

SYMMETRIZED POLY-BERNOULLI NUMBERS AND COMBINATORICS

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ABSTRACT. Poly-Bernoulli numbers are one of generalizations of the classical Bernoulli numbers. Since a negative index poly-Bernoulli number is an integer, it is an interesting problem to study this number from combinatorial viewpoint. In this short article, we give a new combinatorial relation between symmetrized poly-Bernoulli numbers and the Dumont-Foata polynomials.

1. INTRODUCTION

A poly-Bernoulli polynomial $B_m^{(\ell)}(x)$ of index $\ell \in \mathbb{Z}$ is defined by the generating series

$$e^{-xt} \frac{\text{Li}_\ell(1 - e^{-t})}{1 - e^{-t}} = \sum_{m=0}^{\infty} B_m^{(\ell)}(x) \frac{t^m}{m!},$$

where $\text{Li}_\ell(z)$ is the polylogarithm function given by

$$\text{Li}_\ell(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^\ell} \quad (|z| < 1).$$

The polynomial $B_m^{(1)}(x)$ coincides with the classical Bernoulli polynomial $B_m(1-x) = (-1)^m B_m(x)$ since $\text{Li}_1(z) = -\log(1-z)$ holds (see [1, (4.2)]). Following Kaneko [6], the special value $B_m^{(\ell)} := B_m^{(\ell)}(0)$ is particularly called a poly-Bernoulli number of index ℓ .

The aim of this study is to give a combinatorial perspective to the special values of $B_m^{(\ell)}(x)$ at integers $k \in \mathbb{Z}$. We now assume that the index $\ell \leq 0$, so that the values $B_m^{(\ell)}(k)$ are always integers. One of the first such investigations is the coincidence of two numbers, the poly-Bernoulli number $B_m^{(\ell)}(0)$ and the number of 01 lonesum matrices of size $m \times |\ell|$ (see [4]). A recent work by Bényi-Hajnal [3] established more combinatorial relations in this direction.

In this article, we take a step in another direction along with Kaneko-Sakurai-Tsumura [7]. To describe this more precisely, let G_n be the Genocchi number defined by $G_n = 2(2^{n+2} - 1)|B_{n+2}|$, where $B_m = B_m^{(1)}(1)$ is the classical Bernoulli number. Then they showed in [7, Theorem 4.2] that

$$(1) \quad \sum_{\ell=0}^n (-1)^\ell B_{n-\ell}^{(-\ell-1)}(1) = (-1)^{n/2} G_n$$

for any $n \geq 0$. As they mentioned, this equation is an analogue of Arakawa-Kaneko's result [2]

$$(2) \quad \sum_{\ell=0}^n (-1)^\ell B_{n-\ell}^{(-\ell)}(0) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

In addition, Sakurai asked in her master thesis whether we can generalize these equations for any positive integers $x = k$, and give some combinatorial meaning to them. Our main result provides an answer to these two questions in terms of the Dumont-Foata polynomial as follows.

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Theorem 1.1. Let $G_n(x, y, z)$ be the n -th Dumont-Foata polynomial defined in (3), and $\mathcal{B}_m^{(\ell)}(k)$ the symmetrized poly-Bernoulli number defined in (5). Then we have

$$\sum_{\ell=0}^n (-1)^\ell \mathcal{B}_{n-\ell}^{(-\ell)}(k) = k! \cdot (-1)^{n/2} G_n(1, 1, k)$$

for any non-negative integers $n, k \geq 0$. In particular, both sides equal zero for odd n .

This theorem recovers the equations (1) and (2) since $\mathcal{B}_m^{(-\ell)}(0) = B_m^{(-\ell)}(0)$, $\mathcal{B}_m^{(-\ell)}(1) = B_m^{(-\ell-1)}(1)$, and $G_n(1, 1, 1) = G_n$ hold as we see later.

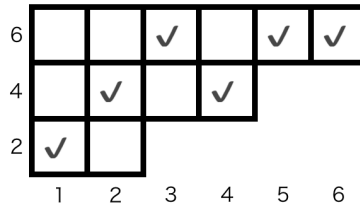
Remark 1.2. Recent work of Bényi-Hajnal [3, Section 6] pointed out that the sequence of diagonal sums

$$\sum_{\ell=0}^n B_{n-\ell}^{(-\ell-1)}(1) \quad (n \geq 0)$$

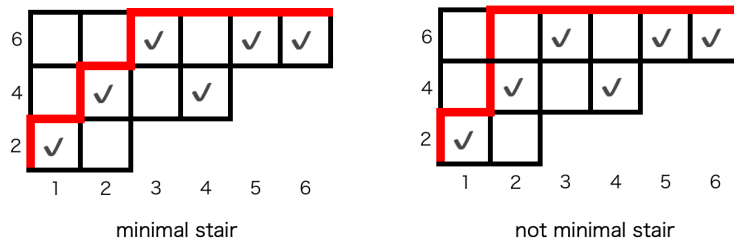
appears in OEIS [8] A136127. It might be an interesting problem to extend our theorem in this direction.

2. DEFINITIONS

2.1. Dumont-Foata polynomials. We review the work of Dumont-Foata [5] here. For a positive even integer $n \in 2\mathbb{Z}$, we consider the surjective map $p : \{1, 2, \dots, n\} \rightarrow \{2, 4, \dots, n\}$ with $p(x) \geq x$ for each $x \in \{1, 2, \dots, n\}$. This map is called (surjective) pistol of size n , and corresponds to the following diagram. Here we draw one example for $n = 6$ given by $p(1) = 2, p(2) = 4, p(3) = 6, p(4) = 4, p(5) = 6, p(6) = 6$.



Let \mathcal{P}_n be the set of all pistols of size n . For each pistol $p \in \mathcal{P}_n$, we define three quantities called bulging, fixed, and maximal points. First, the number $x \in \{1, 2, \dots, n\}$ is a bulging point of $p \in \mathcal{P}_n$ if $p(y) < p(x)$ for any $0 < y < x$. Then we denote by $b(p)$ the number of bulging points of p . In the diagram, $b(p)$ corresponds to the number of steps of the minimal stair covering all check marks. For the above example p , the points $x = 1, 2, 3$ are bulging points, so that $b(p) = 3$.



Next, the point $x \in \{1, 2, \dots, n\}$ is called a fixed point of $p \in \mathcal{P}_n$ if $p(x) = x$. Finally, the point $x \in \{1, 2, \dots, n-1\}$ is a maximal point of $p \in \mathcal{P}_n$ if $p(x) = n$. Then we denote by $f(p)$ and $m(p)$ the numbers of fixed points and maximal points of $p \in \mathcal{P}_n$, respectively. For the above example, we have $f(p) = 2$ and $m(p) = 2$. Under these notations, Dumont-Foata established the following interesting theorem.

Theorem 2.1. [5, Théorème 1a, 2] Let $n \in 2\mathbb{Z}_{>0}$ be a positive even integer. The polynomial defined by

$$(3) \quad G_n(x, y, z) := \sum_{p \in \mathcal{P}_n} x^{b(p)} y^{f(p)} z^{m(p)}$$

is a symmetric polynomial in three variables, and gives the Genocchi number G_n as

$$G_n(1, 1, 1) = G_n.$$

In addition, we put $G_0(x, y, z) = 1$ and $G_n(x, y, z) = 0$ for a positive odd integer $n \in \mathbb{Z}$. The polynomial $G_n(x, y, z)$ is called the n -th Dumont-Foata polynomial. Furthermore, they showed that the polynomial for $n > 0$ has the form $G_n(x, y, z) = xyzF_n(z, y, z)$, and the polynomial $F_n(x, y, z)$ satisfies the recurrence relation

$$F_n(x, y, z) = (x + z)(y + z)F_{n-2}(x, y, z + 1) - z^2F_{n-2}(x, y, z)$$

with initial values $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 1$. This implies that the polynomial $G_n(z) := G_n(1, 1, z)$ called the Gandhi polynomial satisfies

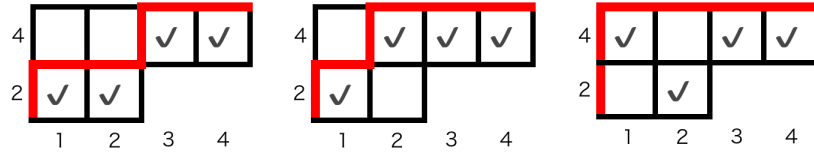
$$(4) \quad G_{n+2}(z) = z(z + 1)G_n(z + 1) - z^2G_n(z)$$

with $G_0(z) = 1, G_1(z) = 0$.

For instance, there exist three pistols of size 4. Each pistol has $(b(p), f(p), m(p)) = (2, 2, 1), (2, 1, 2)$ and $(1, 2, 2)$, so that the Dumont-Foata polynomial is given by

$$G_4(x, y, z) = x^2y^2z + x^2yz^2 + xy^2z^2 = xyz(xy + yz + zx).$$

In fact, $G_4(1, 1, 1) = 3$ coincides the 4-th Genocchi number given by $G_4 = 2(2^6 - 1)|B_6| = 126 \times 1/42 = 3$.



2.2. Symmetrized poly-Bernoulli numbers. The following are the tables of poly-Bernoulli numbers $\{B_m^{(-\ell)}(0)\}$ and $\{B_m^{(-\ell)}(1)\}$ with $m, \ell \geq 0$.

TABLE 1. $B_m^{(-\ell)}(0)$: left, $B_m^{(-\ell)}(1)$: right

$\ell \backslash m$	0	1	2	3	4
0	1	1	1	1	1
1	1	2	4	8	16
2	1	4	14	46	146
3	1	8	46	230	1066
4	1	16	146	1066	6902

$\ell \backslash m$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	3	7	15	31
3	1	7	31	115	391
4	1	15	115	675	3451

We can find the symmetric properties of these numbers at a glance. On the other hand, for $k \geq 2$ it seems unlikely that such a simple symmetric property can be given. In order to reproduce the symmetric properties for any $k \geq 2$, Kaneko-Sakurai-Tsumura [7] considered combinations of $B_m^{(-\ell)}(k)$. To make this precise, let $m, \ell, k \geq 0$ be non-negative integers. We now define the symmetrized poly-Bernoulli number $\mathcal{B}_m^{(-\ell)}(k)$ by

$$(5) \quad \mathcal{B}_m^{(-\ell)}(k) = \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} B_m^{(-\ell-j)}(k),$$

where $\left[\begin{smallmatrix} k \\ j \end{smallmatrix} \right]$ is the Stirling number of the first kind defined in [1, Definition 2.5]. Then this number satisfies $\mathcal{B}_m^{(-\ell)}(k) = \mathcal{B}_\ell^{(-m)}(k)$ for any $m, \ell, k \geq 0$, and

$$\mathcal{B}_m^{(-\ell)}(0) = B_m^{(-\ell)}(0), \quad \mathcal{B}_m^{(-\ell)}(1) = B_m^{(-\ell-1)}(1).$$

TABLE 2. $B_m^{(-\ell)}(2)$: left, $\mathcal{B}_m^{(-\ell)}(2)$: right

$\ell \backslash m$	0	1	2	3	4
0	1	-1	1	-1	1
1	1	0	0	0	0
2	1	2	2	2	2
3	1	6	18	42	90
4	1	14	86	374	1382

$\ell \backslash m$	0	1	2	3	4
0	2	2	2	2	2
1	2	8	20	44	92
2	2	20	104	416	1472
3	2	44	416	2744	15032
4	2	92	1472	15032	120632

Moreover they showed the following explicit formula for $\mathcal{B}_m^{(-\ell)}(k)$.

$$(6) \quad \mathcal{B}_m^{(-\ell)}(k) = \sum_{j=0}^{\min(m,\ell)} j!(k+j)! \left\{ \begin{matrix} m+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+1 \end{matrix} \right\},$$

where $\left\{ \begin{smallmatrix} k \\ j \end{smallmatrix} \right\}$ is the Stirling number of the second kind defined in [1, Definition 2.2]. By using this formula, we prove our main theorem in the next section.

3. PROOF

To prove Theorem 1.1, it suffices to show that the function

$$\tilde{G}_n(k) := \frac{(-1)^{n/2}}{k!} \sum_{\ell=0}^n (-1)^\ell \mathcal{B}_{n-\ell}^{(-\ell)}(k)$$

satisfies the recurrence relation (4) for any integer $k \geq 0$. First we can easily see that $\tilde{G}_0(k) = 1$ and $\tilde{G}_1(k) = 0$, which are initial cases. Moreover for any odd integer n , $\tilde{G}_n(k) = 0$ follows from the symmetric property of $\mathcal{B}_m^{(-\ell)}(k)$. For an even integer $n \geq 2$, by the formula (6) we have

$$(7) \quad \begin{aligned} & (-1)^{n/2} k! \left[k(k+1)\tilde{G}_n(k+1) - k^2\tilde{G}_n(k) - \tilde{G}_{n+2}(k) \right] \\ &= k \sum_{j=0}^{n/2} j!(k+j+1)! \sum_{\ell=j}^{n-j} (-1)^\ell \left\{ \begin{matrix} n-\ell+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+1 \end{matrix} \right\} \\ & (8) \quad - k^2 \sum_{j=0}^{n/2} j!(k+j)! \sum_{\ell=j}^{n-j} (-1)^\ell \left\{ \begin{matrix} n-\ell+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+1 \end{matrix} \right\} \\ & (9) \quad + \sum_{j=0}^{n/2+1} j!(k+j)! \sum_{\ell=j}^{n+2-j} (-1)^\ell \left\{ \begin{matrix} n-\ell+3 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+1 \end{matrix} \right\}. \end{aligned}$$

Since $\left\{ \begin{smallmatrix} k \\ 1 \end{smallmatrix} \right\} = 1$ holds for any $k \geq 1$, the third line (9) is divided according as $j = 0$ or not, which equals

$$k! + \sum_{j=0}^{n/2} (j+1)!(k+j+1)! \sum_{\ell=j+1}^{n+1-j} (-1)^\ell \left\{ \begin{matrix} n+3-\ell \\ j+2 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+2 \end{matrix} \right\}.$$

Let

$$(10) \quad a_{n,j} := \sum_{\ell=j}^{n-j} (-1)^\ell \left\{ \begin{matrix} n-\ell+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+1 \end{matrix} \right\},$$

then the total of (7), (8), and (9) equals

$$(11) \quad k! + \sum_{j=0}^{n/2} j!(k+j)! \left[k(j+1)a_{n,j} + (j+1)(k+j+1)a_{n+2,j+1} \right].$$

Once the sum is 0, the proof completes. By using the generating function given in [1, Proposition 2.6 (8)], we have

$$\frac{t^j}{(1-t)(1-2t)\cdots(1-(j+1)t)} = \sum_{\ell \geq j} \begin{Bmatrix} \ell+1 \\ j+1 \end{Bmatrix} t^\ell = \sum_{\ell \leq n-j} \begin{Bmatrix} n-\ell+1 \\ j+1 \end{Bmatrix} t^{n-\ell}$$

for any non-negative integers $n, j \in \mathbb{Z}_{\geq 0}$. Multiplying these two expressions, we obtain

$$\frac{s^j t^j}{(1-s)(1-t)\cdots(1-(j+1)s)(1-(j+1)t)} = \sum_{\ell \geq j} \sum_{k \leq n-j} \begin{Bmatrix} \ell+1 \\ j+1 \end{Bmatrix} \begin{Bmatrix} n-k+1 \\ j+1 \end{Bmatrix} s^\ell t^{n-k}.$$

By specializing at $s = -x, t = x$,

$$(12) \quad \frac{(-1)^j x^{2j}}{(1-x^2)\cdots(1-(j+1)^2 x^2)} = \sum_{\ell \geq j} \sum_{k \leq n-j} (-1)^\ell \begin{Bmatrix} \ell+1 \\ j+1 \end{Bmatrix} \begin{Bmatrix} n-k+1 \\ j+1 \end{Bmatrix} x^{n+\ell-k}.$$

Thus we see that the number $a_{n,j}$ defined in (10) appears as the n -th coefficient of (12). By the expression of the left-hand side of (12), we easily see that $a_{n,j} = 0$ when n is an odd integer or $2j > n$. Further we get the initial values $a_{2j,j} = (-1)^j, a_{n,0} = 1$ for even n , and the recurrence relation

$$a_{n+2,j} = (j+1)^2 a_{n,j} - a_{n,j-1}.$$

Applying this to the term in (11), we get

$$\begin{aligned} k! + \sum_{j=0}^{n/2} j!(k+j)! \left[k(j+1)a_{n,j} + (j+1)(k+j+1)((j+2)^2 a_{n,j+1} - a_{n,j}) \right] \\ = k! + \sum_{j=0}^{n/2} (j+2)(j+2)!(k+j+1)! a_{n,j+1} - \sum_{j=0}^{n/2} (j+1)(j+1)!(k+j)! a_{n,j}. \end{aligned}$$

Since $a_{n,n/2+1} = 0$ and $a_{n,0} = 1$, this equals 0, which concludes the proof of Theorem 1.1.

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