Points and lines configurations for perpendicular bisectors of convex cyclic polygons

Paul Melotti Sanjay Ramassamy Paul Thévenin

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Abstract

We characterize the topological configurations of points and lines that may arise when placing n points on a circle and drawing the n perpendicular bisectors of the sides of the corresponding convex cyclic n-gon. We also provide a functional central limit theorem describing the shape of a large realizable configuration of points and lines taken uniformly at random among realizable configurations.

1 Introduction

Let $n \geq 3$ and let $P_1, \dots, P_n$ be n distinct points on the unit circle, arranged in the positive cyclic order. For all i between 1 and n denote by L_i the perpendicular bisector of the segment $[P_i, P_{i+1}]$, with indices taken modulo n. We assume that the points are in *generic position*, meaning that these lines are all distinct and no point lies on a line. Then these n lines all go through the center of the circle, hence divide the plane into $2n$ regions. What are the configurations of points and lines that can be realized?

We number the regions in counterclockwise order, the first one being the one containing 1 (or, if 1 is on a boundary, $e^{i\varepsilon}$ for all $\varepsilon > 0$ small enough). For every $1 \leq i \leq 2n$, we set v_i to be the number of points inside the ith region. It is not hard to see that each region contains at most one point. The word $\underline{v} = (v_1, \dots, v_{2n}) \in \{0,1\}^{2n}$ is called the *occupancy word* of the collection of points $P_1, \dots, P_n$. In order to characterize the occupancy words that may arise as one varies the positions of the points, we introduce the notion of *signature* of a word in $\{0,1\}^{2n}$. If $\underline{v} = (v_1, \dots, v_{2n}) \in \{0,1\}^{2n}$ is an arbitrary word, its signature $\underline{\sigma} = (\sigma_1, \dots, \sigma_{2n}) \in \{0,1,2\}^{2n}$ is defined by $\sigma_i = v_i + v_{i+n}$ for every $1 \leq i \leq 2n$, with indices taken modulo $2n$. Clearly, it satisfies $\sigma_i = \sigma_{i+n}$. See an example in Figure 1.

We introduce a notion of cyclic interval of integers. Let $N \geq 1$ be an integer and let $1 \leq i, j \leq N$ be two integers. Define

$$I_N(i,j) = \begin{cases} \{i+1, i+2, \dots, j-1\} & \text{if } i \leq j \\ \{1, 2, \dots, j-1\} \cup \{i+1, i+2, \dots, N\} & \text{if } i > j. \end{cases}$$

A word $\underline{u} = (u_1, \dots, u_{2n}) \in \{0,1,2\}^{2n}$ is called *interlacing* if it satisfies the following two properties:

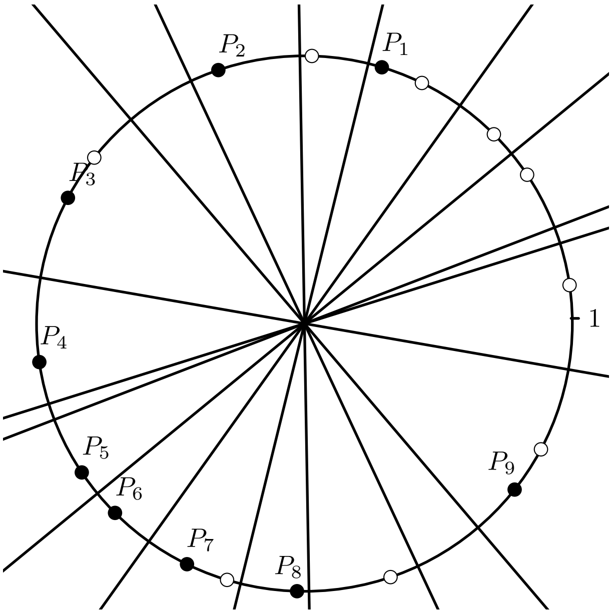

Figure 1: An example with $n = 9$. The black dots are the points $P_1, \dots P_9$ and the white dots are their symmetric relative to the center of the circle. Here $\underline{v} = (0,0,0,0,1,0,1,0,1,1,0,1,1,1,1,0,0,1)$ is the occupancy word (which counts the number of black dots in each region) and its signature is $\underline{\sigma} = (1,0,1,1,2,1,1,0,2,1,0,1,1,2,1,1,0,2)$.

1. there exist two distinct integers $1 \le i, j \le 2n$ such that $u_i = u_j = 0$;
2. for every pair of distinct integers with $1 \le i, j \le 2n$ such that $u_i = u_j = 0$ and $u_k \neq 0$ for all $k \in I_{2n}(i,j)$, there exists a unique $k_0 \in I_{2n}(i,j)$ such that $u_{k_0} = 2$.

Note that an interlacing word takes the values 0 and 2 an equal number of times. We can now characterize the words that may arise as the occupancy word of some collection of points $P_1, \dots, P_n$, we call such words *realizable.*

Theorem 1.1. *A word $\underline{u} = (u_1, \dots, u_{2n}) \in \{0,1\}^{2n}$ is realizable if and only if its signature is interlacing.*

Even in the case $n = 3$ the result seems to be new. We state a finer version of Theorem 1.1 in the case $n = 3$, the proof of which is omitted, since it may be given as an exercise to a bright elementary school student.

Proposition 1.2. *Let A, B, C be three points in the plane with $AB < BC < CA$. Then the three perpendicular bisectors of the triangle ABC divide the plane into six regions satisfying the following properties:*

- *A and B lie in two consecutive regions;*
- *the regions containing B and C are separated by one empty region;*
- *the regions containing A and C are separated by two empty regions.*

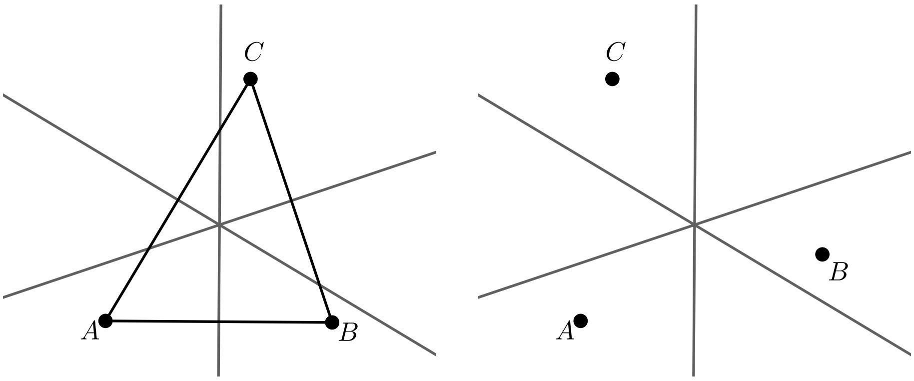

Figure 2: For $n = 3$, there is only one configuration up to cyclic shifts and reversal, shown on the left. The topological configuration on the right is impossible to achieve with lines being perpendicular bisectors of the segments.

In particular, Proposition 1.2 implies that the word $(0, 1, 0, 1, 0, 1)$ is not realizable, see Figure 2.

A *necklace* is an equivalence class of words up to cyclic shifts, and a *bracelet* is an equivalence class of necklaces up to reversal [2]. We denote by $\mathcal{W}_n$ (resp. $\mathcal{N}_n, \mathcal{B}_n$) the set of realizable words (resp. necklaces, bracelets) of length $2n$. We deduce the following asymptotic enumerative result from Theorem 1.1. It is to be compared with the total number of words of length $2n$ containing n ones and n zeros, which is $\binom{2n}{n} = 4^{n(1+o(1))}$.

Corollary 1.3. *The number of realizable words is*

$$\sharp\mathcal{W}_n = 3^n - 2^{n+1} + 1.$$

The exponential growth rate of the number of realizable bracelets and necklaces is equal to 3*, that is,*

$$\lim_{n\to\infty} \frac{1}{n} \log \sharp\mathcal{N}_n = \lim_{n\to\infty} \frac{1}{n} \log \sharp\mathcal{B}_n = 3.$$

The first few values of the sequence $(\mathcal{B}_n)_{n\geq 1}$ are listed in Table 1.

n	3	4	5	6	7	8	9	10
$\mathcal{B}_n$	1	5	9	30	69	203	519	1466

Table 1: First terms of the sequence $(\mathcal{B}_n)_{n\geq 3}$ counting the number of realizable configurations up to cyclic shifts and reversal.

We also study the typical shape of a realizable word, in the following sense. Let $\underline{w}^{(n)}$ be a random word taken uniformly in the set of realizable words of length $2n$. Consider the word $\underline{\hat{w}}^{(n)}$ of length n on the alphabet $\{00, 10, 01, 11\}$, whose letter in position i is the concatenation $w_i^{(n)} w_{i+n}^{(n)}$. For $c \in [0, 1]$ and $a \in \{00, 11, 10, 01\}$, denote by S^a_{cn} the number of letters a in $\underline{\hat{w}}^{(n)}$ between positions 0 and $\lfloor cn \rfloor$. Then the following holds:

Theorem 1.4. *(i) [Law of large numbers] The following holds in probability:*

$$\left(\frac{S_n^{00}}{n}, \frac{S_n^{11}}{n}, \frac{S_n^{10}}{n}, \frac{S_n^{01}}{n}\right) \underset{n\to\infty}{\to} \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right).$$

(ii) [Functional central limit theorem] We have the functional convergence :

$$\frac{2}{\sqrt{n}}\left(S_{cn}^{00} - \frac{cn}{6}, S_{cn}^{11} - \frac{cn}{6}, S_{cn}^{10} - \frac{cn}{3}, S_{cn}^{01} - \frac{cn}{3}\right)_{0\le c\le 1}$$
$$\underset{n\to\infty}{\overset{(d)}{\to}} \left(W_c^{(2)}, W_c^{(2)}, W_c^{(1)} - W_c^{(2)}, -W_c^{(1)} - W_c^{(2)}\right)_{0\le c\le 1}$$

where $W^{(1)}, W^{(2)}$ are two independent Brownian motions of respective variances $2/3$ and $2/9$.

Theorem 1.4 holds also if we replace a uniformly random realizable word by a uniformly random realizable necklace or bracelet (see Remark 5.4).

We emphasize that studying a realizable word taken uniformly at random is *a priori* very different from letting the points $P_1, \dots, P_n$ be i.i.d on the circle and studying their occupancy word. The latter may be deemed a more natural procedure, but we do not have any result in the vein of Theorem 1.4 for it so far.

Organization of the paper

In Section 2, we prove one direction of Theorem 1.1: the interlacement condition is necessary for a realizable word. The converse is proved in Section 3, using an explicit procedure to construct points from a word with interlacing signature. In Section 4 we enumerate realizable words and prove Corollary 1.3. Finally, Section 5 is concerned with the probabilistic aspect of a random realizable word, and the proof of Theorem 1.4.

2 Necessary condition for a realizable word

The unit circle may be identified to the half-open interval $(0, 1]$ via the inverse of the map $x \mapsto e^{2i\pi x}$. Denote by $p_1, \dots, p_n$ the n elements of $(0, 1]$ corresponding to $P_1, \dots, P_n$. For every $1 \le i \le n$ define $l_i = \frac{p_i+p_{i+1}}{2} \mod 1$, where the representative is taken to be in $(0, 1]$ and the indices are considered modulo n. Up to applying a rotation of the circle, one may assume that $l_n = 1$. Then we have

$$0 < p_1 < l_1 < p_2 < l_2 < \dots < l_{n-1} < p_n < l_n = 1.$$

Define also for every $1 \le i \le n$, $p_i' = p_i + \frac{1}{2} \mod 1$ and $l_i' = l_i + \frac{1}{2} \mod 1$. Write $\mathcal{P} = \{p_1, \dots, p_n\}$, $\mathcal{P}' = \{p_1', \dots, p_n'\}$, $\mathcal{L} = \{l_1, \dots, l_n\}$ and $\mathcal{L}' = \{l_1', \dots, l_n'\}$. Let $(m_i)_{1\le i\le 2n}$ be the reordering of the l_i and l_i', that is,

$$\{m_i\}_{1\le i\le 2n} = \mathcal{L} \cup \mathcal{L}'$$

and

$$0 < m_1 < m_2 < \dots < m_{2n-1} < m_{2n} = 1.$$

We also set $m_0 = 0$. Similarly, let $(q_i)_{1\leq i\leq 2n}$ be the reordering of $\mathcal{P}\cup\mathcal{P}'$. For any $1 \leq i \leq 2n$, the signature $\underline{\sigma}$ of the occupancy word associated to $\mathcal{P}$ satisfies

$$\sigma_i = \sharp\left([m_{i-1}, m_i] \cup [m_{i+n-1}, m_{i+n}]\right) \cap \mathcal{P},$$

with indices taken modulo $2n$. Note that for every $1 \leq i \leq n$, $\sigma_i = \sigma_{i+n} = \sharp[m_{i-1}, m_i] \cap (\mathcal{P}\cup\mathcal{P}')$.

For any $(a, b) \in (0, 1]^2$ define

$$d(a, b) = \min(|b - a|, 1 - |b - a|)$$

to be the distance between a and b measured on the circle obtained by identifying the two endpoints of the interval $[0, 1]$. We also introduce a notion of interval on the circle defined as follows. Let a and b be two elements of $(0, 1]^2$ and define

$$I(a, b) = \begin{cases} (a, b) & \text{if } a \leq b \\ (0, b) \cup (a, 1] & \text{if } a > b. \end{cases}$$

We also define the closed and half-closed intervals $I[a, b]$, $I[a, b)$, $I(a, b]$ in the obvious way.

Let $p \in \mathcal{P}$. We define $C(p)$ to be the element $x' \in \mathcal{P}'$ which minimizes $d(p, x')$. By the genericity assumption $C(p)$ is uniquely defined. Similarly, for any $p' \in \mathcal{P}'$, we define $C(p')$ to be the element $x \in \mathcal{P}$ which minimizes $d(p', x)$. For any $q \in \mathcal{P}\cup\mathcal{P}'$, when $C(q)$ belongs to $I(q, q + \frac{1}{2})$ (*resp.* $I(q - \frac{1}{2}, q)$), we say that q *looks* to its right (*resp.* left) and we denote it by $D(q) = R$ (*resp.* $D(q) = L$).

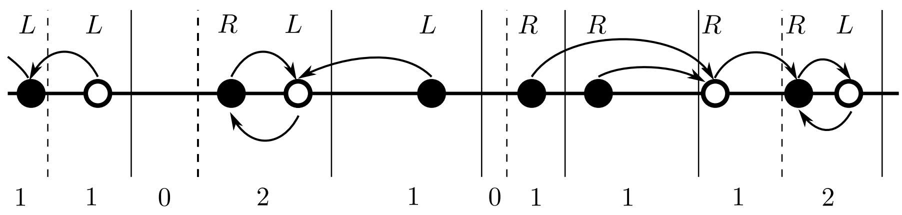

Figure 3: A configuration on a portion of $(0, 1]$. The black (resp. white) dots represent elements of $\mathcal{P}$ (resp. $\mathcal{P}'$), and the vertical solid (resp. dashed) lines represent elements of $\mathcal{L}$ (resp. $\mathcal{L}'$). From each dot q, the arrow is directed towards $C(q)$, and above it is written the value of $D(q)$. Below each region, the corresponding letter in $\underline{\sigma}$ is indicated.

In the remainder of this section, the indices of q and σ will be considered modulo $2n$ and the real numbers of the form $q-\frac{1}{2}$ and $q+\frac{1}{2}$ should be understood as the representative in $(0, 1]$ of an equivalence class modulo 1.

Our aim in this section is to prove the following.

Proposition 2.1. *Let $\underline{u}$ be the occupancy word associated to the collection of points $P_1, \ldots, P_n$. Then its signature $\underline{\sigma}$ is alternating.*

In the next two lemmas, we show that the occurrences of 2 in σ exactly correspond to occurrences of the pattern RL in the successive values of $(D(q_i))_{i=1,\ldots,2n}$.

Lemma 2.2. *Let $1 \leq i \leq 2n$ and assume that $D(q_i) = R$ and $D(q_{i+1}) = L$. Then exactly one of q_i and q_{i+1} belongs to $\mathcal{P}$ and we have $C(q_i) = q_{i+1}$ and $C(q_{i+1}) = q_i$.*

Proof. Up to performing a rotation of the circle, one may assume that $q_i < q_{i+1}$ (this is needed to take into account the case $i = 2n$). Reason by contradiction and assume that both q_i and q_{i+1} are in $\mathcal{P}$. Since $q_{i+1} \in \mathcal{P}$, we cannot have $C(q_i) = q_{i+1}$, and it follows from the fact that $I(q_i, q_{i+1}) \cap \mathcal{P}' = \varnothing$ that $C(q_i) \in I(q_{i+1}, q_{i+1} + \frac{1}{2})$. The element of $\mathcal{P}'$ in $I(q_{i+1}, q_{i+1} + \frac{1}{2})$ which is closest to q_i is also the closest to q_{i+1}, hence $C(q_{i+1}) = C(q_i)$, so that $C(q_{i+1})$ belongs to $I(q_{i+1} - \frac{1}{2}, q_{i+1}) \cap I(q_{i+1}, q_{i+1} + \frac{1}{2}) = \varnothing$, contradiction. Similarly, q_i and q_{i+1} cannot both be in $\mathcal{P}'$. The last two statements of the lemma follow from the fact that $I(q_i, q_{i+1}) \cap (\mathcal{P} \cup \mathcal{P}') = \varnothing$. □

Lemma 2.3. *Let $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$ be such that $p' = C(p)$ and $p = C(p')$. Let also $1 \leq i \leq 2n$ be such that $p \in [m_{i-1}, m_i]$. Then $\sigma_i = 2$. Conversely, let $1 \leq i \leq 2n$ be such that $\sigma_i = 2$ and denote by $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$ the two elements of $[m_{i-1}, m_i] \cap (\mathcal{P} \cup \mathcal{P}')$. Then $p' = C(p)$ and $p = C(p')$.*

Proof. Assume that $p' = C(p)$ and $p = C(p')$ and that i is such that $p \in [m_{i-1}, m_i]$. The point p is the element of $\mathcal{P}$ which is closest to p', hence no element of $\mathcal{L}$ can separate p' from p. Similarly, since p' is the element of $\mathcal{P}'$ which is closest to p, no element of $\mathcal{L}'$ can separate p from p'. Thus $p' \in [m_{i-1}, m_i]$ and $\sigma_i = 2$.

Conversely, let $1 \leq i \leq 2n$ be such that $\sigma_i = 2$ and denote by $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$ the two elements of $[m_{i-1}, m_i] \cap (\mathcal{P} \cup \mathcal{P}')$. If we had $C(p') \neq p$, then the perpendicular bisector of $C(p')$ and p would separate p' from p, which is not the case. So $C(p') = p$ and similarly $C(p) = p'$. □

In the next lemma, we show that the occurrences of 0 in σ exactly correspond to occurrences of LR in the successive values of $(D(q_i))_{i=1,\dots,2n}$.

Lemma 2.4. *Let $1 \leq i \leq 2n$ be such that $D(q_i) = L$ and $D(q_{i+1}) = R$. Then there exists a unique $1 \leq j \leq 2n$ such that $(m_j, m_{j+1}) \in I(q_i, q_{i+1})^2$, and this j satisfies $\sigma_{j+1} = 0$. Conversely, assume $1 \leq j \leq 2n$ is such that $\sigma_{j+1} = 0$. Denote by q_i the largest element of $\mathcal{Q}$ smaller than m_j. Then $D(q_i) = L$ and $D(q_{i+1}) = R$.*

Proof. Let $1 \leq i \leq 2n$ be such that $D(q_i) = L$ and $D(q_{i+1}) = R$. We distinguish three cases.

The first case is when q_i and q_{i+1} are of different types, that is, one belongs to $\mathcal{P}$ and the other to $\mathcal{P}'$. By symmetry we may assume that $q_i \in \mathcal{P}'$ and $q_{i+1} \in \mathcal{P}$. An example can be seen around the left-most empty region in Figure 3. Since $D(q_i) = L$, we have that $C(q_i)$ and q_{i+1} are two consecutive elements in $\mathcal{P}$, hence $M = \frac{C(q_i)+q_{i+1}}{2} \in \mathcal{L}$. Since q_i is closer to $C(q_i)$ than to q_{i+1}, we have that $M \in I(q_i, q_{i+1})$ and M is the only element of $\mathcal{L}$ in $I(q_i, q_{i+1})$. A similar argument shows that $M' = \frac{q_i+C(q_{i+1})}{2}$ is the only element of $\mathcal{L}'$ in $I(q_i, q_{i+1})$. Hence $I(q_i, q_{i+1})$ contains exactly two elements of $\mathcal{L} \cup \mathcal{L}'$, denoting them by m_j and m_{j+1} we conclude that $\sigma_{j+1} = 0$.

The second case is when q_i and q_{i+1} both belong to $\mathcal{P}$ (see for example the configuration around the second 0 in Figure 3). Then $\frac{q_i+q_{i+1}}{2}$ is the only element of $I(q_i, q_{i+1}) \cap \mathcal{L}$. Furthermore, $C(q_i)$ and $C(q_{i+1})$ are consecutive elements in

$\mathcal{P}'$, so $M'' = \frac{C(q_i)+C(q_{i+1})}{2} \in \mathcal{L}'$. Since q_i is closer to $C(q_i)$ than to $C(q_{i+1})$, we have that $M'' \in I(q_i, q_i + \frac{1}{2})$. Since q_{i+1} is closer to $C(q_{i+1})$ than to $C(q_i)$, we have that $M'' \in I(q_{i+1} - \frac{1}{2}, q_{i+1})$. So M'' is the only element of $I(q_i, q_{i+1}) \cap \mathcal{L}'$. The conclusion follows as in the first case.

The third case is when q_i and q_{i+1} both belong to $\mathcal{P}'$, it is treated like the second case.

Conversely, assume $1 \le j \le 2n$ is such that $\sigma_{j+1} = 0$. Since two consecutive elements of $\mathcal{L}$ (resp. of $\mathcal{L}'$) must be separated by an element of $\mathcal{P}$ (resp. of $\mathcal{P}'$), we deduce that among m_j and m_{j+1}, one belongs to $\mathcal{L}$ and the other to $\mathcal{L}'$. Denote by q_i the largest element of $\mathcal{Q}$ smaller than m_j. Then q_{i+1} is bigger than m_{j+1}. If $q_i \in \mathcal{P}$, consider the unique element of $\mathcal{L}' \cap \{m_j, m_{j+1}\}$. It is the bisector of two points of $\mathcal{P}'$, and these points cannot be in $I(q_i, q_{i+1})$, moreover, q_i is to the left of the bisector. This implies that $D(q_i) = L$. This works also in the case $q_i \in \mathcal{P}'$, and similarly, it shows that $D(q_{i+1}) = R$. □

Lemma 2.5. *There exists $1 \le i \le 2n$ such that $\sigma_i = 2$.*

Proof. Consider a pair (p, p') achieving the minimum

$$\min_{\substack{p \in \mathcal{P} \\ p' \in \mathcal{P}'}} d(p - p').$$

Let $1 \le i \le 2n$ be such that $p \in [m_i, m_{i+1}]$. Since $C(p) = p'$ and $C(p') = p$, we deduce from Lemma 2.3 that $\sigma_i = 2$. □

Lemma 2.6. *Assume that $\sigma_1 = 0$ and that there exists $2 \le i \le 2n$ such that $\sigma_i = 0$. Then there exists $2 \le j \le i - 1$ such that $\sigma_j = 2$.*

Proof. Since $\sigma_1 = 0$, we have $q_1 > m_1$ and by Lemma 2.4 we have that $D(q_1) = R$. Denote by q_r the largest element of $\mathcal{Q}$ smaller than m_{i-1}. By Lemma 2.4 we have that $D(q_r) = L$. Denote by k the smallest integer such that $D(q_k) = L$. We have that $2 \le k \le r$. Furthermore, $D(q_{k-1}) = R$, hence by Lemma 2.2 we have that $C(q_{k-1}) = q_k$ and $C(q_k) = q_{k-1}$. Let j be such that $q_k \in [m_{j-1}, m_j]$. Clearly $2 \le j \le i - 1$ and by Lemma 2.3 we have that $\sigma_j = 2$. □

Proof of Proposition 2.1. Let $P_1, \ldots, P_n$ be n points in cyclic order on the circle and let $(\sigma_1, \ldots, \sigma_{2n}) \in \{0, 1, 2\}^{2n}$ be the signature of their occupancy word. Define s_0, s_1 and s_2 to be respectively the number of occurrences of the values 0, 1 and 2 in the signature. Then $2n = s_0 + s_1 + s_2$ and since each point is counted twice in the signature, we also have

$$2n = \sum_{i=1}^{2n} \sigma_i = s_1 + 2s_2.$$

Combining these two equations we obtain that $s_0 = s_2$. From Lemma 2.5 we deduce that $s_2 \ge 1$, and even $s_2 \ge 2$ since $\underline{\sigma}$ is invariant by a translation of n. Therefore $s_0 \ge 2$. Assume that $1 \le i < j \le 2n$ are such that $\sigma_i = \sigma_j = 0$ and $\sigma_k > 0$ for all $i < k < j$. Up to applying a translation, one may assume that $i = 1$. By Lemma 2.6 we deduce the existence of some k such that $i < k < j$ and $\sigma_k = 2$. Given that $s_0 = s_2$, such a k is necessarily unique. Hence we conclude that $\underline{\sigma}$ is interlacing.

□

3 Realizing a word with interlacing signature

In this section we construct an explicit configuration of points from a word whose signature is alternating.

Proposition 3.1. *Let $n \geq 3$ and let $\underline{v} = (v_1, \dots, v_{2n}) \in \{0,1\}^{2n}$ such that its signature $\underline{\sigma} = (\sigma_1, \dots, \sigma_{2n}) \in \{0,1,2\}^{2n}$ is interlacing. Then there exists a configuration of points on the circle having $\underline{v}$ as an occupancy word.*

Proof. We fix $n \geq 3$ and such a word $\underline{v}$. Up to applying a rotation one may assume that $\sigma_1 = 0$.

Denote by T (resp. Z) the subset of all $1 \leq i \leq 2n$ such that $\sigma_i = 2$ (resp. $\sigma_i = 0$) and set $s = \sharp T = \sharp Z$. The set $\{1, \dots, 2n\} \setminus (T \cup Z)$ is composed of several connected components, which are the intervals of integers between two consecutive elements of $T \cup Z$ (note that some of these intervals may be empty). We call such a connected component an *ascending component* (resp. a *descending component*) if it is of the form $I_{2n}(i,j)$ with $i \in Z$ and $j \in T$ (resp. $i \in T$ and $j \in Z$) and for all $k \in I_{2n}(i,j)$ we have $\sigma_k = 1$. Let $i_1 < \cdots < i_s$ be the ordering of T. Let $j_1 < \cdots < j_s$ be the ordering of Z, with $j_1 = 1$.

To each $1 \leq i \leq 2n$ we associate a position p_i in $(0,1]$; in the end this will be the position of a point if $v_i = 1$. First, for all $1 \leq k \leq s$ we set

$$p_{i_k} = \frac{k}{s},$$
$$p_{j_k} = \frac{p_{i_{k-1}} + p_{i_k}}{2} = \frac{2k-1}{2s}.$$

Let $\eta > 0$ be small enough ($\eta < \frac{1}{s2^{n+2}}$ will suffice for our purposes). The following construction is motivated by the definitions of Section 2: on descending (resp. ascending) components, we want to define positions of points that look to the left (resp. right). For $1 \leq k \leq s$, consider the k-th descending component, that is $I_{2n}(i_k, j_{k+1})$. For every $h \in I_{2n}(i_k, j_{k+1})$ we set

$$p_h = p_{i_k} + \eta\left(2^{h-i_k} - 1\right) = \frac{k}{s} + \eta\left(2^{h-i_k} - 1\right).$$

For $1 \leq k \leq s$, consider the k-th ascending component, that is $I_{2n}(j_k, i_k)$. For every $h \in I_{2n}(j_k, i_k)$ we set

$$p_h = p_{i_k} - \eta\left(2^{i_k-h} - 1\right) = \frac{k}{s} - \eta\left(2^{i_k-h} - 1\right).$$

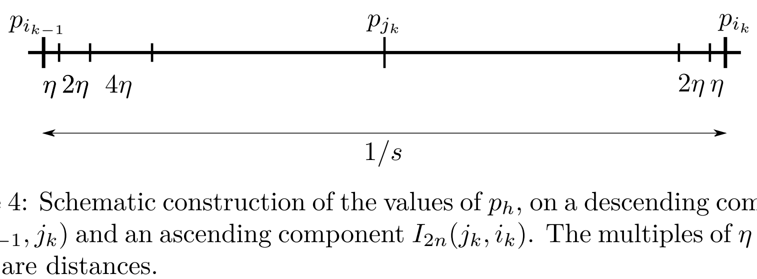

Figure 4: Schematic construction of the values of p_h, on a descending component $I_{2n}(i_{k-1}, j_k)$ and an ascending component $I_{2n}(j_k, i_k)$. The multiples of η written below are distances.

By the symmetry of the word, there cannot be more than n points in a component, therefore in both those cases,

$$d(p_h, p_{i_k}) < \eta(2^n - 1) < \frac{1}{4s}. \tag{3.1}$$

Hence the constructed positions of ascending and descending components lie in disjoint intervals.

Now let $\mathcal{P}$ be the subset of positions $\{p_i \mid 1 \leq i \leq 2n \text{ s.t. } v_i = 1\}$. We claim that this configuration of points has $\underline{v}$ as its occupancy word. As in Section 2, we set $\mathcal{P}' = \{p + \frac{1}{2}, p \in \mathcal{P}\}$, and $\mathcal{L}$ (resp. $\mathcal{L}'$) the positions of the bisectors of $\mathcal{P}$ (resp. $\mathcal{P}'$). We also set $\mathcal{M}$ to be the collection of $\mathcal{L} \cup \mathcal{L}'$, possibly with repetitions. We now characterize the positions of these bisectors.

Lemma 3.2. *For every h in a descending (resp. ascending) component, there is a unique element of $\mathcal{M}$ in $I(p_{h-1}, p_h)$ (resp. in $I(p_h, p_{h+1})$).*

For every $1 \leq k \leq s$, the set $I\left(\frac{2k-1}{2s} - \frac{1}{8s}, \frac{2k-1}{2s} + \frac{1}{8s}\right)$ contains exactly two elements of $\mathcal{M}$.

Moreover, these are all the $2n$ elements of $\mathcal{M}$.

Proof of Lemma 3.2. Let $1 \leq k \leq s$, and let h be in the descending component $I_{2n}(i_k, j_{k+1})$. We distinguish two cases, depending on the value of v_h.

If $v_h = 1$, then $p_h \in \mathcal{P}$. Moreover, $v_{i_k} = 1$ since $i_k \in T$, hence $p_{i_k} \in \mathcal{P}$. Therefore, the rightmost element of $\mathcal{P}$ smaller than p_h belongs to the set $\{p_{i_k}, p_{i_k+1}, \ldots, p_{h-1}\}$. Hence the position l_h of the bisector of this point and p_h satisfies

$$\frac{p_{i_k} + p_h}{2} \leq l_h \leq \frac{p_{h-1} + p_h}{2} < p_h.$$

Now notice that the left-hand side is $p_{i_k} + \eta(2^{i_k - h} - \frac{1}{2})$, which is strictly bigger than p_{h-1}. Hence $l_h \in I(p_{h-1}, p_h)$.

If $v_h = 0$, then as $\sigma_h = 1$, we have $v_{h+n} = 1$. Hence there is an element of $\mathcal{P}$ at position p_{h+n}, and by the invariance under translation of the word by n, we have $p_{h+n} = p_h + \frac{1}{2}$. Therefore, $p_h \in \mathcal{P}'$. Similarly, as $\sigma_{i_k} = 2$, we have $v_{i_k+n} = 1$ so that $p_{i_k+n} \in \mathcal{P}$ and $p_{i_k} \in \mathcal{P}'$. From there, we conclude as in the previous case.

For h in an ascending component, the proof is identical.

For the second point of the Lemma, consider the index $j_k \in Z$. As both $p_{i_{k-1}}$ and p_{i_k} belong to $\mathcal{P}$, the elements of $\mathcal{P}$ directly to the left and right of p_{j_k} belong, respectively, to $\{p_{i_{k-1}}, \ldots, p_{j_k-1}\}$ and to $\{p_{j_k+1}, \ldots, p_{i_k}\}$. Hence the position of their bisector $l_{j_k} \in \mathcal{L}$ satisfies

$$\frac{p_{i_{k-1}} + p_{j_k+1}}{2} \leq l_{j_k} \leq \frac{p_{j_k-1} + p_{i_k}}{2}.$$

As $j_k - 1$ belongs to the descending component $I_{2n}(i_{k-1}, j_k)$, by (3.1) we have $p_{j_k-1} < p_{i_{k-1}} + \frac{1}{4s}$, hence the right-hand side is smaller than $\frac{p_{i_{k-1}}+p_{i_k}}{2} + \frac{1}{8s}$, which is the expected bound. The left-hand side is treated similarly. Then, an identical proof shows that there is an element of $\mathcal{L}'$ in the same interval.

Clearly the elements of $\mathcal{M}$ coming from ascending and descending components are disjoint. Those coming from the second point are also disjoint among themselves, as even if two may share the same position, only one of them will belong to $\mathcal{L}$, and the other to $\mathcal{L}'$. The fact that these two families are disjoint is an easy consequence of (3.1). Hence we constructed two elements of $\mathcal{M}$ for each element of Z, and one for each element of $(T \cup Z)^c$, which is in total $2(\sharp Z) + 2n - (\sharp T + \sharp Z) = 2n$. □

Let $\underline{w}$ be the occupancy word of $\mathcal{P}$. We now have all the tools to prove that $\underline{w} = \underline{v}$. For any $1 \leq k \leq s$, consider the interval $I[p_{i_{k-1}}, p_{i_k})$. In that part, the positions and order of the elements of $\mathcal{M}$ are given by Lemma 3.2: for every h in the descending component $I_{2n}(i_{k-1}, j_k)$ there is one $l_h \in I(p_{h-1}, p_h) \cap \mathcal{M}$; then there are two distinct elements in $l_{j_k}, l'_{j_k} \in I(p_{j_k-1}, p_{j_k+1}) \cap \mathcal{M}$; then for every h in the ascending component $I_{2n}(j_k, i_k)$ there is one $l_h \in I(p_h, p_{h+1})$. Thus the part of $\underline{w}$ corresponding to this interval can be described as: first a 1 (for the region containing $p_{i_{k-1}}$), then for every $h \in I_{2n}(i_{k-1}, j_k)$, either a 1 or a 0 according to the value of v_h (as by definition these are the positions where a point of $\mathcal{P}$ has been put), then a 0 (for the region corresponding to l_{j_k}, l'_{j_k}), then for every $h \in I_{2n}(j_k, i_k)$ either a 0 or a 1 according to the value of v_h. This is clearly the same as $\underline{v}$ at those indices. This being true for any k, by concatenation we get that $\underline{w} = \underline{v}$.

One issue that may arise is that the configuration constructed above is not generic, in the sense that two lines may coincide, which occurs for example when a descending component and the following ascending component are empty. In order to avoid such issues, we perturb the configuration slightly, by fixing $\varepsilon > 0$ and defining for every $1 \leq k \leq 2n$, the point $\widetilde{p}_k = p_k + k\varepsilon$. For ε small enough ($\varepsilon < \frac{\eta}{2n}$ suffices), the relative position of the perturbed points and lines is the same as the unperturbed one, while two lines can no longer coincide.

□

4 Enumerating realizable words, necklaces and bracelets

A realizable word can be represented by a *necklace*, i.e. a word considered up to cyclic shifts, of length $2n$ on the alphabet $\{0, 1\}$ with n times the letter 0 and n times the letter 1. Such necklaces are enumerated by the sequence A003239 of the OEIS (Online Encyclopedia of Integer Sequences [6]). Up to quotienting out necklaces by mirror image, one obtains so-called *bracelets* of length $2n$ with n times the letter 0 and n times the letter 1. Such bracelets are enumerated by the sequence A005648 of the OEIS.

Closed formulas for the number of bracelets of fixed size can notably be found in [2, 4], where they are called self-dual necklaces. However these formulas cannot be directly extended to find the number of realizable bracelets.

Proof of Corollary 1.3. To choose a realizable word $\underline{v}$, one may first choose its alternating signature $\underline{\sigma}$; as $\underline{\sigma}$ is invariant under the shift by n, this amounts to choosing a number $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ such that $\underline{\sigma}$ will have $4p$ letters 0 or 2, then $2\binom{n}{2p}$ choices for their position and whether the first one is a 0 or a 2. Then for every i such that $\sigma_i = 1$ (there are $2n - 4p$ such indices), one has to chose if v_i is 1 or 0, under the condition that $v_{i+n} \neq v_i$. This gives 2^{n-2p} choices. Hence the number of realizable words is

$$\begin{aligned}\sharp\mathcal{W}_n &= 2\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2p}2^{n-2p}\\ &= 2\sum_{2\le k\le n,\ k \text{ even}}\binom{n}{k}2^{n-k}.\end{aligned}$$

Introducing $\mathcal{W}'_n = 2\sum_{1\le k\le n,\ k\text{ odd}}\binom{n}{k}2^{n-k}$, one easily gets $2^{n+1}+\sharp\mathcal{W}_n+\mathcal{W}'_n = 2\times 3^n$ and $2^{n+1}+\sharp\mathcal{W}_n-\mathcal{W}'_n = 2$, and the result follows.

For the number of bracelets and necklaces, they are between $(3^n-2^{n+1}+1)/(4n)$ and $3^n-2^{n+1}+1$, which implies the announced limits. □

We deduce from Corollary 1.3 the following result.

Corollary 4.1. *The proportion of realizable bracelets (resp. necklaces) among all bracelets (resp. necklaces) composed of n 0's and n 1's tends to* 0 *like* $(3/4)^n$ *up to polynomial corrections as n tends to infinity.*

Proof. As the number of bracelets or necklaces with n 0's and n 1's is between $\binom{2n}{n}/(4n)$ and $\binom{2n}{n}$, this is a direct consequence of the bounds given in the proof of Corollary 1.3. □

5 Scaling limit of a uniform realizable word

The aim of this section is to prove Theorem 1.4. Recall that $\underline{w}^{(n)}$ denotes a random word taken uniformly in the set of realizable words of length $2n$, and that we define $\underline{\hat{w}}^{(n)}$ to be the word of length n on the alphabet $\{00, 10, 01, 11\}$, whose letter in position i, denoted $\hat{w}_i^{(n)}$, is the concatenation $w_i^{(n)}w_{i+n}^{(n)}$; we call this the *folded* word obtained from $w^{(n)}$. To simplify notations, we will often drop the dependence in n.

We prove both parts of the theorem at once. The main idea in the proof is to rephrase it in terms of a random walk, and then use a *local limit theorem.* A local limit theorem controls the precise value of a random walk after a large number of steps. Let us state it properly (see e.g. [5, Theorem 6.1] for a proof of this result).

Theorem 5.1 ([5])**.** *Let $j\ge 1$ and $(\boldsymbol{Y}_i)_{i\ge1} := \left((Y_i^{(1)},\dots,Y_i^{(j)})\right)_{i\ge1}$ be i.i.d. random variables in $\mathbb{Z}^j$, such that the covariance matrix Σ of $\mathbf{Y}_1$ is positive definite. Assume in addition that $\boldsymbol{Y}_1$ is aperiodic, and denote by $\boldsymbol{M}$ the mean of $\boldsymbol{Y}_1$. Finally, define for $n\ge1$*

$$\mathbf{T_n} = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\mathbf{Y}_i - n\mathbf{M}\right)\in\mathbb{R}^j.$$

Then, as $n\to\infty$, uniformly for $\mathbf{x}\in\mathbb{R}^j$ such that $\mathbb{P}(\mathbf{T_n}=\mathbf{x})>0$,

$$\mathbb{P}(\mathbf{T_n}=\mathbf{x}) = \frac{1}{(2\pi n)^{j/2}\sqrt{\det\Sigma}}e^{-\frac12{}^t\mathbf{x}\boldsymbol{\Sigma}^{-1}\mathbf{x}} + o\left(n^{-j/2}\right).$$

We now define a walk from a realizable word, in an almost bijective way: to any $a := xy \in \{00, 01, 10, 11\}$, define $f(a) = x - y$. Then, to a folded realizable word $\underline{\hat{w}} := \hat{w}_1 \cdots \hat{w}_n$, we associate the walk S satisfying $S_0 = 0$ and, for all $i \geq 1$, $S_i - S_{i-1} = f(\hat{w}_i)$.

Remark that occurrences of 01 (resp. 10) in $\underline{\hat{w}}$ correspond to jumps by -1 (resp. $+1$) in S. Jumps by 0 in S may correspond to either 00 or 11, but as these two letters shall alternate in a folded realizable word, if one knows whether the first jump 0 corresponds to 00 or 11, then it is possible to recover $\underline{\hat{w}}$ from S. By symmetry, we assume from now on that the first jump by 0 corresponds to 11, so that the map $\underline{\hat{w}} \mapsto S$ is a bijection from $\hat{\mathcal{W}}_n^+$ to $\text{Walks}^+(n)$, where $\hat{\mathcal{W}}_n^+$ is the set of folded realizable words whose first 11 appears before the first 00, and $\text{Walks}^+(n)$ is the set of walks of length n, starting from 0 and with steps in $\{0, +1, -1\}$, with an even nonzero number of steps 0.

Now take n a positive integer. We want to study a uniform element of the set $\text{Walks}^+(n)$. To this end, we first study the set $\text{Walks}(n)$, of walks of length n, starting from 0 and with jumps in $\{0, +1, -1\}$. We define a walk $(T_i)_{0 \leq i \leq n} := (S_i, K_i)_{0 \leq i \leq n}$ on $\mathbb{Z}^2$ as follows: its first coordinate is a uniform element of $\text{Walks}(n)$, $K_0 = 0$ and, for any $0 \leq i \leq n-1$, $K_{i+1} - K_i = \mathbb{1}_{S_{i+1} - S_i = 0}$. In other words, the second coordinate of T enumerates the steps 0 in the walk S. It is clear by definition that $(T_i)_{0 \leq i \leq n}$ is a random walk on $\mathbb{Z}^2$ starting from $(0, 0)$, with i.i.d. jumps $Y_1, \ldots, Y_n$ whose distribution is the following:

$$\mathbb{P}(Y_1 = (1, 0)) = \mathbb{P}(Y_1 = (-1, 0)) = \mathbb{P}(Y_1 = (0, 1)) = \frac{1}{3}.$$

In particular, Y_1 has respective mean and covariance matrix

$$M = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/9 \end{pmatrix}$$

We want to prove the functional convergence of the walk S, along with the process $(K_i)_{0 \leq i \leq n}$ counting the number of "0" jumps in the walk, conditionally on K_n being even and nonzero. Since $\mathbb{P}(K_n = 0) = o(\mathbb{P}(K_n = 0 \mod 2))$, we only need to condition K_n to be even.

In what follows, we define $(S_u)_{u \in [0,n]}$ (resp. $(K_u)_{u \in [0,n]}$) as the linear interpolation of $(S_i)_{i \in [\![0,n]\!]}$ (resp. $(K_i)_{i \in [\![0,n]\!]}$) on the whole interval.

Proposition 5.2. *The following convergence holds in distribution, in* $\mathcal{C}([0, 1], \mathbb{R}^2)$*:*

$$\left(\left(\frac{S_{cn}}{\sqrt{n}}, \frac{K_{cn} - cn/3}{\sqrt{n}} \right)_{0 \leq c \leq 1} \middle| K_n = 0 \mod 2 \right) \xrightarrow[n \to \infty]{(d)} \left(W_c^{(1)}, W_c^{(2)} \right)_{0 \leq c \leq 1}$$

where $W^{(1)}, W^{(2)}$ *are independent Brownian motions of respective variances* $2/3$ *and* $2/9$*.*

The whole proof of this proposition is highly inspired from the one of [7, Lemma 4.1]. Let us start with a result on the corresponding unconditioned random walk.

$$\left(\frac{S_{cn}}{\sqrt{n}}, \frac{K_{cn} - cn/3}{\sqrt{n}} \right)_{0 \leq c \leq 1} \xrightarrow[n \to \infty]{(d)} \left(W_c^{(1)}, W_c^{(2)} \right)_{0 \leq c \leq 1} \tag{5.1}$$

This result is a consequence of Theorem 5.1. Indeed, by [3, Theorem 16.14], it is enough to check that the one-dimensional convergence holds for $t = 1$. One gets this from Theorem 5.1. Uniformly for a, b in a compact subset of $\mathbb{R}$:

$$\mathbb{P}(S_n = \lfloor a\sqrt{n} \rfloor, K_n = \lfloor n/3 + b\sqrt{n} \rfloor) \underset{n\to\infty}{\sim} \frac{1}{2\pi n \sqrt{\det \Sigma}} e^{-\frac{1}{2}\left(\frac{2}{3}a^2 + \frac{2}{9}b^2\right)}.$$

This implies (see e.g. [1, Theorem 7.8]) that $(S_n/\sqrt{n}, (K_n - n/3)/\sqrt{n})$ converges in distribution to $(W_1^{(1)}, W_1^{(2)})$. The convergence (5.1) follows.

We now want a conditioned version of (5.1), taking into account the fact that K_n has to be even. To this end, take $0 < u < 1$ and take $F : \mathcal{C}([0, u], \mathbb{R}^2) \to \mathbb{R}$ a bounded continuous functional. Set

$$E_n = \mathbb{E}\left[F\left(\frac{S_{cn}}{\sqrt{n}}, \frac{K_{cn} - cn/3}{\sqrt{n}} \right)_{0\le c\le u} \middle| K_n = 0 \mod 2 \right].$$

Setting $\varphi_n(i) = \mathbb{P}(K_n = i \mod 2)$ and observing that the (unconditioned) walk until time nu is independent of the walk between nu and n, one can write:

$$E_n = \mathbb{E}\left[F\left(\frac{S_{cn}}{\sqrt{n}}, \frac{K_{cn} - cn/3}{\sqrt{n}} \right)_{0\le c\le u} \frac{\varphi_{n-\lfloor nu \rfloor}(K_{\lfloor nu \rfloor})}{\varphi_n(0)} \right] \tag{5.2}$$

In order to estimate this quantity, simply remark that K_n is distributed as a binomial Bin_n of parameters $(n, 1/3)$. Now, remark by a simple computation that $\mathbb{P}(\mathrm{Bin}_n = 0 \mod 2) + \mathbb{P}(\mathrm{Bin}_n = 1 \mod 2) = 1$, and $\mathbb{P}(\mathrm{Bin}_n = 0 \mod 2) - \mathbb{P}(\mathrm{Bin}_n = 1 \mod 2) = 3^{-n}$, which implies that $\varphi_n(0)$ and $\varphi_n(1)$ both converge to $1/2$ as $n \to \infty$. Thus, (5.2) can be rewritten:

$$\begin{aligned} E_n &= \mathbb{E}\left[F\left(\frac{S_{cn}}{\sqrt{n}}, \frac{K_{cn} - cn/3}{\sqrt{n}} \right)_{0\le c\le u} \frac{1/2 + o(1)}{1/2 + o(1)} \right] \\ &= \mathbb{E}\left[F\left(\frac{S_{cn}}{\sqrt{n}}, \frac{K_{cn} - cn/3}{\sqrt{n}} \right)_{0\le c\le u} \right] + o(1) \end{aligned} \tag{5.3}$$

and we get Proposition 5.2 on $[0, u]$. In order to extend it to the whole interval $[0, 1]$, it now suffices to show that the process is tight on $[0, 1]$.

Proof of the tension on the whole interval. The convergence (5.3) shows notably that, conditionally to the fact that $K_n = 0 \mod 2$, the process

$$(S_{cn}/\sqrt{n}, (K_{cn} - cn/3)/\sqrt{n})_{0\le c\le 1} \tag{5.4}$$

is tight on $[0, u]$ for every $u \in (0, 1)$. To show that it is in addition tight on $[u, 1]$, we only need to check that, for $u \in (0, 1)$, the process

$$(S_{n-cn}/\sqrt{n}, (K_{n-cn} - n(1-c)/3)/\sqrt{n})_{0\le c\le u}$$

is tight conditionally on $K_n = 0 \mod 2$. For this, we use the invariance of the process by time-reversal: the process $(\widehat{S}_i, \widehat{K}_i)_{0\le i\le n} := (S_n - S_{n-i}, K_n -$

$K_{n-i})_{0\le i\le n}$ has the same distribution as $(S_i,K_i)_{0\le i\le n}$, and this is also true under the condition that $K_n = 0 \mod 2$. By definition, we can write

$$\left(\frac{S_{n-cn}}{\sqrt{n}}, \frac{K_{n-cn}-n(1-c)/3}{\sqrt{n}}\right)_{0\le c\le u} = \left(\frac{\widehat{S}_n-\widehat{S}_{cn}}{\sqrt{n}}, \frac{\widehat{K}_n-n/3}{\sqrt{n}} - \frac{\widehat{K}_{cn}-cn/3}{\sqrt{n}}\right)_{0\le c\le u}.$$

Now, letting $\sigma^2 := 2n/9$ be the variance of K_1, we obtain that, uniformly for b in a compact subset of $\mathbb{R}$,

$$\mathbb{P}(K_n = \lfloor n/3 + b\sqrt{n}\rfloor \mid K_n = 0 \mod 2) = \frac{2}{\sqrt{2\pi n}\sigma}e^{-\frac{b^2}{2\sigma^2}} + o\left(\frac{1}{\sqrt{n}}\right)$$

as $n\to\infty$. This implies that, conditionally to $K_n = 0 \mod 2$, $(K_n - n/3)/\sqrt{n}$ converges in distribution. Hence, by (5.3), the initial process (5.4) is tight on $[u,1]$ conditionally on $K_n = 0 \mod 2$.

Finally, the process is tight on $[0,1]$. Furthermore, the convergence of the finite-dimensional marginals is just a consequence of (5.3). This put together implies Proposition 5.2. □

We can now prove the main result of this section, Theorem 1.4, by translating Proposition 5.2 in terms of folded realizable words. For this, we make use of the following lemma, which relates the behaviour of a folded realizable word in $\hat{\mathcal{W}}_n^+$ to the behaviour of the associated element of $\text{Walks}^+(n)$.

Lemma 5.3 (From the walk to the word)**.** *Let $\underline{\hat{w}}$ be a folded realizable word of size n and $(S_i,K_i)_{0\le i\le n}$ the associated walk on $\mathbb{Z}^2$. For any $i\ge 0$, denote by α_i (resp. $\beta_i,\gamma_i,\delta_i$) the number of occurrences of* 11 *(resp.* $00, 10, 01$*) in the word $\underline{\hat{w}}$ up to position i. Then the following holds. For any $i\ge 0$, any $a\in\mathbb{Z}$, any $p\ge 0$:*

$$\begin{cases} S_i = a \\ K_i = p \end{cases} \iff \begin{cases} \alpha_i = \lfloor\frac{p+1}{2}\rfloor, \beta_i = \lfloor\frac{p}{2}\rfloor \\ \gamma_i = \frac{i-p+a}{2}, \delta_i = \frac{i-p-a}{2} \end{cases}$$

This lemma, whose proof is straightforward, implies Theorem 1.4:

Proof of Theorem 1.4. The proof just boils down to putting together Lemma 5.3 and Proposition 5.2. Indeed, Lemma 5.3 (keeping the same notation as in its statement) allows us to write for all $1\le i\le n$:

$$\alpha_i = \frac{i}{6} + \frac{K_i - i/3}{2} + c_1 = \beta_i + c_2$$
$$\gamma_i = \frac{i}{3} + \frac{S_i - (K_i - i/3)}{2}$$
$$\delta_i = \frac{i}{3} + \frac{-S_i - (K_i - i/3)}{2},$$

where c_1, c_2 are bounded in absolute value by 1, independently of n and i. This proves point *(ii)* of the theorem by the convergence of Proposition 5.2. Using the fact that $\sup_{0\le c\le 1}|W_c^{(1)}|$, $\sup_{0\le c\le 1}|W_c^{(2)}|$ are bounded in probability, point *(i)* follows. □

Remark 5.4. Notice that the conclusion of Theorem 1.4 still holds when one considers a uniform realizable necklace instead of a uniform realizable word. Indeed, one just has to check that, with probability going to 1 as $n \to \infty$, a uniform realizable word is not equal to any of its cyclic shifts. To see this, remark that a word equal to one of its cyclic shifts is necessarily periodic, of period at most $n/2$. Thus, there are at most $3^{n/2}$ realizable words with fixed period. Summing over all possible periods, there are at most $n3^{n/2}$ such words, which is $o(\sharp\hat{\mathcal{W}}_n)$. The result follows.

By a similar argument, the conclusion of Theorem 1.4 also holds for a uniform realizable bracelet.

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Département de Mathématiques, Université de Fribourg, PER 12 bu. 0.106, Ch. du Musée 11, 1700 Fribourg, Suisse
E-mail address: `paul.melotti at unifr.ch`

Université Paris-Saclay, CNRS, CEA, Institut de physique théorique, 91191 Gif-sur-Yvette, France
E-mail address: `sanjay.ramassamy at ipht.fr`

Centre de Mathématiques Appliquées, École polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France
E-mail address: `paul.thevenin at polytechnique.edu`