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ON k-LAYERED NUMBERS AND SOME LABELING RELATED TO k-LAYERED NUMBERS

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ABSTRACT. In this paper, first, we define and investigate k-layered numbers, which are a generalization of Zumkeller numbers. After that, we generalize the concept of Zumkeller labeling and Zumkeller cordial labeling to k-layered labeling and k-layered cordial labeling, respectively. Moreover, we prove that every simple graph admits Zumkeller labeling, Zumkeller cordial labeling, 3-layered labeling, 3-layered cordial labeling, 4-layered labeling and 4-layered cordial labeling.

0. INTRODUCTION

A perfect number is a positive integer that is equal to the sum of its proper positive divisors. In 2013, the idea of a Zumkeller numbers, which are generalization of perfect numbers, were first introduced by Zumkeller in Encyclopedia of Integer Sequences [11] A083207 .

Definition 0.1. A positive integer n is said to be Zumkeller if the set of positive divisors of n can be partitioned into two disjoint subsets of equal sum. A Zumkeller partition for a Zumkeller number n is a partition $\{A_1, A_2\}$ of the set of positive divisors of n such that A_1 and A_2 sums to the same value.

Clark et al. [12] announced several results and conjectures related to Zumkeller numbers. In [10], Yujian and K.P.S fund some other results about Zumkeller numbers. They study the relations between practical numbers and Zumkeller numbers. Also, They settle a conjecture from [12]. Moreover, They make substantial contributions regarding the second conjecture from [12].

On the other hand, Balamurgugan et al. [5] introduced k-Zumkeller labeling of graphs.

Definition 0.2. Let $G = (V, E)$ be a graph. An injective function $f : V \to \mathbb{N}$ is called a k-Zumkeller labeling of the graph G if the induced function $f^* : E \to \mathbb{N}$ defined by $f^*(xy) = f(x)f(y)$, $x \in V, y \in V, xy \in E$ satisfies the following two conditions:
(i) $f(xy)$ is a Zumkeller number for all $xy \in E$.
(ii) the number of different Zumkeller numbers used to label the edges of G is k.

They prove that a wide range of graphs admits Zumkeller labeling. After that, In [6] and [7], the concept of Zumkeller cordial was introduced by Murali et al.

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Definition 0.3. Let $G(V,E)$ be a graph. An injection function $f : E \to \mathbb{N}$ is call a Zumkeller cordial labeling of graph G if there exists an induction function $f^* : E \to \{0,1\}$ defined by $f^*(xy) = f^*(x)f^*(y)$ satisfies the following conditions:

(i) For every $xy \in E$

$$f^*(xy) = \begin{cases} 1, & \text{if } f(x)f(y) \text{ is a Zumkeller number;} \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, where $e_{f^*}(1)$ is the number of edges of graph G having label 0 under f^* and $e_{f^*}(1)$ is the number of edges of graph G having label 1 under f^*

They prove that there exist Zumkeller cordial labeling for path, cycles, stars, helm, wheel, flower, crown graphs and etc. Also, in [6] they raised the following open question:

Open Question 0.4. *Does every even flower graph admit Zumkeller cordial labeling?*

In this paper, In section 1, we recall and generalize some results of [10] for k-layered numbers, which are generalization of Zumkeller numbers. Also, in section 2, we find relations between k-multiperfect numbers and k-layered numbers. In addition, in section 3, we investigate the lower density of k-layered number.

At last, in section 4, not only we prove that every simple graph admits Zumkeller and Zumkeller cordial labeling, but also we prove that every simple graph admits some another labeling.

1. k-LAYERED NUMBERS

The definition of Zumkeller numbers motivates us to define k-layered numbers.

Definition 1.1. A positive integer n is said to be k-layered if the set of positive divisors of n can be partitioned into k disjoint subsets of equal sum. A k-layered partition for a k-layered number n is a partition $\{A_1, A_2, \ldots, A_k\}$ of the set of positive divisors of n such that for every $1 \leq i, j \leq k$, each of A_i and A_j sums to the same value.

Remark 1.2. *If n is a 2-layered number, then n is called Zumkeller.*

Let n be a positive integer and $\sigma(n)$ denotes the sum of positive divisors of n. We recall the index of n to be $I(n) = \frac{\sigma(n)}{n}$. Also, n is said to be abundant, perfect and deficient if $I(n) > 2$, $I(n) = 2$ and $I(n) < 2$, respectively.

The proposition 2 from [10] give some necessary condition for a Zumkeller number. We generalize this proposition for k-layered number.

Proposition 1.3. *If n is a k-layered number, then the followings are true:*
(a) $k|\sigma(n)$
(b) $kn \leq \sigma(n)$; *this concludes* $I(n) > k$.

Proof. The proof is identical to proof of the proposition 2 of [10]. □

The following fact gives a necessary and sufficient condition for integer n to be k-layered.

Fact 1.4. The number n is k-layered if and only if we can find $k-1$ disjoint subsets $A_1, A_2, \ldots, A_{k-1}$ of positive divisors of ℓ so that for every $1 \leq i \leq k$, A_i sums to the $\frac{\sigma(n)}{k}$.

Furthermore, we have:

Fact 1.5. If n is a k-layered number and $\ell | k$, then n is $\frac{k}{l}$-layered number.

By 1.5, we can generalize the proposition 13 of [10].

Proposition 1.6. *Let k_1, k_2 and ℓ are positive integers such that $k_1 | k_2$. Let n be a non-k_1-layered number and p a prime number with $gcd(n, p) = 1$. If np^ℓ is k_2-layered, then $p \leq \sigma(n)$.*

Proof. By 1.5, the proof is identical to proof of proposition 13 in [10]. □

Moreover, the three following propositions are generalizations of some propositions in [10].

Proposition 1.7. *Let n and ℓ are positive integers. Suppose that n be a non-k-layered integer and p be a prime number with $gcd(n, p) = 1$. If np^ℓ is k-layered number, then $p < \sigma(n)$.*

Proof. The proof is identical to proof of proposition 13 in [10]. □

Proposition 1.8. *If the integer n is k-layered and w is relatively prime to n, then nw is a k-layered number.*

Proof. The proof is identical to proof of corollary 5 of [10]. □

In addition, we have:

Proposition 1.9. *Let n be a k-layered number and $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ be a prime factorization of n. Then for any non-negative integers $l_1, l_2 \ldots l_m$, the integer*

$$p_1^{k_1 + l_1(k_1+1)} p_2^{k_2 + l_2(k_2+1)} \ldots p_m^{k_m + l_m(k_m+1)}$$

is k-layered.

Proof. The proof is identical to proof of proposition 6 of [10] □

Now, we recall the definition of practical numbers.

Definition 1.10. A positive integer n is said to be a practical number if every positive integer less than n can be represented as a sum of distinct positive divisors of n.

The following proposition gives very worthwhile information about the structure of practical numbers.

Proposition 1.11. *A positive integer n with the prime factorization $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ and $p_1 < p_2 < \cdots < p_m$ is a practical number if and only if $p_1 = 2$ and $p_{i+1} \leq \sigma(p_1^{k_1} \ldots p_i^{k_i}) + 1$ for $1 \leq i \leq m-1$.*

Proof. See [8] □

Also, we have:

Proposition 1.12. *A positive integer n is a practical number if and only if every integer less than or equal to $\sigma(n)$ can be written as a sum of distinct divisors of n.*

Proof. See [8] □

Now, we define almost practical numbers.

Definition 1.13. A positive integer n is called an almost practical number if all of the numbers j which $2 < j < \sigma(n) - 2$ or $j = \sigma(n) - 1$, can be written as a sum of distinct divisors of n.

Remark 1.14. *It is clear that every practical number is an almost practical number.*

We recall some results from [8].

Proposition 1.15. *Let $n \neq 3$ be an odd positive integer and $1 = d_1 < d_2 < \cdots < d_k = n$ are the divisors of n. We also define $\sigma_i = d_1 + d_2 + \cdots + d_i$. Then, n is an almost practical number if and only if $d_2 = 3, d_3 = 5$ and for $i \geq 3$, at least one of the followings are true:*
(a)$d_{i+1} \leq \sigma_i - 2$ *and* $d_{i+1} \neq \sigma_i - 4$.
(b)$d_{i+1} = \sigma_i - 4$ *and* $d_{i+2} = \sigma_i - 2$.

Proof. See [8] □

Remark 1.16. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ be an odd almost practical number in which $p_1 < p_2 < \cdots < p_m$ are all prime factors of n, then by 1.15, it is clear that $p_3 = 7$.*

Now, we state a theorem which has a crucial role in constructing almost practical numbers.

Theorem 1.17. *Let $n \neq 3$ be an almost practical number and p be a prime, then pn is an almost practical number if and only if $2p \leq \sigma(n) - 2$ and $2p \neq \sigma(n) - 4$*

Proof. See [8] □

Proposition 1.18. *Let $n \neq 3$ be an almost practical number and p be a prime dividing n, then pn is an almost practical number.*

Proof. See [8] □

Now, we are going to investigate the relation between almost practical numbers and Zumkeller numbers. The following proposition is a generalization of proposition 10 of [10].

Proposition 1.19. *Let $n \neq 3$ be an almost practical number. Then, n is Zumkeller if and only if $\sigma(n)$ is even.*

Proof. The proof is similar to proof of Proposition 10 of [10]. □

Example 1.20. If s is a positive integer, Then $n = 2^s \times 3$ is a practical number and $\sigma(n)$ is even. Hence, n is Zumkeller.

Example 1.21. It is easy to check that $n = 3^3 \times 5 \times 7$ is an almost practical number. Thus, if $m = 3^{\alpha_1} \times 5^{\alpha_2} \times 7^{\alpha_3}$ such that $\alpha_1 > 2$ and at least one of positive integers α_1, α_2 and α_3 be odd, then n is a Zumkeller number.

Also, we can generalize theorem 11 of [10].

Theorem 1.22. *Let $n \neq 3$ be an almost practical number and p a prime number with $gcd(n,p) = 1$. If $\sigma(n)$ is odd, then pn is a Zumkeller number if and only if $p \leq \sigma(n)$.*

Proof. By 1.19, the proof is identical to proof of theorem 11 of [10] □

In addition, we recall a proposition of [10]:

Proposition 1.23. *Let n be a practical number and p a prime number with $gcd(n,p) = 1$. If $\sigma(n)$ is odd, then pn is a Zumkeller number if and only if $p \leq \sigma(n)$.*

Proof. see [10] □

In the following, we state two crucial theorem about k-layered numbers. For better understanding , first, we state a special case of the theorem.

Proposition 1.24. *Let n be an odd number such that $3|\sigma(n)$. Now, Let A'_1 be a the subset of positive divisors of n so that A'_1 sums to $\frac{2\sigma(n)}{3}$. If α is a positive integer and $A' = \{2^\alpha d | d \in A'_1\}$ such that $2^\alpha n$ is a 3-layered number with 3-layered partition $\{A_1, A_2, A_3\}$ so that $A' \subset A_2 \cup A_3$, then for every integer $\alpha \leq t$, the number $\ell = 2^t n$ is a 3-layered number.*

Proof. Let $2^\alpha n$ be a 3-layered number. Now, we want to prove $k = 2^{\alpha+1} n$ is a 3-layered number. Let D be the set of positive divisors of n. We define:

$$M_1 = A' \cap A_2, M_2 = A' \cap A_3.$$

Now, we define:

$$M'_1 = \{2d | d \in M_1\}, M'_2 = \{2d | d \in M_2\}, M_3 = \{2^{\alpha+1} d | d \in (D \setminus A'_1)\}.$$

It is easy to check that $\{A_1 \cup M_3, (A_2 \setminus M_1) \cup M'_1 \cup M_2, (A_3 \setminus M_2) \cup M'_1 \cup M_1\}$ is a 3-layered partition for k. Thus, by this method, inductively, we can prove that $\ell = 2^t n$ is a 3-layered number for every integer $t \geq \alpha$, □

Thus, we have:

Proposition 1.25. *If $n > 3$ is an odd almost practical number such that $6|\sigma(n)$, then for every positive integer α, the number $\ell = 2^\alpha n$ is a 3-layered.*

Proof. First, we prove that $2n$ is a 3-layered number. Let $A_1 = \{a_1, a_2, \ldots, a_k\}$ is the set of positive divisors of n. By 1.25, n is a Zumkeller number; this concludes that A_1 can be partitioned into two subsets B_1 and B_2 such that each of them sums to $\frac{\sigma(n)}{2}$. Now, we define $A_2 = \{2d | d \in B_2\}$ and $A_3 = \{2d | d \in B_3\}$. We know that n is an odd number. Therefore, for every integer $a \in A_2 \cup A_3$, $2a \notin A_1$. Thus, $\{A_1, A_2, A_3\}$ is a 3-layered partition for n. Also, we know that n is an almost practical number such that $2|\sigma(n)$ $3|\sigma(n)$ and $\frac{2\sigma(n)}{3} \neq 2, n$; this concludes that there exists $A' \subset A_1$ so that A' sums to $\frac{2\sigma(n)}{3}$. On the other hand , let $A'' = \{2d | d \in A'\}$, we know that for every integer $d \in A_1$, $2d \in A_2 \cup A_3$; this concludes that $A'' \subset A_2 \cup A_3$. Then, by 1.24, for every positive integer α, the number $2^\alpha n$ is 3-layered. □

Example 1.26. It is easy to check that $n = 120$ is a 3-layered number with 3-layered partition $\{A_1, A_2, A_3\}$ such that $A_1 = \{20, 40, 60\}$, $A_2 = 120$ and $A_3 = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30, 120\}$. Let $120 = 2^3 n$. It is easy to check that $\frac{2\sigma(n)}{3} = 16$. Now , if we define $A'_1 = \{1, 15\}$, then it is obvious that $A' = \{2^3, 2^3 \times 15\} \subset A_2 \cap A_3$. Thus, by 1.24, for every positive integer $k \geq 3$, the number $2^k \times 3 \times 5$ is a 3-layered number.

Now, we present a proposition like 1.24 for 4-layered numbers.

Proposition 1.27. *Let n be an odd number such that $2|\sigma(n)$. Now, let D be the set of positive divisors of n such that D can be partitioned into two subsets A'_1 and A'_2 so that A'_1 and A'_2 sums to $\frac{\sigma(n)}{2}$. If α is a positive integer, $A''_1 = \{2^\alpha d | d \in A'_1\}$ and $A''_2 = \{2^\alpha d | d \in A'_2\}$ so that $2^\alpha n$ is a 4-layered number with 4-layered partition $\{A_1, A_2, A_3, A_4\}$ such that $A''_1 \subset (A_1 \cup A_2)$ and $A''_2 \subset (A_3 \cup A_4)$, then for every integer $\alpha \leq t$, the number $\ell = 2^t n$ is a 4-layered number.*

Proof. Let $2^\alpha n$ be a 4-layered number. We want to prove $k = 2^{\alpha+1} n$ is a 4-layered number. Let D be the set of positive divisors of n. We define:

$$M_1 = A''_1 \cap A_1, M_2 = A''_1 \cap A_2$$

$$M_3 = A''_2 \cap A_3, M_4 = A''_2 \cap A_4$$

Now, we define:

$$M'_1 = \{2d | d \in M_1\}, M'_2 = \{2d | d \in M_2\}$$

$$M'_3 = \{2d | d \in M_3\},\ M'_4 = \{2d | d \in M_4\}.$$

It is easy to check that $\{(A_1 \setminus M_1) \cup M'_1 \cup M_2, (A_2 \setminus M_2) \cup M'_2 \cup M_1, (A_3 \setminus M_3) \cup M'_3 \cup M_4, (A_4 \setminus M_4) \cup M'_4 \cup M_3)\}$ is a 4-layered partition for k. Thus, by this method, inductively, we can prove that $\ell = 2^t n$ is a 4-layered number for every integer $t \geq \alpha$. $\square$

The two following theorems are a generalization of 1.24 and 1.27, respectively.

Theorem 1.28. *Let n and k be odd positive integers such that n is a k-layered number and $A'_1, A'_2, \ldots, A'_{\frac{k-1}{2}}$ are disjoint subsets of positive divisors of n so that for every integer $1 \leq i \leq \frac{k-1}{2}$, A'_i sums to $\frac{2\sigma(n)}{k}$. Now, let α be a positive integer and for every integer $1 \leq i \leq \frac{k-1}{2}$, $A''_i = \{2^\alpha d | d \in A'_i\}$ such that $2^\alpha n$ is a k-layered number with k-layered partition $\{A_1, A_2, \ldots, A_k\}$ so that for every integer $1 \leq i \leq \frac{k-1}{2}$, $A''_i \subset A_{2i-1} \cup A_{2i}$. Then, for every integer $\alpha \leq t$, the number $\ell = 2^t n$ is a k-layered number.*

Proof. For every positive integers $1 \leq i \leq k-1$, we define $M_i{=}A''_{\lfloor \frac{i+1}{2} \rfloor} \cap A_i$, $M'_i = \{2d | d \in M_i\}$ and $M = \{2^{\alpha+1} d | d \in (D \setminus A'_1 \cup A'_2 \cup \ldots, A'_{\frac{K-1}{2}})\}$. Now, we define :

$$B_i = \begin{cases} (A_i \setminus M_i) \cup M'_i \cup M_{i+1} & \text{if } i \text{ is odd;} \\ (A_j \setminus M_i) \cup M'_i \cup M_{i-1} & \text{if } i \text{ is even.} \end{cases}$$

□

It is easy to check that $\{B_1, B_2, \ldots, B_{k-1}, A_k \cup M\}$ is a k-layered partition for $2^{\alpha+1}n$. Also, by this method, , inductively, we can prove that $\ell = 2^t n$ is k-layered for every integer $t \geq \alpha$.

Theorem 1.29. *Let n be an odd positive integer so that $k|\sigma(n)$, where k is an even positive integer such that n is a k-layered number. Also, let $A'_1, A'_2, \ldots, A'_{\frac{k}{2}}$ be disjoint subsets of positive divisors of n so that for every integer $1 \leq i \leq \frac{k}{2}$, A'_i sums to $\frac{2\sigma(n)}{k}$. Now, let α be a positive integer and for every integer $1 \leq i \leq \frac{k}{2}$, $A''_i = \{2^\alpha d | d \in A'_i\}$ such that $2^\alpha n$ be a k-layered number with k-layered partition $\{A_1, A_2, \ldots, A_k\}$ so that for every integer $1 \leq i \leq \frac{k}{2}$, $A''_i \subset A_{2i-1} \cup A_{2i}$. Then, for every integer t, where $\alpha \leq t$, the number $\ell = 2^t n$ is a k-layered number.*

Proof. For every positive integer $1 \leq i \leq k$, we define:

$$M_i = A''_{\lfloor \frac{i+1}{2} \rfloor} \cap A_i, M'_i = \{2d | d \in M_i\}$$

.

Now, for every $1 \leq i \leq k$, we define:

$$B_i = \begin{cases} (A_i \setminus M_i) \cup M'_i \cup M_{i+1} & \text{if } i \text{ is odd;} \\ (A_j \setminus M_i) \cup M'_i \cup M_{i-1} & \text{if } i \text{ is even.} \end{cases}$$

□

It is easy to check that the set $\{B_1, B_2, \ldots, B_k\}$ is k-layered partition for $2^{\alpha+1}n$. Also, by this method, inductively, we can prove the number $2^t n$ is k-layered for every $t \geq \alpha$.

Example 1.30. Let $\ell = 3 \times 5 \times 7$ and α be positive integers such that $3 \leq \alpha$. Suppose that $p < \frac{\sigma(\ell^\alpha)-4}{2}$ and $p \equiv 2(\mod 3)$. By 1.21 and 1.17, $\ell^\alpha p$ is an almost practical number such that $6|\sigma(\ell)$. Therefore,by 1.25, for every positive integer t, $2^t \ell^\alpha p$ is a 3-layered number.

Now, we state a proposition that we can find a huge set of 4-layered numbers by that.

Proposition 1.31. *Let k,k' be positive numbers, m be a k-layered number, and n be a k'-layered number such that $gcd(m, n) = 1$. Then, mn is a kk'-layered number.*

Proof. Let $A_1, A_2, \ldots, A_k$ be a k-layered partition for m, and $B_1, B_2, \ldots, B'_k$ be a k'-layered partition for n. It is obvious that $\{A_i B_j | 1 \leq i \leq k, 1 \leq j \leq k\}$ is a kk'-layered partition for mn. □

It is clear that the proposition 1.31 can be generalized.

Corollary 1.32. *Let $k_1, k_2, \ldots, k_r$ be positive integers such that for every integer $1 \leq i \leq r$, m_i is a k_i-layered number. Also let for every inetger $1 \leq i \neq j \leq r$, $gcd(m_i, m_j) = 1$. Then, $m_1 m_2 \ldots m_r$ is a $k_1 k_2 \ldots k_r$-layered number.*

The following examples show the power of proposition 1.31 for finding 4-layered numbers.

Example 1.33. By 1.21, $n_1 = 3^{\alpha_1} \times 5^{\alpha_2} \times 7^{\alpha_3}$, in which $\alpha_1 > 2$, α_2, α_3 are positive integers, and at least one of the exponents of its factors is odd, is a Zumkeller number. Let k be a positive integer and p be a prime number such that $p \leq 2^{k+1} - 1$ and $gcd(p, n_1) = 1$. By 1.11, 1.14, and 1.19, for every odd number α_4, the number $n_2 = 2^k \times p^{\alpha_5}$ is Zumkeller. Therefore, by 1.31, $n = n_1 n_2$ is a 4-layered number.

Example 1.34. Let $t \leq 3$ be a positive integer. Now, suppose $p_1, p_2 \leq \sigma(2^t)$ are distinct primes expect for 3, 5 and 7. By 1.22 and 1.19, the number $n_1 = 2^t p_1$ is a Zumkeller number. Also, by definition of p_2, we know $2p_2 < \sigma(3^t \times 5^t \times 7^t) - 4$. Thus, according to 1.21 and 1.17, the number $n_2 = 3^t \times 5^t \times 7^t p_2$ is a Zumkeller number. At last, by 1.31, the number $n_1 n_2 = 2^t \times 3^t \times 5^t \times 7^t p_1 p_2$ is a 4-layered number.

In the following, we want to prove that for every integer $n \geq 11$, the number $n!$ is 3-layered and 4-layered. Before that, we recall a theorem which was proved by Breusch; this theorem is a generalization of Bertrand's postulate theorem.

Theorem 1.35. *For every integer $n \geq 7$, there are primes of the form $3k+1$ and $3k+2$ between n and $2n$.*

Now, we state a theorem.

Theorem 1.36. *If $n \geq 11$ is an integer, then the number $n!$ possesses prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ such that $2 = p_1 < p_2 < \cdots < p_k$ and $\alpha_{k-1} = \alpha_k = 1$. Also, $p_k \leq 2^{\alpha_1}$ and $2p_{k-1} < \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{k-2}^{\alpha_k - 2}) - 4$. In addition, there exists a prime number q such that $q || n!$ and $q \equiv 2(\mod 3)$.*

Proof. If $11 \leq n \leq 16$, then it is easy to check that $n!$ satisfies in the theorem. Now, Let $n \geq 17$ and p be the largest prime factor of $n!$ such that $p^2 | n!$; this concludes $2p | n$. By definition of n, it is clear that $p \geq 7$. Then, by 1.35, there exist at least two distinct prime numbers q_1 and q_2 such that $p < q_1, q_2 < 2p$ and $q_1 \equiv 2(\mod 3)$. Thus, by definition of p, q_1 and q_2 are prime factors of $n!$ with power of one. Furthermore, if $ord_2(n)$ denotes the exponent of the largest power of 2 that divides n, then by Legendre's formula, we have:

$$p_k \leq n < 2^{\lfloor \frac{n}{2} \rfloor} < 2^{ord_2(n)}$$

.

Also, by definition of n, $p_{k-3} \geq 7$. Thus, by Bertrand's postulate theorem, we have:

$$2p_{k-1} < 4p_{k-2} < p_{k-3} p_{k-2} < \sigma(p_2^{\alpha_2} \dots p_{k-2}^{\alpha_{k-2}}) - 4$$

□

Thus, as a consequence of the above theorem, we have the following corollary

Corollary 1.37. *For every integer $n \geq 11$, the number $n!$ is 3-layered.*

Proof. Let $n = 11$ it is easy to check that we can find positive integer α and ℓ such that $2^\alpha \ell$ and ℓ is an almost practical number. Thus, by 1.36 and 1.17, for every integer $n \geq 11$, we can find positive integers α and ℓ such that $n = 2^\alpha \ell$, where ℓ is an odd almost practical number. Therefore, by 1.25, the number $n!$ is a 3-layered number.

□

At last, we close this section by the following corollary.

Corollary 1.38. *For every integer $n \geq 11$, the number $n!$ is 4-layered.*

Proof. Let $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be the prime factorization of $n!$ such that $2 = p_1 < p_2 < \dots < p_k$. By 1.36, 1.11,1.19, and 1.17 the numbers $2^{\alpha_k} p_k$ and $p_2^{\alpha_2} p_3^{\alpha_3} \dots p^{\alpha_{k-1}} k - 1$ are Zumkeller. Thus, by 1.31, $n!$ is 4-layered. □

2. k-MULTIPERFECT NUMBERS AND k-LAYERED NUMBERS

First, we state a proposition that we can find a wide rang of k-layered numbers by that.

Proposition 2.1. *Let $k > 1$, l, t, s be positive integers such that $s|t$. Now, suppose n is a k-layered number such that $\sigma(n) = kl$. If $\sigma(nt) = (k+1)sl$, then $m = nt$ is a $(k+1)$-layered number.*

Proof. Let D be the set of positive divisors of m and let $\{A_1, A_2, \dots, A_k\}$ be k-layered partition for n. Now, for every positive integer $1 \leq i \leq k$, we define $A_i' = \{sd | d \in A_i\}$ and also $A_{k+1}' = D \setminus (A_1' \cup A_2' \cup \dots A_k')$. It is easy to check that $\{A_1', A_2', \dots, A_{k+1}'\}$ is a $(k+1)$-layered partition for m. □

Therefore, we have:

Corollary 2.2. *Let p be a prime number. If n is a p-layered number such that $gcd(n, p) = 1$, then np is a $(p+1)$-layered number*

Example 2.3. If n is an odd Zumkeller number, then by 2.2, $2n$ is a 3-layered number.

Now, we recall the definition of k-multiperfect numbers.

Definition 2.4. Let n and $k \neq 1$ be positive integers. The number n is said to k-multiperfect if $\sigma(n) = kn$.(Note that if n is 2-multiperfect number, then n is said to be perfect.)

We are now ready to state an example, showing a crucial role of proposition 2.1 in finding a huge set of k-layered numbers.

Example 2.5. Let $a_1 = 2\times 3, a_2 = 2^3 \times 3 \times 5, a_3 = 2^5 \times 3^3 \times 5 \times 7, a_4 = 2^{11} \times 3^3 \times 5^2 \times 7^2 \times 13 \times 19 \times 31, a_5 = 2^{19} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13^2 \times 19^2 \times 31^2 \times 37 \times 41 \times 61 \times 127, a_6 = 2^{39} \times 3^{11} \times 5^7 \times 7^3 \times 11 \times 13^2 \times 17 \times 19^2 \times 29 \times 31^2 \times 37 \times 41 \times 61 \times 73 \times 79 \times 83 \times 127 \times 157 \times 313 \times 331 \times 2203 \times 30841 \times 61681$. It was proved that for every integer $1 \leq i \leq 5$, the number a_i is a $(i+1)$-perfect number(See [1].). Also, it is easy to see that for every integer $1 \leq i < 5$ $a_i | a_{i+1}$ and 6 is a Zumkeller number. Thus, by proposition 2.1, for every integer $1 \leq i \leq 5$, a_i is a $(i+1)$-layered number.

Now, we recall a concept of number theory

Definition 2.6. An arithmetical function f is said to be multiplicative if f is not identically zero and also $f(mn) = f(m)f(n)$ whenever $gcd(m, n) = 1$.

Remark 2.7. *We know that the sum divisor function is a multiplicative function (see [14]); this concludes that the function I is multiplicative too.*

Proposition 2.8. *Every perfect number is Zumkeller.*

Proof. Let D be the set of positive divisors of n. We define $A_1 = \{n\}$ $A_2 = D \setminus \{n\}$. It is clear that A_1, A_2 is Zumkeller partition of n. □

The proposition 2.8 lead us to raise the following open question

Open Question 2.9. *For which one of positive integers $k \neq 1$, every k-multiperfect number is k-layered*

Remark 2.10. *It is believed that all k-multiperfect number of index 3,4,5, 6 and 7 are known. Among six* 3*-multiperfect numbers that are fund, the number* 51001180160 *is the largest see [1].*

In the following, we prove that every known 3-multiperfect numbers is 3-layered. Before that, we recall some concept and results in number number theory.

Definition 2.11. The abundant number n is said to be semiperfect if n is equal to all or some of proper divisor of n. Also, the abundant number n which is not semiperfect called weird.

The existence of odd weird numbers is still an open question . The following theorem was proved by W. Fang (see [11]).

Theorem 2.12. *There are no odd weird numbers less than 1.8×10^{19}. In other words, every odd abundant number $a \leq 1.8 \times 10^{19}$ is semiperfect*

Also, the following theorem was proved by Guy:

Theorem 2.13. *Let m be a positive integer, and let p be a prime number such that $2^m \leq p \leq 2^{m+1}$. Then, the number $2^m p$ is a semiperfect number.*

Remark 2.14. *By definition of semiperfec numbers, it is easy to check that every multiple of a semiperfect number is semiperfect.*

Then we have:

Proposition 2.15. *Every known* 3*-multiperfect number is* 3*-layered.*

Proof. Let $\ell \geq 1$ be a positive integer. We know that every known ℓ-multiperfect number is even. Let n be a known 3-multiperfect number and let k and m be positive integers such that $n = 2^k m$ and $gcd(2^k, m) = 1$. Now, suppose that p be the smallest odd prime factor of n. It is easy to check that there exists a positive integer α such that $2^\alpha \leq p \leq 2^{\alpha+1}$ and $2^\alpha | n$. Then, by 2.13 and 2.14, there exists a subset D of the set of proper positive divisor of n such that D sums to n. Now, we define $A_1 = D$, $A_2 = n$. It is obvious that A_1 and A_2 sums $\frac{\sigma(n)}{3}$. Thus, by 1.4, n is 3-layered. □

Remark 2.16. *Let n be a positive integer such that $I(n) \geq 4$. Now, let t be a deficient number such that $n = t^\alpha m$ and $gcd(t, m) = 1$. We know the function I is multiplicative. Therefore, $I(m) > 2$; this concludes that m is an abundant number.*

The theorem 2.12 lead us to the following conjecture.

Conjecture 2.17. *Every odd abundant number is semiperfect.*

Up to now , 36 4-multiperfect numbers are fund [1]. Let n be a 4-multiperfect. According to the reference [1], we know that there exist a positive integer α and an odd positive integer m such that $n = 2^{\alpha}m$. By 2.16, m is an abundant number. One can see that m is a semiperfect number. Then there exists a subset D of the set of proper positive divisors of m such that D sums to m. Now, we define $A_1 = 2^{\alpha}d : d \in D$. Also, it is obvious that $I(\frac{n}{2}) \geq 2$; it concludes that $\sigma(\frac{n}{2}) \geq n$, and also by 1.11 and the reference [1], it is easy to check that $\frac{n}{2}$ is a practical number. Thus, by 1.12, there exists a subest A_2 of the set of positive divisors of $\frac{n}{2}$ such that A_2 sums to $n = \frac{\sigma(n)}{4}$. Now, if we define $A_3 = n$, then for every positive integer i, A_i sums to $\frac{\sigma(n)}{4}$. Then by 1.4, we have the following corollary.

Corollary 2.18. *Every known* 4*-multiperfect number is* 4*-layered.*

Remark 2.19. *Exactly half of known* 4*-multiperfect are divisible by at least a* 3*-multiperfect numbers [1]. Then, once again, by 2.15 and 2.1, at least half of known* 4*-multiperfect are* 4*-layered*

Remark 2.20. *Let* $a_1 = 6, a_2 = 120, a_3 = 30240, a_4 = 14182439040$ *it was proved that for every integer* $1 \leq i \leq 4$*, the number* a_i *is the smallest* $(i+1)$*-perfect number [1]. Also, for every integer* $1 \leq i \leq 3$*,* $a_i | a_(i+1)$*. Then, by 2.1,* a_i *is* $(i+1)$*-layered number for every integer* $1 \leq i \leq 4$

3. Lower density of k-layered

In [10], Yuejian and K.P.S raised the following open question.

Open Question 3.1. *Does the set of Zumkeller numbers possess density?*

In 2010, T.D checked that the 229026 Zumkeller numbers less than 10^6 have a maximum difference of 12; he conjectured that any 12 consecutive numbers include at least one Zumkeller number. At last, in 2019, Charlie presented an easy proof for this conjecture [11]. We here present the proof of this conjecture for completion.

Proposition 3.2. *If* $a < b$ *are two consecutive Zumkeller numbers, then* $|b - a| \leq 12$*; this concludes that the lower density of the set of Zumkeller numbers is at least* $\frac{1}{12}$*.*

Proof. By 1.20 and 1.8, for every positive integer k, the numbers $18k+6$ and $18k+12$ are Zumkeller. Then difference between two consecutive Zumkeller numbers is at most 12. □

Remark 3.3. *There exist consecutive Zumkeller numbers* a *and* b *such that* $b - a = 12$*. For instance,* $a = 222$ *and* $b = 224$ *are consecutive Zumkellers such that* $b - a = 12$*.*

Before finding a lower density for the set of 3-layered numbers and 4-layered numbers, we recall that the number n is said to be superabundant if $I(n) > I(k)$ for all positive integers $k < n$. Also, we have:

Lemma 3.4. *Let* $m_1 < m_2$ *be two consecutive superabundant numbers. For every positive integer* $t < m_2$*,* $I(t) < I(m_1)$*.*

Proof. Let t be a positive integer such that $I(t_1) > I(m_1)$ and $t > m_2$. By definition of m_1, m_2, it is obvious that t fails to be superabundant number; this concludes there exists a positive integer $m_1 < \ell_1 < t < m_2$ such that $I(m_1) > I(\ell_1) > I(t)$. We once again know that ℓ_1 cannot be superabundant so there exists a positive integer ℓ_2 such that $m_1 < \ell_2 < \ell_1 < t < m_2$. Therefore, for every positive integer r, by this algorithm, inductively, we can find distinct positive integers $\ell_1, \ell_2, \ldots, \ell_r$ such that $m_1 < \ell_r < \ell_{r-1} < \cdots < \ell_1 < m_2$ and $I(m_2) > I(\ell_r) > I(\ell_{r-1}) > \cdots > I(\ell_1)$; this contradicts the finiteness of the set $A = \{a | a \in \mathbb{N}, m_1 < a < m_2\}$. □

Now, we find a lower density for the set of 3-layered numbers.

Proposition 3.5. *If $a < b$ are two consecutive 3-layered numbers, then $b - a \leq 360$; this concludes the lower density of the set of 3-layered numbers is at least $\frac{1}{360}$.*

Proof. By 1.26, $n = 120$ is a 3-layered number. Also, it is first superabundant number such that $I(n) \leq 3$ (See [13]). Thus, by 3.4 and 1.3, n is the smallest 3-layered number. Moreover, it is easy to check that at least one of the numbers $t, t+1$, and $t+2$ is not divisible by 3 and 5. Thus by 1.26 and 1.8, one of the numbers tn, $(t+1)n$, or $(t+2)n$ is 3-layered; this concludes that the lower density of 3-layered numbers is at least $\frac{1}{3n} = \frac{1}{360}$ □

If A is a set of positive integer, then we define $S(A)$ as a sum of the integers in A. Now, we find a lower density for the set of 4-layered numbers.

Proposition 3.6. *If $a < b$ are two consecutive 4-layered numbers, then $b - a \leq 249480$; this concludes the lower density of the set of 4-layered numbers is at least $\frac{1}{249480}$.*

Proof. The number $n = 27720 = 2^3 \times 3^2 \times 5 \times 7 \times 11$ is 4-layered because let we define:

$$A_1 = \{\ 2^3 \times 3^2 \times 5, \quad 2^3 \times 3^2 \times 5 \times 7 \times 11\}$$

$$A_2 = \{\ 2 \times 3 \times 5, \quad 2 \times 3 \times 5 \times 11, \quad 2^2 \times 3 \times 5 \times 7 \times 11, \quad 2^3 \times 3 \times 5 \times 7 \times 11, \quad 2^2 \times 3^2 \times 5 \times 7 \times 11\}$$

$$A_3 = \{\ 1, \quad 2 \times 3^2, \quad 2 \times 3 \times 5 \times 7 \times 11, \quad 2^2 \times 3^2 \times 7 \times 11, \quad 2^3 \times 5 \times 7 \times 11, \quad 3^2 \times 5 \times 7 \times 11, \quad 2^3 \times 3^2 \times 5 \times 11, \quad 2^3 \times 3^2 \times 7 \times 11, \quad 2 \times 3^2 \times 5 \times 7 \times 11\}$$

It is easy to check that $S(A_i) = \frac{\sigma(n)}{4}$ for every integer $1 \leq i \leq 3$. Then, by 1.4, n is a 4-layered number. Also, n is the smallest positive integer such that $I(n) \geq 4$ (See [13]). Therefore, by 3.4 and 1.3, 27720 is the smallest 4-layered number. In addition, it is easy to check that for every positive integer k, there exist at least an integer $1 \leq i \leq 9$ such that $gcd(k+i, n) = 1$; by 1.8; this concludes that $(k+i)n$ is a 4-layered numbers so the lower density of 4-layered numbers is at least $\frac{1}{9n} = \frac{1}{249480}$ □

Now, we find the smallest 5-layered number.

Proposition 3.7. *The number $n = 147026880 = 2^6 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17$ is the smallest 5-layered number.*

Proof. We know $t = 122522400$ is the smallest superabundant such that $I(t) \geq 5$ (see [13]). But $5 \nmid \sigma(t)$. Therefore, by 1.3 and 3.4, t fails to be 5-layered and every 5-layered number is larger than t. By a Computational Software like python, it is easy to check that the number $n = 147026880$ is the smallest integer such that $I(n) \geq 5$ and $5 | \sigma(n)$. Now, we want to prove this number is 5-layered. First of all, we know the number $\ell_1 = 120 = 2^3 \times 3 \times 5$ is a 3-multiperfect number and the number $\ell_2 = 32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ is the a 4-perfect number such that $\ell_1 | \ell_2$ (See [1]). Therefore, by 2.15 and 2.1, the number ℓ_2 is a 4-layered number; this concludes that there exists a 4-layered partition $\{A_1, A_2, A_3, A_4\}$ for ℓ_2 such that A_i sums to ℓ_2 for every integer $1 \leq i \leq 4$. Also, for every integer $1 \leq i \leq 4$ we define:

$$A'_i = \{2^3 \times 3 \times 11 \times 17d | d \in A_i\}$$

On the other hand, let $m = 447552$. We know the number $n_1 = 2^6 \times 5 \times 11 \times 13 \times 17$ is a practical number. Also, $m < \sigma(n_1)$. Thus, by definition of practical number, there exist subsets B_1 and B_2 of the set of positive divisors of n_1 such that B_1 and B_2 sum to m and $\frac{m}{7}$, respectively. Now, we define $B'_2 = \{7d | d \in B_1\}$. In addition, we know the number $n_2 = 2^6 \times 3^2 \times 5 \times 7 \times 13 \times 17$ is a practical number such that $\frac{m}{3} < \sigma(n_2)$. Therefore, there exists a subset B_3 of the set of positive divisors for n_2 such that B_3 sums to $\frac{m}{3}$. Now, we define $B'_3 = \{3d | d \in B_3\}$. At last, we know the number $n_3 = 2^2 \times 3^2 \times 5 \times 11 \times 13 \times 17$ is a practical number such that $\frac{m}{21} < \sigma(n_3)$; this concludes that there exists a subset B_4 of the set of positive divisors for n such that B_4 sums to $\frac{m}{21}$. Now, we define $B'_4 = \{21d | d \in B_4\}$. It is easy to check that the sets $B_1, B'_2, B'_3, B'_4, A_1, A_2, A_3, A_4, A'_1, A'_2, A'_3$ and A'_4 are disjoint subsets of the set of positive divisors for n. Now, we define $C_1 = A_1 \cup B_1$ and also for every integer $2 \leq i \leq 4$, we define $C_i = A'_i \cup B'_i$; it is easy to check that every $S(C_i) = \frac{\sigma(n)}{5}$. Then, by 1.4, n is a 5-layered number. □

Theorem 3.8. *The number* $130429015516800 = 2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times$ *is the smallest* 6*-layered number.*

Proof. We can see that the number $n = 130429015516800$ is smallest number such that $I(n) \geq 6$ (see [13]) . Therefore, by 1.3, if m be a 6-layered number, then $m \geq n$. Now, we prove that the number n is a 6-layered number; this concludes that n is the smallest 6-layered number. we define:

$$\begin{aligned} A_1 = \{\ & 2^3, \quad 2^2 \times 5 \times 13 \times 29, \quad 2^4 \times 3 \times 11 \times 13 \times 17 \times 29, \\ & 2^2 \times 3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29, \\ & 2^5 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23, \\ & 2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29\} \end{aligned}$$

$$\begin{aligned} A_2 = \{\ & 2^4 \times 3^3 \times 7 \times 17 \times 19, \quad 2^4 \times 3^3 \times 5^2 \times 7^2 \times 17 \times 19 \times 29, \\ & 2^6 \times 3^3 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^6 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29\} \end{aligned}$$

$$\begin{aligned} A_3 = \{\ & 2^4 \times 3 \times 7 \times 17, \quad 2^7 \times 3 \times 7^2 \times 17 \times 29, \quad 2^7 \times 3^3 \times 5 \times 7^2 \times 11 \times 17 \times 29, \\ & 2^6 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23, \\ & 2^6 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^6 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^7 \times 3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^4 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^7 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^7 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, \\ & 2^5 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29\} \end{aligned}$$

$$\begin{aligned}
A_4 = \{\ & 23, \quad 2\times 3^3\times 5\times 11\times 13, \quad 5^2\times 7\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 17\times 23\times 29,\\
& 2^6\times 3\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^2\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23,\\
& 2^5\times 3^3\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^3\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^4\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 29,\\
& 2^6\times 3^3\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^2\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^5\times 3^3\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 23\times 29,\\
& 2^6\times 3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 11\times 13\times 19\times 23\times 29,\\
& 2^3\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^2\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29,\\
& 2^5\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29\}
\end{aligned}$$

$$\begin{aligned}
A_5 = \{\ & 13, \quad 2\times 11\times 13\times 23, \quad 2^5\times 3^3\times 7^2\times 17\times 23,\\
& 2\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^2\times 5^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 29,\\
& 2^4\times 3^3\times 5^2\times 7^2\times 11\times 13\times 19\times 23\times 29,\\
& 2^7\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7\times 11\times 13\times 17\times 23\times 29,\\
& 2^5\times 3^2\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^3\times 5\times 7^2\times 11\times 17\times 19\times 23\times 29,\\
& 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^4\times 3^2\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^3\times 5^2\times 7\times 11\times 13\times 19\times 23\times 29,\\
& 2^7\times 3\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29,\\
& 2^5\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23,\\
& 2^7\times 3^3\times 5\times 7^2\times 11\times 13\times 17\times 19\times 29,\\
& 2^6\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 23\times 29,\\
& 2^3\times 3^3\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^3\times 5\times 7^2\times 13\times 17\times 19\times 23\times 29,\\
& 2^5\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3^2\times 5\times 7\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^4\times 3^3\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^2\times 5^2\times 7^2\times 11\times 13\times 19\times 23\times 29,\\
& 2^5\times 3^3\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^7\times 3\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29,\\
& 2^6\times 3^3\times 5^2\times 11\times 13\times 17\times 19\times 23\times 29,\\
& 2^2\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29,
\end{aligned}$$

$2^7 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29,$
$2^5 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29,$
$2^7 \times 3^3 \times 5^2 \times 7 \times 11 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^4 \times 3^3 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23,$
$2^7 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29,$
$2^5 \times 3^2 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^3 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^3 \times 5^2 \times 7 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^5 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29,$
$2^7 \times 3^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^4 \times 3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29,$
$2^5 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29,$
$2^6 \times 3^2 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^3 \times 5 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29,$
$2 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^5 \times 3^2 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23,$
$2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29,$
$2^4 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^3 \times 5 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^5 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29,$
$2^6 \times 3^3 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^3 \times 5^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^3 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29,$
$2^7 \times 3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^5 \times 3^3 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^2 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^4 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29,$
$2^5 \times 3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^7 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29,$
$2^7 \times 3^2 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29,$
$2^2 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29\}$

It is easy to check that for every integer $1 \leq i \leq 5$, $S(C_i) = \frac{\sigma(n)}{6}$. Thus, by 1.3, n is a 6-layered number. □

Theorem 3.9. *The number* $19709923047004539052704 00 = 2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53$ *is the smallest* 7*-layered number.*

Proof. We can see that the number 1970992304700453905270400 is smallest number such that $I(n) \geq 7$ (see [13]). Now, we prove that the number $n = 130429015516800$ is a 7-layered number; this concludes that n is the smallest 7-layered number. We define:

$$
\begin{aligned}
A_1 = \{ & 2 \times 3, \quad 3^2 \times 5^2 \times 7^2, \quad 2^6 \times 3^3 \times 19 \times 23 \times 29 \times 37 \times 43, \\
& 3^2 \times 5^2 \times 7 \times 17 \times 19 \times 23 \times 29 \times 37 \times 41 \times 47 \times 53, \\
& 2^5 \times 3 \times 5 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29 \times 31 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53 \}
\end{aligned}
$$

$$
\begin{aligned}
A_2 = \{ & 2 \times 3^2, \quad 2 \times 3 \times 5^2 \times 11 \times 41, \quad 2^3 \times 3^2 \times 5 \times 19 \times 23 \times 29 \times 43 \times 53, \\
& 2^2 \times 3 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 53, \\
& 2^6 \times 3^3 \times 5^2 \times 7^2 \times 13 \times 17 \times 23 \times 29 \times 31 \times 37 \times 41 \times 47 \times 53, \\
& 2^4 \times 3^4 \times 5 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53 \}
\end{aligned}
$$

$$
\begin{aligned}
A_3 = \{ & 2^2 \times 3^2 \times 31, \quad 2^5 \times 3 \times 7 \times 11 \times 19 \times 31 \times 43, \\
& 2^6 \times 3 \times 5 \times 11 \times 13 \times 23 \times 29 \times 31 \times 37 \times 53, \\
& 2^2 \times 3^2 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 37 \times 41 \times 43 \times 53, \\
& 2^3 \times 3^3 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^4 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^4 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^4 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^5 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53 \}
\end{aligned}
$$

$$
\begin{aligned}
A_4 = \{ & 2 \times 7, \quad 2 \times 3 \times 7 \times 11 \times 13 \times 53, \quad 2^2 \times 3^3 \times 7 \times 11 \times 19 \times 29 \times 31 \times 47, \\
& 2^3 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 43 \times 47 \times 53, \\
& 2^7 \times 3 \times 5^2 \times 7 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 53, \\
& 2^6 \times 3^4 \times 5 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^5 \times 3^4 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^4 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^6 \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53, \\
& 2^7 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,
\end{aligned}
$$

$$
\begin{aligned}
&2^5 \times 3^4 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^3 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^5 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53\}
\end{aligned}
$$

$$
\begin{aligned}
A_5 = \{\ &2^3 \times 11 \times 23, \quad 2^2 \times 3 \times 11 \times 13 \times 19 \times 31 \times 47 \times 53,\\
&2 \times 3^4 \times 5^2 \times 11 \times 13 \times 17 \times 29 \times 31 \times 41 \times 47 \times 53,\\
&2^7 \times 3^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^5 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^2 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^4 \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^5 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 43 \times 47 \times 53,\\
&2^7 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^3 \times 3^4 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^5 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^3 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 43 \times 47 \times 53,\\
&2^4 \times 3^2 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^5 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^3 \times 5^2 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^2 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^5 \times 3^3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^4 \times 3^4 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5 \times 7^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47,\\
&2^5 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,\\
&2^6 \times 3^4 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53,
\end{aligned}
$$

$$
\begin{aligned}
&2^7\times 3^4\times 5^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^3\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 53,\\
&2^6\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^2\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^5\times 3^4\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 47\times 53,\\
&2^6\times 3^3\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 43\times 47\times 53,\\
&2^4\times 3^4\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^3\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 41\times 43\times 47\times 53,\\
&2^5\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5\times 7\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^4\times 5^2\times 7^2\times 11\times 13\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^3\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^2\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 37\times 41\times 43\times 47\times 53\}
\end{aligned}
$$

$$
\begin{aligned}
A_6=\{\ &2^2\times 7\times 23,\quad 2\times 17\times 23\times 29\times 31\times 47,\\
&2^7\times 3^4\times 11\times 13\times 17\times 29\times 31\times 41\times 53,\\
&2^5\times 3^4\times 5\times 11\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^5\times 3^3\times 5^2\times 7\times 11\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^4\times 3^4\times 5\times 7^2\times 11\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^4\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 53,\\
&2^5\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 47\times 53,\\
&2^7\times 3\times 5^2\times 7^2\times 11\times 13\times 17\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^3\times 5\times 7^2\times 11\times 13\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^4\times 5^2\times 7^2\times 13\times 17\times 19\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^4\times 3^2\times 5^2\times 7\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^3\times 3^4\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^2\times 5\times 7^2\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^6\times 3^4\times 5^2\times 7^2\times 11\times 17\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 13\times 19\times 23\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^5\times 3^3\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 43\times 47\times 53,\\
&2^6\times 3^4\times 5\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^3\times 5^2\times 7\times 11\times 13\times 17\times 19\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29\times 31\times 41\times 43\times 47\times 53,\\
&2^2\times 3^3\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^2\times 5^2\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47,\\
&2^5\times 3^4\times 5^2\times 7\times 11\times 13\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 7^2\times 11\times 13\times 17\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^4\times 5^2\times 7^2\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 47\times 53,\\
&2^6\times 3^4\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 53,\\
&2^5\times 3^2\times 5^2\times 7^2\times 11\times 17\times 19\times 23\times 29\times 31\times 37\times 41\times 43\times 47\times 53,\\
&2^7\times 3^3\times 5\times 7^2\times 11\times 13\times 17\times 19\times 23\times 29\times 37\times 41\times 43\times 47\times 53,
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$$2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 37 \times 41 \times 43 \times 47 \times 53\}$$

It is easy to check that for every integer $1 \leq i \leq 6$, $S(A_i) = \frac{\sigma(n)}{7}$ □

At last, we close this section by a theorem, which proves that if for positive integer k, there exist a k-layered number, then the set of k-layered numbers possesses a lower density.

Theorem 3.10. *Let n be the smallest k-layered numbers with prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. If $a < b$ are two consecutive k-layered numbers, then $b - a \leq (p_1 p_2 \dots p_k - (p_1 - 1)(p_2 - 1) \dots (p_k - 1))n$.*

Proof. Let s be a non negative integer and d be a positive integer between $sp_1 \dots p_k$ and $(s+1)p_1 \dots p_k$. By proposition 1.8, if $gcd(d, n) = 1$, then dn is a k-layered number. Now, let r be a positive integer which is smaller than $p_1 \dots p_k$. It is clear that $gcd(sp_1 \dots p_k + r, n) = 1$ if and only if $gcd(n, r) = 1$. Thus, there exist at least $\varphi(p_1 \dots p_k)$ numbers d between $sp_1 \dots p_k$ and $(s+1)p_1 \dots p_k$ such that dn is a k-layered number. Therefore, if we ignore $\varphi(p_1 \dots p_k) - 1$ numbers of $p_1 \dots p_k$ numbers between $sp_1 \dots p_k$ and $(s+1)p_1 \dots p_k$, then again we can find a number like d between $sp_1 \dots p_k$ and $(s+1)p_1 \dots p_k$ such that dn is a k-layered number(Note that φ is the euler totient function.). □

4. SOME GRAPH LABELING RELATED TO k-LAYERED NUMBERS

First, we generalize the concept of Zumkeller labeling to k-layered labeling.

Definition 4.1. Let $G = (V, E)$ be a graph. An injective function $f : V \to \mathbb{N}$ is called a l-k-layered labeling of the graph G if the induced function $f^* : E \to \mathbb{N}$ defined by $f^*(xy) = f(x)f(y)$, $x \in V, y \in V, xy \in E$ satisfies the following two conditions:
(i) $f(xy)$ is a k-layered number for all $xy \in E$.
(ii) the number of different k-layered numbers used to label the edges of G is l.

In addition, we generalize the concept of Zumkeller cordial labeling to k-layered labeling.

Definition 4.2. Let $G(V, E)$ be a graph. An injection function $f : E \to \mathbb{N}$ is call a k-layered cordial labeling of graph G if there exists an induced function $f^* : E \to \{0, 1\}$ defined by $f^*(xy) = f^*(x)f^*(y)$ satisfies the following conditions:

(i) For every $xy \in E$

$$f^*(xy) = \begin{cases} 1, & \text{if } f(x)f(y) \text{ is a Zumkeller number;} \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, where $e_{f^*}(1)$ is the number of edges of graph G having label 0 under f^* and $e_{f^*}(1)$ is the number of edges of graph G having label 1 under f^*

Remark 4.3. *B.J. Balamurugan and et al[4] called the graph G Zumkeller, if G admits a Zumkeller labeling. Also, B.J. Murali and et al[6] called the graph G Zumkeller cordial if G admits a Zumkeller cordial labeling. From now on, we call the graph G k-layered if G admits a k-layered labeling and also we called the graph G k-layered cordial if G admits a k-layered cordial labeling.*

Let n be k-layered number. The following proposition states a condition for the integer k, which satisfying that concludes that every graph is k-layered.

Proposition 4.4. *If there exists a k-layered number n with prime factorization* $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ *such that for every positive integer* $1 \leq i \leq t$, α_i *is even, then every graph is k-layered.*

Proof. If for every $1 \leq i \leq$, we label the vertex v_i of graph G with $p_1^{\frac{\alpha_1}{2}+i(\alpha_1+1)} p_2^{\frac{\alpha_2}{2}+i(\alpha_2+1)} \ldots p_t^{\frac{\alpha_t}{2}+i(\alpha_t+1)}$, then, by 1.9 this labeling is a k-layered labeling for G. □

4.1. **Some labeling related to Zumkeller numbers.** .

The following theorem is one the most important theorem of this section:

Theorem 4.5. *Let* K_n *denotes a complete graph on n vertices with vertex set V and edge set E. For every positive integer* $k \leq \frac{n(n-1)}{2}$, *We can find an f-labeling for V such that* $e_f^*(1) = k$. *(*$e_f^*(1)$ *computes Zumkeller edges relative to our labeling)*

Proof. It is easy to check that the statement holds for K_1, K_2 and K_3. Let $m > 3$ be a positive integer. Now, we want to prove that the statement holds for the complete graph K_m. We choose the even number t_0 large enough that there exists a chain of even numbers $t_m < \cdots < t_2 < t_1 < t_0$ which for every positive integer $1 \leq r \leq m$, we can find distinct primes $p_{r,1}, p_{r,2}, \ldots, p_{r,m-1}$ such that for every positive integers, $1 \leq i \leq m$ and $2 \leq j \leq m-1$ we have:

(i) $\sigma(2^{2t_i}) < p_{i,1} < \sigma(2^{t_{i-1}})$.

(ii) $\sigma(2^{2t_i} p_{i,j-1}) < p_{i,j} < \sigma(2^{t_{i-1}})$.

(iii) If $i \neq m$, then $\sigma(2^{2t_i} p_{i,m-1}) < p_{i+1,1}$.

Suppose that $V = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of K_m. We label v_1 with $n_1 = 2^{t_1}$. If we want the edge $v_2 v_1$ to be non-Zumkeller relative to our labeling, then we label v_2 with $n_{2,0} = 2^{t_2} p_{1,1}$ because by (i), for number $n_{2,0} n_1 = 2^{t_1+t_2} p_{1,1}$, we have:

$$\sigma(2^{t_1+t_2}) < \sigma(2^{2t_1}) < p_{1,1}$$

Therefore, according to 1.11 and 1.23, the number $n_{2,0}n_1$, which is labeling of the edge v_2v_1, is non-Zumkeller. Also, if we want the edge v_2v_1 to be Zumkeller, we label v_2 with $n_{2,1} = 2^{t_2}p_{2,1}$ because by (ii), for number $2^{t_1+t_2}p_{2,1}$ we have:

$$p_{2,1} < \sigma(2^{t_1}) < \sigma(2^{t_1+t_2})$$

Thus, according to 1.11 and 1.23, the number $n_{2,1}n_1$, which is labeling of edge v_2v_1 relative to our labeling, is Zumkeller.
After labeling the vertex v_2, we know that the vertex v_2 was labeled with number $n_2 = 2^{t_2}q$ such that $q \in \{p_{1,1}, p_{2,1}\}$. Now, for labeling the vertex v_3, if we want both edges of $\{v_3v_1, v_3v_2\}$ be non-Zumkeller, then we label v_3 with $n_{3,0} = 2^{t_3}p_{1,2}$ because by 1.3, $2^{t_1+t_3}$ is a non-Zumkeller number and by (ii) we have:

$$\sigma(2^{t_1+t_3}) < p_{2,1}$$

Thus, by 1.6, the number $n_{3,0}n_1$, which is labeling of the edge v_3v_1 relative to our labeling, is non-Zumkeller. Moreover, by (i), we have:

$$\sigma(2^{t_2+t_3}) < \sigma(2^{2t_2}) < q,\ \sigma(2^{t_2+t_3}) < \sigma(2^{2t_2}p_{1,1}) < p_{1,2}$$

Thus, by 1.6, the numbers $2^{t_2+t_3}p_{1,2}$, $2^{t_2+t_3}q$ are non-Zumkeller. In addition , by (ii) and (iii), we have:

$$\sigma(2^{t_2+t_3}p_{1,2}) < q, \text{ or } \sigma(2^{t_2+t_3}q) < p_{1,2}$$

Therefore, by 1.6, the number $n_{3,0}n_2 = 2^{t_2+t_3}p_{1,2}q$, which is labeling of the edge v_3v_2, is a non-Zumkeller number. Also, for labeling v_3, if we want one edge of $\{v_3v_1, v_3v_2\}$ be Zumkeller relative to our labeling, then we label v_3 with $n_{3,1} = 2^{t_3}p_{2,2}$ because by (i), for number $n_{3,1}n_1 = 2^{t_1+t_3}p_{2,2}$ we have:

$$p_{2,2} < \sigma(2^{t_1}) < \sigma(2^{t_1+t_3})$$

Therefore, by 1.11 and 1.23, the number $n_{3,1}n_1$, which is labeling of v_3v_1 relative to our labeling, is a Zumkeller number. Also, once again, it is easy to check that the edge v_3v_2 is non-Zumkeller relative to our labeling. At last, for labeling v_3, if we want the both edges of $\{v_3v_1, v_3v_2\}$ to be Zumkeller relative to our labeling, then we label v_3 with $n_{3,2} = 2^{t_3}p_{3,2}$ (By (ii), 1.23, and 1.8 it is clear.). Let $j > 3$ be an integer. By this method, we labeled $j-1$ vertices of complete graph K_m. Now, we want to label the vertex v_j. Suppose that ℓ be an integer such that $0 \le \ell < j \le m$. If we want to have exactly ℓ Zumkeller edges of edges $\{v_jv_1, v_jv_2, \ldots, v_jv_{j-1}\}$ relative to our labeling, we label v_j with $2^{t_j}p_{\ell+1,j}$ because first of all, by (ii), 1.11, 1.19, and 1.8, for every positive integer i which $1 < i < \ell$, v_jv_i is Zumkeller edge relative to our labeling. Moreover, for every positive integer s such that $\ell < s < j$, the edge v_jv_s is non-Zumkeller because we know that the vertex v_s was labeled with $2^{t_s}q'$ in which $q' \in \{p_{i,s-1} : 1 \le i \le s\}$. Then,by (i) and(ii), for the number $2^{t_s+t_j}q'$, we have:

$$\sigma(2^{t_s+t_j}) < \sigma(2^{2t_s}) < q',\ \sigma(2^{t_s+t_j}) < \sigma(2^{2t_s}) < p$$

Thus, by 1.3 and 1.6, the integers $2^{t_s+t_j}q'$ and $2^{t_s+t_j}p_{\ell+1,j}$ are non-Zumkeller. Also, according to (ii) and (iii), we have:

$\sigma(2^{t_s+t_j}q') < \sigma(2^{2t_s}q') < p$, or $\sigma(2^{t_s+t_j})p < \sigma(2^{2t_s}p) < q'$

Therefore, by 1.6, the edge v_jv_s is non-Zumkeller relative to our labeling. □

It is clear that every simple graph is a subgraph of a complete graph. Then, we have:

Corollary 4.6. *Let G be a simple graph with k edges. For every positive integer $t \leq k$, we can find an f-labeling for G that $e_f^*(1) = t$.($e_f^*(1)$ computes Zumkeller edges relative to our labeling)*

By 1.3, we know that every square number fails to be Zumkeller. Therefore, as a consequence of theorem 4.9, we have:

Corollary 4.7. *Let m be a non-negative integer and G be a graph with m loops. Also. let If $m - 1 \leq |G| - m$, then G is a Zumkeller cordial graph.*

4.2. **some labeling related to 3-layered numbers.** .

Now, state a theorem like theorem 4.5 for 3-layered numbers.

Theorem 4.8. *Let K_n be a complete graph on n vertices with vertex set V and edge set E. For every positive integer $k \leq \frac{n(n-1)}{2}$, we can find an f-labeling for V such that $e_f^*(1) = k$. ($e_f^*(1)$ computes 3-layered edges relative to our labeling.) .*

Proof. It is easy to check that the statement holds for K_1, K_2 and K_3. Let $m > 3$ be a positive integer. Now, we want to prove that the statement holds for the complete graph K_m. Let $\ell = 3 \times 5 \times 7$. We choose the even number t_0 large enough that there exists a chain of even numbers $3 < t_m < \cdots < t_2 < t_1 < t_0$ such that first, for every positive integer $1 \leq r \leq m$, $t_i \equiv 1(\mod 3)$. In addition, we can find distinct primes $p_{r,1}, p_{r,2}, \ldots, p_{r,m-1}$ such that for every positive integers, $1 \leq i \leq m$ and $2 \leq j \leq m - 1$ we have:

(i) $p_{i,j} \equiv 2(\mod 3)$

(ii) $\sigma(2^2\ell^{2t_i}) < p_{i,1} < \frac{\sigma(\ell^{t_i-1})-4}{2}$

(iii) $\sigma(2^2\ell^{2t_i}p_{i,j-1}) < p_{i,j} < \frac{\sigma(\ell^{t_i-1})-4}{2}$

(iv) If $i \neq m$, then $\sigma(2^2\ell^{2t_i}p_{i,m-1}) < p_{i+1,1}$.

Suppose that $V = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of K_m. We label v_1 with $n_1 = 2\ell^{t_1}$. If we want the edge v_2v_1 to be non-3-layered relative to our labeling, then we label v_2 with $n_{2,0} = 2\ell^{t_2}p_{1,1}$ because by (ii), for number $n_{2,0}n_1 = 2^2\ell_1^t + t_2p_{1,1}$, we have:

$$\sigma(2^2\ell^{t_1+t_2}) < \sigma(2^2\ell^{2t_1}) < p_{1,1}$$

.

Therefore, by 1.7, the number $n_{2,0}n_1$, which is labeling of edge v_2v_1 relative to our labeling, is non-3-layered. Also, if we want the edge v_2v_1 to be 3-layered relative to our labeling, then we label v_2 with $n_{2,1} = 2\ell^{t_2}p_{2,1}$ because by (iii), for number $n_{2,1}n_1 = 2^2\ell^{t_1+t_2}p_{2,1}$, we have:

$$p_{2,1} < \frac{\sigma(\ell^{t_1}) - 4}{2} < \frac{\sigma(\ell^{t_1+t_2}) - 4}{2}$$

.

Thus, by (i) and 1.30, the number $n_{2,1}n_1$, which is labeling of the edge v_2v_1 relative to our labeling, is 3-layered. After labeling the vertices v_1 and v_2, for labeling the vertex v_3, if we want both edges of $\{v_3v_2, v_3v_1\}$ to be non-3-layered, then we label v_3 with $n_{3,0} = 2\ell^{t_3}p_{1,2}$ because by 1.3, the number $2^2\ell^{t_1+t_3}$ is a non-3-layered number and by (iii), for number $n_{3,0}n_1 = 2^2\ell^{t_1+t_3}p_{1,2}$, we have:

$$\sigma(2^2\ell^{t_1+t_3}) < p_{2,1}$$

Thus, by 1.7, the number $n_{3,0}n_1$, which is labeling of the edge v_3v_1 relative to our labeling, is non-3-layered. In addition, we know the vertex v_2 was labeled with number $2^{t_2}q$ such that $q \in \{p_{1,1}, p_{2,1}\}$ and by (ii), we have :

$$\sigma(2^2\ell^{t_2+t_3}) < \sigma(2^2\ell^{2t_2}) < q,\ \sigma(2^2\ell^{t_2+t_3}) < \sigma(2^2\ell^{2t_2}p_{1,1}) < p_{1,2}$$

Thus, by 1.7, the number $2^2\ell^{t_2+t_3}p_{1,2}$ and $2^2\ell^{t_2+t_3}q$ are non-3-layered. Moreover, by (iii) and (iv), we have:

$$\sigma(2^2\ell^{t_2+t_3}p_{1,2}) < q, \text{ or } \sigma(2^2\ell^{t_2+t_3}q) < p_{1,2}.$$

Therefore, by 1.3 $n_{3,0}n_1$, which is labeling of the edge v_3v_2 relative to our labeling, is non-3-layered. Also, for labeling v_3, if we want one edge of $\{v_3v_1, v_3v_2\}$ to be 3-layered relative to our labeling, then we label v_3 with number $n_{3,1} = 2^{t_3}p_{2,2}$ because by (iii), for number $n_{3,1}n_1 = 2^2\ell^{t_2+t_3}p_{2,2}$, we have:

$$p_{2,2} < \frac{\sigma(\ell^{t_1}) - 4}{2} < \frac{\sigma(\ell^{t_1+t_3}) - 4}{2}$$

.

Therefore, by (i) and 1.30, the number $n_{3,1}n_1$, which is labeling of v_3v_1 relative to our labeling, is 3-layered. In addition, once again by (iii), (iv), and 1.7 the edge v_3v_2 is non-3-layered relative to our labeling. At last, if we want both edges $\{v_3v_1, v_3v_2\}$ to be 3-layered relative to our labeling, then we label v_3 with $2\ell^{t_3}$.(By (i), (iii) and 1.30, it is clear.). Now, Let $j > 3$ be an integer, and by this method, we labeled the $j-1$ vertices of a complete graph. Now, we want to label the vertex v_j. Suppose that l be an integer such that $0 \leq l < j \leq m$. If we want to have exactly l 3-layered edges of $\{v_jv_1, v_jv_2, \dots v_jv_{j-1}\}$ relative to our labeling we label v_j with

$2\ell^{t_j}p_{l+1,j}$ because first of all, by (i), (iii), and 1.30, for every positive integer i such that $1 < i < l$, v_jv_i is a 3-layered edge relative to our labeling. Moreover, for every positive integer s such that $l < s < j$, the edge v_jv_i is non-3-layered because we know the vertex v_s was labeled with $2\ell^{t_s}q'$ in which $q' \in \{p_{i,s-1} : 1 \leq i \leq s\}$. By (i) and (ii), for number $2^2\ell^{t_s+t_j}$, we have:

$$\sigma(2^2\ell^{t_s+t_j}) < \sigma(2^2\ell^{2t_s}) < q', p$$

.

Thus, by 1.7, $2^2\ell^{t_s+t_j}q'$ and $2^2\ell^{t_s+t_j}p_{\ell+1,j}$ are non-3-layered. Also, according to (iii) and (iv), we have:

$$\sigma(2^2\ell^{t_s+t_j}q') < \sigma(2^2\ell^{2t}q') < p \text{ , or } \sigma(2^2\ell^{t_s+t_j}p) < \sigma(2^2\ell^{2t_s}p) < q'.$$

Thus, by 1.7 the edge v_jv_s is non-3-layered relative to our labeling. □

4.3. **some labeling related to 4-layered numbers.** .

Now, we once again state a theorem like 4.5, for 4-layered numbers.

Theorem 4.9. *Let K_n denotes a complete graph on n vertices with vertex set V and edge set E. For every positive integer $k \leq \frac{n(n-1)}{2}$, We can find an f-labeling for V such that $e^*_f(1) = k$. ($e^*_f(1)$ computes 4 -layered edges relative to our labeling.).*

Proof. It is easy to check that the statement holds for K_1, K_2 and K_3. Let $m > 3$ be a positive integer. Now, we want to prove that the statement holds for the complete graph K_m. Let $\ell = 2 \times 3 \times 5 \times 7$. We choose the even number t_0 large enough that there exists a chain of even numbers, $3 < t_m < \cdots < t_2 < t_1 < t_0$ which for every positive integer $1 \leq r \leq m$, we can find distinct primes $p_{r,1}, p_{r,2}, \ldots, p_{r,2m-2}$ such that for every positive integers, $1 \leq i \leq m$ and $2 \leq j \leq 2m-2$ we have:

(i) $\sigma(\ell^{2t_i}) < p_{i,1} < \sigma(2^{t_{i-1}})$

(ii) $\sigma(\ell^{2t_i}p^3_{i,j-1}) < p_{i,j} < \sigma(2^{t_{i-1}})$

(iii) If $i \neq m$, then $\sigma(\ell^{2t_i}p^3_{i,2m-2}) < p_{i+1,1}$.

Suppose that $V = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of K_m. We label v_1 with positive integer $n_1 = \ell^t_1$. Now, If we want the edge v_2v_1 to be non-4-layered relative to our labeling, then we label v_2 with $n_{2,0} = \ell^{t_2}p_{1,1}$ because by 1.3, the number $\ell^{t_1+t_2}$ is non-Zumekeller and by (i), for number $n_{2,0}n_1$, we have:

$$\sigma(\ell^{t_1+t_2}) < \sigma(\ell^{2t_1}) < p_{1,1}$$

Thus, according to 1.6, the number $n_1n_{2,0}$, which is the label of the edge v_2v_1 relative to our labeling, is non-4-layered. Also, if we want the edge v_2v_1 to be

4-layered we label v_2 with $n_{2,1} = \ell^{t_2} p_{2,1} p_{2,2}$ because by (i), for number $\ell^{t_1+t_2} p_{2,1}$, we have:

$$p_{2,1}, p_{2,2} < \sigma(2^{t_1}) < \sigma(\ell^{t_1+t_2})$$

Thus, by 1.34, the number $n_{2,1}n_1$, which is label of v_2v_1 relative to our labeling, is 4-layered. After labeling the vertices v_1 and v_2, for labeling the vertex v_3, if we want both edges of $\{v_3v_1, v_3v_2\}$ to be non-4-layered, then we label v_3 with $n_{3,0}\ell^{t_3} p_{1,2}$ because first of all, by 1.3, we know the number $\ell^{t_1+t_3}$ is a non-4-layered number. In addition, by (i) and (ii), we have:

$$\sigma(\ell^{t_1+t_3}) < \sigma(\ell^{2t_1}) < p_{1,1} < p_{1,2}$$

Thus, by 1.6, the number $n_{3,0}n_1$, which is label of the edge v_3v_1 relative to our labeling, is non-4-layered. Moreover, we know the vertex v_2 is labeled with number $\ell^{t_2}q$ such that either $q \in \{p_{1,1}, p_{2,1}p_{2,2}\}$ and $f = \ell^{t_2+t_3}q$ is non-4-layered number because first of all, we know the number $\ell^{t_2+t_3}$ is non-Zumkeller. In addition, if $q = p_{1,1}$, then by (i), we have:

$$\sigma(\ell^{t_2+t_3}) < \sigma(\ell^{2t_2}) < p_{1,1}$$

Thus, by 1.6, f is non-4-layered. Also, if $q = p_{2,1}p_{2,2}$, then by (ii) , we have:

$$\sigma(\ell^{t_2+t_3}) < p_{2,1}, \sigma(\ell^{t_2+t_3} p_{2,1}) < p_{2,2}$$

Thus, once again, by 1.6, we conclude that f is non-4-layered. At last, by (i) and (ii), we have:

$$\sigma(\ell^{t_2+t_3} q) < \sigma(\ell^{t_2+t_3} p_{1,1}^3) < p_{1,2}$$

Thus, once again, by 1.6, the number $n_{3,0}n_1$, which is label of the edge v_3v_1 relative to our labeling is non-4-layered. Now, if we want one edge of $\{v_3v_2, v_3v_1\}$ to be 4-layered, then we label v_3 with $n_{3,1} = \ell^{t_3} p_{2,3} p_{2,4}$ because in this situation, by (ii) , for number $n_1 n_{3,1} = \ell^{t_1+t_3} p_{2,3} p_{2,4}$ we have :

$$p_{2,3}, p_{2,4} < \sigma(2^{t_1}) < \sigma(2^{t_1+t_3})$$

Therefore, by 1.34, $n_{3,1}n_1$, which is label of edge v_3v_1 relative to our labeling, is 4-layered number. Also, as we said the vertex v_2 is labeled with number $\ell^{t_2}q$ such that the number $\ell^{t_2+t_3}q$ is a non-4-layered and according to (i) and (ii), once again by 1.6 the number $n_{3,1}n_1$, which is label of edge v_3v_2 relative to our labeling, is

non-4-layered. At last, if we want both edges of $\{v_3v_2, v_3v_1\}$ to be 4-layered, then by 1.6, it is sufficient that we label v_3 with number $n_{3,2} = \ell^{t_3}p_{3,1}p_{3,2}$ (Note that it is easy to check that $p_{3,1}, p_{3,2} < \sigma(2^{t_1+t_3}), \sigma(2^{t_1+t_2})$.).
Let $j > 3$ be an integer and by this method, we labeled $j-1$ vertices of a complete graph K_m. Now, we want to label the vertex v_j. Suppose that h be an integer such that $0 \leq h < j \leq m$. Now, let we want to have exactly h 4-layered edges of edges $\{v_jv_1, v_jv_2, \ldots, v_jv_{j-1}\}$ relative to our labeling. If $h = 0$, then by (i) and 1.6, it is sufficient that we label v_j with number $\ell^{t_j}p_{j-1}$. Also, if $h \neq 0$, then we label v_j with $\ell^{t_j}p_{2h+1}p_{2h+2}$ because first of all, by (ii) and 1.34, for every integer i which $1 < i < h$, v_jv_i is 4-layered edge relative to our labeling. In addition, once again, it is easy to check that for every positive integer s such that $h < s < j$, the edge v_jv_s is non-4-layered. □

Remark 4.10. *It is easy to check that we can state something like 4.6 and 4.7 for 4-layered graphs.*

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