Convolution identities of poly-Cauchy numbers with level 2

Takao Komatsu
Department of Mathematical Sciences, School of Science
Zhejiang Sci-Tech University
Hangzhou 310018 China
komatsu@zstu.edu.cn

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Abstract

Poly-Cauchy numbers with level 2 are defined by inverse sine hyperbolic functions with the inverse relation from sine hyperbolic functions. In this paper, we show several convolution identities of poly-Cauchy numbers with level 2. In particular, that of three poly-Cauchy numbers with level 2 can be expressed as a simple form. In the sequel, we introduce the Stirling numbers of the first kind with level 2.
Keywords: Poly-Cauchy numbers, hyperbolic functions, inverse hyperbolic functions, convolutions, Stirling numbers of the first kind

1 Introduction

Poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ are defined as

$$\mathrm{Lif}_k\bigl(\log(1+t)\bigr)=\sum_{n=0}^\infty c_n^{(k)}\frac{t^n}{n!}\,, \tag{1}$$

where $\mathrm{Lif}_k(z)$ is the *polylogarithm factorial* or *polyfactorial* function, defined by

$$\mathrm{Lif}_k(z)=\sum_{m=0}^\infty\frac{z^m}{m!(m+1)^k}$$

([6, 7]). The concept of poly-Cauchy numbers with the polylogarithm factorial function is an anagloues of poly-Bernoulli numbers with the polylogarithm function ([4]).

There are many papers on poly-xxx numbers and most of them are just generalizations for generalization's sake, but this paper does not add another example. For, most generalizations or variations of so-called poly numbers or polynomials are just with level 1, but we consider poly numbers with level 2.

Poly-Cauchy numbers $\mathfrak{C}_n^{(k)}$ *with level* 2 [10] are defined by

$$\mathrm{Lif}_{2,k}(\mathrm{arcsinh}t) = \sum_{n=0}^\infty \mathfrak{C}_n^{(k)} \frac{t^n}{n!}\,, \tag{2}$$

where arcsinh$t$ is the inverse hyperbolic sine function and

$$\mathrm{Lif}_{2,k}(z) = \sum_{m=0}^\infty \frac{z^{2m}}{(2m)!(2m+1)^k}\,.$$

which may be called the *polylogarithm factorial function with level* 2. Notice that poly-Cauchy numbers with level 2 are not simple generalizations of poly-Cauchy numbers or the original Cauchy numbers $c_n := c_n^{(1)}$ defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^\infty c_n \frac{t^n}{n!}$$

because poly-Cauchy numbers with level 2 are based on hyperbolic functions but so-called poly-Cauchy (or the original Cauchy) numbers with level 1 are based upon logarithm functions. In this sense, most generalizations of poly-Cauchy numbers are still based upon the same logarithm functions, but our poly-Cauchy numbers are constructed by a different function. In a similar sense, there are many generalizations of poly-Bernoulli numbers as with level 1, but poly-cosecant numbers [5] are those with level 2.

The original Cauchy numbers, poly-Cauchy numbers and most of their generalizations are realted with the Stirling numbers of the first kind. On the contrary, poly-Cauchy numbers with level 2 are related with poly-Cauchy numbers with level 2, which are not simple generalizations but essentially different from the original Stirling numbers of the first kind.

In fact, $\mathfrak{C}_n^{(k)}$ has an expression in terms of $(2m+1)^k$ $(m=1,2,\dots,n)$ by using the Stirling numbers of the first kind with level 2. Note that $\mathfrak{C}_n = 0$ for odd n.

Theorem 1. *For integers n and k with $n \ge 1$,*

$$\mathfrak{C}_{2n}^{(k)} = \sum_{m=1}^n \frac{(-4)^{n-m}}{(2m+1)^k} \left[\!\left[{n \atop m} \right]\!\right]\,,$$

where for $m = 1, 2, \dots, n$

$$\left[\!\left[n \atop m \right]\!\right] = \left[n \atop m \right]^2 - 2 \left[n \atop m-1 \right] \left[n \atop m+1 \right] + 2 \left[n \atop m-2 \right] \left[n \atop m+2 \right] - \cdots + 2(-1)^{m-1} \left[n \atop 1 \right] \left[n \atop 2m-1 \right] \quad (3)$$

with $\left[\!\left[n \atop 0 \right]\!\right] = 0$.

Remark. Poly-Cauchy numbers can be expressed by using the Stirling numbers of the first kind:

$$c_n^{(k)} = \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(m+1)^k} \left[n \atop m \right]$$

([6]), where $\left[n \atop m \right]$ are the (unsigned) Stirling numbers of the first kind arise as coefficient of the rising factorial

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{m=0}^{n} \left[n \atop m \right] x^m .$$

The Stirling numbers of the first kind $\left[\!\left[n \atop m \right]\!\right]$ with level 2 arise as coefficient of the rising factorial

$$x(x+1^2)(x+2^2)\cdots(x+(n-1)^2) = \sum_{m=0}^{n} \left[\!\left[n \atop m \right]\!\right] x^m \quad (4)$$

(see, e.g., [11, p.213–217],[2][1]), So, they can also be written as

$$\left[\!\left[n \atop m \right]\!\right] = \sum_{1 \le i_1 < \cdots < i_{n-m} \le n-1} (i_1 \cdots i_{n-m})^2 .$$

Thus, the following relation holds:

$$\left[\!\left[n \atop m \right]\!\right] = \left[\!\left[n-1 \atop m-1 \right]\!\right] + (n-1)^2 \left[\!\left[n-1 \atop m \right]\!\right] \quad (5)$$

[1] There is a relation $\left[\!\left[n \atop m \right]\!\right] = (-1)^{n-m} t(2n, 2m)$, where $t(n,m)$ are the central factorial numbers of the first kind, defined by $x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2)\cdots(x - \frac{n}{2} + 1) = \sum_{m=0}^{n} t(n,m) x^m$.

($Cf.$[3][2]). In this sense, the numbers $\left[\!\left[{n \atop m}\right]\!\right]$ are suitable to be called the *Stirling numbers of the first kind with level* 2, because the (unsigned) Stirling numbers of the first kind satisfy the recurrence relation

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}.$$

Notice that concerning the Stirling numbers of the first kind we see

$$\begin{aligned}
&\begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \quad (n \ge 1), \quad \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!, \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! H_{n-1}, \\
&\begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}, \quad \begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{3n-1}{4}\binom{n}{3}, \quad \begin{bmatrix} n \\ n-3 \end{bmatrix} = \binom{n}{2}\binom{n}{4}, \\
&\begin{bmatrix} n \\ n-4 \end{bmatrix} = \frac{15n^3 - 30n^2 + 5n - 2}{48}\binom{n}{5}, \quad \begin{bmatrix} n \\ n-5 \end{bmatrix} = \frac{3n^2 - 7n - 2}{8}\binom{n}{2}\binom{n}{6}, \\
&\begin{bmatrix} n \\ n-6 \end{bmatrix} = \frac{63n^5 - 315n^4 + 315n^3 + 91n^2 - 42n - 16}{576}\binom{n}{7}.
\end{aligned}$$

In particular, for $m = 1, 2, 3$

$$\begin{aligned}
\left[\!\!\left[{n \atop 1}\right]\!\!\right] &= \bigl((n-1)!\bigr)^2, \\
\left[\!\!\left[{n \atop 2}\right]\!\!\right] &= \bigl((n-1)!\bigr)^2 H_{n-1}^{(2)} \quad [12, \text{A001819}], \\
\left[\!\!\left[{n \atop 3}\right]\!\!\right] &= \bigl((n-1)!\bigr)^2 \frac{(H_{n-1}^{(2)})^2 - H_{n-1}^{(4)}}{2} \quad [12, \text{A001820}],
\end{aligned}$$

where

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k}$$

is the generalized harmonic number of order k. The numbers $\left[\!\left[{n \atop 4}\right]\!\right]$ can be found in [12, A001821]. Since $\left[{n \atop k}\right] = 0$ for $k > n > 0$,

$$\begin{aligned}
\left[\!\!\left[{n \atop n}\right]\!\!\right] &= \begin{bmatrix} n \\ n \end{bmatrix}^2 = 1, \\
\left[\!\!\left[{n \atop n-1}\right]\!\!\right] &= \begin{bmatrix} n \\ n-1 \end{bmatrix}^2 - 2\begin{bmatrix} n \\ n-2 \end{bmatrix}\begin{bmatrix} n \\ n \end{bmatrix} = \frac{1}{2^2}\binom{2n}{3}, \\
\left[\!\!\left[{n \atop n-2}\right]\!\!\right] &= \begin{bmatrix} n \\ n-2 \end{bmatrix}^2 - 2\begin{bmatrix} n \\ n-3 \end{bmatrix}\begin{bmatrix} n \\ n-1 \end{bmatrix} + 2\begin{bmatrix} n \\ n-4 \end{bmatrix}\begin{bmatrix} n \\ n \end{bmatrix}
\end{aligned}$$

[2] In [3] $u(n,m)$ are used as $u(n,m) = (-1)^{n-m}\left[\!\left[{n \atop m}\right]\!\right]$.

$$= \frac{5n+1}{3\cdot 2^3}\binom{2n}{5},$$

$$\left[\!\left[{n \atop n-3}\right]\!\right] = \left[{n \atop n-3}\right]^2 - 2\left[{n \atop n-4}\right]\left[{n \atop n-2}\right] + 2\left[{n \atop n-5}\right]\left[{n \atop n-1}\right] - 2\left[{n \atop n-6}\right]\left[{n \atop n}\right]$$

$$= \frac{35n^2+21n+4}{9\cdot 2^4}\binom{2n}{7},$$

$$\left[\!\left[{n \atop n-4}\right]\!\right] = \frac{(5n+2)(35n^2+28n+9)}{15\cdot 2^5}\binom{2n}{9},$$

$$\left[\!\left[{n \atop n-5}\right]\!\right] = \frac{385n^4+770n^3+671n^2+286n+48}{9\cdot 2^6}\binom{2n}{11}.$$

Proof of Theorem 1. First, notice that the expression of $\left[\!\left[{n \atop m}\right]\!\right]$ in terms of $\left[{n \atop k}\right]$ can satisfy the same recurrence relation in (5). From (3), since

$$\left[{n-1 \atop m}\right]\left[{n-1 \atop m-1}\right] - \left[{n-1 \atop m-1}\right]\left[{n-1 \atop m}\right] + \left[{n-1 \atop m-2}\right]\left[{n-1 \atop m+1}\right] - \cdots$$
$$- \left[{n-1 \atop m+1}\right]\left[{n-1 \atop m-2}\right] + \left[{n-1 \atop m+2}\right]\left[{n-1 \atop m-3}\right] - \cdots = 0,$$

we can also see that

$$\left[\!\left[{n \atop m}\right]\!\right] = \left((n-1)\left[{n-1 \atop m}\right] + \left[{n-1 \atop m-1}\right]\right)^2$$
$$- 2\left((n-1)\left[{n-1 \atop m-1}\right] + \left[{n-1 \atop m-2}\right]\right)\left((n-1)\left[{n-1 \atop m+1}\right] + \left[{n-1 \atop m}\right]\right)$$
$$+ 2\left((n-1)\left[{n-1 \atop m-2}\right] + \left[{n-1 \atop m-3}\right]\right)\left((n-1)\left[{n-1 \atop m+2}\right] + \left[{n-1 \atop m+1}\right]\right) - \cdots$$
$$= \left[\!\left[{n-1 \atop m-1}\right]\!\right] + (n-1)^2\left[\!\left[{n-1 \atop m}\right]\!\right]$$
$$+ 2(n-1)\left(\left[{n-1 \atop m}\right]\left[{n-1 \atop m-1}\right] - \left[{n-1 \atop m-1}\right]\left[{n-1 \atop m}\right] + \left[{n-1 \atop m-2}\right]\left[{n-1 \atop m+1}\right] - \cdots\right.$$
$$\left. - \left[{n-1 \atop m+1}\right]\left[{n-1 \atop m-2}\right] + \left[{n-1 \atop m+2}\right]\left[{n-1 \atop m-3}\right] - \cdots\right)$$
$$= \left[\!\left[{n-1 \atop m-1}\right]\!\right] + (n-1)^2\left[\!\left[{n-1 \atop m}\right]\!\right].$$

Now, because

$$\frac{(\mathrm{arcsinh}t)^{2m}}{(2m)!} = \sum_{n=m}^\infty (-4)^{n-m}\left[\!\left[{n \atop m}\right]\!\right]\frac{t^{2n}}{(2n)!}$$

(see, e.g.[2, (4.1.4)][3]), we have

$$\begin{aligned}\sum_{n=0}^\infty \mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!}&=\sum_{m=0}^\infty \frac{(\mathrm{arcsinh} t)^{2m}}{(2m)!(2m+1)^k}\\ &=\sum_{m=0}^\infty \frac{1}{(2m+1)^k}\sum_{n=m}^\infty (-4)^{n-m}\left[\!\!\left[{n \atop m} \right]\!\!\right]\frac{t^{2n}}{(2n)!}\\ &=\sum_{n=0}^\infty\sum_{m=0}^n \frac{(-4)^{n-m}}{(2m+1)^k}\left[\!\!\left[{n \atop m} \right]\!\!\right]\frac{t^{2n}}{(2n)!}\,.\end{aligned}$$

Comparing the coefficients on both sides, we get the result. □

Poly-Cauchy numbers have an expression of integrals

$$c_n^{(k)}=n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{x_1x_2\cdots x_k}{n}dx_1dx_2\dots dx_k$$

([6]). Poly-Cauchy numbers with level 2 also have a similar expression (or a kind of definition).

Corollary 1. *For $n\ge 0$ and $k\ge 1$, we have*

$$\mathfrak{C}_{2n}^{(k)}=(-4)^n(n!)^2\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{\frac{x_1x_2\cdots x_k}{2}}{n}\binom{-\frac{x_1x_2\cdots x_k}{2}}{n}dx_1dx_2\dots dx_k\,.$$

Proof. By Theorem 1 and the expression in (4),

$$\begin{aligned}&(-4)^n(n!)^2\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{\frac{x_1x_2\cdots x_k}{2}}{n}\binom{-\frac{x_1x_2\cdots x_k}{2}}{n}dx_1dx_2\dots dx_k\\ &=\underbrace{\int_0^1\cdots\int_0^1}_{k}\sum_{m=0}^n(-4)^{n-m}\left[\!\!\left[{n \atop m} \right]\!\!\right](x_1x_2\cdots x_k)^{2m}dx_1dx_2\dots dx_k\\ &=\sum_{m=1}^n\frac{(-4)^{n-m}}{(2m+1)^k}\left[\!\!\left[{n \atop m} \right]\!\!\right]=\mathfrak{C}_{2n}^{(k)}\,.\end{aligned}$$

□

[3]This proof is based upon the inverse relation between sin and arcsin with the orthogonal property of the central factorial numbers of both kinds.

2 Convolution

When $k = 1$, several initial values of $\mathfrak{C}_n = \mathfrak{C}_n^{(1)}$ are as follows.

$$\{\mathfrak{C}_{2n}\}_{n\ge 0} = 1, \frac{1}{3}, -\frac{17}{15}, \frac{367}{21}, -\frac{27859}{45}, \frac{1295803}{33}, -\frac{5329242827}{1365}, \dots .$$

In [?], the convolution identity for Cauchy numbers are given as

$$\sum_{k=0}^{n}\binom{n}{k}c_k c_{n-k} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n\ge 0)\,.$$

A more general case $\sum_{k=0}^{n}\binom{n}{k}c_{k+l}c_{n-k+m}$ for some fixed nonnegative integers l and m is treated in [8]. In [9], the convolution identities for Cauchy numbers of the second kind $\hat{c}_n$, defined by

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty}\hat{c}_n\frac{t^n}{n!}$$

have been studied. In this section, we give the convolution identity for Cauchy numbers with level 2 is given. For simplicity, we use the conventional convolution notation

$$(\mathfrak{C}_{2j_1} + \cdots + \mathfrak{C}_{2j_k})^n := \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}}\binom{2n}{2i_1,\dots,2i_k}\mathfrak{C}_{2i_1+2j_1}\cdots\mathfrak{C}_{2i_k+2j_k}\,,$$

where

$$\binom{2n}{2i_1,\dots,2i_k} = \frac{(2n)!}{(2i_1)!\cdots(2i_k)!}$$

is the multinomial coefficient.

Theorem 2. *For $n\ge 0$*

$$(\mathfrak{C}_0 + \mathfrak{C}_0)^n = (2n)!\sum_{l=0}^{n}\frac{(-1)^{n-l}(2n-2l-3)!!(2l-1)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\,.$$

Here $(2i-1)!! = (2i-1)(2i-3)\cdots 1$ $(i\ge 1)$ with $\bigl(-(2i+1)\bigr)!! = \frac{(-1)^i}{(2i-1)!!}$ $(i\ge 1)$ and $(-1)!! = 1$.

Proof. For simplicity, put

$$L(t) := \frac{t}{\mathrm{arcsinh} t} = \sum_{n=0}^{\infty}\mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!}\,.$$

Since

$$L'(t) = \frac{1}{\text{arcsinh}t} - \frac{t}{\sqrt{1+t^2}(\text{arcsinh}t)^2}$$
$$= \frac{1}{t}L(t) - \frac{1}{t\sqrt{1+t^2}}L(t)^2\,,$$

we have

$$L(t)^2 = -t\sqrt{1+t^2}L(t)' + \sqrt{1+t^2}L(t)\,. \tag{6}$$

Because

$$\sqrt{1+t^2} = \sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}(t^2)^j$$
$$= \sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}$$

and

$$tL(t)' = \sum_{l=0}^{\infty}(2n)\mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!}\,,$$

we have

$$t\sqrt{1+t^2}L(t)' = \left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^{\infty}(2n)\mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l-1}(2n-2l-3)!!(2l)}{2^{n-l}(n-l)!}\frac{\mathfrak{C}_{2l}}{(2l)!}\right)t^{2n}$$

and

$$t\sqrt{1+t^2}L(t) = \left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^{\infty}\mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l-1}(2n-2l-3)!!}{2^{n-l}(n-l)!}\frac{\mathfrak{C}_{2l}}{(2l)!}\right)t^{2n}$$

Comparing the coefficients with

$$L(t)^2 = \sum_{n=0}^{\infty}(\mathfrak{C}_0+\mathfrak{C}_0)^n\frac{t^{2n}}{(2n)!}\,,$$

we get the result. □

Since

$$L(t)L''(t) = \left(\frac{1}{2(1+t^2)^{3/2}} - \frac{1}{6\sqrt{1+t^2}}\right) L(t)$$
$$+ \left(\frac{\sqrt{1+t^2}}{6t} + \frac{1}{2t(1+t^2)^{3/2}} - \frac{2}{3t\sqrt{1+t^2}}\right) L'(t)$$
$$+ \frac{1}{2}\left(\frac{1}{\sqrt{1+t^2}} - \sqrt{1+t^2}\right) L''(t) - \frac{t\sqrt{1+t^2}}{3} L^{(3)}(t)\,, \qquad (7)$$

together with

$$\frac{1}{\sqrt{1+t^2}} = \sum_{j=0}^{\infty} \frac{(-1)^j (2j-1)!!}{2^j \cdot j!} t^{2j}$$

and

$$\frac{1}{(1+t^2)^{3/2}} = \sum_{j=0}^{\infty} \frac{(-1)^j (2j+1)!!}{2^j \cdot j!} t^{2j}$$

we have

$$\left(\frac{1}{2(1+t^2)^{3/2}} - \frac{1}{6\sqrt{1+t^2}}\right) L(t)$$
$$= \left(\sum_{j=0}^{\infty} \frac{(-1)^j (3j+1)(2j-1)!!}{3 \cdot 2^j \cdot j!} t^{2j}\right)$$
$$\times \left(\sum_{l=0}^{\infty} \mathfrak{C}_{2l} \frac{t^{2l}}{(2l)!}\right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{(-1)^{n-l}(2n-2l-1)!!(3n-3l+1)}{3 \cdot 2^{n-l}(n-l)!(2l)!} \mathfrak{C}_{2l}\right) t^{2n}\,,$$

$$\left(\frac{\sqrt{1+t^2}}{6t} + \frac{1}{2t(1+t^2)^{3/2}} - \frac{2}{3t\sqrt{1+t^2}}\right) L'(t)$$
$$= \left(\sum_{j=0}^{\infty} \frac{(-1)^j \big(-1 + 3(2j+1)(2j-1) - 4(2j-1)\big)(2j-3)!!}{6 \cdot 2^j \cdot j!} t^{2j}\right) \left(\sum_{l=0}^{\infty} \mathfrak{C}_{2l+2} \frac{t^{2l}}{(2l+1)!}\right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{2(-1)^{n-l}(2n-2l-3)!!(n-l)(3n-3l-2)}{3 \cdot 2^{n-l}(n-l)!(2l+1)!} \mathfrak{C}_{2l+2}\right) t^{2n}\,,$$

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{1}{\sqrt{1+t^2}}-\sqrt{1+t^2}\right)L''(t)\\
&=\left(\sum_{j=0}^{\infty}\frac{(-1)^j j(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^{\infty}\mathfrak{C}_{2l+2}\frac{t^{2l}}{(2l)!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l}(2n-2l-3)!!(n-l)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+2}\right)t^{2n}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{t\sqrt{1+t^2}}{3}L^{(3)}(t)\\
&=\left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{3\cdot 2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^{\infty}(2l)\mathfrak{C}_{2l+2}\frac{t^{2l}}{(2l)!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l-1}(2n-2l-3)!!(n-l)(2n)}{3\cdot 2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+2}\right)t^{2n}\,.
\end{aligned}
$$

Thus, the right-hand side of (7) is equal to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l}(2n-2l-1)!!(3n-3l+1)}{3\cdot 2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\right)t^{2n}\\
&\quad+\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^{n-l}(6n^2-6nl+4l^2-n+3l)(2n-2l-3)!!}{3\cdot 2^{n-l}(n-l)!(2l+1)!}\mathfrak{C}_{2l+2}t^{2n}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l}(2n-2l-1)!!(3n-3l+1)}{3\cdot 2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\right)t^{2n}\\
&\quad+\sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{n-l-1}(6n^2-6nl+4l^2+5n-5l+1)(2n-2l-1)!!}{3\cdot 2^{n-l+1}(n-l+1)!(2l-1)!}\mathfrak{C}_{2l}t^{2n}\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n+1}\frac{(-1)^{n-l-1}(2l-1)(3n^2-3nl+2l^2+4n-3l+1)(2n-2l-1)!!}{3\cdot 2^{n-l}(n-l+1)!(2l)!}\mathfrak{C}_{2l}t^{2n}\,.
\end{aligned}
$$

Since the left-hand side of (7) is equal to

$$
\sum_{n=0}^{\infty}(\mathfrak{C}_0+\mathfrak{C}_2)^n\frac{t^{2n}}{(2n)!}\,,
$$

comparing the coefficients on both sides, we get the result of $(c_0+c_2)^n$.

Theorem 3. *For $n \ge 0$,*

$$
\begin{aligned}
&(\mathfrak{C}_0+\mathfrak{C}_2)^n\\
&=(2n)!\sum_{l=0}^{n+1}\frac{(-1)^{n-l-1}(2l-1)(3n^2-3nl+2l^2+4n-3l+1)(2n-2l-1)!!}{3\cdot 2^{n-l}(n-l+1)!(2l)!}\mathfrak{C}_{2l}\,.
\end{aligned}
$$

Similarly, the convolution of $(c_2+c_2)^n$ can be given as follows.

Theorem 4. *For $n \ge 0$,*

$$
\begin{aligned}
&(\mathfrak{C}_2+\mathfrak{C}_2)^n\\
&=\frac{(2n)!}{30}\sum_{l=0}^{n}\frac{(-1)^{n-l}(10n-8l+5)(2n-2l-3)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+4}\\
&\quad-\frac{(2n)!}{3}\sum_{l=0}^{n}\frac{(-1)^{n-l}(6l+1)(2n-2l+1)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+2}\\
&\quad-\frac{(2n)!}{30}\sum_{l=0}^{n}\frac{(-1)^{n-l}(160l^3-220l^2+72l-1)(2n-2l+1)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\,.
\end{aligned}
$$

Proof. We know that

$$
\begin{aligned}
\bigl(L''(t)\bigr)^2&=-\frac{t\sqrt{1+t^2}}{30}L^{(5)}(t)+\frac{1}{6}\left(\frac{1}{\sqrt{1+t^2}}-2\sqrt{1+t^2}\right)L^{(4)}(t)\\
&\quad-\frac{3t+2t^2}{3(1+t^2)^{3/2}}L^{(3)}(t)-\frac{2+t^2}{6(1+t^2)^{3/2}}L''(t)-\frac{t}{30(1+t^2)^{3/2}}L'(t)\\
&\quad+\frac{1}{30(1+t^2)^{3/2}}L(t)\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
&-\frac{t\sqrt{1+t^2}}{30}L^{(5)}(t)\\
&=-\frac{1}{30}\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\sum_{l=3}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l-4}}{(2l-5)!}\\
&=\frac{1}{30}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l-3)!!(2l)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+4}\right)t^{2l}\,,
\end{aligned}
$$

$$\frac{1}{6}\left(\frac{1}{\sqrt{1+t^2}}-2\sqrt{1+t^2}\right)L^{(4)}(t)$$
$$=-\frac{1}{6}\left(\sum_{j=0}^{\infty}\frac{(-1)^j(2j-1)!!}{2^j\cdot j!}t^{2j}\right.$$
$$\left.-\sum_{j=0}^{\infty}\frac{(-1)^{j-1}2(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\sum_{l=2}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l-4}}{(2l-4)!}$$
$$=\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l-3)!!(2n-2l+1)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+4}\right)t^{2l},$$

$$-\frac{3t+2t^2}{3(1+t^2)^{3/2}}L^{(3)}(t)$$
$$=-\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)!!}{2^j\cdot j!}t^{2j}\sum_{l=2}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l-2}}{(2l-3)!}$$
$$-\frac{2}{3}\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)!!}{2^j\cdot j!}t^{2j}\sum_{l=2}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l-3)!}$$
$$=-\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l+1)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+2}\right)t^{2l}$$
$$-\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l+1)!!(2l)(2l-1)(2l-2)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\right)t^{2l},$$

$$--\frac{2+t^2}{6(1+t^2)^{3/2}}L''(t)$$
$$=-\frac{2}{6}\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)!!}{2^j\cdot j!}t^{2j}\sum_{l=1}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l-2}}{(2l-2)!}$$
$$-\frac{1}{6}\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)!!}{2^j\cdot j!}t^{2j}\sum_{l=1}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l-2)!}$$
$$=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l+1)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+2}\right)t^{2l}$$
$$-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l+1)!!(2l)(2l-1)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\right)t^{2l}$$

and

$$
\begin{aligned}
&-\frac{t}{30(1+t^2)^{3/2}}L'(t)+\frac{1}{30(1+t^2)^{3/2}}L(t)\\
&=-\frac{1}{30}\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)!!}{2^j\cdot j!}t^{2j}\sum_{l=1}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l-1)!}\\
&\quad+\frac{1}{30}\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)!!}{2^j\cdot j!}t^{2j}\sum_{l=0}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l)!}\\
&=-\frac{1}{30}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-l}(2n-2l+1)!!(2l-1)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\right)t^{2l}\,,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\bigl(L''(t)\bigr)^2\\
&=\frac{1}{30}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l}(10n-8l+5)(2n-2l-3)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+4}\right)t^{2l}\\
&\quad-\frac{1}{3}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l}(6l+1)(2n-2l+1)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l+2}\right)t^{2l}\\
&\quad-\frac{1}{30}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^{n-l}(160l^3-220l^2+72l-1)(2n-2l+1)!!}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\right)t^{2l}\,.
\end{aligned}
$$

Since

$$
\bigl(L''(t)\bigr)^2=\sum_{n=0}^{\infty}(\mathfrak{C}_2+\mathfrak{C}_2)^n\frac{t^{2n}}{(2n)!}\,,
$$

comparing the coefficients on both sides, we get the desired result. □

2.1 Higher-order convolutions

The convolution identity for three Cauchy numbers with level 2 can be given as follows.

Theorem 5. *For $n\ge 1$,*

$$
\begin{aligned}
&(\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0)^n\\
&=(2n-1)(n-1)\mathfrak{C}_{2n}+n(2n-1)(2n-3)^2\mathfrak{C}_{2n-2}\,.
\end{aligned}
$$

Proof. From (6),

$$L(t)L(t)' = \frac{t}{2\sqrt{1+t^2}}L(t) - \frac{t^2}{2\sqrt{1+t^2}}L(t)' - \frac{t\sqrt{1+t^2}}{2}L(t)''\,.$$

So,

$$\begin{aligned}
L(t)^3 &= -t\sqrt{1+t^2}L(t)L(t)' + \sqrt{1+t^2}L(t)^2 \\
&= -t\sqrt{1+t^2}\left(\frac{t}{2\sqrt{1+t^2}}L(t) - \frac{t^2}{2\sqrt{1+t^2}}L(t)' - \frac{t\sqrt{1+t^2}}{2}L(t)''\right) \\
&\quad + \sqrt{1+t^2}\big(-t\sqrt{1+t^2}L(t)' + \sqrt{1+t^2}L(t)\big) \\
&= \left(1+\frac{t^2}{2}\right)L(t) - \left(t+\frac{t^3}{2}\right)L(t)' + \frac{t^2(1+t^2)}{2}L(t)'' \\
&= \sum_{n=0}^\infty \mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!} + \frac{1}{2}\sum_{n=0}^\infty (2n+2)(2n+1)\mathfrak{C}_{2n}\frac{t^{2n+2}}{(2n+2)!} \\
&\quad - \sum_{n=1}^\infty \mathfrak{C}_{2n}\frac{t^{2n}}{(2n-1)!} - \frac{1}{2}\sum_{n=0}^\infty (2n+2)(2n+1)(2n)\mathfrak{C}_{2n}\frac{t^{2n+2}}{(2n+2)!} \\
&\quad + \frac{1}{2}\sum_{n=1}^\infty \mathfrak{C}_{2n}\frac{t^{2n}}{(2n-2)!} \\
&\quad + \frac{1}{2}\sum_{n=0}^\infty (2n+2)(2n+1)(2n)(2n-1)\mathfrak{C}_{2n}\frac{t^{2n+2}}{(2n+2)!} \\
&= \sum_{n=0}^\infty \left(1-2n+\frac{(2n)(2n-1)}{2}\right)\mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!} \\
&\quad + \frac{1}{2}\sum_{n=1}^\infty (2n)(2n-1)\big(1-(2n-2)+(2n-2)(2n-3)\big)\mathfrak{C}_{2n-2}\frac{t^{2n}}{(2n)!} \\
&= \sum_{n=0}^\infty (2n-1)(n-1)\mathfrak{C}_{2n}\frac{t^{2n}}{(2n)!} + \sum_{n=1}^\infty n(2n-1)(2n-3)^2\mathfrak{C}_{2n-2}\frac{t^{2n}}{(2n)!}\,.
\end{aligned}$$

Comparing the coefficients with

$$L(t)^3 = \sum_{n=0}^\infty (\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0)^n\frac{t^{2n}}{(2n)!}\,,$$

we get the result. □

The convolution identity for four Cauchy numbers with level 2 can be given as follows.

Theorem 6. *For $n \ge 1$,*

$$
\begin{aligned}
&(\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0)^n\\
&=\frac{(2n)!}{6}\sum_{l=0}^{n}\frac{(-1)^{n-l}(2n-2l-3)!!(2l-1)(2l-2)(2l-3)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}\\
&\quad+\frac{(2n)!}{6}\sum_{l=1}^{n}\frac{(-1)^{n-l}(2n-2l-3)!!(2l)(2l-1)(2l-3)^3}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l-2}\,.
\end{aligned}
$$

Proof. From the proof of Theorem 5 and Theorem 3, we get

$$
\begin{aligned}
L(t)^4&=\left(1+\frac{t^2}{2}\right)L(t)-\left(t+\frac{t^3}{2}\right)L(t)'+\frac{t^2(1+t^2)}{2}L(t)''\\
&=\frac{(6+t^2)\sqrt{1+t^2}}{6}L(t)-\frac{t(6+t^2)\sqrt{1+t^2}}{6}L'(t)\\
&\quad+\frac{t^2\sqrt{1+t^2}}{2}L''(t)-\frac{(t^3+t^5)\sqrt{1+t^2}}{6}L^{(3)}\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{(6+t^2)\sqrt{1+t^2}}{6}L(t)\\
&=\left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l)!}\right)\\
&\quad+\frac{1}{6}\left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=1}^{\infty}(2l)(2l-1)\mathfrak{C}_{2l-2}\frac{t^{2l}}{(2l)!}\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{t(6+t^2)\sqrt{1+t^2}}{6}L'(t)\\
&=-\left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^{\infty}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l)!}\right)\\
&\quad-\frac{1}{6}\left(\sum_{j=0}^{\infty}\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=1}^{\infty}(2l)(2l-1)\mathfrak{C}_{2l-2}\frac{t^{2l}}{(2l)!}\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{t^2\sqrt{1+t^2}}{2}L''(t)\\
&=\frac{1}{2}\left(\sum_{j=0}^\infty\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^\infty(2l)(2l-1)\mathfrak{C}_{2l}\frac{t^{2l}}{(2l)!}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{(t^3+t^5)\sqrt{1+t^2}}{6}L^{(3)}\\
&=-\frac{1}{6}\left(\sum_{j=0}^\infty\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^\infty\frac{(2l)!}{(2l-3)!}\mathfrak{C}_{2l}\frac{t^{2l}}{(2l)!}\right)\\
&\quad-\frac{1}{6}\left(\sum_{j=0}^\infty\frac{(-1)^{j-1}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=1}^\infty\frac{(2l)!}{(2l-5)!}\mathfrak{C}_{2l-2}\frac{t^{2l}}{(2l)!}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
L(t)^4&=\frac{1}{6}\left(\sum_{j=0}^\infty\frac{(-1)^{j}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=0}^\infty(2l-1)(2l-2)(2l-3)\mathfrak{C}_{2l}\frac{t^{2l}}{(2l)!}\right)\\
&\quad+\frac{1}{6}\left(\sum_{j=0}^\infty\frac{(-1)^{j}(2j-3)!!}{2^j\cdot j!}t^{2j}\right)\left(\sum_{l=1}^\infty(2l)(2l-1)(2l-3)^3\mathfrak{C}_{2l-2}\frac{t^{2l}}{(2l)!}\right)\\
&=\frac{1}{6}\sum_{n=0}^\infty\sum_{l=0}^n\frac{(-1)^{n-l}(2n-2l-3)!!(2l-1)(2l-2)(2l-3)}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l}t^{2n}\\
&\quad+\frac{1}{6}\sum_{n=1}^\infty\sum_{l=1}^n\frac{(-1)^{n-l}(2n-2l-3)!!(2l)(2l-1)(2l-3)^3}{2^{n-l}(n-l)!(2l)!}\mathfrak{C}_{2l-2}t^{2n},
\end{aligned}
$$

Comparing the coefficients with

$$
L(t)^4=\sum_{n=0}^\infty(\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0)^n\frac{t^{2n}}{(2n)!},
$$

we get the result. □

As seen, convolution identities of the odd number of Cauchy numbers are simpler than those of the even number. Similarly, from

$$
L(x)^5=\frac{x^4(1+x^2)^2}{4!}L^{(4)}(x)+\frac{x^3(x^2-2)(1+x^2)}{12}L^{(3)}(x)
$$

$$
\begin{aligned}
&+\frac{x^2(x^4+10x^2+12)}{4!}L''(x)-\frac{x(x^4+20x^2+24)}{4!}L'(x)\\
&+\frac{x^4+20x^2+24}{4!}L(x)\,,
\end{aligned}
$$

we have for $n\ge 2$

$$
\begin{aligned}
&(\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0+\mathfrak{C}_0)^n\\
&=\binom{2n-1}{4}\mathfrak{C}_{2n}+\frac{4n^2-16n+17}{3}\binom{2n}{2}\binom{2n-3}{2}\mathfrak{C}_{2n-2}\\
&\quad+\binom{2n}{4}(2n-5)^4\mathfrak{C}_{2n-4}\,.
\end{aligned}
$$

From

$$
\begin{aligned}
L(x)^7&=\frac{x^6(1+x^2)^3}{6!}L^{(6)}(x)+\frac{x^5(3x^2-2)(1+x^2)^2}{2\cdot 5!}L^{(5)}(x)\\
&\quad+\frac{x^4(4x^4+x^2+6)(1+x^2)}{3!4!}L^{(4)}(x)\\
&\quad+\frac{x^3(2x^2+3)(x^4-4x^2-8)}{4!3!}L^{(3)}(x)\\
&\quad+\frac{x^2(x^6+91x^4+420x^2+360)}{6!}L''(x)\\
&\quad+\frac{x(x^6+182x^4+840x^2+720)}{6!}L'(x)\\
&\quad+\frac{x^6+182x^4+840x^2+720}{6!}L(x)\,,
\end{aligned}
$$

we have for $n\ge 3$

$$
\begin{aligned}
&(\underbrace{\mathfrak{C}_0+\cdots+\mathfrak{C}_0}_{7})^n\\
&=\binom{2n-1}{6}\mathfrak{C}_{2n}+\frac{12n^2-60n+83}{15}\binom{2n}{2}\binom{2n-3}{4}\mathfrak{C}_{2n-2}\\
&\quad+\frac{(4n^2-24n+39)(12n^2-72n+109)}{15}\binom{2n}{4}\binom{2n-5}{2}\mathfrak{C}_{2n-4}\\
&\quad+\binom{2n}{6}(2n-7)^6\mathfrak{C}_{2n-6}\,.
\end{aligned}
$$

Nevertheless, the higher-order cases seem to be more complicated when the number of Cauchy numbers increases. It is expected that for any integer

$r \geq 1$

$$(\underbrace{\mathfrak{C}_0 + \cdots + \mathfrak{C}_0}_{2r+1})^n = \sum_{k=0}^{r} P_{r,2k}(n) \binom{2n}{2k} \binom{2n-2k-1}{2r-2k} \mathfrak{C}_{2n-2k}\,,$$

where $P_{r,2k}(n)$ are the polynomials of n with degree $2k$ $(0 \leq k \leq r)$. In particular, $P_{r,0}(n) = 1$ and $P_{r,2r} = (2n-2r-1)^{2r}$.

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