta)}(x)p_m^{(\alpha,\beta)}(x) = \sum_{j=0}^{m+n} R(\alpha,\beta,j)p_j^{(\alpha,\beta)}(x),$$

where $R(\alpha,\beta,j)\geqslant 0$, for $j=0,\dots,m+n$. Moreover, [8, Theorem 3.1] implies that the Jacobi expansion of the polynomial

$$x \mapsto \frac{p_n^{(\alpha,\beta)}(x)}{\prod_{j=1}^{\ell}(x-x_{j,n})}, \quad (1\leqslant \ell\leqslant n)$$

has nonnegative coefficients. Together these results directly imply the claim. Since, for any fixed ℓ, $x_{\ell,n}\to 1$ as $n\to\infty$, one can set $F_- := f - f(1)g_-$, and check that $F_- \in \mathcal{B}_-^0(I;\alpha,\beta)\setminus\{\mathbf{0}\}$, $k_-(\widehat{F}_-)=k_-(\widehat{f})$, $r(F_-;I)<r(f;I)$, provided n is chosen sufficiently large.

Case $s=+1$. Let $f\in\mathcal{B}_+(I;\alpha,\beta)\setminus\{\mathbf{0}\}$, and consider the auxiliary polynomial g_+,

$$g_+(x) = \frac{(1-x_{1,n})(1-x_{2,n})}{p_n^{(\alpha,\beta)}(1)^2}\frac{p_n^{(\alpha,\beta)}(x)^2}{(x-x_{1,n})(x-x_{2,n})}.$$

Similarly to the case $s=-1$, we have that $g_+(1)=1$, $g_+(x)\geqslant 0$ if $-1\leqslant x\leqslant x_{2,n}$, $\widehat{g}_+(0)=0$, and $\widehat{g}_+(n)\geqslant 0$ for all $n\geqslant 1$. Letting $F_+ := f-f(1)g_+$, we check that $F_+\in\mathcal{B}_+^0(I;\alpha,\beta)\setminus\{\mathbf{0}\}$, satisfies $k_+(\widehat{F}_+)=k_+(\widehat{f})$, $r(F_+;I)<r(f;I)$, provided n is chosen sufficiently large. □

5.6. **Proof of Theorem 2.16.** We present the proof for the polynomial P only, since it proceeds analogously for Q. For simplicity, we write $x_0=1$ and $\{x_m < \dots < x_1\}\subset(-1,1)$

[11]More generally, let $-1<x_{n,n}<x_{n-1,n}<\ldots<x_{1,n}<1$ denote the zeros of the polynomial $p_n^{(\alpha,\beta)}$.

for the zeros of the polynomial $p_m^{(\alpha+1,\beta)}$. The crux of the matter boils down to the following simple result.

Lemma 5.1. *Let $f \in \mathcal{B}_s(I;\alpha,\beta)\setminus\{\mathbf{0}\}$ be a polynomial of degree at most $2m$, and further assume that $f(1)=0$ if $s=+1$. Then $r(f;I)\geqslant 1-x_1$, where equality is attained if and only if f is a positive multiple of the polynomial P in* (2.16)*.*

Proof of Lemma 5.1. Aiming at a contradiction, assume that $r(f;I)<1-x_1$. Then $f(x)\geqslant 0$ if $-1\leqslant x\leqslant x_1$, whence

$$0\leqslant \lambda_0 f(1)+\sum_{j=1}^{m}\lambda_j f(x_j)=\int_{-1}^{1} f(x)w_{\alpha,\beta}(x)\,\mathrm{d}x=\widehat{f}(0)\leqslant 0.$$

Thus $f(x_j)=0$ for $j=0,\dots,m$, and $f'(x_j)=0$ for $j=1,\dots,m$. Moreover, f necessarily vanishes at $x=1-r(f;I)$. We conclude that $\deg(f)\geqslant 2m+2$, which is absurd. The preceding argument further shows that if $r(f;I)=1-x_1$, then f must coincide with a positive multiple of the polynomial (2.16). ☐

Proof of Theorem 2.16. Set $k:=k_s(\widehat{P})$. Note that $k\geqslant 2$, and that $s\widehat{P}(k-1)<0$. Moreover, since P is monic of degree $2m$, then $k=2m+1$ if $s=-1$. Set $\delta:=-\frac{1}{2}s\widehat{P}(k-1)$, and let $h\in\mathcal{B}_s^0(I;\alpha,\beta)\setminus\{\mathbf{0}\}$ be such that $\|ch-P\|_{L^\infty(I)}<\delta$, for some $c>0$. Estimate:

$$|c\widehat{h}(k-1)-\widehat{P}(k-1)|\leqslant \|ch-P\|_{L^2(I)}\leqslant \|ch-P\|_{L^\infty(I)}<\delta=-\frac{1}{2}s\widehat{P}(k-1).$$

Thus $sc\widehat{h}(k-1)<\frac{1}{2}s\widehat{P}(k-1)<0$, and $k_s(\widehat{h})\geqslant k$. Lemma 5.1 implies that if h is not a multiple of P (i.e. $\inf_{c>0}\|ch-P\|_{L^\infty(I)}>0$), then $r(P;I)<r(h;I)$. Therefore $r(P;I)k_s(\widehat{P})^2<r(h;I)k_s(\widehat{h})^2$, as desired. ☐

6. Numerical Evidence

6.1. **Discrete Fourier Transform.** Conjecture 3.4 implies the existence of a well-defined jump function $k\mapsto q_s(k)$, which records the smallest value of q for which (k,q) is s-feasible but $(k-1,q)$ is not; in other words, $k=\mathbb{A}^{\mathrm{disc}}(q_s(k))$, and no other $q<q_s(k)$ has this property. We strongly believe that the first few values of $q_s(k)$ coincide with the ones displayed in Table 1, although we cannot claim its correctness in any rigorous way since all the computations were performed using floating-point arithmetic. In the case $s=-1$, the pattern of $q_s(k)$ in Table 1 is easy to guess, since for $k>3$ it is in perfect accordance with the sequence

$$\left\lfloor\frac{k^2-2k+2}{2}\right\rfloor_{k\geqslant 4}=5,8,13,18,25,32,41,50,61,72,\dots.$$

In the case $s=+1$, the pattern is not so easy to guess, although it seems to grow quadratically with k. Surprisingly, typing the numbers $6,14,25,40,58$ into the *On-Line Encyclopedia* *of Integer Sequences* [28] returns precisely one hit, which reveals that our numerical approximation of $q_+(k)$ agrees for $k \in \{3,4,5,6,7\}$ with

$$\lfloor (k-1)^2 \varphi \rfloor_{k \geqslant 3} = 6, 14, 25, 40, 58, 79, 103, 131, 161, 195, \dots, \tag{6.1}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio. Unfortunately, this coincidence stops at $k = 7$, and from then onwards our numerical value of $q_+(k)$ seems to be slightly larger than that of (6.1). Nevertheless, one can check that, for $8 \leqslant k \leqslant 43$, we have $q_+(k-1) < \lfloor (k-1)^2 \varphi \rfloor < q_+(k)$, which means that $k = \mathbb{A}^{\text{disc}}(\lfloor (k-1)^2 \varphi \rfloor) + 1$ for $8 \leqslant k \leqslant 43$, in support of Conjecture 3.5. Moreover, the first author together with Henry Cohn and David de Laat have unpublished numerical data in strong support of an upper bound for $\mathbb{A}_+(1)$ which starts with $0.558\dots$ The function attaining the latter bound is a polynomial multiple of a Gaussian, and exhibits a shape which is remarkably akin to the plot in Figure 1; in particular, it appears to vanish identically in similar intervals. Analogously, plotting the minimizer $f(x) = \frac{\sin^2(\pi x)}{(\pi x)^2} - (1-|x|)_+$ attaining $\mathbb{A}_-(1) = 1$ against the corresponding discrete approximation yields Figure 2. It is worth pointing out that, since $q_s(k)$ seems to grow quadratically with k, the error of $k(2q_s(k)+1)^{-\frac{1}{2}}$ is of the order $O(k^{-1})$. Therefore, in order to obtain a 3-digit approximation of the limit of $k(2q_s(k)+1)^{-\frac{1}{2}}$, as $k \to \infty$, one would have to set $k \approx 10^3$ and run several linear programs with $q \approx 10^6$, which lies at the computational limit of what the current best linear programming solvers can accomplish in a reasonable time frame. For some reason which is unclear to us, the $+1$ uncertainty principle consistently seems to be computationally harder than the -1 uncertainty principle.

k	q_-	$\frac{k}{\sqrt{2q_-+1}}$	q_+	$\frac{k}{\sqrt{2q_++1}}$	k	q_-	$\frac{k}{\sqrt{2q_-+1}}$	q_+	$\frac{k}{\sqrt{2q_++1}}$
3	3	1.1339	6	0.8321	24	265	1.0415	871	0.5749
4	5	1.2060	14	0.7428	25	288	1.0408	948	0.5740
5	8	1.2127	25	0.7001	26	313	1.0383	1029	0.5730
6	13	1.1547	40	0.6667	27	338	1.0377	1113	0.5721
7	18	1.1508	58	0.6472	28	365	1.0356	1200	0.5714
8	25	1.1202	80	0.6305	29	392	1.0351	1291	0.5706
9	32	1.1163	104	0.6225	30	421	1.0333	1385	0.5699
10	41	1.0976	133	0.6120	31	450	1.0328	1482	0.5693
11	50	1.0945	164	0.6064	32	481	1.0312	1583	0.5686
12	61	1.0820	198	0.6023	33	512	1.0307	1687	0.5680
13	72	1.0796	236	0.5977	34	545	1.0294	1794	0.5675
14	85	1.0706	277	0.5943	35	578	1.0290	1904	0.5671
15	98	1.0687	322	0.5906	36	613	1.0277	2018	0.5666
16	113	1.0620	370	0.5878	37	648	1.0274	2135	0.5662
17	128	1.0604	420	0.5862	38	685	1.0263	2256	0.5657
18	145	1.0552	475	0.5837	39	722	1.0260	2379	0.5653
19	162	1.0539	533	0.5817	40	761	1.0250	2506	0.5650
20	181	1.0497	594	0.5800	41	800	1.0247	2637	0.5645
21	200	1.0487	658	0.5787	42	841	1.0238	2770	0.5642
22	221	1.0453	726	0.5772	43	882	1.0235	2907	0.5639
23	242	1.0444	797	0.5759					

TABLE 1. The table displays pairs $(k, q_-), (k, q_+)$ which are numerically -1- and $+1$-feasible, respectively. Recall that, according to Definition 3.2, a pair (k, q) is s-feasible if there exists $f \in \mathcal{A}_s^{\text{disc}}(q)$, such that $k_{f,s} \leqslant k$. We produced this table using *Gurobi* [22] with quad precision and barrier method; *Mathematica* [36] was used as an interface for *Gurobi*. We have checked numerically that, for any given pair $(k, q_\pm)$ from the table, the pairs $(k', q_s), (k, q_s')$ are always s-feasible, for any $k' \geqslant k$ and $q_s' \leqslant q_s$. We also verified numerically that the set of integers q, for which (k, q) is s-feasible but $(k-1, q)$ is not, coincides with the interval $[q_s(k), q_s(k+1)-1]$, where $k \mapsto q_s(k)$ is the function given by the table. Thus the table seems to indeed record the jumps of the function $q \mapsto \mathbb{A}_s^{\text{disc}}(q)$.

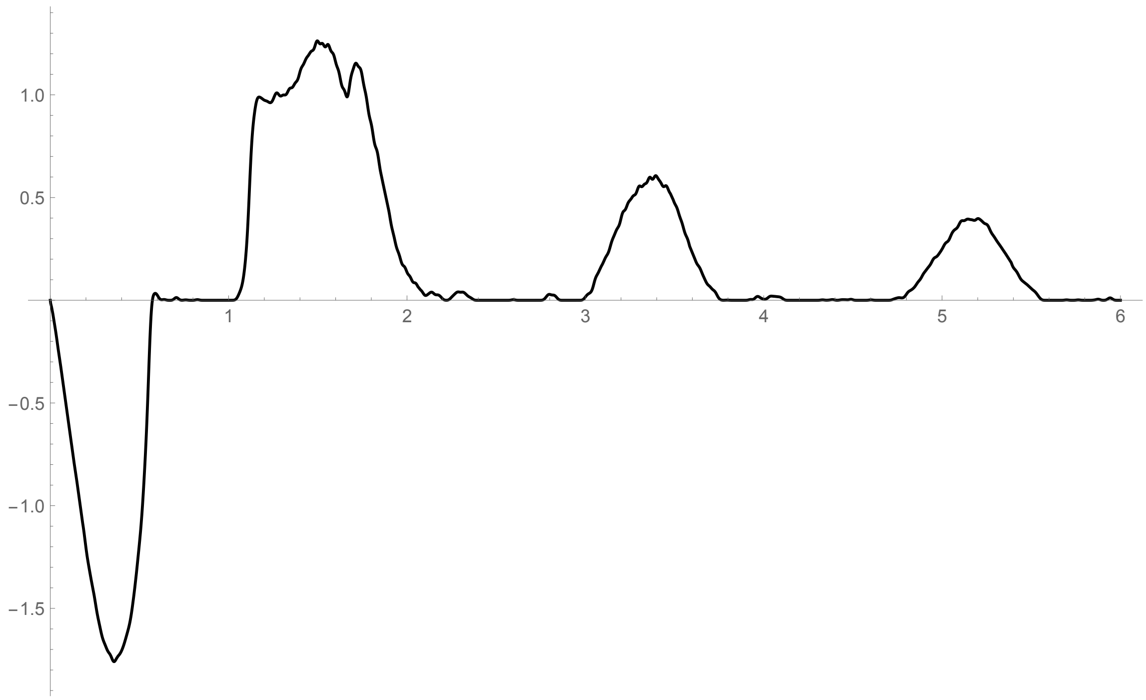

FIGURE 1. This is the plot of $F(x) = \frac{x^{\frac{1}{2}}}{2q+1} \sum_{n=-q}^{q} J(\frac{2n}{q+1}) f(n) e^{2\pi i n \frac{x}{\sqrt{2q+1}}}$, where $k_{f,+} = 60$, $q = 5\,692$, and the vector f is a feasible answer to Problem 3.1, as delivered by *Gurobi*'s linear programming solver. J is the Jackson kernel, used to significantly reduce the Gibbs phenomenon in F. It seems sensible to plot F in this way, since the entries of the vector $(f(n))_{n=-q}^{q}$ can be interpreted as the values of a function discretized at the nodes $x = n(2q+1)^{-\frac{1}{2}}$. One can only wonder whether the flatter areas in the plot indicate that minimizers for $\mathbb{A}_+(1)$ may vanish identically in certain intervals.

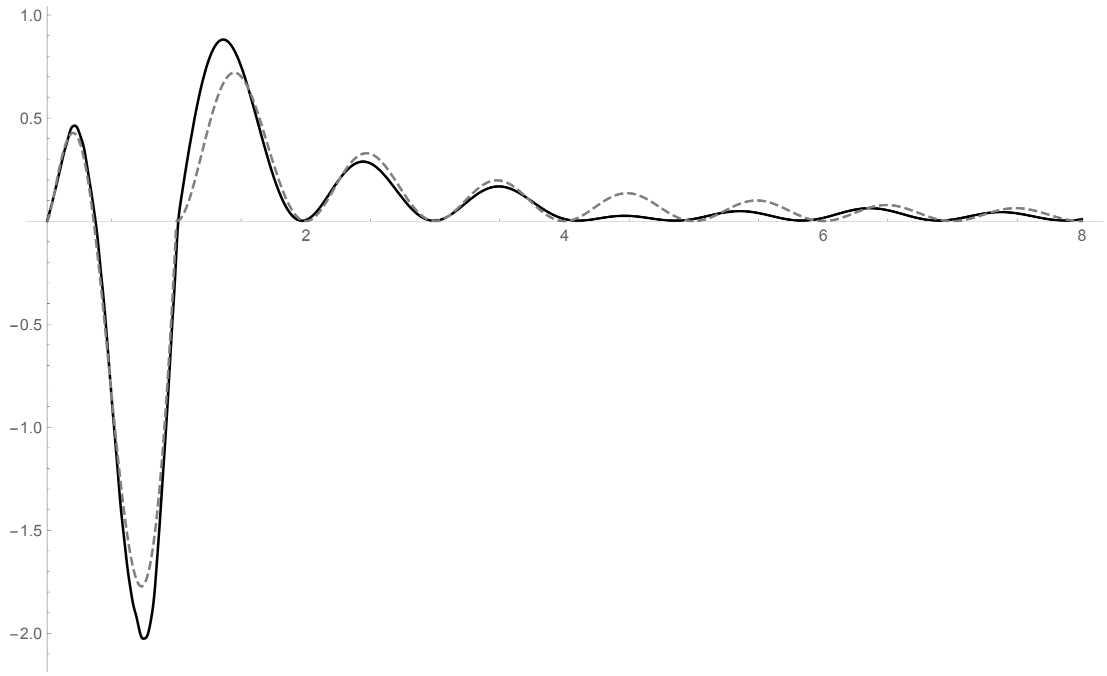

FIGURE 2. These are the plots of $G(x) = x^{\frac{1}{2}}\sin^2(\pi x)(\pi x)^{-2} - (1-|x|)_+$ (dashed) and $F(x) = \frac{x^{\frac{1}{2}}}{2q+1}\sum_{n=-q}^{q} J(\frac{2n}{q+1})f(n)e^{2\pi i n \frac{x}{\sqrt{2q+1}}}$ (thick), where $k_{f,-} = 105$, $q = \lfloor (k_{f,-}^2 - 2k_{f,-} + 2)/2 \rfloor = 5\,408$, and the vector f is a feasible answer to Problem 3.1, as delivered by *Gurobi*'s linear programming solver. J is the Jackson kernel, used to significantly reduce the Gibbs phenomenon in F. The plots of F and G are not too similar, hinting that F might be close to some unknown minimizer for $\mathbb{A}_-(1)$. We did try to plot other known minimizers G for $\mathbb{A}_-(1)$ against F, but this choice was by far the better fit. It is still possible that, for much larger values of q, the plots look more similar.

6.2. **Discrete Hankel Transform.** Tables 2 and 3 display numerical data relative to the sign uncertainty principles for the discrete Hankel transform. For each sign $s \in \{+,-\}$, dimension d, and parameter k, the pair (k, q_s) is numerically $(s, \frac{d}{2}-1)$-feasible, in the sense of Definition 3.6. We used floating-point arithmetic, and therefore we cannot claim these numbers to be correct in the proof theoretical sense, but we believe they are. We have checked numerically that, for any given pair (k, q_s) in these tables, the pairs $(k', q_s), (k, q_s')$ are always s-feasible, for any $k' \geqslant k$ and $q_s' \leqslant q_s$. We have also numerically verified that the set of integers q, for which (k, q) is $(s, \frac{d}{2}-1)$-feasible but $(k-1, q)$ is not, coincides with the interval $[q_s(k;d), q_s(k+1;d)-1]$, where $k \mapsto q_s(k;d)$ denotes the function given by Tables 2 and 3. Hence these tables seem to record the jumps of the function $q \mapsto \mathbb{A}_s^{\mathrm{disc}}(q, \frac{d}{2}-1)$.

It does not seem easy to detect any distinguishable patterns in the entries of Tables 2 and 3, except for the special cases $d \in \{2, 8, 24\}$ when $s = -1$, and $d = 12$ when $s = +1$. In these cases, one can indeed spot a pattern in the first few entries of the corresponding columns, which in turn motivated Conjecture 3.8. If $(s,d)=(-,2)$, then the sequence

$$\left\lfloor \frac{\sqrt{3}(k^2-2k+2)}{4} \right\rfloor_{k\geqslant 4} = 4,7,11,16,21,28,35,43,52,62,\ldots \tag{6.2}$$

matches the data from Table 2 for $k \in \{4,5,6,7,8\}$, and seems to be slightly below the values from that table if $k>8$. In particular, this means that $\left(k, \lfloor\frac{\sqrt{3}(k^2-2k+2)}{4}\rfloor\right)$ should be $(s, 2/1-1)$-feasible, for all $k \geqslant 4$. Similarly, if $(s,d)=(-,8),(-,24),(+,12)$ respectively, then the data match the sequences

$$\begin{aligned} \left\lfloor \frac{k^2}{4} \right\rfloor_{k\geqslant 4} &= 4,6,9,12,16,20,25,30,36,42,\ldots, \\ \left\lfloor \frac{k^2+6k-8}{8} \right\rfloor_{k\geqslant 4} &= 4,5,8,10,13,15,19,22,26,29,\ldots, \\ \left\lfloor \frac{k^2+2k-1}{4} \right\rfloor_{k\geqslant 3} &= 3,5,8,11,15,19,24,29,35,41,\ldots, \end{aligned} \tag{6.3}$$

for $k \in \{4,5,6,7,8,9,10,11,12\}$, $k \in \{4,5,6,7,8\}$, and $k \in \{3,4,5,6,7,8,9,10,11\}$.

Similarly to what was already observed in §6.1, the $+1$ problem seems to be computationally harder than the -1 problem. Nevertheless, one can check that the sequences in (6.2) and (6.3) always belong to the interval $(q_s(k-1;d), q_s(k;d)]$ for $k \leqslant 30$ and $(s,d)=(-,2),(-,8),(-,24),(+,12)$, respectively. This means that $k-1$ coincides with the quantities

$$\mathbb{A}_-^{\mathrm{disc}}\left(\left\lfloor \frac{\sqrt{3}(k^2-2k+2)}{4} \right\rfloor, \frac{2}{2}-1\right), \quad \mathbb{A}_-^{\mathrm{disc}}\left(\left\lfloor \frac{k^2}{4} \right\rfloor, \frac{8}{2}-1\right),$$
$$\mathbb{A}_-^{\mathrm{disc}}\left(\left\lfloor \frac{k^2+6k-8}{8} \right\rfloor, \frac{24}{2}-1\right), \quad \mathbb{A}_+^{\mathrm{disc}}\left(\left\lfloor \frac{k^2+2k-1}{4} \right\rfloor, \frac{12}{2}-1\right),$$

and provides further evidence towards Conjecture 3.8.

k \ d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	7	7	7	7	6	6	6	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5
6	11	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	16	15	15	14	13	13	12	12	11	11	11	11	11	11	11	11	11	11	11	11	11	11	10
8	21	19	18	17	17	16	16	15	15	15	15	14	14	14	14	14	14	14	14	14	14	13	13
9	29	27	25	23	22	21	20	20	19	19	19	18	18	18	18	17	17	17	17	17	17	17	16
10	35	31	29	28	27	26	25	24	24	23	23	23	22	22	22	21	21	21	21	20	20	20	20
11	45	41	38	35	33	31	30	30	29	28	27	27	26	26	26	25	25	25	25	24	24	24	24
12	53	47	43	41	39	38	36	35	34	33	33	32	31	31	30	30	29	29	29	28	28	28	28
13	64	58	53	49	46	44	43	41	40	39	38	37	36	36	35	35	34	33	33	33	32	32	32
14	74	65	60	57	54	52	49	48	46	45	44	43	42	41	40	40	39	38	38	37	37	36	36
15	87	79	72	66	62	59	57	55	53	52	50	49	48	47	46	45	44	44	43	42	42	41	41
16	98	87	80	75	71	68	65	62	60	58	57	55	54	53	52	51	50	49	48	48	47	46	46
17	114	102	93	85	80	76	73	70	68	66	64	62	61	59	58	57	56	55	54	53	52	52	51
18	126	111	102	96	90	86	82	79	76	73	71	69	67	66	64	63	62	61	60	59	58	57	56
19	143	129	117	107	101	96	91	88	84	81	79	77	75	73	71	70	68	67	66	65	64	63	62
20	157	139	128	119	112	106	101	97	93	90	87	85	82	80	78	77	75	74	72	71	70	69	68
21	177	158	143	132	124	117	112	107	103	99	96	93	90	88	86	84	82	81	79	78	76	75	74
22	192	169	155	145	136	129	123	117	113	108	105	102	99	96	94	92	90	88	86	85	83	82	80
23	213	191	173	159	149	141	134	128	123	118	114	111	108	105	102	99	97	95	94	92	90	89	87
24	231	203	186	173	162	153	146	139	134	129	124	120	117	113	111	108	105	103	101	99	97	96	94
25	254	227	205	188	176	166	158	151	145	139	134	130	126	122	119	116	114	111	109	107	105	103	101
26	272	240	220	204	191	180	171	163	156	150	145	140	136	132	128	125	122	120	117	115	112	111	109
27	297	266	239	220	206	194	184	176	168	162	156	151	146	142	138	134	131	128	126	123	121	118	116
28	318	280	256	237	222	209	198	189	181	174	167	162	157	152	148	144	140	137	134	132	129	127	124
29	344	308	277	255	238	224	213	203	194	186	179	173	168	163	158	154	150	147	143	140	138	135	133
30	367	323	295	273	255	240	228	217	207	199	191	185	179	173	168	164	160	156	153	149	146	144	141

TABLE 2. Numerical data for the discrete Hankel transform -1 uncertainty principle. If q_- is an entry in the table, then (k, q_-) is numerically $(-1, \frac{d}{2}-1)$-feasible. The *Gurobi* solver [22] was used with *Mathematica* [36] as interface.

k \ d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	9	7	7	6	6	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	4
5	14	12	10	9	9	8	8	8	8	8	8	8	7	7	7	7	7	7	7	7	7	7	7
6	24	18	16	15	14	13	12	12	12	11	11	11	11	10	10	10	10	10	10	10	10	10	10
7	33	24	21	19	18	17	17	16	16	15	15	15	14	14	14	14	13	13	13	13	13	13	13
8	46	35	30	27	24	23	22	21	20	20	19	19	18	18	18	18	17	17	17	17	16	16	16
9	58	42	36	33	31	29	28	27	26	25	24	23	23	22	22	22	21	21	21	21	20	20	20
10	75	56	48	42	39	36	34	33	31	30	29	29	28	27	27	26	26	25	25	25	24	24	24
11	90	66	56	51	47	44	42	40	38	37	35	34	33	33	32	31	31	30	30	29	29	28	28
12	111	82	70	61	56	52	50	47	45	43	42	40	39	38	37	37	36	35	35	34	33	33	32
13	129	94	80	72	66	62	58	55	53	50	49	47	46	44	43	42	41	40	40	39	39	38	37
14	153	114	97	85	77	72	67	64	61	58	56	54	52	51	50	48	47	46	45	44	44	43	43
15	175	127	109	97	89	83	77	73	70	66	64	62	60	58	56	55	54	52	51	50	49	48	48
16	203	150	128	111	102	94	88	83	79	75	72	69	67	65	63	62	60	59	58	56	55	54	54
17	229	166	141	126	115	106	99	93	89	85	81	78	75	73	71	69	67	65	64	63	62	61	60
18	260	192	163	142	129	119	111	105	99	95	90	87	84	81	79	77	75	73	71	69	68	67	66
19	289	209	178	159	144	133	124	116	110	105	100	96	93	90	87	84	82	80	78	77	75	74	72
20	324	238	202	177	160	147	137	129	122	116	111	106	102	99	96	93	90	88	86	84	82	81	79
21	355	258	220	195	177	163	151	142	134	127	122	117	112	108	105	102	99	96	94	92	90	88	86
22	395	290	245	215	194	179	166	155	147	139	133	127	122	118	114	111	108	105	102	100	97	96	94
23	430	311	265	235	213	195	181	170	160	152	145	139	133	128	124	120	117	114	111	108	106	103	101
24	472	347	293	257	232	213	197	185	174	165	157	150	145	139	134	130	126	123	120	117	114	112	109
25	511	370	315	279	252	231	214	200	188	179	170	163	156	150	145	140	136	133	129	126	123	120	118
26	558	409	345	302	273	250	231	216	204	193	183	175	168	162	156	151	147	142	139	135	132	129	126
27	600	434	369	326	294	269	249	233	219	207	197	188	181	174	168	162	157	153	149	145	141	138	135
28	649	476	401	352	317	290	268	250	235	223	212	202	194	186	180	174	168	163	159	155	151	148	144
29	695	503	427	378	340	311	288	268	252	238	227	216	207	199	192	186	179	174	170	165	161	157	154
30	748	548	462	405	364	333	308	287	270	255	242	231	221	212	205	198	191	186	181	176	171	167	163

TABLE 3. If q_+ is an entry in the table, then (k, q_+) is numerically $(+1, \frac{d}{2} - 1)$-feasible.

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