MULTISCALE SUBSTITUTION TILINGS

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ABSTRACT. We introduce multiscale substitution tilings, which are a new family of tilings of Euclidean space. These tilings are generated by substitution schemes on a finite set of prototiles, in which multiple distinct scaling constants are allowed. This is in contrast to the standard case of the well-studied substitution tilings which includes examples such as the Penrose and the pinwheel tilings. Under an additional irrationality assumption on the scaling constants, our construction defines a new class of tilings and tiling spaces, which are intrinsically different from those that arise in the standard setup. We study various structural, geometric, statistical and dynamical aspects of these new objects and establish a wide variety of properties. Among our main results are explicit density formulas and the unique ergodicity of the associated tiling dynamical systems.

1. INTRODUCTION

In the construction of substitution tilings, which are a classical object of study within the field of aperiodic order and mathematical models of quasicrystals, a standing assumption is that the substitution rule that generates the tiling is associated with a single unique scaling constant. More precisely, given a set of initial tiles, also known as prototiles and usually assumed to be finite, the substitution rule describes a tessellation of each prototile by rescaled copies of prototiles, where the applied scaling constant is unique, and hence we refer to it as a fixed scale substitution rule. This implies that a uniform inflation by the reciprocal constant defines a patch of tiles, each of which is a copy of a prototile under some isometry of the space. By repeating this process countably many times, a tiling of the space can be defined, with the property that all of the tiles that appear in the tiling are copies of the original prototiles. The beautiful examples of the Penrose and pinwheel tilings, see [Pe] and [Rad], among other well-studied examples, can be constructed in this way, with isometry groups of translations and rigid motions, respectively. For a comprehensive discussion and additional examples see [BG] and references therein.

When considering the construction of substitution tilings, a natural question that may arise concerns the scaling constant: what kind of tilings emerge if the standard assumption of a single scaling constant is relaxed?

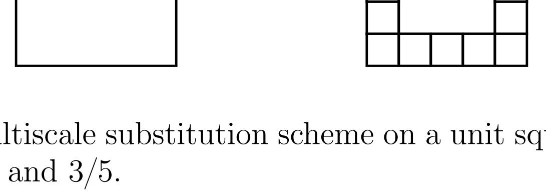

FIGURE 1. A multiscale substitution scheme on a unit square S, with scaling constants $1/5$ and $3/5$.

Clearly, if one is to use multiple distinct scaling constants, as in the example illustrated in Figure 1, an attempt to define tilings by consecutive substitutions and inflations in the way described above may result in tilings with either arbitrarily small or arbitrarily large tiles. These tilings would not naturally induce Delone sets, which are often used to model physical structures. In order to generate tilings with tiles of bounded scale, a different procedure must be used.

We propose here a general procedure to overcome the above mentioned difficulty. In the present introductory section we shall describe and introduce the new process in a somewhat intuitive way, a precise and detailed presentation will be given in the next sections. Start with a single tile, and inflate it continuously. When its volume reaches a certain threshold, which is set *a priori* to be the unit volume, substitute the tile according to the prescribed substitution rule. Continue to inflate, and substitute any tile when its volume reaches the threshold. This defines a continuous family of patches of tiles within some bounded interval of scales, and from which tilings of the entire space can be defined as limiting objects with respect to a suitable topology. For an example of such a patch see Figure 2, and note that it can also be defined by considering an inflation of the original tile, which is then substituted until all tiles are of volume smaller than the threshold. The collection of all tilings generated by a given substitution scheme defines the associated multiscale tiling space, which is a compact space of tilings and is closed under translations.

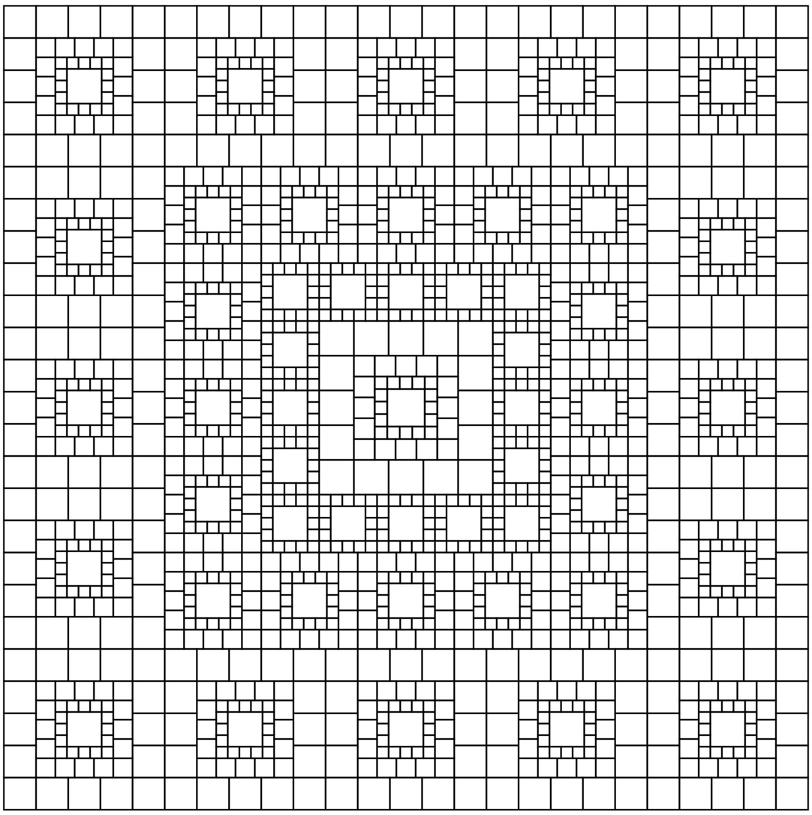

FIGURE 2. A patch of a multiscale substitution tiling.

The aforementioned semi-flow, which we call the substitution flow, is shown to be well defined on the space of tilings. The substitution flow has periodic orbits which give rise to stationary points in this space, and these stationary tilings can be represented as limits of sequences of nested patches. As we will see, in the fixed scale case, this procedure generates tilings that can be defined by the standard substitution and inflation procedure.

Our focus in this paper is on tilings generated by incommensurable multiscale substitution schemes on a finite set of prototiles, which are classified by a certain irrationality assumption, as will be precisely defined in §3. For example, the substitution scheme illustrated in Figure 1 is incommensurable. In this example incommensurability follows from the fact that the two participating scaling constants $1/5$ and $3/5$ are such that $(1/5)^m \neq (3/5)^n$ for any $0 \neq m, n \in \mathbb{Z}$, which is true because 3 and 5 are co-prime. In general, incommensurability amounts to the existence of at least two periodic orbits of the substitution flow for which the logarithms of their lengths are linearly independent over the rationals. As we will see, there are more natural ways to define incommensurability, and incommensurability can be considered a typical property of multiscale substitution schemes. We remark that as in the fixed scale case, for non-incommensurable schemes, which we refer to as commensurable, the generated multiscale tilings can be defined also in the standard way, though perhaps a larger set of prototiles is required. In some aspects, fixed scale, commensurable and incommensurable schemes are analogous to integer, rational and irrational real numbers. In particular, fixed scale schemes are generalized by commensurable schemes, while incommensurability is a complement, disjoint property.

We present a thorough study of this new class of incommensurable tilings and their properties. Such tilings consist of tiles of finitely many types that appear in infinitely many scales, and are therefore of infinite local complexity. The standard tools of the substitution matrix and the theory of Perron-Frobenius are no longer applicable, and the new tools of the associated directed weighted graph and the recent results of [KSS] on the distribution of paths on incommensurable graphs are introduced. We show that incommensurability replaces the role of primitivity, and present explicit formulas for the asymptotic density of tiles, as well as for frequencies of tiles of certain types and scales and the volume they occupy. In addition we discuss a form of scale complexity and show that the Delone sets associated with incommensurable tilings are not uniformly spread, in the sense that associated point sets are never bounded displacement equivalent to lattices. All the relevant terms will be precisely defined in the coming sections.

Although the construction is well defined and many of our results hold also for the case where tiles are substituted by rescaled copies of prototiles under some isometry, when considering the associated tiling dynamical system, which is defined with respect to the action by translations, we focus on the case where only translations of rescaled copies are allowed in the substitution rule. In such a case, the tiling dynamical system is shown to be minimal. Minimality, combined with the properties of the substitution flow on the tiling space, allows for the existence of supertiles for any tiling in the tiling space, which form an extremely useful hierarchical structure on tilings. The appropriate variants of uniform patch frequencies, in which patches are counted together with their dilations in some non-trivial interval of scales, are shown to hold. Finally, we show that as in the fixed scale case, the existence of uniform patch frequencies can be used to establish unique ergodicity of the tiling dynamical system.

A detailed introduction of multiscale substitution schemes, the associated graphs and the precise notion of incommensurability, appears in §2 and §3. Two large illustration of fragments of multiscale substitution tilings are included as an appendix. The main results and the structure of the paper are summarized as follows, where the section number indicates the place in the paper where the result is properly stated and proved.

Structural results. *An incommensurable multiscale substitution scheme generates a multiscale tiling space. The substitution flow acts on the multiscale tiling space, and* *periodic orbits give rise to stationary tilings §4. Each tiling is equipped with a hierarchical structure of supertiles, though not necessarily in a unique way §6.*

Geometric results. *For every tiling in the multiscale tiling space, all tiles are similar to rescaled copies of prototiles §4. They appear in a dense set of scales within certain intervals of possible scales, and the same holds for any legal patch. Scale complexity is defined for stationary tilings, and the existence of "Sturmian" tilings is established §5. Appropriate variants of almost repetitivity and almost local indistinguishability are shown to hold §6. Tilings are not uniformly spread §8.*

Statistical results. *Explicit asymptotic formulas for the number of tiles of a given type and scales within a given interval, that appear in large supertiles, are given §7. A variant of uniform patch frequencies is established, where patches are counted together with dilations. For any non-trivial interval of dilations, legal patches have positive patch frequencies §9. When considered without dilations, all patches have uniform frequency zero.*

Dynamical results. *The multiscale tiling space is equipped with an action by translations to form a tiling dynamical system, which is minimal §6, and in fact uniquely ergodic §10.*

The definition and study of multiscale substitution tilings is part of the recent resurgence of interest in tilings that are not assumed to be of finite local complexity, but still possess rich structure, hierarchies and symmetries. See among others [FRi, FRo, FS] and [LSo], and the earlier [D, Ke], and [Sa], where generalized pinwheel tilings of the plane, which can be viewed as multiscale substitution tilings with a single prototile, are introduced. It is our hope that this new class of tilings and the examples it produces will become an object of study in the community of aperiodic order, as there are still many interesting questions to consider, both of geometric and of dynamical flavor.

The construction of incommensurable tilings and the associated multiscale tiling spaces may prove interesting in relation to various other research directions. Incommensurable α-Kakutani substitution schemes, studied in [Sm] in the context of Kakutani sequences of partitions [Ka], generate one dimensional tilings. Every tile is a segment, and by identifying the endpoints of segments with point masses, the unique translation invariant measure on the tiling dynamical space defines a new type of point process on the real line, see [BBM, BL] for more on spectral theory, diffraction and point processes in the context of aperiodic order. Another example is the relation between the substitution flow and the theory of hyperbolic dynamics and dynamical zeta functions, see [PP]. The study of uniform distribution and discrepancy of sequences of partitions is also closely related [Sm], as is the study of self-similar non-lattice fractal strings and sprays, see [LV] and references therein.

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2. Substitution schemes and the substitution flow

We begin with an introduction to multiscale substitution schemes and their associated graphs, and with detailed definitions of the basic objects required for the definition and study of multiscale substitution tilings, which will be defined in §4.

2.1. Tiles and multiscale substitution schemes.

A *tile* T in $\mathbb{R}^d$ is a bounded Lebesgue measurable set of positive measure, which is denoted by vol$T$ and referred to as the *volume* of T, and with a boundary of measure zero. A *tessellation* $\mathcal{P}$ of a set $U \subset \mathbb{R}^d$ is a collection of tiles with pairwise disjoint interiors so that the union of their support is U. For the sake of clarity, a tessellation of a bounded set will be called a *patch* and a tessellation of the entire space will be called a *tiling*. Given a patch $\mathcal{P}$, a tiling $\mathcal{T}$ and a subset B of their support, the sub-patch consisting of all tiles in $\mathcal{P}$ that intersect B is denoted by $[B]^{\mathcal{P}}$, and the sub-patch $[B]^{\mathcal{T}}$ is similarly defined. Note that we view tiles as embedded subsets of $\mathbb{R}^d$, though the location of the origin usually does not matter to us. We will specify the location of the origin when it is important.

Definition 2.1. A *multiscale substitution scheme* $\sigma = (\tau_\sigma, \varrho_\sigma)$ in $\mathbb{R}^d$ consists of a finite list of labeled tiles $\tau_\sigma = (T_1, \ldots, T_n)$ in $\mathbb{R}^d$ called *prototiles*, and a *substitution rule* defining a tessellation $\varrho_\sigma(T_i)$ of each prototile T_i, so that every tile in $\varrho_\sigma(T_i)$ is a translation of a rescaled copy of a prototile in τ_σ. We denote by $\omega_\sigma(T_i)$ the list of rescaled prototiles whose translations appear in the patch $\varrho_\sigma(T_i)$, presented as

$$\omega_\sigma(T_i) = \left(\alpha_{ij}^{(k)} T_j : j = 1, \ldots, n, \; k = 1, \ldots, k_{ij}\right),$$

and referred to as the *tiles of substitution*. Here $\alpha_{ij}^{(k)}$ are positive constants, and k_{ij} is the number of *tiles of type* j in $\varrho_\sigma(T_i)$, that is, the number of rescaled copies of T_j in $\varrho_\sigma(T_i)$. We assume throughout that $T_i \neq \alpha T_j$ for any $i \neq j$ and all $\alpha > 0$.

It is helpful to consider the analogy to jigsaw puzzles, where the prototiles τ_σ are puzzles to be solved using the pieces in ω_σ, and ϱ_σ gives a solution $\varrho_\sigma(T_i)$ to each of the puzzles. We will refer to multiscale substitution schemes also as *substitution schemes*, and occasionally simply as *schemes*. Unless otherwise stated, all schemes are in $\mathbb{R}^d$ and τ_σ consists of n prototiles $T_1, \ldots, T_n$.

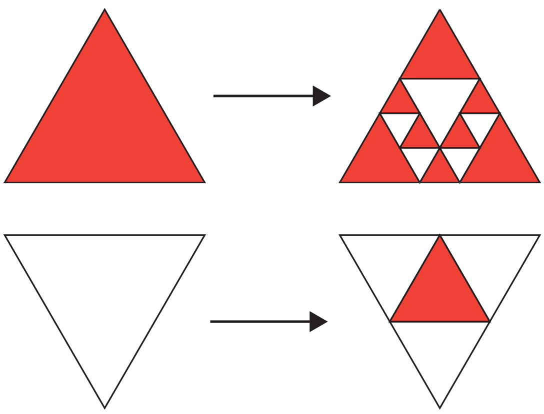

Figure 3. A multiscale substitution scheme on two triangles U and D.

Example 2.2. Figure 3 describes a multiscale substitution scheme σ in $\mathbb{R}^2$. The set of $n = 2$ prototiles $\tau_\sigma = (U, D)$ consists of an equilateral triangle U directed up and an equilateral D triangle directed down, both of volume 1. The tessellations $\varrho_\sigma(U), \varrho_\sigma(D)$ are illustrated on the right-hand side of the figure. For example, $\omega_\sigma(U)$ consists of three copies of $\frac{2}{5}U$, five copies of $\frac{1}{5}U$, four copies of $\frac{1}{5}D$ and a single copy of $\frac{2}{5}D$.

Remark 2.3. Substitution schemes can also be defined so that tiles in $\varrho_\sigma(T_i)$ are only assumed to be isometric to the tiles of substitution, instead of the stronger restriction of being their translations. Examples include the substitution schemes that generate Sadun's generalized pinwheel tilings, see [Sa], and we note that many of the constructions and results discussed below may be extended also to this more general construction.

Definition 2.4. A substitution scheme is *normalized* if all prototiles are of unit volume. Two substitution schemes are said to be *equivalent* if their prototile sets consist of the same tiles up to scale changes, and the tessellations of the prototiles prescribed by the substitution rules are the same up to an appropriate change of scales.

Geometric objects such as patches, tilings and sequences of partitions that are defined using equivalent substitution schemes are identical up to a rescaling by some positive constant. Clearly, every equivalence class of schemes contains a unique normalized scheme. The geometric nature of our construction implies the following result.

Proposition 2.5. *Let σ be a normalized substitution scheme. For every $i = 1, \ldots, n$ the constants α_{ij}^k satisfy the following algebraic equations:*

$$\sum_{j=1}^{n} \sum_{k=1}^{k_{ij}} \left(\alpha_{ij}^{(k)}\right)^d = \sum_{j=1}^{n} \sum_{k=1}^{k_{ij}} \mathrm{vol}\left(\alpha_{ij}^{(k)} T_j\right) = \mathrm{vol}(T_i) = 1.$$

Given a substitution scheme σ, the constants $\alpha_{ij}^{(k)}$ associated with the equivalent normalized scheme are called the *constants of substitution*. From here on, **all substitution schemes are assumed to be normalized**.

2.2. **Graphs associated with multiscale substitution schemes.** Denote by $G = (\mathcal{V}, \mathcal{E}, l)$ a directed weighted multigraph with a set of vertices $\mathcal{V}$ and a set of weighted edges $\mathcal{E}$, with positive weights which are regarded as lengths. A *path* in G is a directed walk on the edges of G that originates and terminates at vertices of G. A *metric path* in G is a directed walk on edges of G that does not necessarily originate or terminate at vertices of G. An edge of weight a is equipped with a linear parametrization by the interval $[0, a]$, and the parametrization is used to define the *path distance* l on edges, paths and metric paths in G. In our terminology an edge is assumed to contain its terminal vertex but not its initial one, that is, every vertex is seen as a point contained in each of its incoming edges. The path of length zero with initial vertex i is assumed to consist only of the vertex i.

We now define the graph associated with a substitution scheme.

Definition 2.6. Given a substitution scheme σ, the *associated graph* G_σ is the following directed weighted graph. The vertices $\mathcal{V} = \{1, \ldots, n\}$ are defined according to the prototiles $\tau_\sigma = (T_1, \ldots, T_n)$, where the vertex $i \in \mathcal{V}$ is associated with the prototile $T_i \in \tau_\sigma$. The edges $\mathcal{E}$ are defined according to the tiles of substitution in ω_σ, where the edge $\varepsilon \in \mathcal{E}$ associated with the tile $\alpha T_j \in \omega_\sigma(T_i)$ has initial vertex i, terminal vertex j and is of length

$$l(\varepsilon) := \log \tfrac{1}{\alpha}.$$

Note that G_σ depends only on the elements of $\omega_\sigma(T_i)$ for $T_i \in \tau_\sigma$, and not on the specific configuration in which they appear in the patches $\varrho_\sigma(T_i)$. In other words, G_σ can be thought of as the abelianization of σ, just as the substitution matrix is the abelianization of a fixed scale scheme, see e.g. [BG]. We also remark that if σ is a fixed scale scheme, the lengths of all the edges in G_σ are the same and the adjacency matrix of G_σ, when thought of as a combinatorial graph, is precisely the substitution matrix of σ.

Example 2.7. The graph associated with the square substitution scheme described in the introduction is illustrated in Figure 4. It consists of a single vertex associated with the single prototile S, and self loops associated with the tiles of substitution. There is a single loop of length $\log\frac{5}{3}$ associated with the larger square $\frac{3}{5}S \in \omega_\sigma(S)$ and sixteen distinct loops of length $\log 5$ associated with the sixteen smaller squares $\frac{1}{5}S \in \omega_\sigma(S)$.

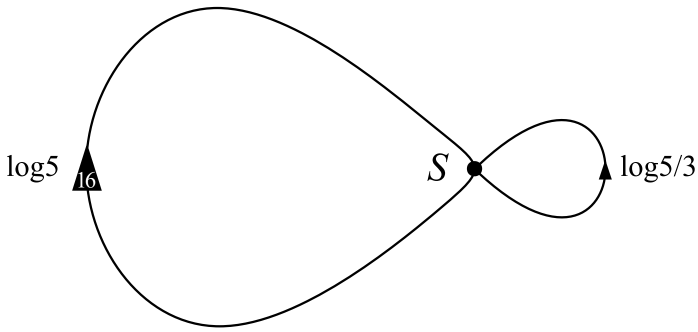

FIGURE 4. The graph associated with the substitution scheme on the square S. The large arrow stands for 16 distinct loops of length $\log 5$.

Example 2.8. Figure 5 illustrates the graph associated with the triangle substitution scheme described in Figure 3. It consists of $n = 2$ vertices associated with the prototiles (U, D). Edges associated with tiles in $\omega_\sigma(U)$ initiate from the vertex on the left, and edges associated with tiles that are rescaled copies of D terminate at the vertex on the right.

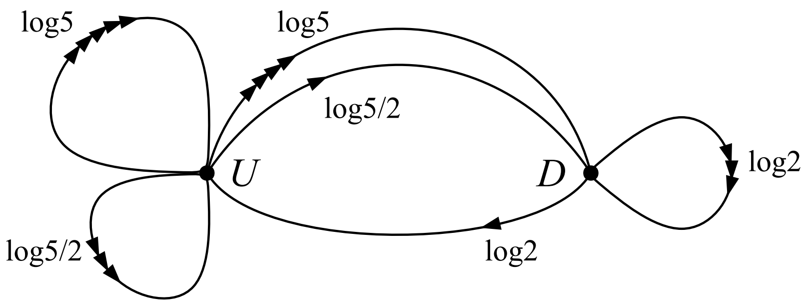

FIGURE 5. The graph associated with the substitution scheme on the triangles U and D. Multiple arrows stand for multiple distinct edges.

Remark 2.9. Given a substitution scheme σ, the graph associated with any equivalent scheme can be derived from G_σ by sliding its vertices along its paths so that the lengths of all closed paths is not changed. For a general discussion and additional examples of graphs associated with substitution schemes see [Sm, §4].

2.3. **The substitution flow and the generating patches.** We now introduce important elements in the definition and study of multiscale substitution tilings.

Definition 2.10. Let σ be a substitution scheme. The *substitution flow* $F_t(T_i)$, where $t \in \mathbb{R} \geqslant 0$ is referred to as *time*, defines a family of patches in the following way. At $t = 0$, set $F_0(T_i) = T_i$, which is a patch consisting of a single tile. As t increases, inflate the patch by a factor e^t, and substitute tiles of volume larger than 1 according to the substitution rule ϱ_σ. Equivalently, $F_t(T_i)$ is the patch supported on $e^t T_i$, which is the result of the repeated substitution of $e^t T_i$ and all subsequent tiles with volume greater than 1 in $e^t T_i$, until all tiles in the patch are of unit volume or less.

Fix a position of T_i so that the origin of $\mathbb{R}^d$ is an interior point, and denote

$$\mathscr{P}_i := \left\{ F_t(T_i) : t \in \mathbb{R}^+ \right\}, \tag{2.1}$$

where $\mathbb{R}^+ := \{t \geqslant 0 : t \in \mathbb{R}\}$. Note that $\mathscr{P}_i$ exhausts $\mathbb{R}^d$. The patches $\mathscr{P}_\sigma := \bigcup_{i=1}^n \mathscr{P}_i$ are called the *generating patches* of the scheme σ.

Example 2.11. Consider the substitution flow associated with the substitution scheme on the two triangles U, D as illustrated in Figure 3. The patches illustrated in Figure 6 are the first elements of $\mathscr{P}_\sigma$ with the property that a tile of unit volume appears in the patch. The times $t \in \mathbb{R}^+$ in which they appear are also given, and note that the patch $F_t(U)$ is supported on a triangle of side length e^t.

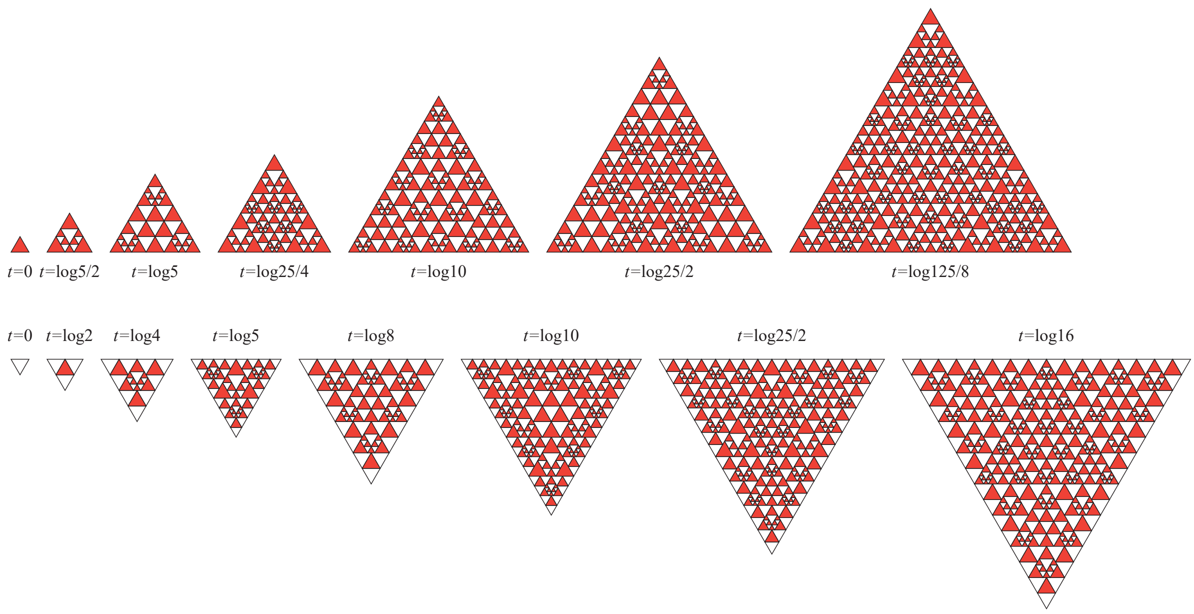

FIGURE 6. The patches $F_t(U)$ and $F_t(D)$ for the first values of t for which the patches contain tiles of unit volume.

A very useful observation is that the substitution flow can be modeled by the flow along the edges of the associated graph G_σ, and that tiles in a patch in $\mathscr{P}_\sigma$ correspond to metric paths in G_σ. This correspondence is summarized in the following proposition, which follows directly from our definitions.

Proposition 2.12. *Let σ be a substitution scheme with an associated graph G_σ. Let $T_i \in \tau_\sigma$, fix $t \in \mathbb{R}^+$ and consider a tile T in the patch $F_t(T_i)$.*

(1) *T corresponds to a unique metric path γ_T in G_σ that originates at vertex i and is of length t. The edges in γ_T are determined according to the sequence of ancestors of T under the substitution flow.*
(2) *If T is of type j and scale $\alpha = e^{-\delta}$, then γ_T terminates at a point on an edge ε with terminal vertex j, and the termination point is of distance $\delta = \log \frac{1}{\alpha}$ from j.*
(3) *If γ_T is of length t and terminates exactly at vertex j, then T is a tile of type j and unit volume.*

Corollary 2.13. *For every $t \in \mathbb{R}^+$, the patch $F_t(T_i) \in \mathscr{P}_i$ corresponds to the family of all metric paths of length t that originate at vertex i, and every path in G_σ that originates at vertex i and is of length $s > t$ is the continuation of exactly one of these paths.*

Example 2.14. A very simple but fundamental family of multiscale substitution schemes are the so-called α-Kakutani schemes in $\mathbb{R}^1$, for $\alpha \in (0,1)$, which can be shown to generate the α-Kakutani sequences of partitions of the unit interval, first introduced in [Ka], see also [FS, Appendix 5] for a related construction. The unit interval I is the single prototile, and it is substituted by two intervals, one of length α and the other of length $1-\alpha$. For $\alpha = \frac{1}{3}$, the associated graph consists of a single vertex associated with the unit interval I, and two loops - the longer one, of length $\log 3$, is associated with the interval $\frac{1}{3}I \in \varrho_\sigma(I)$, and the shorter, of length $\log \frac{3}{2}$, is associated with $\frac{2}{3}I \in \varrho_\sigma(I)$.

Figure 7 concerns with the $\frac{1}{3}$-Kakutani substitution scheme, and illustrates the first three elements of $\mathscr{P}_\sigma$ with the property that a tile (interval) of unit volume appears in the patch, together with the metric paths associated with the tiles comprising the patches. The right-most interval in each patch corresponds to the top path beneath it. Note that the patch on the left is $F_0(I)$ and so the single tile in it corresponds to the single metric path of zero length. Also note that tiles of volume 1 correspond to metric paths that terminate at the vertex.

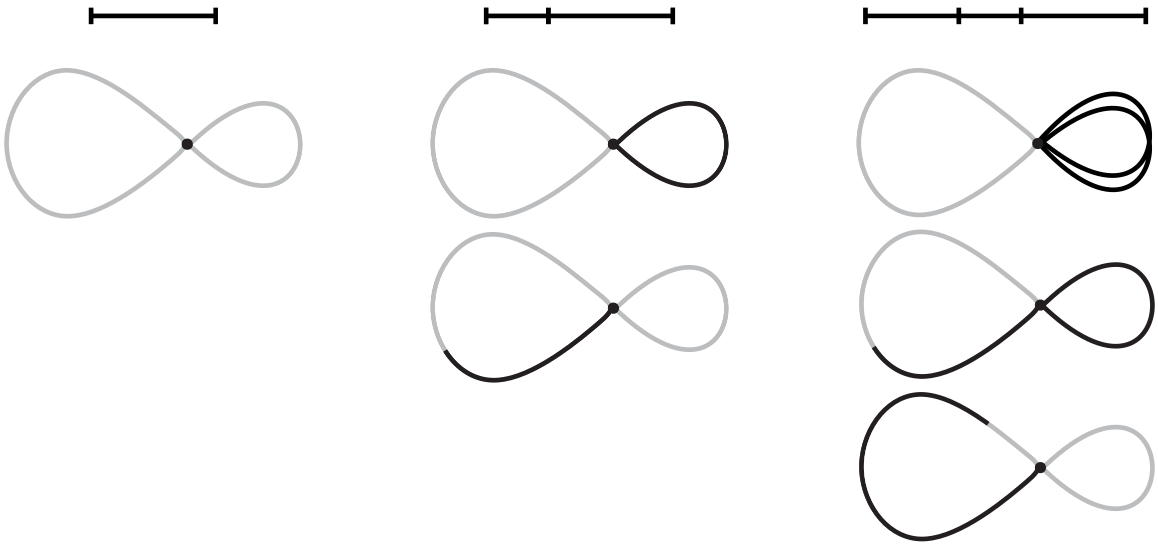

FIGURE 7. The patches $F_0(I)$, $F_{\log 3/2}(I)$ and $F_{2\log 3/2}(I)$ and the metric paths associated with the tiles that comprise them.

3. Incommensurable substitution schemes

The following fundamental properties of substitution schemes are easier to present in terms of the substitution flow and the associated graph.

Definition 3.1. A substitution scheme σ is *irreducible* if for any $i, j = 1, \ldots, n$ there exists $t > 0$ so that $F_t(T_i)$ contains a tile of type j.

In terms of the associated graph G_σ, this is equivalent to the statement that G_σ is *strongly connected*, that is, for any $i, j \in \mathcal{V} = \{1, \ldots, n\}$ there exists a path with initial vertex i and terminal vertex j. We remark that this definition coincides with the definition of this notion in the fixed scale setup, see e.g. [BG, §2.4]. From here on **all schemes are assumed to be irreducible**.

Definition 3.2. A substitution scheme σ is *incommensurable* if there exist $T_i, T_j \in \tau_\sigma$ and two tiles of type i and j in patches in $\mathscr{P}_i$ and $\mathscr{P}_j$, respectively, which are of unit volume at times t_1 and t_2, with $t_1 \notin \mathbb{Q}t_2$. Otherwise, the scheme is called *commensurable.*

In terms of the associated graph G_σ, incommensurability is equivalent to having two closed paths in G_σ of lengths a and b, with $a \notin \mathbb{Q}b$, in which case G_σ is said to be an *incommensurable graph.* This follows from part (3) of Proposition 2.12, because a tile T of type i and unit volume in $F_t(T_i)$ corresponds to a closed path in G_σ with initial and terminal vertex i and with length t. For equivalent definitions and more on incommensurability see §3.1.

In view of Remark 2.9, incommensurability does not depend on the choice of representative of the substitution scheme equivalence class, in the sense of Definition 2.4. Note that examples of incommensurable schemes are quite easy to come up with. In fact, incommensurability can be thought of as typical property of substitution schemes, in the sense that for a naive choice of substitution rules, the resulting scheme is incommensurable.

Example 3.3. The substitution schemes illustrated in Figures 2 and 3 are both normalized, since all prototiles are assumed to be of unit volume. In addition they are both irreducible and incommensurable. This can be easily verified by the graphs illustrated in Figures 4 and 5, as both graphs are strongly connected, and both contain pairs of loops of incommensurable lengths.

Remark 3.4. Commensurable schemes include all *fixed scale* substitution schemes, which are the schemes in which all constants of substitution are equal. The Rauzy fractal scheme introduced in [Rau] can be viewed as an example of a commensurable multiscale substitution scheme on a single prototile, which is not of fixed scale. For a further discussion and illustrated examples see [Sm].

3.1. **Equivalent definitions of incommensurability.**

As we focus in the coming sections on incommensurable substitution schemes, we now present some useful equivalent conditions to incommensurability of graphs and schemes.

Lemma 3.5. *Let G be a strongly connected directed weighted graph. The following are equivalent:*

1. *The graph G is incommensurable.*
2. *Every vertex in G is contained in two closed paths of incommensurable lengths.*
3. *The set of lengths of closed paths in G is not a uniformly discrete[1] subset of $\mathbb{R}$.*

[1] A set S in $\mathbb{R}^d$ is *uniformly discrete* if $\inf\{\mathbf{dist}(x, y) : x, y \in S\} > 0$, where **dist** is Euclidean distance.

Proof. (1) $\iff$ (2) It is enough to show that there exist two closed paths of incommensurable lengths that pass through all vertices of G. Let α and β be two closed paths of lengths $a, b \in \mathbb{R}$ with $a \notin \mathbb{Q}b$, and assume that α passes through vertex i and β passes through vertex j. Since G is strongly connected, there exists a closed path γ that passes through all the vertices in G, and we denote the length of this path by c.

Using the closed paths α, β and γ we can construct closed paths that pass through all vertices of G and are of lengths $a + c, a + 2c$ and $b + c$. By direct calculation

$$\frac{a+c}{b+c} \in \mathbb{Q} \Longrightarrow \frac{a+2c}{b+c} \notin \mathbb{Q},$$

for otherwise $\frac{a}{b} \in \mathbb{Q}$, which is a contradiction. We conclude that either the paths of lengths $a + c$ and $b + c$ or those of lengths $a + 2c$ and $b + c$ constitute a pair of closed paths of incommensurable lengths that pass through all vertices of G. The converse is trivial.

(1) $\iff$ (3) If there exist two closed paths in G of lengths a, b such that $a \notin \mathbb{Q}b$, then for every $\varepsilon > 0$ there exist $p, q \in \mathbb{N}$ such that $|aq - pb| < \varepsilon$, and so the set of lengths of closed paths in G is not uniformly discrete.

Conversely, assume that G is commensurable, that is, $a \in \mathbb{Q}b$ for any two lengths a, b of closed paths in G. Since G is a finite graph, there is a finite set L of lengths for which the length of any closed path in G is a linear combination with integer coefficients of elements in L. It follows that there exists some $c > 0$ so that every closed path is of length which is an integer multiple of c, and so the set of lengths of closed paths is uniformly discrete. ☐

Definition 3.6. Given a substitution scheme σ, denote

$$\mathscr{S}_{i\to j} := \left\{t \in \mathbb{R}^+ \,:\, \text{a tile of type } j \text{ and unit volume appears in } F_t\left(T_i\right)\right\}, \qquad (3.1)$$

and set $\mathscr{S}_i := \bigcup_{j=1}^{n} \mathscr{S}_{i\to j}$.

Irreducibility of σ implies that the sets $\mathscr{S}_{i\to j}$ are all infinite. Note that by Proposition 2.12, an equivalent definition is

$$\mathscr{S}_{i\to j} = \left\{t \in \mathbb{R}^+ \,:\, \text{a path of length } t, \text{origin } i \text{ and termination } j \text{ appears in } G_\sigma\right\}.$$

Example 3.7. The times t that appear in Figure 6 are the smallest elements in the sets $\mathscr{S}_1$ and $\mathscr{S}_2$, where T_1 is the triangle U, and T_2 is the triangle D.

Lemma 3.8. *Let σ be an irreducible substitution scheme, let $T_i \in \tau_\sigma$ and let $s_1 < s_2 < \dots$ be an increasing enumeration of the set $\mathscr{S}_{i\to i}$. Then σ is incommensurable if and only if*

$$\lim_{m\to\infty} s_{m+1} - s_m = 0.$$

Proof. Assume that σ is incommensurable, and let $\varepsilon > 0$. By Lemma 3.5, the associated graph G_σ has two closed paths α, β through vertex i, which are of lengths $a < b$ with $a \notin \mathbb{Q}b$. Therefore, there exists some $M \in \mathbb{N}$ so that the set

$$\{ma (\mathrm{mod}\ b) \,:\, m \in \{1, \dots, M\}\}$$

is ε-dense in $[0, b)$.

For each $m = 1, \dots, M$ let $k_m \in \mathbb{N}$ be such that $0 \leqslant ma - k_m b < b$. Since $a < b$ we have $k_m < M$ for every m. Then the set

$$\{Mb + (ma - k_m b) \,:\, m \in \{1, \dots, M\}\}$$

is ε-dense in $[Mb, (M + 1) b)$, and hence the set

$$A := \{Nb + (ma - k_m b) \,:\, m \in \{1, \dots, M\}, M \leqslant N \in \mathbb{N}\}$$

is ε-dense in the ray $[Mb, \infty)$. On the other hand, $A \subset \mathscr{S}_{i\to i}$, since $Nb + (ma - k_m b)$ is the length of a closed path in G_σ that consists of m walks along α and $N - k_m > 0$ walks along β. Since ε is arbitrary, the assertion follows.

The converse follows directly from Lemma 3.5. □

Corollary 3.9. *Fix $T_i, T_j \in \tau_\sigma$, and let $s_1 < s_2 < \dots$ be an increasing enumeration of the set $\mathscr{S}_{i\to j}$. Then σ is incommensurable if and only if*

$$\lim_{m\to\infty} s_{m+1} - s_m = 0.$$

Proof. Irreducibility implies that for every $i, j \in \{1, \dots, n\}$ there is some $t > 0$ so that a tile of type j and unit volume appears in $F_t(T_i)$. So $s + t \in \mathscr{S}_{i\to j}$ wherever $s \in \mathscr{S}_{i\to i}$, and the assertion follows. □

4. Construction of multiscale substitution tilings and tiling spaces

Let $\mathrm{Cl}(\mathbb{R}^d)$ be the space of closed subsets of the metric space $(\mathbb{R}^d, \mathbf{dist})$, where $\mathbf{dist}$ is the Euclidean distance. Define a metric D on $\mathrm{Cl}(\mathbb{R}^d)$ by

$$D(A_1, A_2) := \inf\left(\left\{ r > 0 : \begin{array}{c} A_1 \cap B(0, 1/r) \subset A_2^{+r} \\ A_2 \cap B(0, 1/r) \subset A_1^{+r} \end{array} \right\} \cup \{1\}\right), \tag{4.1}$$

where A^{+r} stands for the r-neighborhood of the set $A \subset \mathbb{R}^d$, and $B(x, R)$ is the open ball of radius $R > 0$ centered at $x \in \mathbb{R}^d$, both with respect to the metric $\mathbf{dist}$. The topology induced by the metric D is called the *Chabauty-Fell topology*, and in the context of tiling spaces it is often called the *local rubber topology*. It follows that D is a complete metric on $\mathrm{Cl}(\mathbb{R}^d)$, and the space $\big(\mathrm{Cl}(\mathbb{R}^d), D\big)$ is compact, see e.g. [dH], [LSt] for this and more concerning this topology.

Given a patch $\mathcal{P}$ or a tiling $\mathcal{T}$ in $\mathbb{R}^d$, it can be identified with the union of the boundaries of its tiles, denoted by $\partial\mathcal{P}$ and $\partial\mathcal{T}$, respectively. With this identification, the elements of $\mathscr{P}_\sigma$, as well as tilings of unbounded regions, are viewed as elements of $\mathrm{Cl}(\mathbb{R}^d)$, and the metric D can be applied. The following result is straightforward from the definitions.

Proposition 4.1. *Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{Cl}(\mathbb{R}^d)$ be two closed subsets of $\mathbb{R}^d$, and assume that*

$$D(\mathcal{T}_1, \mathcal{T}_2) < \varepsilon$$

for some $\varepsilon > 0$. If $v \in \mathbb{R}^d$ is a vector of Euclidean norm $\|v\| \leqslant \frac{1}{2\varepsilon}$, then

$$D(\mathcal{T}_1 - v, \mathcal{T}_2 - v) < 2\varepsilon.$$

Note that for any $t \in \mathbb{R}^+$ and $T_i \in \tau_\sigma$, the patch $F_t(T_i)$ contains finitely many tiles, all of which are of volume at most 1. We deduce the following result from our definition of the substitution flow.

Proposition 4.2. *Fix $i \in \{1, \dots, n\}$ and let $t_0 \in \mathbb{R}^+$. The function $t \mapsto D(F_t(T_i), F_{t_0}(T_i))$, defined on $\mathbb{R}^+$, is left-continuous at t_0, that is,*

$$\lim_{t\to t_0^-} D(F_t(T_i), F_{t_0}(T_i)) = 0.$$

4.1. **The multiscale tiling space.** The *multiscale tiling space* generated by a substitution scheme σ is the space of all tilings $\mathcal{T}$ of $\mathbb{R}^d$ with the property that every sub-patch of $\mathcal{T}$ is a limit of translated sub-patches of elements in $\mathscr{P}_\sigma$ with respect to the metric D. The multiscale tiling space is denoted by $\mathbb{X}_\sigma^F$ and its elements are called *multiscale substitution tilings*. We will often refer to $\mathbb{X}_\sigma^F$ simply as the *tiling space*. Following the terminology of [FS], we refer to patches of tilings in $\mathbb{X}_\sigma^F$ that are sub-patches of elements of $\mathscr{P}_\sigma$ as *legal* patches, and to the rest of the patches as *admitted in the limit*.

The following Proposition 4.3 provides an equivalent definition for the tiling space $\mathbb{X}_\sigma^F$.

Proposition 4.3. *Let σ be a substitution scheme. Then*

$$\mathbb{X}_\sigma^F = \left\{\mathcal{T} = \lim_{k\to\infty} \mathcal{P}_k + v_k : \mathcal{P}_k \in \mathscr{P}_\sigma,\ v_k \in \mathbb{R}^d,\ \text{and } \mathcal{T} \text{ tiles } \mathbb{R}^d\right\}, \tag{4.2}$$

where limits are taken with respect to the metric D.

Proof. Denote by Y the right-hand side of (4.2). To see that $\mathbb{X}_\sigma^F \subset Y$, take $\mathcal{T} \in \mathbb{X}_\sigma^F$ and set $\mathcal{Q}_k = [B(0,k)]^{\mathcal{T}}$, the patch that consists of all tiles in $\mathcal{T}$ that intersect the ball of radius k around the origin. On the one hand, clearly $D(\mathcal{T}, \mathcal{Q}_k) \leqslant 1/k$ by definition of the metric D. On the other hand, by definition of the space $\mathbb{X}_\sigma^F$, for any $k \in \mathbb{N}$ the patch $\mathcal{Q}_k$ is obtained as a limit of the form $\lim_{j\to\infty} \mathcal{R}_j + u_j$, where $\mathcal{R}_j$ are sub-patches of patches $\mathcal{P}_j \in \mathscr{P}_\sigma$ and $u_j \in \mathbb{R}^d$. It follows that there exists $j_k \in \mathbb{N}$ such that $D(\mathcal{Q}_k, \mathcal{P}_j + u_j) < 1/k$ for any $j \geqslant j_k$. Combining the above we get $\mathcal{T} = \lim_{k\to\infty} \mathcal{P}_{j_k} + u_{j_k}$, and so $\mathcal{T} \in Y$.

Conversely, every sub-patch of a limiting object of the form $\lim_{k\to\infty} \mathcal{P}_k + v_k$ for $\mathcal{P}_k \in \mathscr{P}_\sigma$, $v_k \in \mathbb{R}^d$, is a limit of sub-patches of the patches $P_k + v_k$. Assuming additionally that $\lim_{k\to\infty} \mathcal{P}_k + v_k$ tiles $\mathbb{R}^d$ implies that the limit is in $\mathbb{X}_\sigma^F$. □

Corollary 4.4. *The space $\mathbb{X}_\sigma^F$ is a closed, non-empty subset of* $\mathrm{Cl}(\mathbb{R}^d)$.

Proof. To see that $\mathbb{X}_\sigma^F \neq \varnothing$ one simply takes a limit of a converging sequence of patches $\mathcal{P}_k \in \mathscr{P}_\sigma$ whose supports exhaust $\mathbb{R}^d$. By compactness of the space $(\mathrm{Cl}(\mathbb{R}^d), D)$ such sequences exist. To see that it is closed, let $\mathcal{T}_k \in \mathbb{X}_\sigma^F$ be a sequence so that $\mathcal{T} = \lim_{k\to\infty} \mathcal{T}_k$. For each k let $\mathcal{P}_k \in \mathscr{P}_\sigma$, $v_k \in \mathbb{R}^d$ be so that $D(\mathcal{T}_k, \mathcal{P}_k + v_k) < 1/k$. Then $\mathcal{T} = \lim_{k\to\infty} \mathcal{P}_k + v_k$ and so $\mathcal{T} \in \mathbb{X}_\sigma^F$. □

The following proposition is a simple exercise in convergence of compact sets with respect to the Hausdorff metric, and establishes a first simple result about tiles that appear in multiscale substitution tilings.

Proposition 4.5. *Let σ be a substitution scheme. Every tile of every $\mathcal{T} \in \mathbb{X}_\sigma^F$ is similar to one of the prototiles in τ_σ.*

4.2. **Stationary tilings.** Consider a substitution scheme σ and a tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$. By Proposition 4.5, every tile T in $\mathcal{T}$ is a rescaled copy of a prototile in τ_σ. The substitution flow F_t, which was defined for prototiles in Definition 2.10, can thus be naturally extended to tiles in $\mathcal{T}$ and to patches in $\mathcal{T}$, and so to the entire tiling $\mathcal{T}$. It follows that for every $\mathcal{T} \in \mathbb{X}_\sigma^F$ one has:

(1) For any $t \in \mathbb{R}^+$, $F_t(\mathcal{T}) \in \mathbb{X}_\sigma^F$.
(2) For any $t, s \in \mathbb{R}^+$, $F_t(F_s(\mathcal{T})) = F_{t+s}(\mathcal{T})$.

Thus, the substitution flow defines a semi-flow on $\mathbb{X}_\sigma^F$. Theorem 4.6 below establishes the existence of non-trivial periodic orbits.

Theorem 4.6. *Let σ be an irreducible substitution scheme. There exist an $s \in \mathbb{R}^+$ and a tiling $\mathcal{S} \in \mathbb{X}_\sigma^F$ so that $F_s(\mathcal{S}) = \mathcal{S}$.*

Proof. Irreducibility of σ implies that for every i there are infinitely many $t \in \mathbb{R}^+$ for which the patch $F_t(T_i)$ contains a tile of type i that does not share its boundary. It follows that there are infinitely many $s \in \mathscr{S}_{i\to i}$ for which the patch $F_s(T_i)$ contains a translate of T_i that does not share its boundary. For any such $s \in \mathscr{S}_{i\to i}$ there exists a unique location in which T_i can be initially positioned around the origin, so that the inflated patch $F_s(T_i)$ contains the patch $T_i = F_0(T_i)$ as a sub-patch with support in that very same location. Note that under our assumption, this *control point* is an interior point of T_i. In such a case also $F_{2s}(T_i)$ contains $F_s(T_i)$ as a sub-patch, and more generally the patch $F_{ks}(T_i)$ contains $F_{(k-1)s}(T_i)$ for every $k \in \mathbb{N}$, that is, the patches $F_{ks}(T_i)$ define a nested sequence of patches. Therefore, the union

$$\mathcal{S} := \bigcup_{k=0}^{\infty} F_{ks}(T_i)$$

is a tiling in $\mathbb{X}_\sigma^F$, and it clearly satisfies $F_s(\mathcal{S}) = \mathcal{S}$. □

Let $\mathcal{S}$ be as above, then for any $a \in \mathbb{R}^+$

$$F_s(F_a(\mathcal{S})) = F_a(F_s(\mathcal{S})) = F_a(\mathcal{S}) \tag{4.3}$$

and $F_a(\mathcal{S}) = \bigcup_{k=0}^{\infty} F_{a+ks}(T_i)$, which leads us to the following definition.

Definition 4.7. A tiling $\mathcal{S} \in \mathbb{X}_\sigma^F$ of the form

$$\mathcal{S} = \bigcup_{k=0}^{\infty} F_{a+ks}(T_i), \tag{4.4}$$

where $s \in \mathscr{S}_{i\to i}$, $a \in \mathbb{R}^+$ and $F_{a+ks}(T_i)$ define a nested sequence of patches, is called a *stationary tiling*. As in the construction in the proof of Theorem 4.6, we assume throughout that the origin is an interior point of T_i. We also assume that $s \in \mathscr{S}_{i\to i}$ is minimal in the sense that the associated closed path in G_σ is a prime orbit, that is, not the concatenation of multiple copies of a single closed orbit.

We remark that in fact there are infinitely many values of $s \in \mathbb{R}^+$ for which there are tilings $\mathcal{S} \in \mathbb{X}_\sigma^F$ that satisfy $F_s(\mathcal{S}) = \mathcal{S}$. Moreover, in view of (4.3), for every $\mathcal{P} \in \mathscr{P}_\sigma$ there exists a stationary tiling that contains a translate of $\mathcal{P}$ as a sub-patch. In addition, every stationary tiling can be represented as

$$\mathcal{S} = \bigcup_{k=0}^{\infty} F_{ks}(T),$$

where T is some rescaled copy of a prototile.

Example 4.8. Once again we consider the square substitution scheme with $\tau_\sigma = (S)$, illustrated in Figure 1 and discussed in previous examples. The large square in the middle of the patch $\varrho_\sigma(S)$ is associated with the single loop of length $\log\frac{5}{3}$ in the associated graph G_σ illustrated in Figure 4. It follows that for $s = \log\frac{5}{3}$ and the initial positioning of S so that the origin is in the center of the square, we can define a sequence of patches $F_{ks}(S) = F_{k\log(5/3)}(S)$ with support $e^{ks} = (5/3)^k$ so that the patch $F_{ks}(S)$ contains $F_{(k-1)s}(S)$ for every $k \in \mathbb{N}$. For example, the patches illustrated in Figure 8 are the $k = 0, 1, \ldots, 6$ elements of the nested sequence.

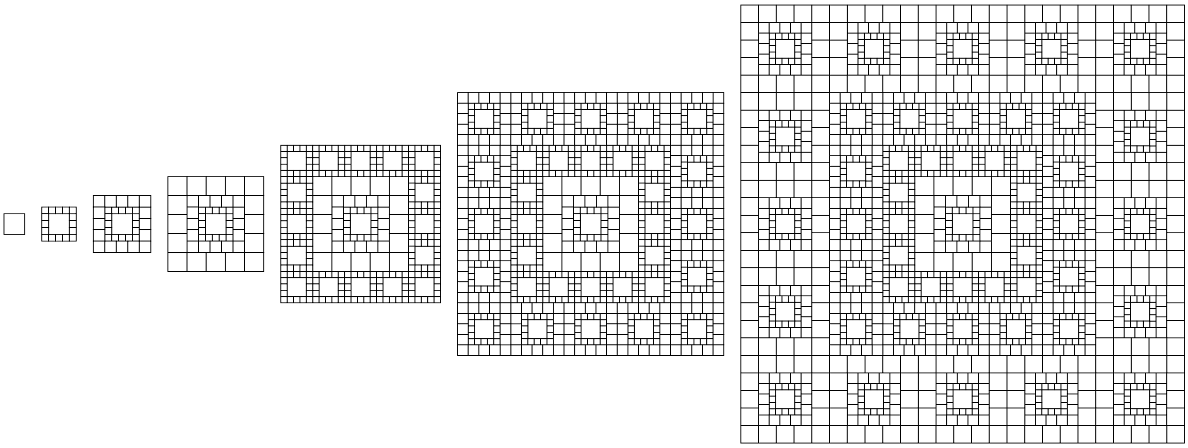

FIGURE 8. The patches $F_0(S), F_{1\cdot\log 5/3}(S), \ldots, F_{6\cdot\log 5/3}(S)$, the first seven elements of the nested sequence of patches that define a stationary tiling.

Example 4.9. The three patches illustrated in Figure 9 are the $k = 0, 1$ and 2 elements of the nested sequence of patches of the stationary tiling construction associated with the choice of the central copy of $\frac{1}{5}U$ in $\varrho_\sigma(U)$, where σ is the substitution scheme on two triangles illustrated in Figure 3.

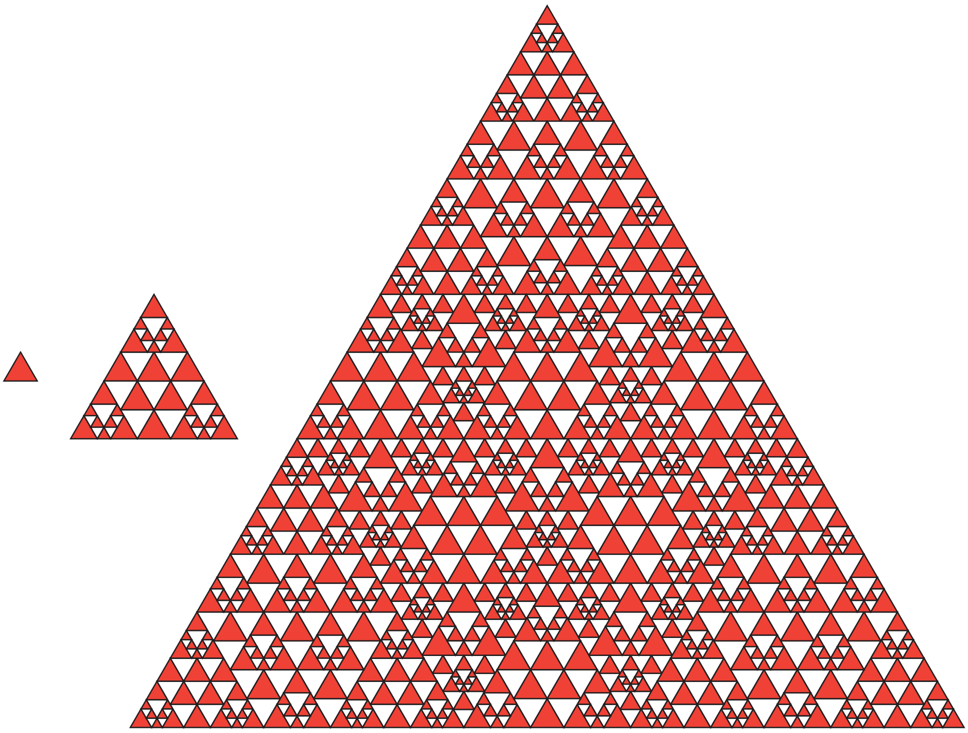

FIGURE 9. The patches $F_0(U)$, $F_{1\cdot\log 5}(U)$ and $F_{2\cdot\log 5}(U)$, the first three elements of the nested sequence of patches that define a stationary tiling.

Remark 4.10. Stationary tilings can be generated using the more general definition of substitution schemes in which isometries are allowed in the substitution rule, and not only translations, see Remark 2.3. Assume $F_s(T_i)$ contains a copy of $\varphi(T_i)$, where φ is an isometry of $\mathbb{R}^d$. The patch $\varphi^{-1}(F_s(T_i))$ contains a copy of T_i as a sub-patch, and similarly as described above, we get a sequence of patches $\varphi^{-k}(F_{ks}(T_i))$ which define a stationary tiling $\mathcal{T}$ in the sense that $\varphi^{-1}(F_s(\mathcal{T})) = \mathcal{T}$. We note that Sadun's generalized pinwheel tilings in [Sa] can be formulated as stationary tilings in this way.

We end this section with a remark on tilings generated by commensurable schemes.

Remark 4.11. Let σ be a (not necessarily normalized) fixed scale scheme with scaling constant α, then all edges in the associated graph G_σ are of length $\log \frac{1}{\alpha}$. Define a type of substitution flow $\widetilde{F}_t$ according to the flow on G_σ, in which the tiles are substituted simultaneously when they all reach the volumes of the original prototiles they are associated with. The patches $F_{k\log(1/\alpha)}(T_i)$ are simply the patches $(1/\alpha)^k \varrho_\sigma^k(T_i)$, and all patches in $\mathscr{P}_i$ are inflations of such patches for some $k \in \mathbb{N}$.

Recall that in the classical theory of fixed scale tilings, the well studied tiling space, which we denote here by $\mathbb{X}_\sigma$, can be defined using the patches $(1/\alpha)^k \varrho_\sigma^k(T_i)$, see [BG] and references within. It follows that in such a case, the space $\mathbb{X}_\sigma^{\widetilde{F}}$, defined using the flow $\widetilde{F}_t$, can be expressed simply as the product $\mathbb{X}_\sigma \times (\alpha, 1]$. It was shown in [Sm] that Kakutani sequences of partitions generated by commensurable schemes can be represented as subsequences of generation sequences of partitions generated by fixed scale schemes. In our setting and language, this simply means that given a commensurable scheme σ there exists a fixed scale scheme $\widetilde{\sigma}$ so that $\mathbb{X}_\sigma^F = \mathbb{X}_{\widetilde{\sigma}}^{\widetilde{F}}$, and so for the study of commensurable multiscale tilings, one should refer to results on standard fixed scale substitution tilings and the standard spaces of tilings. As we will see below, incommensurable tilings and multiscale tiling spaces differ from the classical setup in various ways.

5. Scales and complexity in stationary tilings

From here on **all schemes are assumed to be incommensurable**. We show that an incommensurable stationary tiling has tiles in a set of scales which is dense within an interval of possible scales. In particular, it is of infinite local complexity, see e.g. [BG].

Definition 5.1. Given a substitution scheme σ and a prototile $T_j \in \tau_\sigma$, denote

$$\beta_j^{\min} := \min \{\alpha : \exists i \in \{1, \ldots, n\}, \varrho_\sigma(T_i) \text{ contains a copy of } \alpha T_j\}. \tag{5.1}$$

The interval $(\beta_j^{\min}, 1]$ is called the *interval of possible scales of tiles of type* j. Note that the prototiles $T_i, T_j \in \tau_\sigma$ on the right-hand side of (5.1) are assumed to be of unit volume. A *tile of possible type and scale* is any translated copy of αT_j for $T_j \in \tau_\sigma$ and $\alpha \in (\beta_j^{\min}, 1]$.

Recall that a fixed scale substitution scheme is called *primitive* if there exists $k \in \mathbb{N}$ so that all types of tiles belong to the patch that is the result of k applications of the substitution rule on any initial prototile. The following theorem demonstrates how incommensurability takes the part of primitivity in multiscale substitution tilings. It also plays an important role in the proof of minimality of the tiling dynamical system in §6.

Theorem 5.2. *Let σ be an irreducible incommensurable substitution scheme, let $\mathcal{S} \in \mathbb{X}_\sigma^F$ and $s \in \mathbb{R}^+$ be so that $\mathcal{S} = F_s(\mathcal{S})$, and let $\varepsilon > 0$. Then there exists $K \in \mathbb{N}$ so that for every integer $k \geqslant K$, a tile T in $\mathcal{S}$ and $j \in \{1, \ldots, n\}$, the set*

$$\mathrm{Scales}(e^{ks}T, j) := \left\{\alpha : \textit{the patch supported on } e^{ks}T \textit{ in } F_{ks}(\mathcal{S}) = \mathcal{S} \textit{ contains a copy of } \alpha T_j\right\}$$

is ε-dense in $(\beta_j^{\min}, 1]$. In particular, the set of scales in which tiles of each type appear in stationary tilings is dense within their intervals of possible scales.

Proof. Since every tile T in $\mathcal{S}$ is a translated copy of αT_i for some $T_i \in \tau_\sigma$ and some $\alpha \in (\beta_i^{\min}, 1]$, the patch supported on $e^{ks}T$ is a translated copy of $F_{ks}(\alpha T_i) = F_{ks-\log(1/\alpha)}(T_i)$. Note that since $F_s(\mathcal{S}) = \mathcal{S}$, the set $e^{ks}T$ is indeed a support of a patch in $\mathcal{S}$.

Fix $\varepsilon > 0$ and $j \in \{1\ldots, n\}$. We show that there exists $K_j \in \mathbb{N}$ so that $\text{Scales}(e^{ks}T, j) \cap (c, c+\varepsilon] \neq \varnothing$ for every $c \in (\beta_j^{\min}, 1-\varepsilon]$, every tile T in $\mathcal{S}$, and every integer $k \geqslant K_j$. Set $\ell = \ell(j) \in \{1\ldots, n\}$ so that $\varrho_\sigma(T_\ell)$ contains a copy of $\beta_j^{\min} T_j$. First observe that for every $c \in (\beta_j^{\min}, 1-\varepsilon]$ and $t \in \mathbb{R}^+$, the following implication holds:

$$F_t(T_i) \text{ contains a copy of } T_\ell \qquad \Longrightarrow \tag{5.2}$$
$$F_{t+\log c - \log \beta_j^{\min}}(T_i) \text{ contains a copy of } F_{\log c - \log \beta_j^{\min}}(\beta_j^{\min} T_j) = cT_j$$

For $i \in \{1\ldots, n\}$ let $s_1(i) < s_2(i) < \ldots$ be an increasing enumeration of $\mathscr{S}_{i\to\ell}$, then by Corollary 3.9

$$\lim_{m\to\infty} s_{m+1}(i) - s_m(i) = 0.$$

It follows that for every $\delta > 0$ there exists $K_j \in \mathbb{N}$ so that for every $k \geqslant K_j$, $i \in \{1, \ldots, n\}$, $\alpha \in (\beta_i^{\min}, 1]$ and $c \in (\beta_j^{\min}, 1-\varepsilon]$ there exists some $m \in \mathbb{N}$ with

$$ks - \log c + \log \beta_j^{\min} - \delta < s_m(i) + \log(1/\alpha) \leqslant ks - \log c + \log \beta_j^{\min}. \tag{5.3}$$

Put $h := s_m(i) + \log(1/\alpha) + \log c - \log \beta_j^{\min}$, then (5.3) simply says $h \in (ks-\delta, ks]$.

Suppose that T is of type i and scale α. Since $s_m(i) \in \mathscr{S}_{i\to\ell}$, the patch $F_{s_m(i)+\log(1/\alpha)}(T) = F_{s_m(i)}(T_i)$ contains a copy of T_ℓ. By the implication in (5.2), $F_h(T)$ contains a copy of cT_j. Write $\eta := ks - h$, then $\eta < \delta$ and $F_{ks}(T) = F_\eta(F_h(T))$ contains a copy of $F_\eta(cT_j) = e^\eta c T_j$, where the equality holds whenever δ is small enough. In particular, for $\delta \leqslant \log(1+\varepsilon/\beta)$, where $\beta := \min_j \beta_j^{\min}$, we have $e^\eta c \in (c, c+\varepsilon]$, because $c \geqslant \beta$. Taking $K = \max_j K_j$ completes the proof. □

5.1. **Tile complexity.** Let σ be an irreducible substitution scheme, and consider a stationary tiling $\mathcal{S} = \bigcup_{k=0}^\infty F_{ks}(T) \in \mathbb{X}_\sigma^F$. For every $k \geqslant 0$, denote by $c_{\mathcal{S},j}(k)$ the number of distinct scales in which a tile of type j appears in $F_{ks}(T)$, and set $c_{\mathcal{S}}(k) = \sum_{j=1}^n c_{\mathcal{S},j}(k)$. So $c_{\mathcal{S}}(k)$ is the number of distinct tiles up to translation in $F_{ks}(T)$, and $c_{\mathcal{S}}(k)$ is called the *tile complexity function* of $\mathcal{S}$. Note that it is well define because s is assumed to be minimal with the property $F_s(\mathcal{S}) = \mathcal{S}$. Since $F_{ks}(T)$ is a nested sequence of patches, clearly $c_{\mathcal{S}}(k)$ is non-decreasing.

Theorem 5.3. *Let σ be an irreducible substitution scheme and let $\mathcal{S} = \bigcup_{k=0}^\infty F_{ks}(T) \in \mathbb{X}_\sigma^F$ be a stationary tiling. If $c_{\mathcal{S}}(\ell+1) = c_{\mathcal{S}}(\ell)$ for some $\ell \in \mathbb{N}$, then $c_{\mathcal{S}}(k) = c_{\mathcal{S}}(\ell)$ for all $k \geqslant \ell$. Moreover, this is the case if and only if the scheme σ is commensurable.*

Proof. Assume $c_{\mathcal{S}}(\ell+1) = c_{\mathcal{S}}(\ell)$. This means that every type and scale that appears in the patch $F_{(\ell+1)s}(T)$ also appears in the patch $F_{\ell s}(T)$. Since the tiles in $F_{(\ell+1)s}(T)$ are all sub-tiles in patches defined by applying the substitution flow F_s to the tiles of $F_{\ell s}(T)$, we deduce that applying F_s to the tiles of $F_{(\ell+1)s}(T)$ defines a patch with tiles of the same set of types and scales. Therefore, repeated applications of F_s will always result with patches whose tiles are of the same types and scales, and so $c_{\mathcal{S}}(k) = c_{\mathcal{S}}(\ell)$ for every $k \geqslant \ell$.

For the second part of the theorem, if σ is incommensurable then Theorem 5.2 in particular implies that $c_{\mathcal{S}}(k)$ tends to infinity. Conversely, assume $c_{\mathcal{S}}(k)$ is strictly increasing, then in particular it is unbounded as an increasing sequence of integers. If σ is commensurable, then Remark 4.11 implies that there exists a standard substitution tiling $\mathcal{T}$ generated by a fixed scale substitution scheme, so that for any $k \in \mathbb{N}$ the patches $F_{ks}(T)$ are sub-patches of $\mathcal{T}$ (see also [Sm, Theorem 7.2]). Since the fixed scale substitution tiling $\mathcal{T}$ has only finitely many tiles up to translation, the complexity function $c_{\mathcal{S}}(k)$ is bounded, contradicting our assumption. □

Given a finite alphabet $\mathcal{A}$ and an infinite sequence $u \in \mathcal{A}^{\mathbb{N}}$, the cardinality $p_k(u)$ of the set of *words* in u of length k defines the *complexity function* of u. The statement of Theorem 5.3 resembles that of the well-known result that a sequence $u \in \mathcal{A}^{\mathbb{N}}$ is periodic if and only if there exists $k \in \mathbb{N}$ for which $p_k(u) = p_{k+1}(u)$, see e.g. [BG, Proposition 4.11]. Non-periodic sequences with minimal complexity function $p_k(u) = k + 1$ for all $k \in \mathbb{N}$ are called *Sturmian sequences*. The following Proposition 5.4 establishes the existence of "Sturmian" incommensurable stationary tilings.

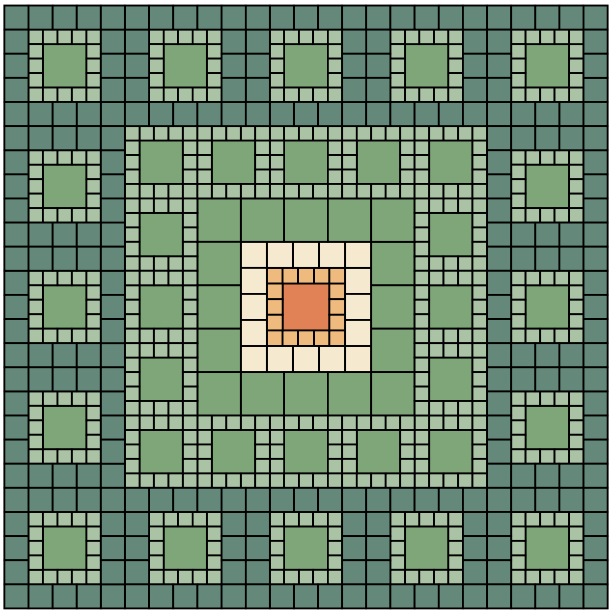

FIGURE 10. The patch $F_{5s}(S)$. Squares are colored according to the $c_{\mathcal{S}}(5) = 6$ distinct scales in which the appear in $F_{5s}(S)$.

Proposition 5.4. *There exist incommensurable stationary tilings with $c_{\mathcal{S}}(k) = c_{\mathcal{S}}(k+1)$ for all $k \in \mathbb{N}$.*

Proof. Let $\mathcal{S}$ be the stationary tiling with square tiles, constructed in Example 4.8 using the substitution scheme on the unit square S. Here $s = \log(5/3)$ and $T = S$, and we denote by $\mathcal{S}_k$ the patch $F_{k\log(5/3)}(S)$. Clearly $c_{\mathcal{S}}(0) = 1$, and from Figure 8 we deduce $c_{\mathcal{S}}(1) = 2$. Assuming, by induction, that $c_{\mathcal{S}}(m) = m + 1$ for all $1 \leqslant m < k$, we prove that $c_{\mathcal{S}}(k) = k + 1$.

Since $\mathcal{S}$ is stationary, for every m the set of $c_{\mathcal{S}}(m)$ different scales that appear in $F_{ms}(T_i)$ contains the $c_{\mathcal{S}}(m-1)$ scales that appear in $F_{(m-1)s}(T_i)$. In particular, using the induction hypothesis, the set of k scales that appear in $\mathcal{S}_{k-1}$ consists of the $k-1$ scales that appear in $\mathcal{S}_{k-2}$, and one additional scale $\alpha \in (\frac{1}{5}, 1]$. Therefore, squares in $\mathcal{S}_k$ that are descendants of squares of these $k-1$ scales in $\mathcal{S}_{k-1}$, appear in k scales. Regarding the scale α, we consider two cases: If $\alpha \leqslant \frac{3}{5}$, then applying F_s to squares of scale α results in squares of side length $e^s\alpha = \frac{5}{3}\alpha \leqslant 1$, and so at most one new scale appears in $\mathcal{S}_k$. Otherwise $\alpha > \frac{3}{5}$, and so applying F_s to squares of scale α results in a patch consisting of a single central square of scale $\frac{3}{5} \cdot e^s\alpha = \alpha$, and 16 smaller squares of scale $\frac{1}{5} \cdot e^s\alpha$. And so once again, at most one new scale appears in $\mathcal{S}_k$. Since the scheme is incommensurable, by Theorem 5.3, at least one new scale must appear in $\mathcal{S}_k$, finishing the proof. □

5.2. **Patches and scales.** Recall that given a substitution scheme σ, a patch is called legal if it appears as a sub-patch of some element of $\mathscr{P}_\sigma$. We show here that if σ is irreducible and incommensurable, then for every legal patch, every stationary tiling contains rescaled copies of the patch, with a dense set of scales.

Lemma 5.5. *Let σ be an irreducible incommensurable substitution scheme, and let $\mathcal{P}$ be a legal patch. Then every stationary tiling $\mathcal{S} \in \mathbb{X}_\sigma^F$ contains a translated copy of $\alpha\mathcal{P}$ for some scale $\alpha > 0$.*

Proof. Since $\mathcal{P}$ is legal, there is $t \in \mathbb{R}^+$ so that $\mathcal{P}$ is a sub-patch of $F_t(T_i)$ for some $T_i \in \tau_\sigma$. Since $\mathcal{P}$ contains finitely many tiles, there are $t_{\min} < t \leqslant t_{\max}$ so that every element of $\{F_t(T_i) : t_{\min} < t \leqslant t_{\max}\}$ contains a rescaled copy of $\mathcal{P}$ as a sub-patch. We remark that in $F_{t_{\max}}(T_i)$ the dilation of $\mathcal{P}$ is such that the tiles of maximal volume are of volume 1, and that in $F_{t_{\min}}(T_i)$ an ancestor of at least one of the tiles of the patch $\mathcal{P}$ appears at volume exactly 1.

Let $\mathcal{S} \in \mathbb{X}_\sigma^F$ be a stationary tiling for which $F_s(\mathcal{S}) = \mathcal{S}$. Fix $m \in \mathbb{N}$ so that $ms > t_{\max}$. By Theorem 5.2 the tiles in $\mathcal{S}$ appear in a dense set of scales within the intervals of possible scales. In particular, by irreducibility, there exists a tile T in $\mathcal{S}$ so that $F_{ms-t_{\max}}(T)$ contains a translated copy of αT_i for some $\alpha \in (e^{t_{\min}-t_{\max}}, 1]$. This can be deduced by considering a segment associated with the interval of scales $(e^{t_{\min}-t_{\max}}, 1]$, which is a subset of an edge of the associated graph G_σ that terminates at vertex $i \in \mathcal{V}$. This segment can be mapped to another segment on the edges of G_σ by choosing a "reverse walk" of length $ms - t_{\max}$. Any point in the resulting segment corresponds to a choice of some rescaled prototiles, that is, some tile of possible type and scale. Since tiles appear in $\mathcal{S}$ in a dense set of scales within all possible scales, it is guaranteed that indeed there is a tile T in $\mathcal{S}$ which is a copy of αT_i with $\alpha \in (e^{t_{\min}-t_{\max}}, 1]$.

It follows that $F_{ms}(T) = F_{t_{\max}}\left(F_{ms-t_{\max}}(T)\right)$ contains $F_{t_{\max}}(\alpha T_i) = F_{t_{\max}-\log(1/\alpha)}(T_i)$ as a sub-patch. Clearly

$$t_{\min} \leqslant t_{\max} - \log(1/\alpha) \leqslant t_{\max},$$

and so $F_{ms}(T)$ contains a rescaled copy of $\mathcal{P}$. Since $F_{ms}(T)$ is a patch in $F_{ms}(\mathcal{S}) = \mathcal{S}$, the result follows. □

Let $\mathcal{S} \in \mathbb{X}_\sigma^F$ be a stationary tiling with $\mathcal{S} = \bigcup_{k=0}^{\infty} F_{ks}(T)$. For any fixed patch $\mathcal{P}_0$ in $\mathcal{S}$, there exists a smallest $k \geqslant 1$ for which $\mathcal{P}_0$ is a sub-patch of $F_{ks}(T)$. The patch $\mathcal{P}_0$ can also be considered in the context of the continuous family $\{F_t(T) : t \in \mathbb{R}^+\}$. As in the proof of Lemma 5.5, we define a maximal non-trivial interval $t_{\mathcal{P}_0} := (t_{\min}, t_{\max}]$ so that $ks \in t_{\mathcal{P}_0}$, and such that for any $t \in t_{\mathcal{P}_0}$ the patch $F_t(T)$ contains a rescaled copy of $\mathcal{P}_0$ as a sub-patch. If the original patch $\mathcal{P}_0$ is thought of as being of scale 1, then these rescaled copies of $\mathcal{P}_0$ in $\{F_t(T) : t \in t_{\mathcal{P}_0}\}$ appear in scales within an interval $I'_{\mathcal{P}_0}$ of scales that contains 1, and by left continuity also some left neighborhood of 1.

Now let $\mathcal{P}$ be a patch in $\mathcal{S}$, consider all of its translated copies in $\mathcal{S}$ and let $(\mathcal{P}_j)_{j\geqslant 1}$ be an enumeration of them. Each $\mathcal{P}_j$ is a patch in $\mathcal{S}$ and has its own interval of times $t_{\mathcal{P}_j}$ and interval of scales $I'_{\mathcal{P}_j}$ as described above. Clearly all the intervals of the form $I'_{\mathcal{P}_j}$ have the same maximum, and the associated rescaled copies of these patches have maximal tiles of volume 1. Also note that since there are only finitely many tiles in a patch, and since tile scales are bounded, the intervals $I'_{\mathcal{P}_j}$ are all bounded uniformly from below. We can thus define the *interval of possible scales* for the patch $\mathcal{P}$ in $\mathcal{S}$ as the union of all these intervals, and denote it by $I_{\mathcal{P}}$. Clearly $I_{\mathcal{P}}$ contains 1 and a left neighborhood of 1.

Lemma 5.6. *Let σ be an irreducible incommensurable substitution scheme, and let $\mathcal{S} \in \mathbb{X}_\sigma^F$ be a stationary tiling. Let $\mathcal{P}$ be a patch in $\mathcal{S}$, and let $I_{\mathcal{P}} = (\beta_{\mathcal{P}}^{\min}, \beta_{\mathcal{P}}^{\max}]$ be the interval of scales in which $\mathcal{P}$ appears in $\mathcal{S}$. Then the set*

$$\mathrm{Scales}(\mathcal{S},\mathcal{P}) := \{\alpha : \ \mathcal{S} \textit{ contains a copy of } \alpha\mathcal{P}\}$$

is dense in $I_{\mathcal{P}}$.

Proof. Let $(\mathcal{P}_j)_{j\geqslant 1}$ be an enumeration of translated copies of $\mathcal{P}$ in $\mathcal{S}$. Fix a patch $\mathcal{P}_j$, and as in the discussion above let $t_{\mathcal{P}_j}$ and $I'_{\mathcal{P}_j}$ be the associated intervals of times and of scales, respectively.

Assume $s \in \mathbb{R}^+$ is such that $F_s(\mathcal{S}) = \mathcal{S}$ and write $t_{\mathcal{P}_j} = (t_{\min}, t_{\max}]$. As in the proof of Lemma 5.5, if we pick $m \in \mathbb{N}$ so that $ms > t_{\max}$, then $F_{ms}(\mathcal{S}) = \mathcal{S}$ contains a rescaled copy of $\mathcal{P}_j$. In fact, it follows from the proof of Lemma 5.5, that $\mathcal{S}$ contains rescaled copies of $\mathcal{P}_j$ with scales which are dense in the interval $I'_{\mathcal{P}_j}$. Since $I_{\mathcal{P}}$ is defined as the union of the intervals $I'_{\mathcal{P}_j}$, the proof is complete. ☐

Corollary 5.7. *Every stationary tiling contains rescaled copies of every legal patch, and the patches appear in a dense set of scale.*

It will follow from Corollary 9.6 that the above holds for any tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$.

6. The tiling dynamical system

A *dynamical system* is a pair (X, G) where X is a topological space and G is a group that acts on X. The *orbit* of a point $x \in X$ is the set $\mathcal{O}(x) := \{g.x \, : \, g \in G\}$. A *subsystem* of (X, G) is a closed, non-empty and G-invariant set $Y \subset X$. Examples of subsystems include orbit closures $\overline{\mathcal{O}(x)}$, for $x \in X$.

We study the dynamical system $\left(\mathbb{X}_\sigma^F, \mathbb{R}^d\right)$, where the group $\mathbb{R}^d$ acts on tilings of $\mathbb{R}^d$ by translations, and the topology on $\mathbb{X}_\sigma^F$ is determined by the metric D, defined in (4.1). Traditionally, we call it the *tiling dynamical system*.

6.1. **Minimality.** Recall that a dynamical system (X, G) is called *minimal* if it contains no proper subsystems, or equivalently, if every orbit is dense.

Theorem 6.1. *Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$. Then the dynamical system $\left(\mathbb{X}_\sigma^F, \mathbb{R}^d\right)$ is minimal.*

The following result is a key step in the proof of Theorem 6.1.

Lemma 6.2. *Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$, and let $\mathcal{S} = \bigcup_{k=0}^\infty F_{ks}(T_i) \in \mathbb{X}_\sigma^F$ be a stationary tiling. Then $\mathcal{O}(\mathcal{S})$ is dense in $\mathbb{X}_\sigma^F$.*

Proof. Let $\varepsilon > 0$. We need to show that for every $\mathcal{T} \in \mathbb{X}_\sigma^F$, there exists $u \in \mathbb{R}^d$ so that

$$D\left(\mathcal{T}, \mathcal{S} - u\right) < \varepsilon. \tag{6.1}$$

First, in view of Proposition 4.3, the tiling $\mathcal{T}$ may be described as the limit $\lim_{\ell\to\infty}\mathcal{P}_\ell - v_\ell$, where $\mathcal{P}_\ell \in \mathscr{P}_\sigma$ and $v_\ell \in \mathbb{R}^d$. Let $\mathcal{P}_\ell \in \mathscr{P}_\sigma$ and $v_\ell \in \mathbb{R}^d$ be such that

$$D(\mathcal{T}, \mathcal{P}_\ell - v_\ell) < \varepsilon/2. \tag{6.2}$$

Note that in particular, the support of $\mathcal{P}_\ell - v_\ell$ contains the ball $B(0, 2/\varepsilon)$.

Write $\mathcal{P}_\ell = F_a(T_j)$ for some $T_j \in \tau_\sigma$ and $a \in \mathbb{R}^+$. Let $b \in \mathscr{S}_{i\to j}$, and let $s_1 < s_2 < \dots$ be an increasing enumeration of the set $\mathscr{S}_{i\to i}$. By Lemma 3.8, for any $\delta > 0$ there exists M so that $s_{m+1} - s_m < \delta$, for every $m \geqslant M$. Choose a minimal $k \in \mathbb{N}$ such that $(ks - b - s_M) - a \geqslant 0$. Since $s_m \to \infty$ and the gap between two sequential elements in $\mathscr{S}_{i\to i}$ with indices greater than M is smaller than δ, there exists some $m \geqslant M$ for which

$$0 \leqslant a - (ks - b - s_m) < \delta$$

holds. Since the patch $\mathcal{P}_\ell$ has finitely many tiles, for small enough values of δ the patch $\mathcal{P}_\ell$ is a small inflation of the patch $F_{ks-b-s_m}(T_j)$, and by taking small values of δ the inflation can be made arbitrarily small. In particular, by appropriate choices of δ there exist choices of m for which the two patches have arbitrarily small Hausdorff distance, and this is also true for their translations by v_ℓ. Since the support of $\mathcal{P}_\ell - v_\ell$ contains $B(0, 2/\varepsilon)$, it follows that

$$D(\mathcal{P}_\ell - v_\ell, F_{ks-b-s_m}(T_j) - v_\ell) < \varepsilon/2.$$

By our choice of b and s_m we have $b + s_m \in \mathscr{S}_{i\to j}$, and so $F_{b+s_m}(T_i)$ contains a translated copy of T_j. Hence $F_{ks}(T_i) = F_{ks-b-s_m}(F_{b+s_m}(T_i))$ contains a translated copy of the patch $F_{ks-b-s_m}(T_j)$ as a sub-patch. In turn, $F_{ks}(T_i) - v_\ell$ contains a translated copy of the patch $F_{ks-b-s_m}(T_j) - v_\ell$ as a sub-patch. Therefore, there exists some $w_\ell \in \mathbb{R}^d$ with

$$D(\mathcal{P}_\ell - v_\ell, F_{ks}(T_i) - v_\ell - w_\ell) < \varepsilon/2.$$

Then setting $u := v_\ell + w_\ell$, we have

$$D(\mathcal{P}_\ell - v_\ell, \mathcal{S} - u) < \varepsilon/2. \tag{6.3}$$

Combining (6.2) and (6.3), the triangle inequality implies (6.1), finishing the proof. □

Proof of Theorem 6.1. Given any $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{X}_\sigma^F$, we show that $\mathcal{T}_2 \in \overline{\mathcal{O}(\mathcal{T}_1)}$. Let $\mathcal{S} = \bigcup_{k=0}^{\infty} F_{ks}(T_i)$ be a stationary tiling, for some $s > 0$ and $T_i \in \tau_\sigma$. By Lemma 6.2, $\mathcal{T}_2 \in \overline{\mathcal{O}(\mathcal{S})}$ and hence it suffices to show that $\mathcal{S} \in \overline{\mathcal{O}(\mathcal{T}_1)}$. We show that for every $\varepsilon > 0$ there exists $u \in \mathbb{R}^d$ such that $D(\mathcal{T}_1 - u, S) < \varepsilon$.

Let $\varepsilon > 0$. Fix $m \in \mathbb{N}$ such that $\mathrm{supp}(F_{ms}(T_i)) \supset B(0, 3/\varepsilon)$. By left-continuity (see Proposition 4.2), there exists $0 < \delta < \log(\varepsilon/2)$ so that for every $0 \leqslant \eta < \delta$ the support of $F_{ms-\eta}(T_i)$ contains $B(0, 2/\varepsilon)$ and

$$D(F_{ms-\eta}(T_i), F_{ms}(T_i)) < \varepsilon/2. \tag{6.4}$$

We first show that there exists some $R = R(\varepsilon) > 0$ so that for every $z \in \mathbb{R}^d$ the patch $[B(z,R)]^{\mathcal{S}}$ contains a translated copy of the patch $F_{ms-\eta}(T_i)$, for some $0 \leqslant \eta < \delta$. By Theorem 5.2, in particular, there exists $K > 0$ so that for every tile T is $\mathcal{S}$ the set

$$\left\{\alpha : \text{the patch supported on } e^{Ks}T \text{ in } \mathcal{S} \text{ contains a copy of } \alpha T_i\right\}$$

is δ-dense in the interval $(\beta_i^{\min}, 1]$. In particular, for every tile T in $\mathcal{S}$, the patch supported on $e^{Ks}(T)$ contains a copy of αT_i for $\alpha \in (1-\delta, 1]$, and hence the patch supported on

$e^{(K+m)s}(T)$ contains a copy of $F_{ms-\eta}(T_i)$, where here $\eta = 1-\alpha \in [0,\delta)$. Set

$$R = R(\varepsilon) := e^{(K+m)s} \cdot \max_j \{\text{diameter}(T_j)\}.$$

Then every patch $[B(z,R)]^{\mathcal{S}}$, for $z \in \mathbb{R}^d$, contains a translated copy of the patch $F_{ms-\eta}(T_i)$, for some $0 \leqslant \eta < \delta$.

Invoking Lemma 6.2 once again, we find some $z \in \mathbb{R}^d$ so that $D(\mathcal{S} - z, \mathcal{T}_1) < 1/R$. This implies that $[B(z,R)]^{\mathcal{S}}$ and $[B(0,R)]^{\mathcal{T}_1}$ are $1/R$-close, with respect to the Hausdorff metric. Since $[B(z,R)]^{\mathcal{S}}$ contains a copy of $F_{ms-\eta}(T_i)$, for $0 \leqslant \eta < \delta$, the patch $[B(0,R)]^{\mathcal{T}_1}$ contains a patch $\mathcal{P}$ whose Hausdorff distance is less than $1/R$ from $F_{ms-\eta}(T_i)$. Recall that $F_{ms-\eta}(T_i)$ contains a ball of radius $2/\varepsilon$, then in particular there exists some $u \in B(0,R)$, with $B(u, 2/\varepsilon) \subset B(0,R)$, such that

$$D(\mathcal{T}_1 - u, F_{ms-\eta}(T_i)) \leqslant D(\mathcal{P} - u, F_{ms-\eta}(T_i)) < \varepsilon/2. \tag{6.5}$$

Combining (6.4) and (6.5) we obtain $D(\mathcal{T}_1 - u, F_{ms}(T_i)) < \varepsilon$. But since $\mathcal{S} \supset F_{ms}(T_i)$, this implies $D(\mathcal{T}_1 - u, \mathcal{S}) \leqslant D(\mathcal{T}_1 - u, F_{ms}(T_i)) < \varepsilon$, and the proof is complete. □

6.2. **Supertiles.** Consider an irreducible incommensurable substitution scheme σ in $\mathbb{R}^d$. The following Proposition 6.3 implies the existence of the powerful hierarchical structure known as supertiles on all tilings in the multiscale tiling space $\mathbb{X}_\sigma^F$.

Proposition 6.3. *Let σ be an irreducible incommensurable substitution scheme, let $\mathcal{S} \in \mathbb{X}_\sigma^F$ and let $s \in \mathbb{R}^+$ be so that $F_s(\mathcal{S}) = \mathcal{S}$. Then the map $F_s : \mathbb{X}_\sigma^F \to \mathbb{X}_\sigma^F$ is continuous and surjective.*

Proof. The proof is very similar to [LSo, Proposition 4.9]. First, continuity follows directly from the definition of the flow F_s and the metric D on $\mathbb{X}_\sigma^F$. To see that it is surjective, first note that for every $\mathcal{T} \in \mathbb{X}_\sigma^F$ and $x \in \mathbb{R}^d$

$$F_s(\mathcal{T} - x) = F_s(\mathcal{T}) - e^s x. \tag{6.6}$$

Since $F_s(\mathcal{S}) = \mathcal{S}$, by (6.6) the orbit of $\mathcal{S}$ is mapped onto the itself. The minimality of $(\mathbb{X}_\sigma^F, \mathbb{R}^d)$, established in Theorem 6.1, implies that the space $\mathbb{X}_\sigma^F$ is equal to the closure of this orbit. Combined with the continuity of F_s we deduce that $F_s(\mathbb{X}_\sigma^F) = \mathbb{X}_\sigma^F$. □

Definition 6.4. Let $\mathcal{S} \in \mathbb{X}_\sigma^F$ and $s \in \mathbb{R}^+$ be so that $F_s(\mathcal{S}) = \mathcal{S}$, and let $\mathcal{T} \in \mathbb{X}_\sigma^F$. For every $m \in \mathbb{N}$, let $\mathcal{T}^{-m} \in \mathbb{X}_\sigma^F$ be a choice of a tiling so that $F_{ms}(\mathcal{T}^{-m}) = \mathcal{T}$. Then $\{e^{ms}T : T \text{ is a tile in } \mathcal{T}^{-m}\}$ are *order m supertiles* of $\mathcal{T}$, or simply *m-supertiles*, and are denoted by $T^{(m)}$. The type of the supertile $e^{ms}T$ is inherited from the type of T, and for every $\ell \geqslant m$ every ℓ-supertile can be decomposed into a union of m-supertiles.

Note that a priori the map F_s may not be a bijection, and so the composition of the tiles in $\mathcal{T}$ into supertiles with respect to F_s is not necessarily unique. When considering such supertiles in the stationary tiling $\mathcal{S}$, it is natural to choose $\mathcal{S}^{-m}$ to be $\mathcal{S}$ for every $m \in \mathbb{N}$. This means that supertiles are sets of the form $e^{ms}T$ for some tile T of $\mathcal{S}$ itself. In general, for a supertile $e^{ms}T$ of a tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$, since $\mathcal{T}^{-m}$ is not necessarily $\mathcal{T}$, the tile T cannot be assumed to be a tile of $\mathcal{T}$ itself. Nevertheless, supertiles defined with respect to F_s are always of the form $e^{ms}T$ for some tile T of possible type and scale, that is, a copy of αT_i for some prototile $T_i \in \tau_\sigma$ and scale $\alpha \in (\beta_i^{\min}, 1]$.

Example 6.5. Figure 11 illustrates tiles, thought of as 0-supertiles, within 1-supertiles and a single 2-supertile, in a stationary tiling generated by the scheme on two triangles illustrated in Figure 3.

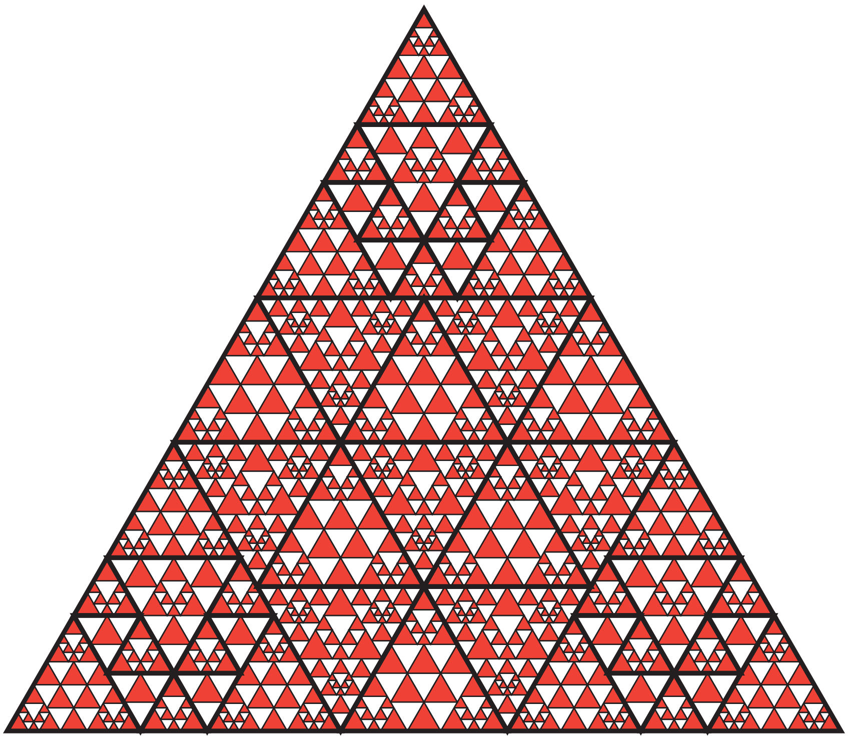

FIGURE 11. Tiles as 0-supertiles, within 1-supertiles (in bold boundaries), all within a single 2-supertile. Compare also with Figure 9.

6.3. **Geometric interpretation of dynamical properties.** A finite local complexity tiling $\mathcal{T}$ of $\mathbb{R}^d$ is called *repetitive* if for every $r > 0$ there is a radius $R := R(r) > 0$, so that for every $x \in \mathbb{R}^d$ the patch $[B(x,R)]^{\mathcal{T}}$ contains translated copies of all the patches of the form $[B(y,r)]^{\mathcal{T}}$, $y \in \mathbb{R}^d$ (see e.g. [BG, Section 5.3], [bS1], and also [FRi] and [LP] for a clearer distinction between similar, common, definitions of repetitivity).

Recall that a set $S \subset \mathbb{R}^d$ is called *relatively dense*, or *syndetic* in the topological dynamics setup, if there exists some $R > 0$ so that S intersects every ball of radius R in $\mathbb{R}^d$. A relatively dense set with parameter R is called *R-dense*. Note that repetitivity means that for every $r > 0$ and every $y \in \mathbb{R}^d$, the set

$$Z_y := \left\{ t \in \mathbb{R}^d \,:\, [B(y,r)]^{\mathcal{T}} - y = [B(t,r)]^{\mathcal{T}} - t \right\}$$

of *return times* to $[B(y,r)]^{\mathcal{T}}$ is relative dense. Since $[B(y,r)]^{\mathcal{T}} - y = [B(0,r)]^{\mathcal{T}-y}$, the set Z_y is simply the collection of $t \in \mathbb{R}^d$ for which $\mathcal{T} - y$ and $\mathcal{T} - t$ agree on the ball of radius r around the origin.

An immediate corollary of Theorems 6.1 and 5.2 is that if σ is incommensurable then every $\mathcal{T} \in \mathbb{X}_\sigma^F$ is of infinite local complexity. Since repetitivity is clearly impossible for such tilings, we consider the following suitable variant.

Definition 6.6. Given $\varepsilon > 0$, a tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$ is called *ε-repetitive* if there exists some $R = R(\varepsilon) > 0$ such that for every $y \in \mathbb{R}^d$ the set

$$A_y := \left\{ x \in \mathbb{R}^d \,:\, D(\mathcal{T} - y, \mathcal{T} - x) < \varepsilon \right\}$$

of return times to the ε-neighborhood of $\mathcal{T} - y$ is R-dense. $\mathcal{T}$ is *almost repetitive* if it is ε-repetitive for every $\varepsilon > 0$ (compare [FRi, Definitions 2.13 and 3.5]).

Recall that two tilings $\mathcal{T}, \mathcal{T}'$ of $\mathbb{R}^d$ are called *locally indistinguishable* (LI) if $\mathcal{T}$ and $\mathcal{T}'$ have the same collection of patches of compact support (see e.g. [BG, Definition 5.5], [LP, Definition 1.6]). This notion induces an equivalence relation on tilings of $\mathbb{R}^d$ and one denotes by $\mathrm{LI}(\mathcal{T})$ the equivalence class of the tiling $\mathcal{T}$. As in the case of repetitivity, we consider the suitable variant of local indistinguishability.

Definition 6.7. Given $\varepsilon > 0$, we say that two tilings $\mathcal{T}, \mathcal{T}' \in \mathbb{X}_\sigma$ are *ε-locally indistinguishable* (ε-LI), if for every $y \in \mathbb{R}^d$ there exist $x_1, x_2 \in \mathbb{R}^d$ such that

$$D\left(\mathcal{T} - y, \mathcal{T}' - x_1\right), D\left(\mathcal{T}' - y, \mathcal{T} - x_2\right) < \varepsilon.$$

$\mathcal{T}, \mathcal{T}' \in \mathbb{X}_\sigma^F$ are called *almost locally indistinguishable* (ALI), if they are ε-LI for every $\varepsilon > 0$ (compare [FRi, Definition 3.9]). We denote by $\mathrm{ALI}(\mathcal{T})$ the collection of tilings in $\mathbb{X}_\sigma^F$ that are ALI with $\mathcal{T}$.

A standard theorem in the theory of tilings, or discrete patterns, states that repetitivity, having a closed LI equivalence class, and having a minimal orbit closure with respect to translations are all equivalent properties (see e.g. [BG, Theorem 5.4], [LP, Theorem 3.2]). A variant of this theorem for r-separated point sets in a setting that resembles the one studied here is given in [FRi, Theorem 3.11]. Our proof of the multiscale substitution tilings variant is similar, and included here due to small differences in the definitions.

Theorem 6.8. *Let σ be an irreducible incommensurable substitution scheme and let $\mathcal{T} \in \mathbb{X}_\sigma^F$. The following are equivalent:*

1. The orbit closure $\overline{\mathcal{O}(\mathcal{T})} := \overline{\{\mathcal{T} + x \,:\, x \in \mathbb{R}^d\}}$ of $\mathcal{T}$ is minimal.
2. $\mathrm{ALI}(\mathcal{T})$ is closed in $\left(\mathbb{X}_\sigma^F, D\right)$.
3. $\mathcal{T}$ is almost repetitive.

Proof. (1) $\Rightarrow$ (2) : It suffices to show that $\overline{\mathcal{O}(\mathcal{T})} = \mathrm{ALI}(\mathcal{T})$. First, clearly $\overline{\mathcal{O}(\mathcal{T})} \supset \mathrm{ALI}(\mathcal{T})$, because if $\mathcal{T}' \in \mathrm{ALI}(\mathcal{T})$, putting $y = 0$ in Definition 6.7 implies that for every $k \in \mathbb{N}$ there exists some $z_k \in \mathbb{R}^d$ with $D\left(\mathcal{T} - z_k, \mathcal{T}'\right) < 1/k$, and so $\mathcal{T}' \in \overline{\mathcal{O}(\mathcal{T})}$.

Let $\mathcal{T}' \in \overline{\mathcal{O}(\mathcal{T})}$, $\varepsilon > 0$ and $y \in \mathbb{R}^d$. Then $\mathcal{T}' - y \in \overline{\mathcal{O}(\mathcal{T})}$, hence there exists some $x_1 \in \mathbb{R}^d$ so that $D\left(\mathcal{T}' - y, \mathcal{T} - x_1\right) < \varepsilon$. By minimality $\mathcal{T}$, and hence $\mathcal{T} - y$, belong to $\overline{\mathcal{O}(\mathcal{T}')}$, so there exists some $x_2 \in \mathbb{R}^d$ with $D\left(\mathcal{T} - y, \mathcal{T}' - x_2\right) < \varepsilon$. We conclude that $\mathcal{T}' \in \mathrm{ALI}(\mathcal{T})$.

(2) $\Rightarrow$ (3) : Assume that (3) doesn't hold. Then there is some $\varepsilon > 0$ for which $\mathcal{T}$ is not ε-repetitive. Namely, there is some $y \in \mathbb{R}^d$ so that the set of return times to the ε-neighborhood of $\mathcal{T} - y$ is not R-dense for any $R > 0$. Then there is a sequence $x_k \in \mathbb{R}^d$ for which the sequence of patches $\left[B(x_k, k)\right]^{\mathcal{T}}$ does not contain a patch that, after translation, is ε-close to $\mathcal{T} - y$. The inclusion

$$\overline{\mathcal{O}(\mathcal{T})} \supset \mathrm{ALI}(\mathcal{T}) \supset \mathcal{O}(\mathcal{T}),$$

combined with the assumption that $\mathrm{ALI}(\mathcal{T})$ is closed, implies that $\overline{\mathcal{O}(\mathcal{T})} = \mathrm{ALI}(\mathcal{T})$.

Let $\mathcal{T}' \in \mathbb{X}_\sigma^F$ be a limit of some subsequence of $\left[B(x_k, k)\right]^{\mathcal{T}} - x_k$. Then no translation of $\mathcal{T}'$ is ε-close to $\mathcal{T} - y$. Since $\mathcal{T}'$ is a limit of the sequence $\mathcal{T} - x_k$, then $\mathcal{T}' \in \overline{\mathcal{O}(\mathcal{T})} = \mathrm{ALI}(\mathcal{T})$. This means that there is some $x \in \mathbb{R}^d$ so that $D\left(\mathcal{T} - y, \mathcal{T}' - x\right) < \varepsilon$, a contradiction.

(3) $\Rightarrow$ (1) : Let $\mathcal{T}' \in \overline{\mathcal{O}(\mathcal{T})}$ and let $\varepsilon > 0$, then there exist $y_k \in \mathbb{R}^d$ with $D\left(\mathcal{T} - y_k, \mathcal{T}'\right) < 1/k$. By (3), $\mathcal{T}$ is $\frac{\varepsilon}{2}$-repetitive. In particular, for $y = 0$ in Definition 6.6, there is some

$R > 0$ for which the set

$$A_0 = \left\{x \in \mathbb{R}^d \,:\, D\left(\mathcal{T}, \mathcal{T} - x\right) < \varepsilon/2\right\}$$

is R-dense, and we may assume that $R > 2/\varepsilon$. Fix $K \geqslant 2R$. Since

$$D\left(\mathcal{T} - y_K, \mathcal{T}'\right) < 1/K \leqslant 1/2R, \tag{6.7}$$

by the definition of D, the patches supported on the $2R$-ball around the origin in $\mathcal{T}'$ and in $\mathcal{T} - y_K$ are $1/2R$-close. Since A_0 is R-dense and $R > 2/\varepsilon$, there is some $x_0 \in B(y_K, R)$ so that $D\left(\mathcal{T}, \mathcal{T} - x_0\right) < \varepsilon/2$ and

$$B(x_0, 2/\varepsilon) \subset B(y_K, 2R). \tag{6.8}$$

Set $x_1 = x_0 - y_K$. Since $||x_1|| < R$, combining (6.7) and (6.8) implies that

$$D\left(\mathcal{T} - x_0, \mathcal{T}' - x_1\right) = D\left(\mathcal{T} - y_K - x_1, \mathcal{T}' - x_1\right) < \varepsilon/2,$$

and by the triangle inequality this yields

$$D\left(\mathcal{T}, \mathcal{T}' - x_1\right) \leqslant D\left(\mathcal{T}, \mathcal{T} - x_0\right) + D\left(\mathcal{T} - x_0, \mathcal{T}' - x_1\right) < \varepsilon.$$

It follows that $\mathcal{T} \in \overline{\mathcal{O}(\mathcal{T}')}$, and minimality is established. □

Corollary 6.9. *Let σ be an irreducible incommensurable substitution scheme. Then every $\mathcal{T} \in \mathbb{X}_\sigma^F$ is almost repetitive, and every $\mathcal{T}, \mathcal{T}' \in \mathbb{X}_\sigma^F$ are* ALI*-equivalent.*

7. Explicit tile counting formulas

This section contains various explicit asymptotic counting formulas for tiles in incommensurable multiscale tilings, according to their types and scales. Results on the distribution of metric paths on incommensurable graphs [KSS] play an important role.

Definition 7.1. Let G be a directed weighted graph with a set of vertices $\mathcal{V} = \{1, \ldots, n\}$. Let $i, j \in \mathcal{V}$ be a pair of vertices in G, and assume that there are $k_{ij} \geqslant 0$ edges $\varepsilon_1, \ldots, \varepsilon_{k_{ij}}$ with initial vertex i and terminal vertex j. The *graph matrix function* of G is the matrix valued function $M : \mathbb{C} \to M_n(\mathbb{C})$, its (i, j) entry the exponential polynomial

$$M_{ij}(s) = e^{-s \cdot l(\varepsilon_1)} + \cdots + e^{-s \cdot l\left(\varepsilon_{k_{ij}}\right)}.$$

If i is not connected to j by an edge, put $M_{ij}(s) = 0$. Note that the restriction of M to $\mathbb{R}$ maps $\mathbb{R}$ to real valued matrices.

Note that given a substitution scheme σ, the (i, j) entry of $M_\sigma(s)$, where M_σ is the graph matrix function of the associated graph G_σ, is given by

$$\left(\alpha_{ij}^{(1)}\right)^s + \cdots + \left(\alpha_{ij}^{(k_{ij})}\right)^s.$$

7.1. **The number of tiles of given types and scales.**

First we need the following counting result on metric paths in graphs associated with incommensurable schemes.

Theorem 7.2. *Let σ be a normalized irreducible incommensurable scheme in $\mathbb{R}^d$, and let G_σ be the associated graph. Let I be an interval that is contained in an edge $\varepsilon \in \mathcal{E}$ with initial vertex $h \in \mathcal{V}$, and assume that I is of length $\delta > 0$ and of distance α_0 from the vertex h. Then the number of metric paths in G_σ of length x, with initial fixed vertex $i \in \mathcal{V}$, that terminate at a point in I, grows as*

$$e^{-\alpha_0 d} \frac{1 - e^{-\delta d}}{d} q_h e^{dx} + o\left(e^{dx}\right), \quad x \to \infty,$$

independent of i, where $q_h \cdot \mathbf{1}$ is column h of the rank 1 matrix

$$Q_\sigma = \frac{\operatorname{adj}\left(I - M_\sigma\left(d\right)\right)}{-\operatorname{tr}(\operatorname{adj}\left(I - M_\sigma\left(d\right)\right) \cdot M'_\sigma\left(d\right))},$$

with $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$, and M_σ is the graph matrix function of G_σ.

Sketch of proof. The proof follows from a slight adjustment of the proof of Theorem 1 in [KSS], and so we only give an outline of the proof. A main tool is the Wiener-Ikehara Tauberian theorem, by which the exponential growth rate of our counting function follows from special properties of its Laplace transform, and in particular the location of its poles, see [MV]. Roughly speaking, if the there exists $\lambda \in \mathbb{R}$ for which the Laplace transform has a simple pole at $s = \lambda$, and there are no other poles in the half plane $\operatorname{Re}(s) \geqslant \lambda$, then the counting function grows exponentially with exponent λ.

The counting function we are interested in can be written as

$$B_{i,I}\left(x\right) = \sum_{\gamma \in \Gamma(i,h)} \chi_{(l(\gamma)+\alpha_0, l(\gamma)+\alpha_0+\delta)}(x), \tag{7.1}$$

where $\Gamma\left(i, h\right)$ is the set of paths in G_σ with initial vertex i and terminal vertex h. Note that although (7.1) is written as if I is assumed to be open, the arguments and results presented here are not changed if I is assumed to contain one or both of its endpoints, as the Laplace transform is not changed if only countably many values of the counting function are varied. A direct computation, details of which can be found in [KSS], shows that the Laplace transform is

$$\mathcal{L}\left\{B_{i,I}\left(x\right)\right\}\left(s\right) := \int_0^\infty B_{i,I}\left(x\right) e^{-xs} dx = e^{-\alpha_0 s} \frac{1 - e^{-\delta s}}{s} \cdot \frac{\left(\operatorname{adj}\left(I - M_\sigma\left(s\right)\right)\right)_{ih}}{\det\left(I - M_\sigma\left(s\right)\right)}.$$

As a result of Proposition 2.5, since G_σ is associated with a substitution scheme, the value of λ, which due to the form of the Laplace transform is the maximal real value for which the spectral radius of $M_\sigma(\lambda)$ is exactly 1, is the dimension d, see [Sm, Lemma 8.1]. The Perron-Frobenius eigenvalue of the non-negative matrix $M_\sigma(d)$ is indeed 1, and a corresponding eigenvector can be chosen to be the vector of volumes of prototiles in τ_σ. Since σ is normalized, this vector is $\mathbf{1}$, from which it follows that Q_σ is a positive matrix. Since the columns of Q_σ are multiples of the eigenvector $\mathbf{1}$, all the rows of Q_σ are identical, from which the independence of i stems in the statement of the result.

The rest of the proof concerns the careful analysis of the poles of the Laplace transform, and more specifically the zeros of the exponential polynomial $\det(I - M_\sigma(s))$, to which the bulk of [KSS] is dedicated. The steps taken there provide proof that there are no poles in the half plane $\operatorname{Re}(s) \geqslant d$, except for a simple pole located at $s = d$, establishing the properties required for the application of the Wiener-Ikehara theorem. □

Theorem 7.3. *Let σ be an irreducible incommensurable scheme in $\mathbb{R}^d$. For every $j = 1, \ldots, n$ and $0 \leqslant a < b \leqslant 1$, the number of tiles of type j in $F_t(T_i)$ with scale in $[a, b]$, or equivalently with volume in $[a^d, b^d]$, grows as*

$$\varphi_{j,[a,b]} e^{dt} + o\left(e^{dt}\right), \quad t \to \infty,$$

independent of i, where $\varphi_{j,[a,b]} := \sum_{h=1}^n c_{hj,[a,b]} q_h$ and q_h is as in Theorem 7.2,

$$c_{hj,[a,b]} := \frac{1}{d} \sum_{k=1}^{k_{hj}} \left(\alpha_{hj}^{(k)}\right)^d \left(\left(\eta_{hj}^{(k)}(a)\right)^{-d} - \left(\mu_{hj}^{(k)}(b)\right)^{-d}\right), \tag{7.2}$$

$\alpha_{hj}^{(k)}$ *are the constants of substitution as in Definition 2.1, and*

$$\eta_{hj}^{(k)}(a) := \max\left\{a, \alpha_{hj}^{(k)}\right\}, \ \mu_{hj}^{(k)}(b) := \max\left\{b, \alpha_{hj}^{(k)}\right\}. \tag{7.3}$$

Although the formulas given in Theorem 7.3 seem complicated, in fact they are easily evaluated in examples, as will be demonstrated in Example 7.8 at the end of this section.

Proof. Tiles of type j in $F_t(T_i)$ correspond to metric paths of length t that terminate on edges with end point j. Let $\varepsilon \in \mathcal{E}$ be such an edge in G_σ, and assume ε is of length $l(\varepsilon) = \log\frac{1}{\alpha}$, corresponding to tiles of scale in $(\alpha, 1]$. Let δ and α_0 be as in the statement of Theorem 7.2. There are three distinct cases:

(1) If $\alpha < a < b$ then

$$\alpha_0 = \log\tfrac{1}{\alpha} - \log\tfrac{1}{a} = \log\tfrac{a}{\alpha}$$
$$\delta = \log\tfrac{1}{\alpha} - \log\tfrac{1}{b} - \alpha_0 = \log\tfrac{b}{a}.$$

Therefore

$$e^{-\alpha_0 d}\frac{1-e^{-\delta d}}{d} = \frac{1}{d}\alpha^d\left(a^{-d} - b^{-d}\right).$$

(2) If $a \leqslant \alpha < b$ then $\alpha_0 = 0$ and

$$\delta = \log\tfrac{1}{\alpha} - \log\tfrac{1}{b} - \alpha_0 = \log\tfrac{b}{\alpha}.$$

Therefore

$$e^{-\alpha_0 d}\frac{1-e^{-\delta d}}{d} = \frac{1}{d}\alpha^d\left(\alpha^{-d} - b^{-d}\right).$$

(3) If $a < b \leqslant \alpha$ then $\alpha_0 = 0$ and $\delta = 0$ and there is no contribution to the counting.

Summing the contributions from all edges in G_σ that terminate at the vertex j, we establish the required formula. □

Note that by Theorem 7.2 and its proof, the exact same formula arises for scales in the interval $[a,b], (a,b]$ or $[a,b)$. In particular, $\varphi_{j,[a,a]} = 0$ and we deduce the following corollary.

Corollary 7.4. *For every $j = 1,\dots,n$ and $\alpha \in (\beta_j^{\min}, 1]$, the number of tiles of type j and scale exactly α in $F_t(T_i)$ is $o\left(e^{dt}\right)$ as t tends to infinity.*

The results stated above can be described in terms of supertiles.

Corollary 7.5. *Under the notation of Theorem 7.3, for every tile T of possible type and scale (see Definition 5.1), the number of tiles of type j in $F_t(T)$ with scale in $[a,b]$ grows as*

$$\varphi_{j,[a,b]}\mathrm{vol}\left(e^t T\right) + o\left(\mathrm{vol}\left(e^t T\right)\right), \quad t \to \infty.$$

In particular, for every sequence $\left(T^{(m)}\right)_{m\geqslant 0}$ of supertiles of growing order in $\mathcal{T} \in \mathbb{X}_\sigma^F$, the number of tiles of type j with scale in $[a,b]$ in the patch $[T^{(m)}]^{\mathcal{T}}$, grows as

$$\varphi_{j,[a,b]}\mathrm{vol}\left(T^{(m)}\right) + o\left(\mathrm{vol}\left(T^{(m)}\right)\right), \quad m \to \infty.$$

Proof. If T is a rescaled copy of αT_i, then

$$F_t(T) = F_t(\alpha T_i) = F_{t-\log(1/\alpha)}(T_i),$$

and by Theorem 7.3 the number of tiles we are interested in grows as

$$\varphi_{j,[a,b]}e^{d(t-\log(1/\alpha))} + o\left(e^{dt}\right), \quad t \to \infty.$$

But $e^{d(t-\log(1/\alpha))} = e^{dt}\alpha = \mathrm{vol}(e^tT)$, and the result follows. $\square$

We remark that Corollary 7.5 gives an alternative proof that the scales in which tiles appear in incommensurable tilings are dense within the intervals of possible scales, see Theorem 5.2. For the next results, recall that for a tiling $\mathcal{T}$ and a set $B \subset \mathbb{R}^d$ we denote $[B]^{\mathcal{T}}$ to be the patch of all tiles in $\mathcal{T}$ that intersect B.

Corollary 7.6. *Let $\left(T^{(m)}\right)_{m\geqslant 0}$ be a sequence of supertiles of growing order in $\mathcal{T} \in \mathbb{X}_\sigma^F$. The number of tiles of type j in $[T^{(m)}]^{\mathcal{T}}$ grows as*

$$\varphi_{j,(\beta_j^{\min},1]}\mathrm{vol}\left(T^{(m)}\right) + o\left(\mathrm{vol}\left(T^{(m)}\right)\right), \quad m \to \infty$$

with

$$\varphi_{j,(\beta_j^{\min},1]} = \sum_{h=1}^{n} \frac{1}{d} \sum_{k=1}^{k_{hj}} \left(1 - \left(\alpha_{hj}^{(k)}\right)^d\right) q_h. \tag{7.4}$$

In particular, the total number of tiles in $[T^{(m)}]^{\mathcal{T}}$ grows as

$$\sum_{j=1}^{n} \varphi_{j,(\beta_j^{\min},1]}\mathrm{vol}\left(T^{(m)}\right) + o\left(\mathrm{vol}(T^{(m)})\right), \quad t \to \infty, \tag{7.5}$$

independent of i.

Proof. Simply note that the parameters in (7.3) satisfy $\eta_{hj}^{(k)}(0) = \alpha_{hj}^{(k)}$ and $\mu_{hj}^{(k)}(1) = 1$, for all i, j, k. $\square$

Manipulations of the formulas given above allow for computations of other quantities. For example, we give below the asymptotic formula for the relative number of tiles of a given type and scales within a fixed interval of scales, within all tiles in supertiles. Various other quantities can be similarly derived.

Corollary 7.7. *For every $j = 1, \dots, n$ and $0 \leqslant a < b \leqslant 1$, the relative number of tiles of type j with scale in $[a, b]$ within the total number of tiles in a patch supported on an m-supertile, tends to*

$$\frac{\varphi_{j,[a,b]}}{\sum_{j=1}^n \varphi_{j,(\beta_j^{\min},1]}} + o(1), \quad m \to \infty,$$

where $\varphi_{j,[a,b]}$ and $\varphi_{j,(\beta_j^{\min},1]}$ are as in Theorem 7.2 and Corollary 7.6, respectively.

Example 7.8. Let σ be the substitution scheme in $\mathbb{R}^2$ on $T_1 = U$ and $T_2 = D$ the two equilateral triangles, as described in Figure 3. By a direct computation

$$Q_\sigma = \frac{\mathrm{adj}\left(I - M_\sigma(2)\right)}{-\mathrm{tr}(\mathrm{adj}\left(I - M_\sigma(2)\right) \cdot M_\sigma'(2))} = \frac{1}{\frac{4}{25}\log 2 + \frac{1}{4}\log 5} \begin{pmatrix} \frac{1}{4} & \frac{8}{25} \\ \frac{1}{4} & \frac{8}{25} \end{pmatrix}$$

and so

$$q_1 = \frac{1}{4} \cdot \frac{1}{\frac{4}{25}\log 2 + \frac{1}{4}\log 5} \quad \text{and} \quad q_2 = \frac{8}{25} \cdot \frac{1}{\frac{4}{25}\log 2 + \frac{1}{4}\log 5}.$$

Consider rescaled copies of U (tiles of type 1) with scales within the interval $\left[\frac{3}{5}, \frac{4}{5}\right]$. Since $\alpha \leqslant \frac{1}{2}$ for any scale α in which tiles appear in the substitution scheme σ, we have $\eta = \frac{3}{5}$ and $\mu = \frac{4}{5}$ in (7.3). Plugging this into (7.2) we get

$$c_{11,\left[\frac{3}{5},\frac{4}{5}\right]} = \frac{119}{288} \quad \text{and} \quad c_{21,\left[\frac{3}{5},\frac{4}{5}\right]} = \frac{175}{1152},$$

and so

$$\varphi_{1,\left[\frac{3}{5},\frac{4}{5}\right]} = c_{11,\left[\frac{3}{5},\frac{4}{5}\right]}q_1 + c_{21,\left[\frac{3}{5},\frac{4}{5}\right]}q_2 \approx 0.296.$$

We now use the above computation to derive a couple of results about tiles that are rescaled copies of U with scales in the interval $\left[\frac{3}{5},\frac{4}{5}\right]$. First, according to Corollary 7.5 , for every tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$ and every sequence $\left(T^{(m)}\right)_{m\geqslant 0}$ of supertiles of growing order in $\mathcal{T}$, the number of such tiles in the patch supported on $T^{(m)}$ grows approximately as

$$0.296 \cdot \mathrm{vol}\left(T^{(m)}\right) + o\left(\mathrm{vol}\left(T^{(m)}\right)\right), \quad m \to \infty.$$

Next, using (7.4) we have

$$\varphi_{1,(\beta_1^{\min},1]} \approx 2.016 \quad \text{and} \quad \varphi_{2,(\beta_2^{\min},1]} \approx 1.841,$$

and by Corollary 7.7 we deduce that the relative number of such tiles within the total number of tiles in a patch supported on an m-supertile, tends to

$$\frac{\varphi_{1,\left[\frac{3}{5},\frac{4}{5}\right]}}{\varphi_{1,(\beta_1^{\min},1]} + \varphi_{2,(\beta_2^{\min},1]}} + o(1) \approx 0.076 + o(1), \quad m \to \infty.$$

7.2. **The volume occupied by tiles of given types and scales.** For the main results of this section we turn to probabilistic results on graphs.

Theorem 7.9. *Let σ be a normalized irreducible incommensurable scheme in $\mathbb{R}^d$, and let G_σ be the associated graph. For any $i \in \mathcal{V}$ and $\varepsilon \in \mathcal{E}$ with initial vertex i let $p_{i\varepsilon}$ be the probability that a walker who is passing through vertex i chooses to continue his walk through the edge ε, and assume that the sum of the probabilities over all edges originating at any vertex is equal to 1. Let I be an interval that is contained in some edge $\varepsilon \in \mathcal{E}$ with initial vertex $h \in \mathcal{V}$, and assume that I is of length $\delta > 0$ and distance α_0 from the vertex h. Then the probability that a walker originating at vertex $i \in \mathcal{V}$ and advancing at unit speed is on the interval I after walking along a metric path of length x, tends to*

$$p_{h\varepsilon}\delta q_h + o(1), \quad x \to \infty,$$

independent of i and of α_0, where q_h is as in Theorem 7.2.

Proof. The proof is very similar to that of Theorem 7.2 given above for counting paths terminating in given intervals. Here we adjust Theorem 2 in [KSS] in the same way Theorem 1 in [KSS] is adjusted for the proof of Theorem 7.2. □

Theorem 7.10. *Let σ be an irreducible incommensurable scheme in $\mathbb{R}^d$. For every $j = 1, \ldots, n$ and $0 \leqslant a < b \leqslant 1$, the volume covered by tiles of type j in $F_t(T_i)$ with scale in $[a,b]$, or equivalently with volume in $[a^d, b^d]$, grows as*

$$\nu_{j,[a,b]}e^{dt} + o\left(e^{dt}\right), \quad t \to \infty,$$

where $\nu_{j,[a,b]} := \sum_{h=1}^n d_{hj,[a,b]}q_h$ and q_h is as in Theorem 7.2,

$$d_{hj,[a,b]} := \sum_{k=1}^{k_{hj}} \left(\alpha_{hj}^{(k)}\right)^d \log \frac{\eta_{hj}^{(k)}(a)}{\mu_{hj}^{(k)}(b)},$$

with $\eta_{hj}^{(k)}(a)$ and $\mu_{hj}^{(k)}(b)$ as in (7.3), and independent of i.

Proof. We assign probabilities to the graph G_σ in the following way. Assume ε is an edge with initial vertex h that is associated to a tile $T = \alpha T_j \in \omega_\sigma(T_h)$. Then put $p_{h\varepsilon} = \mathrm{vol} T = \alpha^d$, which is the probability for a point in T_i to belong to the tile T after the substitution rule ϱ_σ is applied to T_i. By (2.1), for every vertex the sum of the probabilities on its outgoing edges is 1.

Let ε be an edge as above, and recall that $l(\varepsilon) = \log \frac{1}{\alpha}$. We wish to calculate the probability that a metric path terminates at a point on ε associated with a tile of scale in $[a, b]$. Once again there are three distinct cases:

(1) If $\alpha < a < b$ then $\delta = \log \frac{b}{a}$, and so

$$p_{h\varepsilon}\delta = \alpha^d \log \tfrac{b}{a}.$$

(2) If $a \leqslant \alpha < b$ then $\delta = \log \frac{b}{\alpha}$, and so

$$p_{h\varepsilon}\delta = \alpha^d \log \tfrac{b}{\alpha}.$$

(3) If $a < b \leqslant \alpha$ then $\delta = 0$ and there is no contribution to the counting.

Summing the contributions from all edges in G_σ that terminate at the vertex j, we establish that the probability for a point in the patch $F_t(T_i)$ to be in a tile of type j and scale in $[a, b]$ tends to

$$\sum_{h=1}^{n} d_{hj,[a,b]} q_h + o\left(1\right), \quad t \to \infty,$$

and since the volume of this patch is e^{dt} we arrive at the required formula. □

The proof of Corollary 7.6 yields also the following analogous formula.

Corollary 7.11. *Let* $\left(T^{(m)}\right)_{m \geqslant 0}$ *be a sequence of supertiles of growing order in* $\mathcal{T} \in \mathbb{X}_\sigma^F$. *The volume of the region covered by tiles of type* j *in* $[T^{(m)}]^{\mathcal{T}}$ *grows as*

$$\nu_{j,(\beta_j^{\min},1]} \mathrm{vol}\left(T^{(m)}\right) + o\left(\mathrm{vol}\left(T^{(m)}\right)\right), \quad m \to \infty.$$

where

$$\nu_{j,(\beta_j^{\min},1]} = \sum_{h=1}^{n} \sum_{k=1}^{k_{hj}} \left(\alpha_{hj}^{(k)}\right)^d \log \tfrac{1}{\alpha_{hj}^{(k)}} q_h.$$

Remark 7.12. The results and formulas stated above can be used to recover the formulas given in [Sa, Theorem 8] for irrational generalized pinwheel tilings, as these tilings can also be described as stationary tilings generated by incommensurable substitution schemes on a single right triangle, as described in Remark 4.10. We note that the computations for Corollaries 7.6 and 7.11 appear also in [Sm, §2, §8], which includes additional explicit formulas for various examples in the context of Kakutani sequences of partitions.

8. Multiscale tilings are not uniformly spread

Following Laczkovich [L], we say that a point set $Y \subset \mathbb{R}^d$ is *uniformly spread* if there exists some $\alpha > 0$ and a bijection $\phi : Y \to \alpha\mathbb{Z}^d$ satisfying

$$\sup_{y \in Y} \|y - \phi(y)\| < \infty.$$

Such a mapping ϕ is called a *bounded displacement (BD)*. Given a tiling $\mathcal{T}$ of $\mathbb{R}^d$, by tiles of uniformly bounded diameter, we say that $\mathcal{T}$ is *uniformly spread* if there exists a point set $Y_{\mathcal{T}}$, that is obtained by picking a point from each tile in $\mathcal{T}$, which is uniformly spread.

Note that since the tiles are of uniformly bounded diameter, if $\mathcal{T}$ is uniformly spread then every point set $Y_{\mathcal{T}}$ obtained in the above manner is uniformly spread.

The following criterion, phrased here for tilings instead of point sets, was proved by Laczkovich, see [L, Theorem 1.1]. Recall that for a set $A \subset \mathbb{R}^d$ and a constant $r > 0$, we denote $A^{+r} = \{x \in \mathbb{R}^d : \ \mathbf{dist}(x, A) \leqslant r\}$.

Theorem 8.1 ([L]). *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ by tiles of uniformly bounded diameter, then $\mathcal{T}$ is uniformly spread if and only if there exist positive constants $c, \alpha > 0$ so that*

$$\left|\#[U]^{\mathcal{T}} - \alpha \cdot \mathrm{vol}(U)\right| \leqslant c \cdot \mathrm{vol}((\partial U)^{+1}) \tag{8.1}$$

holds for every bounded, measurable set U.

The question whether fixed scale substitution tilings are uniformly spread was studied in [ACG, FSS, yS1] and [yS2], where it was shown that certain conditions on the eigenvalues and eigenvectors of the substitution matrix imply a positive answer to this question. Our goal in this section is to show that incommensurable multiscale substitution tilings are never uniformly spread, emphasizing once again the difference between the tilings in the standard setup and the tilings being studied here.

Theorem 8.2. *Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$. Then every $\mathcal{T} \in \mathbb{X}_\sigma^F$ is not uniformly spread.*

Using Laczkovich's criterion, Theorem 8.2 is a consequence of the following lemma, which is standard in the context of the dynamical and the fractal zeta functions, to which the books [PP] and [LV] are respectively dedicated.

Lemma 8.3. *Let σ be an irreducible, incommensurable substitution scheme in $\mathbb{R}^d$. Let $E(t)$ denote the error term in Theorem 7.3. Then for no constant $\beta < d$ do we have*

$$E(t) = O\left(e^{\beta t}\right).$$

In particular, if E_m denotes the error term in (7.5), for no $\varepsilon > 0$ do we have

$$E_m = O\left(\left(\mathrm{vol}\left(T^{(m)}\right)\right)^{1-\varepsilon}\right).$$

Sketch of proof. The result follows from information on the location of the poles of the Laplace transform of the counting function as appears in (7.1), which are the zeroes of the exponential polynomial $\det(I - M_\sigma(s))$.

First, assume by contradiction that the error term is bounded by $e^{\beta t}$ for some $\beta < d$. Then by inverse Laplace transform theory, all poles $s \in \mathbb{C}$ of the Laplace transform other than $s = d$ have real part less or equal to $\beta < d$, see [Po, Proposition 6]. Therefore, they must be bounded away from the vertical line $\mathrm{Re}(s) = d$.

On the other hand, incommensurability of σ implies that d is a limit point of the real parts of zeroes of $\det(I - M_\sigma(s))$. Similarly to [LV, Theorem 3.23], this is deduced by considering rational approximations to the exponential polynomial $\det(I - M_\sigma(s))$, and using Rouché's theorem. See also [Po, Proposition 7]. $\square$

Proof of Theorem 8.2. Let $\mathcal{T} \in \mathbb{X}_\sigma^F$ and let $\left(T^{(m)}\right)_{m\geqslant 0}$ be a sequence of supertiles of growing order in $\mathcal{T}$. Denote by $\#[T^{(m)}]^{\mathcal{T}}$ the number of tiles in $[T^{(m)}]^{\mathcal{T}}$. By Corollary 7.6, $\#[T^{(m)}]^{\mathcal{T}}$ grows as $C\mathrm{vol}(T^{(m)}) + E_m$ for some constant C. Hence for any $\alpha \neq C$, the inequality (8.1) clearly fails for $U = T^{(m)}$. For $\alpha = C$ we obtain that

$$\left|\#[T^{(m)}]^{\mathcal{T}} - \alpha \cdot \mathrm{vol}\left(T^{(m)}\right)\right| = E_m.$$

$T^{(m)}$ are m-supertiles and hence

$$\mathrm{vol}\left(\left(\partial T^{(m)}\right)^{+1}\right) = C_\partial \left(\mathrm{vol}\left(T^{(m)}\right)\right)^{(d-1)/d}$$

for some constant C_∂. In view of Lemma 8.3 we obtain that for any constant c

$$\left|\#[T^{(m)}]^{\mathcal{T}} - \alpha \cdot \mathrm{vol}\left(T^{(m)}\right)\right| > c \cdot \mathrm{vol}\left(\left(\partial T^{(m)}\right)^{+1}\right)$$

for m large enough. We deduce that for any constant c the inequality (8.1) in Laczkovich's criterion fails for $U = T^{(m)}$, for m large enough, and the proof is complete. □

Remark 8.4. Let $d \geqslant 2$. A point set $Y \subset \mathbb{R}^d$ is called *rectifiable*, or *bi-Lipschitz equivalent to a lattice*, if there exists a bi-Lipschitz bijection $\phi : Y \to \mathbb{Z}^d$. Namely, a bijection ϕ for which there is some constant $L \geqslant 1$ that satisfies

$$\frac{1}{L} \leqslant \frac{\|\phi(y_1) - \phi(y_2)\|}{\|y_1 - y_2\|} \leqslant L$$

for every two distinct points $y_1, y_2 \in Y$.

Rectifiability is often established using a sufficient condition of Burago and Kleiner which requires an appropriate upper bound on the discrepancy $\left|\#[U]^{\mathcal{T}} - \alpha \cdot \mathrm{vol}(U)\right|$ for large cubes U, see [BK2]. Improving the lower bound of $E(t)$ from Lemma 8.3 to a polynomial error term of the form e^{dt}/t^δ for some $\delta \leqslant 1$, would imply the failure of Burago-Kleiner condition for all point sets that arise from multiscale substitution tilings. Lohöfer and Mayer claim this with $\delta = 1$ for what in our context would be constructions associated with the golden ratio, which heuristically is the case with smallest error term, see [LM, p. 5]. Unfortunately, this appears without proof.

We remark that the existence of non-rectifiable point sets in $\mathbb{R}^d$ is a non-trivial result, see [BK1] and [McM]. Finding concrete non-rectifiable examples, which currently include only those described in [CN] and in [G], is a very interesting problem.

9. Uniform patch frequencies

This section is dedicated to the study of patch frequencies in incommensurable tilings. We define a multiscale tiling variant of patch frequency, where instead of counting appearances of patches up to translation equivalence as done in standard constructions, we group together patches that are dilations of each other. This is crucial in order to establish the existence of positive uniform patch frequencies for dilations of legal patches, which is key in our proof of unique ergodicity of incommensurable tiling dynamical system in §10.

Our proof follows the framework suggested in appendix A.1 of [LMS2] for the existence of uniform patch frequencies of fixed scale substitution tilings. Indeed, some of the steps are identical to those of Lee, Moody and Solomyak, while others require new ideas and results that strongly depend on the incommensurability of the underlying substitution scheme. In particular, the assumption of primitivity and the theory of Perron-Frobenius are replaced by our results on patches in incommensurable tilings and their scales developed in Sections 5 and 7.

Definition 9.1. A sequence $(A_q)_{q\geqslant 1}$ of bounded measurable subsets of $\mathbb{R}^d$ is *van Hove* if

$$\lim_{q\to\infty} \frac{\mathrm{vol}\left((\partial A_q)^{+r}\right)}{\mathrm{vol}(A_q)} = 0$$

for all $r > 0$, where as before $A^{+r} = \{x \in \mathbb{R}^d : \mathbf{dist}(x, A) \leqslant r\}$. In addition, we denote

$$A^{-r} := \{x \in A : \mathbf{dist}(x, \partial A) \geqslant r\}$$

for any $r > 0$ and a bounded set $A \subset \mathbb{R}^d$.

Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$. By our definition of tiles and by Proposition 4.5, all tiles in a tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$ have boundary of measure zero, and so every sequence of supertiles with growing order is van Hove. Given a patch $\mathcal{P}$ in a tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$, a bounded interval $I \subset \mathbb{R}$ and a bounded set $A \subset \mathbb{R}^d$, denote

$$L_{\mathcal{P},I}(A, \mathcal{T}) := \#\{g \in \mathbb{R}^d : \exists \alpha \in I \text{ s.t. } g + \alpha\mathcal{P} \subset \mathcal{T}, (g + \operatorname{supp}(\alpha\mathcal{P})) \subset A\},$$
$$N_{\mathcal{P},I}(A, \mathcal{T}) := \#\{g \in \mathbb{R}^d : \exists \alpha \in I \text{ s.t. } g + \alpha\mathcal{P} \subset \mathcal{T}, (g + \operatorname{supp}(\alpha\mathcal{P})) \cap A \neq \varnothing\},$$

where once again $\#B$ denotes the number of elements in a finite set B.

Theorem 9.2. *Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$, and let $\mathcal{S} \in \mathbb{X}_\sigma^F$ be a stationary tiling. Let $\mathcal{P}$ be a patch in $\mathcal{S}$ and let I be a bounded interval that contains 1 and a left neighborhood of 1. Then for every van Hove sequence $(A_q)_{q\geqslant 1}$ in $\mathbb{R}^d$*

$$\operatorname{freq}(\mathcal{P}, I, \mathcal{S}) := \lim_{q\to\infty} \frac{L_{\mathcal{P},I}(A_q + h, \mathcal{S})}{\operatorname{vol}(A_q)} \tag{9.1}$$

exists uniformly in $h \in \mathbb{R}^d$, and is positive.

Let $s \in \mathbb{R}^+$ be so that $\mathcal{S} = F_s(\mathcal{S})$. Since $\mathcal{S}$ is fixed, we simplify notation and set $L_{\mathcal{P},I}(A) := L_{\mathcal{P},I}(A, \mathcal{S})$ and $N_{\mathcal{P},I}(A) := N_{\mathcal{P},I}(A, \mathcal{S})$.

Lemma 9.3. *Let $A \subset \mathbb{R}^d$ be a bounded set and let $(A_q)_{q\geqslant 1}$ be a van Hove sequence in $\mathbb{R}^d$. Let $\mathcal{P}$ be a patch in $\mathcal{S}$ and let $I \subset \mathbb{R}$ be a bounded interval that contains 1 and a left neighborhood of 1. Then*

(1) *there exists $c_1 > 0$, which depends only on σ, so that*
$$L_{\mathcal{P},I}(A) \leqslant c_1 \operatorname{vol}(A).$$
(2) *there exist $c_2, q_0 > 0$, which depend on $\mathcal{P}, I$ and $(A_q)_{q\geqslant 1}$, so that for all $q \geqslant q_0$*
$$L_{\mathcal{P},I}(A_q + h) \geqslant c_2 \operatorname{vol}(A_q).$$
(3) $\lim_{q\to\infty} N_{\mathcal{P},I}(\partial(A_q + h))/L_{\mathcal{P},I}(A_q + h) = 0$ *uniformly in $h \in \mathbb{R}^d$.*

Proof. We follow the proof of Lemma A.4 in [LMS2]. Parts (1) and (3) are identical but are short and so we include them here. Lemma 5.6 replaces primitivity in part (2).

(1) Select a tile from the patch $\mathcal{P}$. Distinct patches of the form $\alpha\mathcal{P}$ in $\mathcal{S}$ will have distinct selected tiles, therefore
$$L_{\mathcal{P},I}(A) \leqslant V_{\min}^{-1} \operatorname{vol}(A),$$
where $V_{\min}$ is the infimum over tile volumes in $\mathcal{S}$.
(2) By Lemma 5.6, and since $\mathcal{S}$ is stationary, there exists an $\ell \in \mathbb{N}$ so that for every tile T in $\mathcal{S}$, the patch $F_{\ell s}(T)$ contains a copy of the patch $\alpha\mathcal{P}$ as a sub-patch, for some $\alpha \in I$. It follows that for any set A, the number $L_{\mathcal{P},I}(A)$ is at least the number of ℓ-supertiles with support contained in A. Since tiles in $\mathcal{S}$ are of volume

at most 1, the volume of an ℓ-supertile is at most $e^{\ell sd}$. Let r be the supremum of the diameters of ℓ-supertiles. We have obtained

$$\begin{aligned} L_{\mathcal{P},I}(A_q+h) &\geqslant e^{-\ell sd}\mathrm{vol}(A_q^{-r}+h) \\ &= e^{-\ell sd}\mathrm{vol}(A_q^{-r}) \\ &\geqslant e^{-\ell sd}\left(\mathrm{vol}(A_q)-\mathrm{vol}\left((\partial A_q)^{+r}\right)\right). \end{aligned}$$

Since $(A_q)_{q\geqslant 1}$ is van Hove, the proof is complete.

(3) Let r denote the supremum of the diameters of the supports of $\alpha\mathcal{P}$, for $\alpha \in I$. In view of parts (1) and (2) of this lemma

$$\frac{N_{\mathcal{P},I}(\partial A_q+h)}{L_{\mathcal{P},I}(A_q+h)} \leqslant \frac{L_{\mathcal{P},I}\left((\partial A_q)^{+r}+h\right)}{L_{\mathcal{P},I}(A_q+h)} \leqslant \frac{c_1\mathrm{vol}((\partial A_q)^{+r}+h)}{c_2\mathrm{vol}(A_q+h)} = \frac{c_1\mathrm{vol}((\partial A_q)^{+r})}{c_2\mathrm{vol}(A_q)} \to 0,$$

uniformly in $h \in \mathbb{R}^d$, as required. ☐

Consider (2) for q-supertiles, that is, $A_q = e^{qs}\mathrm{supp}T$, where T is a tile in $\mathcal{S}$. Since there are only finitely many types of tiles and the set of scales in which they appear is bounded, we obtain the following.

Corollary 9.4. *Under the assumptions of Lemma 9.3, if $A_q = e^{qs}\mathrm{supp}T$, where T is a tile in $\mathcal{S}$, then there exists a constant for which* (2) *holds for all tiles T in $\mathcal{S}$, and the convergence in* (3) *is uniform in the choice of T.*

The main step of the proof of Theorem 9.2 is given by the following Lemma.

Lemma 9.5. *Let $\mathcal{S}, \mathcal{P}$ and I be as above. Then*

$$c_{\mathcal{P},I} := \lim_{q\to\infty} \frac{L_{\mathcal{P},I}(e^{qs}\mathrm{supp}T)}{\mathrm{vol}(e^{qs}\mathrm{supp}T)} > 0 \tag{9.2}$$

exists uniformly in tiles T in $\mathcal{S}$.

Proof. Fix $\varepsilon > 0$. By Lemma 9.3 and Corollary 9.4, there exists $m_0 \in \mathbb{N}$ so that for every $m \geqslant m_0$ and every tile T in $\mathcal{S}$

$$N_{\mathcal{P},I}(\partial e^{ms}\mathrm{supp}T) \leqslant \varepsilon\, L_{\mathcal{P},I}(e^{ms}\mathrm{supp}T). \tag{9.3}$$

Choose a tile T in $\mathcal{S}$, and assume it is of type j and scale α, that is, T is a translated copy of αT_j, and set $A = \mathrm{supp}T$. For $q > m > m_0$, consider the decomposition of the q-supertile $e^{qs}T$ into m-supertiles. The support of an m-supertile of type i is of the form $e^{ms}\beta A_i$, where A_i is a translation of the support of the prototile $T_i \in \tau_\sigma$, and $\beta \in (\beta_i^{\min}, 1]$ is the scale in which the corresponding tile appears in $F_{(q-m)s}(\alpha T_j)$.

Observe that the set $\left\{F_{ms}(\beta T_i) : \beta \in (\beta_i^{\min}, 1]\right\}$ consists of a finite number of patches up to rescaling, because $F_{ms}(\beta T_i) = F_{ms-\log(1/\beta)}(T_i)$ and

$$\mathscr{S}_i \cap \left\{ms - \log(1/\beta) : \beta \in (\beta_i^{\min}, 1]\right\}$$

is a finite set. Therefore, there is a finite number of tile substitutions under the flow $F_t(T_i)$ for $ms - \log(1/\beta_i^{\min}) < t \leqslant ms$. It follows that $L_{\mathcal{P},I}(e^{ms}\beta A_i)$ is a piecewise constant function of the variable $\beta \in (\beta_i^{\min}, 1]$. Namely, there is a finite partition of the interval $(\beta_i^{\min}, 1]$ into M_i sub-intervals $\{I_{i,1}, \ldots, I_{i,M_i}\}$, depending only on $\mathcal{P}, I$ and m, so that for every choice of representatives $\{\beta_{i,\ell} \in I_{i,\ell} : \ell = 1, \ldots, M_i\}$ the equality

$$L_{\mathcal{P},I}(e^{ms}\beta A_i) = L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i).$$

holds for every $\beta \in I_{i,\ell}$.

Denote by $n_{i,\ell,q}$ the number of m-supertiles with support $e^{ms}\beta A_i$ with $\beta \in I_{i,\ell}$ in the decomposition of $e^{qs}T$ into m-supertiles. Together with (9.3), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{\ell=1}^{M_i} L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i)n_{i,\ell,q} &\leqslant L_{\mathcal{P},I}(e^{qs}A) \\ &\leqslant \sum_{i=1}^{n}\sum_{\ell=1}^{M_i} L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i)n_{i,\ell,q} + N_{\mathcal{P},I}(\partial e^{ms}\beta_{i,\ell}A_i)n_{i,\ell,q} \qquad (9.4) \\ &\leqslant (1+\varepsilon)\sum_{i=1}^{n}\sum_{\ell=1}^{M_i} L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i)n_{i,\ell,q}.\end{aligned}$$

Note that $n_{i,\ell,q}$ is exactly the number of tiles of type i and scale in the interval $I_{i,\ell}$ in the patch $F_{(q-m)s-\log(1/\alpha)}(T_j)$. By Theorem 7.3, there exists a constant $\varphi_{i,I_{i,\ell}} > 0$ independent of j so that

$$\lim_{q\to\infty} \frac{n_{i,\ell,q}}{e^{((q-m)s-\log(1/\alpha))d}} = \varphi_{i,I_{i,\ell}}.$$

Since $\mathrm{vol}(e^{qs}A) = e^{(qs-\log(1/\alpha))d}$, this yields

$$\lim_{q\to\infty} \frac{n_{i,\ell,q}}{\mathrm{vol}(e^{qs}A)} = \varphi_{i,I_{i,\ell}}e^{-msd}. \qquad (9.5)$$

Dividing (9.4) by $\mathrm{vol}(e^{qs}A)$ and letting $q \to \infty$ as in (9.5), we obtain

$$\limsup_{q\to\infty} \frac{L_{\mathcal{P},I}(e^{qs}A)}{\mathrm{vol}(e^{qs}A)} - \liminf_{q\to\infty} \frac{L_{\mathcal{P},I}(e^{qs}A)}{\mathrm{vol}(e^{qs}A)} \leqslant \varepsilon \sum_{i=1}^{n}\sum_{\ell=1}^{M_i} L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i)\varphi_{i,I_{i,\ell}}e^{-msd}. \qquad (9.6)$$

By (1) of Lemma 9.3, and since tiles in $\mathcal{S}$ are of volume at most 1, we have

$$L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i) \leqslant c_1\mathrm{vol}(e^{ms}\beta_{i,\ell}A_i) \leqslant c_1 e^{msd}.$$

The right-hand side of (9.6) is thus bounded by

$$\varepsilon \sum_{i=1}^{n}\sum_{\ell=1}^{M_i} c_1\varphi_{i,I_{i,\ell}} \leqslant \varepsilon\, \widetilde{c},$$

with $\widetilde{c}$ independent of q and of m. Since $\varepsilon > 0$ is arbitrarily, the limit in (9.2) exists. This limit, denoted by $c_{\mathcal{P},I}$, satisfies

$$\sum_{i=1}^{n}\sum_{\ell=1}^{M_i} L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i)\varphi_{i,I_{i,\ell}}e^{-msd} \leqslant c_{\mathcal{P},I} \leqslant (1+\varepsilon)\sum_{i=1}^{n}\sum_{\ell=1}^{M_i} L_{\mathcal{P},I}(e^{ms}\beta_{i,\ell}A_i)\varphi_{i,I_{i,\ell}}e^{-msd},$$

independent of the choice of tile $T \in \mathcal{S}$. By (2) of Lemma 9.3, for large enough values of m the left-hand side is positive, and so $c_{\mathcal{P},I}$ is positive. □

The proof of Theorem 9.2 now follows by the same arguments as those given in detail in the proof of Lemma A.6 in [LMS2].

Sketch of proof of Theorem 9.2. Consider the decomposition of $\mathbb{R}^d$ into the m-supertiles $e^{ms}T$ for T in $\mathcal{S}$. $L_{\mathcal{P},I}(A_q + h)$ can be approximated by sums of $L_{\mathcal{P},I}(e^{ms}\mathrm{supp}T)$, using Lemma 9.5. The van Hove property is then used to show that boundary effects are negligible. □

In fact, the arguments used above can be used to establish the existence of uniform patch frequencies for general tilings $\mathcal{T} \in \mathbb{X}_{\sigma}^{F}$.

Corollary 9.6. *Let $\mathcal{T} \in \mathbb{X}_\sigma^F$. For any legal patch $\mathcal{P}$ in $\mathcal{T}$ and a bounded interval $I \subset \mathbb{R}$ that contains a left neighborhood of* 1

$$\mathrm{freq}(\mathcal{P}, I, \mathcal{T}) = \mathrm{freq}(\mathcal{P}, I, \mathcal{S}) > 0,$$

and in particular, Corollary 5.7 holds for $\mathcal{T}$. Non-legal patches have zero frequency.

Proof. We only sketch the proof, since it follows from standard arguments about supertiles as in the proof of [LSo, Theorem 4.11 (ii)]. If $\mathcal{P}$ is a legal patch and I is an interval as above, then for any tiling $\mathcal{T} \in \mathbb{X}_\sigma^F$ the associated frequency in supertiles approaches $c_{\mathcal{P},I}$, and when considering general sets the difference amounts to boundary effects. In the non-legal case, since such patches must intersect the boundaries of supertiles of arbitrarily large order, the van Hove property of sequences of supertiles deem the frequencies negligible. □

We end this section by showing that in the classical sense of uniform patch frequency, all patches in tiling $\mathbb{X}_\sigma^F$ have zero frequency.

Theorem 9.7. *Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$, let $\mathcal{S} \in \mathbb{X}_\sigma^F$ be a stationary tiling, and let $\mathcal{P}$ be a patch in $\mathcal{T} \in \mathbb{X}_\sigma^F$. For any van Hove sequence $(A_q)_{q \geqslant 1}$ in $\mathbb{R}^d$*

$$\mathrm{freq}(\mathcal{P}, \mathcal{T}) := \lim_{q \to \infty} \frac{L_{\mathcal{P}}(A_q + h)}{\mathrm{vol}(A_q)} = 0$$

uniformly in $h \in \mathbb{R}^d$, where

$$L_{\mathcal{P}}(A) := L_{\mathcal{P},\{1\}} = \#\{g \in \mathbb{R}^d : \ g + \mathcal{P} \subset \mathcal{T}, (g + \mathrm{supp}(\mathcal{P})) \subset A\}.$$

Proof. The proof follows from the fact that for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ so that for every $m \geqslant m_0$

$$\frac{L_{\mathcal{P}}(e^{ms}A)}{\mathrm{vol}(e^{ms}A)} < \varepsilon$$

holds, for every set A that is the support of a tile T. This is true because the patch $\mathcal{P}$ contains at least one tile, say a translated copy of βT_i for some prototile $T_i \in \tau_\sigma$ and $\beta \in (\beta_i^{\min}, 1]$. Clearly, $L_{\mathcal{P}}(e^{ms}A)$ is smaller than the number of copies of βT_i contained in $F_{ms}(A)$, which is $o(\mathrm{vol}(e^{ms}A))$ by Corollary 7.4. As in Theorem 9.2, the proof now follows from standard approximation arguments for supertiles and the van Hove property. □

10. Unique ergodicity

This section is dedicated to the proof of unique ergodicity of incommensurable tiling dynamical systems.

Theorem 10.1. *Let σ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$. Then the dynamical system $(\mathbb{X}_\sigma^F, \mathbb{R}^d)$ is uniquely ergodic.*

Our proof draws inspiration from that given in [LSo] for the case of tiling spaces associated with fixed scale substitution tilings of infinite local complexity. Although the approach and framework follow that of Lee and Solomyak, since Theorem 9.7 implies that the standard patch frequencies are all zero, their arguments cannot be applied directly. The main innovation is given in Lemma 10.3, in which we take advantage of the fact that patch frequencies are non-zero only when patches are counted with their rescalings in some non-trivial interval of scales. This allows us to use the theory of Riemann-Stieltjes integration in order to evaluate the countable sum to which the ergodic averages converge.

10.1. **Cylinder sets and partitions of the space of tilings.** Let σ be an irreducible incommensurable multiscale substitution scheme in $\mathbb{R}^d$, and let

$$\mathbb{X}_{\mathcal{S}} := \overline{\mathcal{O}(\mathcal{S})} = \overline{\{\mathcal{S} - x \,:\, x \in \mathbb{R}^d\}},$$

where $\mathcal{S} \in \mathbb{X}_\sigma^F$ is a stationary tiling. As shown in §6, the dynamical system $(\mathbb{X}_\sigma^F, \mathbb{R}^d)$ is minimal, and so $\mathbb{X}_{\mathcal{S}} = \mathbb{X}_\sigma^F$. Nevertheless, we use the notation $\mathbb{X}_{\mathcal{S}}$ to emphasize that the space is the orbit closure of a specified stationary tiling $\mathcal{S}$. We begin by defining a sequence of partitions of $\mathbb{X}_{\mathcal{S}}$ into finitely many cylinder sets.

The idea is to create "pixelized" images from patches and to define cylinders as the collection of tilings with the same "pixelized" image on a large centered cube

$$C_m := [-2^m, 2^m)^d.$$

Subdivide C_m into small cubes c_ω of side length $\frac{1}{2^m}$ taken to be products of half open intervals (this are the "pixels"), and write

$$C_m = \bigsqcup_{\omega=1}^{2^{2m+1}d} c_\omega.$$

Let $\mathcal{T} \in \mathbb{X}_{\mathcal{S}}$, and consider the patch $[C_m]^{\mathcal{T}}$, which consists of all tiles in $\mathcal{T}$ that intersect C_m. We use this patch to color the small cubes with colors $\{0, 1, \ldots, n\}$ according to the following rule. Recall that by Proposition 4.5, every tile in every tiling in $\mathbb{X}_\sigma^F$ is similar to one of the prototiles in τ_σ and can be assigned a type. If c_ω is contained in the interior of the support of a tile of type $i \in \{1, \ldots, n\}$, we color the cube c_ω in color i. Otherwise, c_ω intersects the boundary of the support of a tile in $\mathcal{T}$. In such a case we color c_ω in the color 0. This coloring of the small cubes in C_m according to $[C_m]^{\mathcal{T}}$ is the *m-pixelization of $\mathcal{T}$*, and is denoted by $\Pi_m(\mathcal{T})$, see Figure 12 for an illustration.

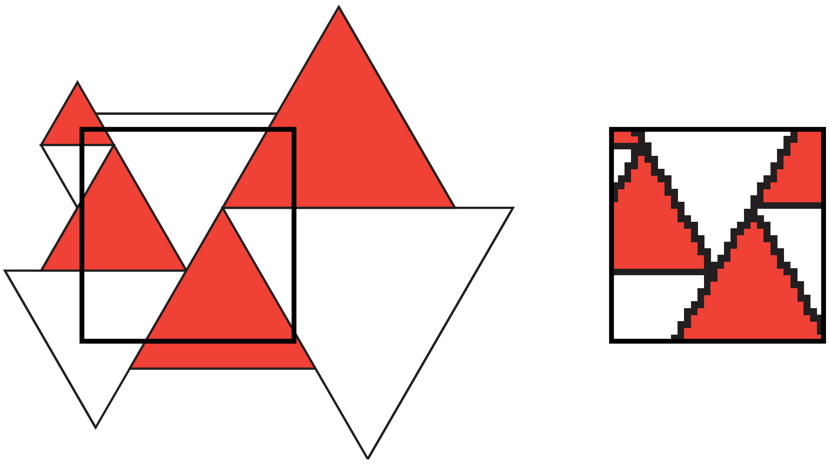

FIGURE 12. A patch $[C_m]^{\mathcal{T}}$ and its m-pixelization.

Next, denote by $\{U_{m,\ell}\}_{\ell=1}^{N_m}$ the finite set of all different colorings of the small cubes in C_m using the colors $0, 1, \ldots, n$. For $\ell \in \{1, \ldots, N_m\}$, let

$$X(U_{m,\ell}) := \{\mathcal{T} \in \mathbb{X}_{\mathcal{S}} \,:\, \Pi_m(\mathcal{T}) = U_{m,\ell}\}$$

be the set of tilings in $\mathbb{X}_{\mathcal{S}}$ whose m-pixelization is $U_{m,\ell}$. This is a cylinder set in $\mathbb{X}_{\mathcal{S}}$, and clearly for every $m \in \mathbb{N}$

$$\mathbb{X}_{\mathcal{S}} = \bigsqcup_{\ell=1}^{N_m} X(U_{m,\ell}), \tag{10.1}$$

where $\bigsqcup$ denotes a disjoint union.

Fix $m \in \mathbb{N}$ and $\ell \in \{1, \ldots, N_m\}$, and consider the set $\mathcal{G}_{m,\ell}$ of all patches of the form $[C_m]^{\mathcal{S}-x}$ with $\mathcal{S} - x \in X(U_{m,\ell})$ and $x \in \mathbb{R}^d$. For every patch $\mathcal{P} \in \mathcal{G}_{m,\ell}$, there is a Borel set $V_{\mathcal{P}} \subset \mathbb{R}^d$ of *maximal wiggle*, for which $\Pi_m(\mathcal{P} - x) = U_{m,\ell}$ for all $x \in V_{\mathcal{P}}$ and $V_{\mathcal{P}}$ is maximal with this property with respect to inclusion. Since the small cubes c_ω are products of half open intervals, for every patch $\mathcal{P} \in \mathcal{G}_{m,\ell}$ the set $V_{\mathcal{P}}$ contains an open set. Note that although $V_{\mathcal{P}}$ always contains the origin, it is not necessarily an interior point of $V_{\mathcal{P}}$.

The tiling $\mathcal{S}$ has countably many patches up to translation equivalence, and so there are countably many patches of the form $[C_m]^{\mathcal{S}-x}$ in $\mathcal{G}_{m,\ell}$, modulo translation equivalence. We choose a set $(\mathcal{G}_{m,\ell})_{\equiv}$ of representatives in the following way. First, note that given a patch $\mathcal{P} \in \mathcal{G}_{m,\ell}$, there exists an interval of scales in which dilations of $\mathcal{P}$ have $U_{m,\ell}$ as their m-pixelization, where the m-pixelization of a patch with support that covers C_m is defined similarly to the m-pixelization of a tiling. If this interval of scales is degenerate and contains only 1, we can take advantage of the fact that the set $V_{\mathcal{P}}$ of maximal wiggle contains an open set, and so for some small translation of $\mathcal{P}$ the corresponding set of scales is non-degenerate. In addition, recall that by Lemma 5.6, patches appear in $\mathcal{S}$ in a dense set of scales. Combining the above we conclude that it is always possible to choose a set $(\mathcal{G}_{m,\ell})_{\equiv}$ of representatives modulo translation equivalence of the form

$$(\mathcal{G}_{m,\ell})_{\equiv} = \bigcup_{j \geqslant 1} \{s_{ij}\mathcal{P}_j\}_{i \geqslant 1}, \tag{10.2}$$

with the property that for every $j \in \mathbb{N}$, the scaling constants $\{s_{ij}\}_{i \geqslant 1}$ are dense in some interval I_j that contains 1 and a left neighborhood of 1.

Given a patch $\mathcal{P}$ and a set $V \subset \mathbb{R}^d$, define

$$X(\mathcal{P}, V) := \{\mathcal{T} \in \mathbb{X}_{\mathcal{S}} : \exists x \in V \text{ s.t. } \mathcal{P} - x \subset \mathcal{T}\}.$$

It follows that

$$X(U_{m,\ell}) = \bigsqcup_{i,j \geqslant 1} X(s_{i,j}\mathcal{P}_j, V_{s_{i,j}\mathcal{P}_j}) \bigsqcup \widetilde{X}_{m,\ell},$$

where $\mathcal{P}_j$, $\{s_{ij}\}_{i \geqslant 1}$ and I_j as in (10.2), $V_{s_{ij}\mathcal{P}_j}$ are the sets of maximal wiggle for the patches $s_{ij}\mathcal{P}_j$, and

$$\widetilde{X}_{m,\ell} := \left\{\mathcal{T} \in \mathbb{X}_{\mathcal{S}} : \Pi_m(\mathcal{T}) = U_{m,\ell}, \nexists x \in \mathbb{R}^d \text{ s.t. } [C_m]^{\mathcal{T}} + x \subset \mathcal{S}\right\}$$

are the tilings so that the patch $[C_m]^{\mathcal{T}}$ is only admitted in the limit, compare [LSo].

10.2. **Proof of unique ergodicity.** Denote by $\chi_{m,\ell}$ the characteristic function of $X(U_{m,\ell})$. Using the decomposition (10.1), in order to establish unique ergodicity for $(\mathbb{X}_{\mathcal{S}}, \mathbb{R}^d)$ it is enough to show that for every van Hove sequence $(A_q)_{q \geqslant 1}$ and every pair (m, ℓ)

$$\lim_{q \to \infty} \frac{1}{\operatorname{vol}(A_q)} \int_{A_q} \chi_{m,\ell}(\mathcal{S} - x - h)dx = u_{m,\ell} \tag{10.3}$$

uniformly in $h \in \mathbb{R}^d$, where $u_{m,\ell}$ is a constant depending only on m and on ℓ. We note that relying on the decomposition of the space (10.1) into finitely many pairwise disjoint cylinders, which can be made arbitrarily small, the sufficiency of (10.3) is standard, see e.g. [LMS1, Theorem 2.6]. This is also the approach in [LSo, Theorem 3.2], where it is shown that

$$\lim_{q\to\infty}\left(\frac{1}{\mathrm{vol}(A_q)}\int_{A_q}\chi_{m,\ell}(\mathcal{S}-x-h)dx-\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{\mathrm{vol}(V_{s_{i,j}\mathcal{P}_j})L_{s_{i,j}\mathcal{P}_j}(A_q+h)}{\mathrm{vol}(A_q)}\right)=0.$$

Here $L_{\mathcal{P}}(A) := L_{\mathcal{P},\{1\}}(A)$, where as in §9 we simplify notation and put $L_{\mathcal{P},I}(A) = L_{\mathcal{P},I}(A,\mathcal{S})$. Combined with (10.3), unique ergodicity of $(\mathbb{X}_{\mathcal{S}},\mathbb{R}^d)$ will follow from the next result.

Proposition 10.2. *There exist constants* $\widetilde{v}_j$ *so that*

$$\lim_{q\to\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{\mathrm{vol}(V_{s_{i,j}\mathcal{P}_j})L_{s_{i,j}\mathcal{P}_j}(A_q+h)}{\mathrm{vol}(A_q)}=\sum_{j=1}^{\infty}\widetilde{v}_j\mathrm{freq}(\mathcal{P}_j,I_j,\mathcal{S})<\infty,$$

uniformly in $h\in\mathbb{R}^d$*, with* $\mathcal{P}_j$*,* $\{s_{ij}\}_{i\geqslant 1}$ *and* I_j *as in* (10.2) *and* $\mathrm{freq}(\mathcal{P},I,\mathcal{S})$ *as in* (9.1).

For the proof of Proposition 10.2 we establish the following two lemmas. Lemma 10.3 in particular is a key step in the proof, and the density of scales in which patches appear in incommensurable tilings plays an important role, together with the existence of uniform patch frequencies as established in §9.

Lemma 10.3. *For every* $j\geqslant 1$ *there exists a constant* $\widetilde{v}_j$ *so that*

$$\lim_{q\to\infty}\sum_{i=1}^{\infty}\frac{\mathrm{vol}(V_{s_{i,j}\mathcal{P}_j})L_{s_{i,j}\mathcal{P}_j}(A_q+h)}{\mathrm{vol}(A_q)}=\widetilde{v}_j\mathrm{freq}(\mathcal{P}_j,I_j,\mathcal{S}),$$

uniformly in $h\in\mathbb{R}^d$.

Proof. Fix j, and write $a := \inf I_j$ and $b := \sup I_j$. For every $x\in[a,b]$ denote $I(x) := [a,x]$. Define the functions $f, g_{q,h} : [a,b]\to\mathbb{R}$ by

$$f(x)=\mathrm{vol}(V_{x\mathcal{P}_j})$$
$$g_{q,h}(x)=\frac{L_{\mathcal{P}_j,I(x)}(A_q+h)}{\mathrm{vol}(A_q)},$$

where $f(a)=0$ if $\Pi_m(a\mathcal{P}_j)\neq U_{m,\ell}$. By definition, f is continuous and $g_{q,h}$ is monotone for every $q\in\mathbb{N}$ and $h\in\mathbb{R}^d$, and so the Riemann-Stieltjes integral

$$\int_a^b f dg_{q,h}$$

exists (see [A, §7]). Fix $N\in\mathbb{N}$, and relabel the elements of $\{s_{i,j}\}_{i=1}^N$ in an increasing order. Let $\{x_i\}_{i=1}^N$ be so that

$$a=x_0<x_1<\cdots<x_N=b$$
$$x_{i-1}\leqslant s_{i,j}\leqslant x_i \qquad \forall 1\leqslant i\leqslant N.$$

By definition of the Riemann-Stieltjes integral and by our construction, we have

$$\begin{aligned}\int_a^b f dg_{q,h} &= \lim_{N\to\infty}\sum_{i=1}^{N}f(s_{i,j})\left(g_{q,h}(x_i)-g_{q,h}(x_{i-1})\right)\\ &=\sum_{i=1}^{\infty}\frac{\mathrm{vol}(V_{s_{i,j}\mathcal{P}_j})L_{s_{i,j}\mathcal{P}_j}(A_q+h)}{\mathrm{vol}(A_q)},\end{aligned}\tag{10.4}$$

where the second equality holds because for fixed q and h the sum on the right-hand side is finite. We remark that even though almost every term in the sum established in (10.4) is equal to 0, as q increases and h varies the quantity $L_{s_{i,j}\mathcal{P}_j}(A_q + h)$ gets positive values for every i.

By Theorems 9.2 and 9.7, the function

$$g_\infty(x) := \lim_{q\to\infty} g_{q,h}(x) = \operatorname{freq}(\mathcal{P}_j, I(x), \mathcal{S})$$

is well defined, and the convergence is uniform in h. Observe that f is bounded and that g_∞ is monotone. By the first mean value theorem for Riemann-Stieltjes integrals (see [A, Theorem 7.30]) and Theorem 9.7, there exists a positive constant $\inf\{f(x)\} \leqslant \widetilde{v}_j \leqslant \sup\{f(x)\}$ for which

$$\int_a^b f dg_\infty = \widetilde{v}_j(g_\infty(b) - g_\infty(a)) = \widetilde{v}_j \operatorname{freq}(\mathcal{P}_j, I_j, \mathcal{S}). \tag{10.5}$$

In view of (10.4) and (10.5), to finish the proof we therefore must show that the order of the limit and the integration can be switched, that is

$$\lim_{q\to\infty} \int_a^b f dg_{q,h} = \int_a^b f d\left(\lim_{q\to\infty} g_{q,h}\right) = \int_a^b f dg_\infty. \tag{10.6}$$

Indeed, let us fix $q \in \mathbb{N}$. Using integration by parts (see [A, §7.5]) we obtain

$$\int_a^b f dg_{q,h} = f(b)g_{q,h}(b) - f(a)g_{q,h}(a) - \int_a^b g_{q,h} df. \tag{10.7}$$

Note that $g_{q,h}$ is a monotone, piecewise-constant function, and that f is continuous. Then the integral on the right-hand side of (10.7) exists, and can be viewed as a Lebesgue integral. Taking limits as $q \to \infty$ in (10.7) we obtain

$$\lim_{q\to\infty} \int_a^b f dg_{q,h} = f(b)g_\infty(b) - f(a)g_\infty(a) - \lim_{q\to\infty} \int_a^b g_{q,h} df. \tag{10.8}$$

Since the patches $s_{i,j}\mathcal{P}_j$ are of bounded diameter ρ (which is related to m), we have

$$g_{q,h} \leqslant C(\rho),$$

where $C(\rho)$ is a constant that depends only on the tiling $\mathcal{S}$ and on ρ. Therefore, by the Lebesgue's dominated convergence theorem and another integration by parts

$$\lim_{q\to\infty} \int_a^b g_{q,h} df = \int_a^b g_\infty df = g_\infty(b)f(b) - g_\infty(a)f(a) - \int_a^b f dg_\infty. \tag{10.9}$$

Combining (10.8) and (10.9) we arrive at (10.6), thus finishing the proof. □

Lemma 10.4. *Assume that for every $\varepsilon > 0$, there exists $q_0 \in \mathbb{N}$ so that for every $q \geqslant q_0$ and every $h \in \mathbb{R}^d$*

$$\sum_{j=q_0}^{\infty} \sum_{i=1}^{\infty} \frac{\operatorname{vol}(V_{s_{i,j}\mathcal{P}_j}) L_{s_{i,j}\mathcal{P}_j}(A_q + h)}{\operatorname{vol}(A_q)} < \varepsilon.$$

Then Proposition 10.2 holds.

Proof. The proof is very similar to the proof of [LSo, Corollary 3.3]. The key step is to use Lemma 10.3 in order to apply Fatou's lemma and bound from below the limit inferior of the expression on the left-hand side of (10.2). The limit superior is then bounded from above using the assumed inequality. □

We now use Lemmas 10.3 and 10.4 to establish Proposition 10.2, from which unique ergodicity of the dynamical system follows, thus proving Theorem 10.1.

Proof of Proposition 10.2. Fix $r > 0$ and enumerate all patches in $\mathcal{S}$ that have diameter less than r. A patch $\mathcal{P}$ is called *k-special* if it occurs as a sub-patch of a k-supertile, with k minimal, and we denote $\mathrm{sp}(\mathcal{P}) = k$. As shown in the proof of Lemma 9.5, up to dilation there are finitely many patches that can be supported on a k-supertile. Group all inflations of a given patch together. By the above and by Lemma 10.4, it is enough to show that

$$\sum_{\mathrm{sp}(\mathcal{P})>k} \sum_{i=1}^{\infty} \frac{L_{s_i\mathcal{P}}(A_q + h)}{\mathrm{vol}(A_q)} \tag{10.10}$$

can be made arbitrarily small, where $(s_i)_{i\geqslant 1}$ is an enumeration of the set of possible scales in which $\mathcal{P}$ can appear. Namely, we need to show that for every $\varepsilon > 0$ there exist $k_1, q_1 > 0$ such that for every $k \geqslant k_1$ and $q \geqslant q_1$ the quantity in (10.10) is less than ε.

Indeed, since the support of every k-special patch intersects the boundary of some k-supertile, we have

$$\sum_{\mathrm{sp}(\mathcal{P})>k} \sum_{i=1}^{\infty} L_{s_i\mathcal{P}}(A_q + h) \leqslant \# \left\{ \mathcal{P} \,:\, \mathrm{diam}(\mathcal{P}) \leqslant r, \mathrm{supp}(\mathcal{P}) \subset \bigcup_{T\in\mathcal{A}} \left(\partial(e^{ks}\mathrm{supp}T)\right)^{+r} \right\} \tag{10.11}$$

where the union is taken over the set $\mathcal{A}$ of all tiles $T \in \mathcal{S}$ such that $e^{ks}\mathrm{supp}(T) \subset (A_q+h)^{+r}$. The rest of the proof is now identical to that of [LSo, Theorem 4.14]. There exists a constant C_r such that the right-hand side of (10.11) is bounded by

$$C_r \sum_{T\in\mathcal{A}} \mathrm{vol}\left(\partial(e^{ks}\mathrm{supp}T)\right)^{+r}. \tag{10.12}$$

Given $\delta > 0$, since $\left(e^{ks}\mathrm{supp}T\right)_{k\geqslant 1}$ is van Hove, (10.12) can be bounded by

$$C_r\delta\mathrm{vol}\left((A_q + h)^{+r}\right) = C_r\delta\mathrm{vol}\left((A_q)^{+r}\right)$$

for sufficiently large k. The sequence $(A_q)_{q\geqslant 1}$ is also van Hove, and so

$$\mathrm{vol}\left((A_q)^{+r}\right) \leqslant (1+\delta)\mathrm{vol}A_q$$

for sufficiently large q. Therefore, for any sufficiently large k, for sufficiently large q

$$\sum_{\mathrm{sp}(\mathcal{P})>k} \sum_{i=1}^{\infty} \frac{L_{s_i\mathcal{P}}(A_q + h)}{\mathrm{vol}(A_q)} \leqslant C_r\delta(1+\delta).$$

Since the right-hand side is arbitrarily small, the proposition follows. □

Remark 10.5. We believe that the arguments in §9 and §10 can be extended to the case of schemes with "incommensurability of orientations", in the sense of the pinwheel tilings. Presumably, an additional layer of isometries would be added to the information carried by the associated graph, and in the definition of patch frequencies, patches would be counted together with isometric copies within a neighborhood of the identity in the associated isometry group. More on this will appear in future work.

Appendix 1. Patches of multiscale substitution tilings

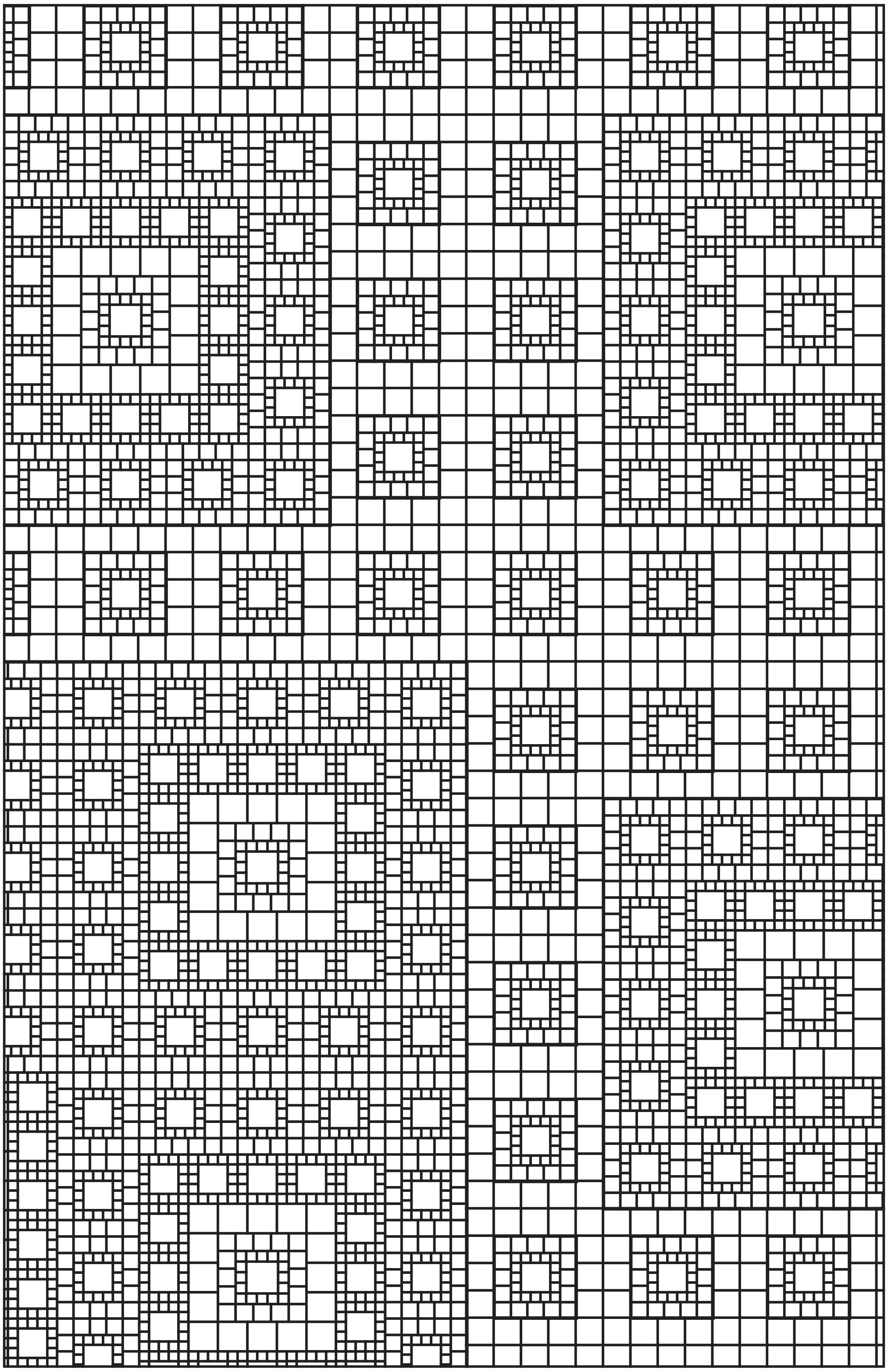

Figure 13. A fragment of an incommensurable multiscale substitution tiling of $\mathbb{R}^2$, generated by the square scheme illustrated in Figure 1.

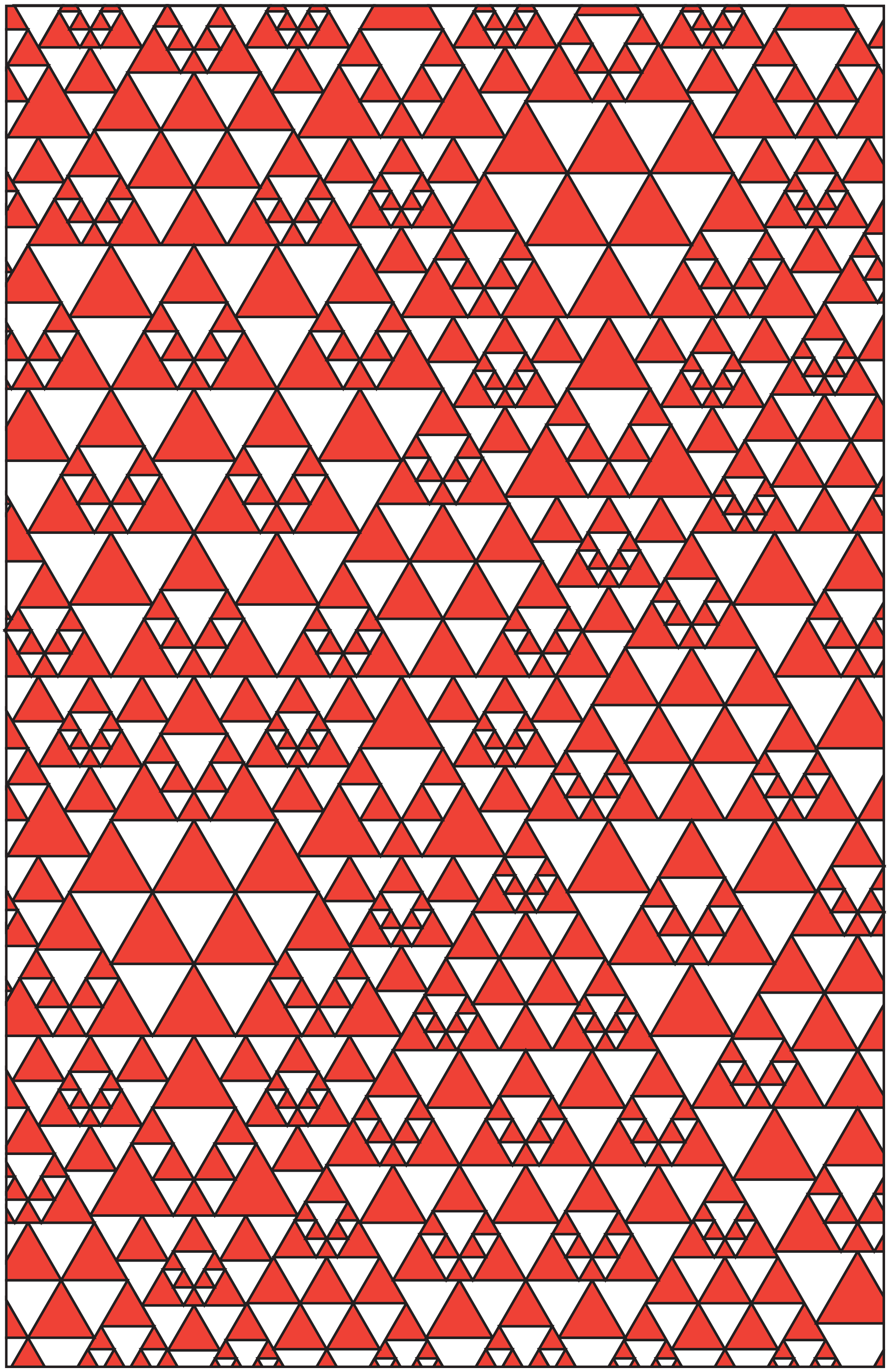

FIGURE 14. A fragment of an incommensurable multiscale substitution tiling of $\mathbb{R}^2$, generated by the triangles scheme illustrated in Figure 3.

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