An Extension of Furstenberg's Theorem of the Infinitude of Primes

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Abstract

The usual product $m \cdot n$ on $\mathbb{Z}$ can be viewed as the sum of n terms of an arithmetic progression whose first term is $a_1 = m - n + 1$ and whose difference is $d = 2$. Generalizing this idea, we define new similar product mappings, and we consider new arithmetics that enable us to extend Furstenberg's theorem of the infinitude of primes. We also review the classic conjectures in the new arithmetics. Finally, we make important extensions of the main idea. We see that given any integer sequence, the approach generates an arithmetic on integers.

1 Introduction

In 1955, H. Furstenberg proposed a topological proof of the infinitude of primes. He considered the arithmetic progression topology on the integers, where a basis of open sets is given by subsets of the following form: for $a, b \in \mathbb{Z}, a > 0$,

$$S(a,b) = \{n \cdot a + b : n \in \mathbb{Z}\}.$$

Arithmetic progressions themselves are by definition open.
He studied this family of open sets and reached the following expression:

$$\bigcup_{p\,\text{prime}} S(p,0) = \mathbb{Z} \setminus \{-1,\ 1\}. \tag{1}$$

With (1), he concluded that the set of primes is infinite; see [8] for more details.

On the other hand, the usual product $m \cdot n$ on $\mathbb{Z}$ can be viewed as the sum of n terms of an arithmetic progression (a_n) whose first term is $a_1 = m - n + 1$ and whose difference is $d = 2$.

Example.

$$7 \cdot 5 = (7-5+\underbrace{1)+}_{+2}\underbrace{5+}_{+2}\underbrace{7+}_{+2}\underbrace{9+}_{+2}11$$

$$5 \cdot 7 = (5-7+\underbrace{1)+}_{+2}\underbrace{1+}_{+2}\underbrace{3+}_{+2}\underbrace{5+}_{+2}\underbrace{7+}_{+2}\underbrace{9+}_{+2}11$$

It is thus natural to consider the following questions:
I. What if one were to consider product mappings, similar to the usual case, by varying the first term a_1?
II. What if one were to consider product mappings by varying the distance $d = k$?
For instance, could we do the following in the previous example?

$$7 \odot_3 5 = (7-5+\underbrace{1)+}_{+3}\underbrace{6+}_{+3}\underbrace{9+}_{+3}\underbrace{12+}_{+3}15 = 45$$

$$5 \odot_3 7 = (5-7+\underbrace{1)+}_{+3}\underbrace{2+}_{+3}\underbrace{5+}_{+3}\underbrace{8+}_{+3}\underbrace{11+}_{+3}\underbrace{14+}_{+3}17 = 56$$

Why is $+2$ so special in the usual product?
Question II is more interesting than the first. By studying this question, we obtain the definition of a family of product mappings on integers.

Definition. Given $m, n, k \in \mathbb{Z}$, we define

$$m \odot_k n = (m-n+1) + (m-n+1+k) + \ldots + (m-n+1+k+ \overset{(n-1)}{\cdots} +k)$$

as the *k-arithmetic* product.

In connection with the above result, for each $k \in \mathbb{Z}$, the expression "*given a k-arithmetic*" refers to the fact that we are going to work with integers, the sum, the new product and the usual order. This means that we are going to work on

$$\mathcal{Z}_k = \{\mathbb{Z}, +, \odot_k, <\}.$$

Clearly, $\mathcal{Z}_2$, the $2-$*arithmetic*, will be the usual arithmetic.
Characterizing the concept of divisor and prime number in the new arithmetics, we obtain the following important theorem:

Theorem. *Given a k-arithmetic, the primes* $(arith\ k)$ *are:*

- *The usual primes if* $k \in E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$.
- *The usual powers of two if* $k \in O = \{\ldots, -3, -1, 1, 3, \ldots\}$.

Now, using the previous result and similar to Furstenberg's proof, we can define

$$S_k(a, b) = \{n \odot_k a + b : n \in \mathbb{Z}\}$$

and obtain

$$k \in E \Rightarrow \bigcup_{p\,\text{prime (arith k)}} S_k(p, 0) = \mathbb{Z} \setminus \{-1,\ 1\}$$

$$k \in O \Rightarrow \bigcup_{p\,\text{prime (arith k)}} S_k(p, 0) = \mathbb{Z} \setminus \{0\}$$

As we will see, the above expression relates the usual primes and the powers of two.

Another interesting point is to study the classical conjectures in the new arithmetics. One interesting result is the following.

Theorem. *If Goldbach's conjecture is true in the usual arithmetic, then the conjecture must be true in* $\mathcal{Z}_k,\ k \in E$.

In other words, if k is even, the conjecture is equivalent in all *k-arithmetics.*
This result is notable because the *k-arithmetic* product ($\odot_k$) is not commutative and not associative if $k \neq 2$. We will also revisit Collatz conjecture.

Finally, we make important extensions of the main idea. We see that given any sequence, the approach will generate an arithmetic on integers. A result in this line is the following.

Theorem. *If we consider the arithmetic generated by* $(a_n) = 2, 6, 10, 14, \ldots$, *the primes will be the usual powers of three.*

In this part of the paper, new sequences and connections appear.

We have briefly presented the idea of this paper. We now develop the new arithmetics and their connections with Furstenberg's theorem and the classic conjectures in the usual arithmetic.

2 Basic definitions and properties

We now start to construct the definition of $\mathcal{Z}_k$, that is, the set of integers with the sum, usual order and a product mapping similar to the usual one.

Definition 1 (*k-arithmetic product* $\odot_k$)**.** Given $m, k \in \mathbb{Z}$, for all positive integers n, we define the following expression

$$m \odot_k n = (m - n + 1) + (m - n + 1 + k) + \ldots + (m - n + 1 + k + \overset{(n-1)}{\cdots} + k)$$

as the *k-arithmetic* product.

This arithmetic progression can be added to obtain the following formula:

$$m \odot_k n = (m - n + 1) \cdot n + \frac{n \cdot (n-1) \cdot k}{2}. \tag{2}$$

We take (2) as Definition (1) and consider $n \in \mathbb{Z}$. Observe that the usual product is used to define the *k-arithmetic product* (which includes the usual one if $k = 2$). This is not a contradiction; for example, recall that human sight is used to study the eyes and note the Beltrami-Klein model in Geometry: we can use the euclidean plane to work with this non-euclidean geometry.

Our product can be formalized with the successor operation in Peano arithmetic.

Recall the definition of the product between two natural numbers m, n using the successor operation $S(n)$: $P(m, n)$ is a number such that

- $P(m, 1) = m$.
- $P(m, S(n)) = m + P(m, n)$. (3)

Similarly to (3), we can consider $\mathcal{N} = \{1, S, +, <\}$ (Peano arithmetic with only the successor operation, the sum and the usual order) and define the following product operation:

Definition 2 (*t-Peano product*)**.** Given $t \in \mathbb{N}$, to every pair of numbers $m, n \in \mathbb{N}$, we may assign in exactly one way a natural number, called $P_t(m, n)$, such that

- $P_t(m, 1) = m$.
- $P_t(m, S(n)) = m + P_t(m + t, n)$.

$P_t(m, n)$ is called the *t-Peano product* of m and n.

Now, we can relate the *k-arithmetic* product and the $t - Peano\ product$.

Proposition 3. *Given $m, n, t \in \mathcal{N}$, then $P_t(m, n) = m \odot_{t+2} n$.*

Proof. We need only Definitions (2) and (1).

$$P_t(m, n) = P_t(m, S(n-1)) = m + P_t(m + t, n - 1) =$$
$$= m + P_t(m + t, S(n-2)) = m + m + t + P_t(m + t + t, n - 2) = ... =$$
$$= m + (m + t) + (m + t + t) + \ldots + (m + (n-1)t) = mn + \frac{n(n-1)t}{2}.$$

On the other hand:

$$m \odot_{t+2} n = (m - n + 1)n + \frac{n(n-1)(t+2)}{2} = mn + \frac{n(n-1)t}{2}.$$

The result follows. □

We have studied this proposition to show that the new products are similar to the usual one and could have arisen from Peano arithmetic. Later, this result will motivate us to make interesting conjectures and heuristic reasoning. However, the following theorem provides a better understanding of the *k-arithmetic* product.

Theorem 4 (*k-arithmetic polygonal theorem*)**.** *The product $n \odot_k n$ is the nth $(k+2)$-gonal number.*

Proof. The formula of the *nth* l-gonal is $\frac{1}{2}n((l-2)n-(l-4))$. We have, by hypothesis, $k+2=l$; hence,

$$\frac{n((l-2)n-(l-4))}{2} = \frac{n(kn-(k-2))}{2} = \frac{n \cdot k(n-1)}{2} + n = n \odot_k n \qquad \square$$

This theorem is easy to prove and has great significance. The *k-arithmetic* product generalizes the usual product in the following way: the *square of a number* is a square (polygon) in the usual arithmetic but a triangle in *1-arithmetic*, a pentagon in *3-arithmetic*, a hexagon in *4-arithmetic*, etc. That is, $4 \odot_2 4 = 16$ is the fourth square, whereas $4 \odot_1 4 = 10$ is the fourth triangular number and $4 \odot_3 4 = 22$ is the fourth pentagonal number.
If we want to calculate the 5th pentagonal number, we can calculate $5 \odot_3 5 = 35$.

From now on, we will work on $\mathcal{Z}_k = \{\mathbb{Z}, +, \odot_k, <\}$. If $k \neq 2$, the *k-arithmetic* product $\odot_k$ is not commutative and not associative; thus, the group or ring structures are not considered in this paper. However, we have interesting algebraic properties that connect the usual product with the others. For instance:

Proposition 5. *Given $a, b, c, d, k \in \mathbb{Z}$, the following properties are satisfied:*

1. *The k-arithmetic product is not associative in general.*
   If $k \neq 2$ and $c \neq \{0, 1\}$, then $(a \odot_k b) \odot_k c \neq a \odot_k (b \odot_k c)$.
2. *The k-arithmetic product is not commutative in general but*
   $a \odot_k (1-a) = (1-a) \odot_k a$.
3. $(a-b) \cdot (c+d) = a \odot_k c + a \odot_k d - b \odot_k c - b \odot_k d$.
4. $(a+b) \odot_k (a+b) = a \odot_k a + b \odot_k b + k \cdot a \cdot b$.
5. $(a-b)^2 = a \odot_k a + b \odot_k b - b \odot_k a - a \odot_k b$.
6. $a \odot_k (-b) = (k-2-a) \odot_k b$.

Proof. Only we have to use Formula (2). $\square$

It is not the purpose of this paper to study these types of properties. In the following chapter, we will extend the definition of divisor and prime number and obtain the fundamental theorem that will allow us to extend Furstenberg's theorem of the infinitude of primes.

3 Divisors and primes.

Definition 6 (*k-arithmetic divisor*)**.** Given a *k-arithmetic*, an integer $d > 0$ is called a divisor of a $(arith\ k)$ if there exists some integer b such that $a = b \odot_k d$. We can write:

$$d \mid a\ (artih\ k) \ \Leftrightarrow\ \exists b \in \mathbb{Z}\ such\ that\ b \odot_k d = a$$

In other words, d is the number of terms of the summation that represents the *k-arithmetic* product (see the following example).

Example 7. Consider the following expression:

$$\underbrace{6}_{where\ to\ begin} \odot_3 \underbrace{5}_{number\ of\ terms} = 2 + 5 + 8 + 11 + 14 = 40$$

The number of terms is 5; hence, we can say that 5 is a divisor of 40 in $3-arithmetic$, that is, 5 is a divisor of 40 $(arith\ 3)$.
Notably, a divisor is always a positive number, and the number 6 indicates where we should start the summation. However, we cannot be sure that 6 is a divisor of 40 $(arith\ 3)$. Another point is to consider the expression $6 \odot_3 (-5)$. Is -5 a divisor? We can use point 6 of Proposition (5):

$$6 \odot_3 (-5) = (3 - 2 - 6) \odot_3 5 = -15.$$

We can say that 5 is a divisor of -15 $(arith\ 3)$.

To characterize the set of divisors, we define the *k-arithmetic* quotient:

Definition 8 (*k-arithmetic quotient* $\oslash_k$)**.** Given a *k-arithmetic*, an integer c is called a quotient of a divided by b $(arith\ k)$ if and only if $c \odot_k b = a$. We write:

$$a \oslash_k b = c \Leftrightarrow c \odot_k b = a$$

By means of the following proposition, we can use the usual quotient to study the new one.

Proposition 9. *Given a k-arithmetic and* $a, b, k \in \mathbb{Z}$ $(b \neq 0)$,

$$a \oslash_k b = \frac{a}{b} + (b-1) \cdot (1 - \frac{k}{2}).$$

Proof. $a \oslash_k b = c \Leftrightarrow c \odot_k b = a \Leftrightarrow (c - b + 1)b + \frac{1}{2}b(b-1)k = a \Leftrightarrow c = \frac{a}{b} + (b-1) \cdot (1 - \frac{k}{2})$ □

We must consider $\oslash_k$ in the following manner. If we want to write a as the sum of b terms of an arithmetic progression, then the quotient will give us the place to start the summation (see the following example).

Example 10. Express 81 as the sum of 6 terms of an arithmetic progression whose difference is 3.

We can then obtain $81 \oslash_3 6 = \frac{81}{6} + 5 \cdot (1 - \frac{3}{2}) = 11$. Hence, $81 = 11 \odot_3 6$. The first term is $11 - 6 + 1 = 6$, and the solution is $6 + 9 + 12 + 15 + 18 + 21 = 81$.
Clearly, 6 is a divisor of 81 (*arith* 3).

Corollary 11. *$a \oslash_k b$ is an integer $\Leftrightarrow$ b is a divisor of a (arith k).*

Proof.
$$a \oslash_k b = c \in \mathbb{Z} \overset{Def.(8)}{\Leftrightarrow} c \odot_k b = a \overset{Def.(6)}{\Leftrightarrow} b \mid a \ (arith\ k)$$
$\square$

Consider the Example (7): $40 = 6 \odot_3 5$ but 6 is not a divisor of 40 (*arith* 3) because $40 \oslash_3 6 = \frac{40}{6} + 5 \cdot (1 - \frac{3}{2}) = \frac{25}{6} \notin \mathbb{Z}$.

For the upcoming Lemma and the rest of this paper, we use the following notation for even and odd numbers.

Notation 12. We write the set of even and odd numbers as follows:

- $E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$.
- $O = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$.

Lemma 13. *Given a k-arithmetic and $a \in \mathbb{Z}$, the divisors of a (arith k) are:*

1. *The usual divisors of a if $k \in E$.*
2. *The usual divisors of $2a$ except the even usual divisors of a if $k \in O$.*

Proof. We use Proposition (9) and Corollary (11) in the following cases:

1) $k \in E$. $d \mid a \Leftrightarrow d \mid a \ (arith\ k)$.

2a) $k \in O$. If d is odd: $d \mid 2a \Leftrightarrow d \mid a \ (arith\ k)$.

2b) $k \in O$. If d is even: $d \mid 2a$ and $d \nmid a \Leftrightarrow d \mid a \ (arith\ k)$.

1. $k \in E$. *Suppose d is a usual divisor of a*:

$$a \oslash_k d = \underbrace{\frac{a}{d}}_{\in \mathbb{Z}} + (d-1)(1 - \underbrace{\frac{k}{2}}_{\in \mathbb{Z}}) \in \mathbb{Z} \Rightarrow d \mid a \ (arith\ k).$$

Hence, d is a divisor of a (*arith* k).

$k \in E$. *Suppose d is a divisor of a (arith k)*:

$$d \mid a \ (arith\ k) \Leftrightarrow \exists b \in \mathbb{Z} \text{ such that } b \odot_k d = a \Leftrightarrow (b - d + 1) + \underbrace{\frac{(d-1)k}{2}}_{\in \mathbb{Z}} = \frac{a}{d} \in \mathbb{Z}.$$

Hence, $\frac{a}{d} \in \mathbb{Z}$, and d is a usual divisor of a.

2. We consider two cases:
a) $k \in O$. *Suppose d is an odd usual divisor of $2a$*:

If $d \mid 2a \Rightarrow d \mid a$ because d is an odd number. Then, $a \oslash_k d = \underbrace{\frac{a}{d}}_{\in\mathbb{Z}} + \underbrace{(d-1)}_{even}(1-\frac{k}{2}) \in \mathbb{Z}$.

Hence, a$\oslash_k d \in \mathbb{Z}$, and d is a divisor of a $(arith\ k)$.

$k \in O$. *Suppose d is an odd divisor of a $(arith\ k)$:*

$d \mid a\ (arith\ k) \Leftrightarrow \exists b \in \mathbb{Z}$ such that $b \odot_k d = a \Leftrightarrow (b-d+1) + \underbrace{(d-1)}_{even}\frac{k}{2} = \frac{a}{d} \in \mathbb{Z}$.

Then, $\frac{a}{d} \in \mathbb{Z}$, and d is a usual divisor of a. Hence, d is a usual divisor of $2a$.

b) $k \in O$. *Suppose d is an even usual divisor of $2a$ but d is not a divisor of a:*

$d \mid 2a \Rightarrow \exists h \in \mathbb{Z}$ such that $2a = dh \Rightarrow \frac{a}{d} = \frac{h}{2}$. By hypothesis $d \nmid a$; hence, $\frac{h}{2} \notin \mathbb{Z}$ and h is odd.

Then, $a \oslash_k d = \frac{a}{d} + (d-1)(1-\frac{k}{2}) = \frac{h}{2} + (d-1)(1-\frac{k}{2}) = \frac{1}{2}(\underbrace{\underbrace{h}_{odd} - \underbrace{(d-1)k}_{odd}}_{even}) + d - 1 \in \mathbb{Z}$.

Hence, d is a divisor of a $(arith\ k)$.

$k \in O$. *Suppose d is an even number and d is a divisor of a $(arith\ k)$:*

$$d \mid a\ (arith\ k) \Leftrightarrow \exists b \in \mathbb{Z} \text{ such that } b \odot_k d = a \Leftrightarrow \begin{cases} (b-d+1) + \underbrace{\frac{(d-1)k}{2}}_{\notin\mathbb{Z}} = \frac{a}{d} \notin \mathbb{Z}. \\ \\ 2(b-d+1) + (d-1)k = \frac{2a}{d} \in \mathbb{Z}. \end{cases}$$

Hence, $d \nmid a$ and $d \mid 2a$ in the usual sense. $\square$

The following example is an interesting application of the lemma.

Example 14. Express the number 12 in all possible ways as a sum of an arithmetic progression whose difference is 3.

The divisors of 12 $(arith\ 3)$ are the usual divisors of 24 except the even usual divisors of 12: $\{1,\ \cancel{2},\ 3,\ \cancel{4},\ \cancel{6},\ 8,\ \cancel{12},\ 24\}$.

- $d = 1 \Rightarrow a = \frac{12}{1} + (1-1)(1-\frac{3}{2}) = 12 \Rightarrow 12 = 12 \odot_3 1 \Rightarrow 12 = 12$.
- $d = 3 \Rightarrow a = \frac{12}{3} + (3-1)(1-\frac{3}{2}) = 3 \Rightarrow 12 = 3 \odot_3 3 \Rightarrow 12 = 1 + 4 + 7$.
- $d = 8 \Rightarrow a = \frac{12}{8} + (8-1)(1-\frac{3}{2}) = -2 \Rightarrow 12 = -2 \odot_3 8 \Rightarrow 12 = -9-6-3+0+3+6+9+12$.
- $d = 24 \Rightarrow a = \frac{12}{24} + (24-1)(1-\frac{3}{2}) = -11 \Rightarrow 12 = -11 \odot_3 24 \Rightarrow 12 = -34-31-28-\ldots+29+32+35$.

We can use this example to study the number of nondecreasing arithmetic progressions of positive integers with sum n. See A049988 and [11]. However, we leave this approach for another time.

Let us now consider the primes ($arith\ k$). Following the results above, an integer $a > 1$ always has two divisors in any *k-arithmetic*:

- If $k \in E$, 1 and a are divisors of a ($arith\ k$).
- If $k \in O$, 1 and $2a$ are divisors of a ($arith\ k$).

Thus, we can write the following definition.

Definition 15 (*k-arithmetic prime*)**.** An integer $p > 1$ is called a prime ($arith\ k$), or simply a $k - prime$, if it has only two divisors ($arith\ k$).
An integer greater than 1 that is not a prime ($arith\ k$) is termed a composite ($arith\ k$).

With Lemma (13), it is easy to characterize the primes ($arith\ k$).

Theorem 16 (*Fundamental k-arithmetic theorem*)**.** *Given a k-arithmetic, the k-primes are:*

1. *The usual primes if* $k \in E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$.
2. *The powers of two if* $k \in O = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$.

Proof. 1. By Lemma (13), if $k \in E$, then $d \mid a \Leftrightarrow d \mid a$ ($arith\ k$):

p usual prime $\Leftrightarrow 1 \mid p$ & $p \mid p \Leftrightarrow 1 \mid p$ ($arith\ k$) & $p \mid p$ ($arith\ k$) $\Leftrightarrow p$ is prime ($arith\ k$).

2. $k \in O$. If $a > 1$ is not a power of two, then $a = 2^s \cdot b$ (b odd and $s \in \{0, 1, 2, \ldots\}$). By Lemma (13), $b \mid a$ ($arith\ k$). In conclusion, $1, 2a, b$ are divisors of a ($arith\ k$); hence, a is not prime ($arith\ k$).

$k \in O$. If $a > 1$ is a power of two, then $a = 2^s$ ($s \in \{1, 2, \ldots\}$). By Lemma (13), the divisors of a ($arith\ k$) are the usual divisors of $2a$ except the even usual divisors of a. Hence, the divisors of a ($arith\ k$) are:

$$\{1,\ \cancel{2},\ \cancel{2^2}, \ldots,\ \cancel{2^s},\ 2 \cdot 2^s\}.$$

1 and 2^{s+1} are the unic divisors of a ($arith\ k$); thus, a is prime ($arith\ k$). ☐

We do not consider representation of an integer as a product of primes ($arith\ k$). Because we are not in a unique factorization domain, we will have cases like the following:

$$15 = 8 \odot_1 2 = [(2 \odot_1 2) \odot_1 (2 \odot_1 2)] \odot_1 (2 \odot_1 2).$$

Now, we can extend Furstenbergs theorem of the infinitude of primes.

4 The extension of Furstenbergs theorem

Attempting to adapt classic arguments about the infinity of primes, we observe an interesting extension of Furstenbergs theorem of the infinitude of primes. We adapt the version of the original proof [8] in [1].

Theorem 17. *For all integer k, there are infinitely many primes on $\mathcal{Z}_k$.*

Proof. For each $k \in \mathbb{Z}$, we are going to define a topology on $\mathcal{Z}_k$.
For $a, b \in \mathbb{Z}, a > 0$, we set

$$S_k(a, b) = \{n \odot_k a + b : n \in \mathbb{Z}\}.$$

Each set $S_k(a, b)$ is a infinite arithmetic progression whose difference is a for all k. Let's see it: if we fix a, b and k,

$$n \odot_k a + b = n \cdot a + b' \text{ where } b' = \frac{1}{2}a(a-1)(k-2) + b. \tag{4}$$

Now call a set $U \subseteq \mathbb{Z}$ open if either U is empty, or if to every $h \in U$ there exists some $a, b \in \mathbb{Z}, a > 0$ with $h \in S_k(a, b) \subseteq U$.
Clearly, the union of open sets is open again.
If U_1 and U_2 are open and $h \in U_1 \cap U_2$ with $h \in S_k(a_1, b_1) \subseteq U_1$ and $h \in S_k(a_2, b_2) \subseteq U_2$, that is, $h \in \{\dots, -a_1 + h,\ h\ , a_1 + h, \dots\} \subseteq U_1$ and $h \in \{\dots, -a_2 + h,\ h\ , a_2 + h, \dots\} \subseteq U_2$, then $h \in \{\dots, -a_1 \cdot a_2 + h,\ h\ , a_1 \cdot a_2 + h, \dots\} \subseteq U_1 \cap U_2$. That is, for some b_3, $h \in S_k(a_1 \cdot a_2, b_3) \subseteq U_1 \cap U_2$.
Then, any finite intersection of open sets is open. Thus, for each $k \in \mathbb{Z}$, we have a topology on $\mathcal{Z}_k$. Let us note two facts:
(A) Any nonempty open set is infinite.
(B) Any set $S_k(a, b)$ is closed.
Point (A) is clear. For (B), we observe that $S_k(a, b) = \mathbb{Z} \setminus \bigcup_{i=1}^{a-1} S_k(a, b + i)$. Hence, $S_k(a, b)$ is the complement of an open set.
Now, we consider $\bigcup_p S_k(p, 0)$, where p is prime ($arith\ k$). There are two possibilities:

$$\begin{aligned} k \in E &\Rightarrow \bigcup_{p\,\text{prime (arith k)}} S_k(p, 0) = \mathbb{Z} \setminus \{-1,\ 1\} \\ k \in O &\Rightarrow \bigcup_{p\,\text{prime (arith k)}} S_k(p, 0) = \mathbb{Z} \setminus \{0\} \end{aligned} \tag{5}$$

The first possibility is easy to check. If $k \in E$, primes ($arith\ k$) are the usual ones. Moreover, $S_k(p, 0) = S_2(p, 0)$: if $k \in E$, in Formula (4) we can see that $b' \equiv b \pmod a$.
Since any number $h \neq 1, -1$ has a prime divisor p (remember Lemma (13)) and hence is contained in $S_k(p, 0)$, the first possibility is proved.
If $k \in O$, primes ($arith\ k$) are the powers of two. If $h = \pm\, 2^s \cdot m$, (m *odd*, $s \in \{0, 1, \dots\}$), then $h \in S_k(2^{s+1}, 0)$:

$$S_k(2^{s+1}, 0) = \{n \odot_k 2^{s+1} : n \in \mathbb{Z}\} = \{2^s \cdot (2n + \underbrace{(2^{s+1} - 1)(k-2)}_{odd}) : n \in \mathbb{Z}\}.$$

Additionally, if we suppose that $0 \in S_k(2^t, 0)$ for some $t \in \{1, 2, \ldots\}$, then there must exist $c \in \mathbb{Z}$ such that $c \odot_k 2^t = 0$. However,

$$c \odot_k 2^t = 0 \Leftrightarrow (c - 2^t + 1) \cdot 2^t + \frac{2^t(2^t-1)k}{2} = 0 \Leftrightarrow \underbrace{(2^t-1)(k-2)}_{odd} + \underbrace{2c}_{even} = 0 \ (!)$$

This contradiction proves the second possibility.
Now, if primes ($artith\ k$) were finite, then $\bigcup_p S_k(p, 0)$ would be a finite union of closed sets (by (B)), and hence closed. Consequently, $\{-1,\ 1\}$ and $\{0\}$ would be an open set, in violation of (A). $\square$

Interestingly, the extension of the theorem consists of proving that there are infinitely many powers of two. Regardless, Formula (5) is important and completes or extends Furstenberg's theorem of the infinitude of primes.

As we will see later, the previous argument will be interesting to adopt when we study new arithmetics.

5 Classic problems revisited.

We now study some classic conjectures of number theory. We start with Goldbach's Conjecture and observe an interesting property. If $k \in E$, the conjecture is equivalent in all *k-arithmetics*. If $k \in O$, the conjecture is false. The same result occurs with more important conjectures, which reminds us of what happens in geometry with the postulate of parallels: the concept of "line between two points" depends on the space we are considering. Understanding this idea, the euclidean parallel postulate is solved with simplicity (it can be true or false).

Definition 18 (*k-Goldbach property*)**.** Let $H = \{6, 8, 10, \ldots\}$. We say that $\mathcal{Z}_k$ has the $k-Goldbach$ property, denoted by $\mathcal{Z}_k \vDash G_k$, if for all $h \in H$, there exist p_1, p_2 primes ($arith\ k$) such that $p_1 + p_2 = h$.

Clearly, the usual Golbach conjecture could be translated as follows: $\mathcal{Z}_2 \vDash G_2$. Now, we have an interesting theorem.

Theorem 19 (*Relation with Goldbach's conjecture*)**.** *If Goldbach's conjecture is true in the usual arithmetic, then the conjecture must be true in $\mathcal{Z}_k$, $k \in E$. Additionally, Goldbach's conjecture is false in $\mathcal{Z}_k$, $k \in O$. That is:*

- *If $k \in O$, then $\mathcal{Z}_k \vDash \neg G_k$.*
- *If $k \in E$, then $\mathcal{Z}_2 \vDash G_2 \Leftrightarrow \mathcal{Z}_k \vDash G_k$.*

Proof. The result is an obvious consequence of the "Fundamental *k-arithmetic* theorem" (16). If $k \in O$, primes (*arith* k) are the powers of two, and the conjecture is clearly false. For instance, 14 in not the sum of two powers of two. If $k \in E$, primes (*arith* k) are the usual primes, and the conjecture is equivalent in $\mathcal{Z}_k,\ k \in E$. $\square$

This is an important fact because if $k \neq 2$, then $\odot_{_k}$ is not commutative and not associative.

We could have approached this result in a different way:

Definition 20 ($k-$*Goldbach property**)**.** We say that $\mathcal{Z}_k$ has the $k-$*Goldbach property**, denoted by $\mathcal{Z}_k \ \vDash G_k^*$, if for all $a \odot_k 2 > 4$, $a \in \mathbb{Z}$, there exist p_1, p_2 primes (*arith* k) such that $a \odot_k 2 = p_1 + p_2$.

Now, Theorem (19) is satisfied because

$$\begin{array}{l} \text{If } k \in E,\ \{a \odot_k 2 : a \in \mathbb{Z}\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}. \\ \text{If } k \in O,\ \{a \odot_k 2 : a \in \mathbb{Z}\} = \{\ldots, -3, -1, 1, 3, \ldots\}. \end{array} \tag{6}$$

Clearly, if $k \in O$, a usual odd number cannot be obtained as the sum of two powers of two. If $k \in E$, the conjecture is clearly equivalent in $\mathcal{Z}_k,\ k \in E$. Thus, we have:

- If $k \in O$, then $\mathcal{Z}_k \ \vDash \neg G_k^*$.
- If $k \in E$, then $\mathcal{Z}_2 \vDash G_2^* \Leftrightarrow \mathcal{Z}_k \ \vDash G_k^*$.

This type of argument can be made similarly for the twin prime conjecture, Sophie Germain conjecture and Euclid primes conjecture. We leave the formalization of these problems for another time and focus on the Collatz conjecture.

Problem 21 (*Relation with Collatz conjecture*)**.** The *k-Collatz problem* is similar to the usual one but uses $\odot_k$ and $\oslash_k$.

$$f_k(n) = \begin{cases} n \oslash_k 2, & \text{if 2 is a divisor of } n \ (arith\ k) \\ n \odot_{_k} 3 + 1, & \text{if 2 is not a divisor of } n \ (arith\ k) \end{cases}$$

We consider the orbit of an integer n: $f_k(n) \to f_k(f_k(n)) \to \ldots$

Example 22. In this example, we consider the 17-orbit varying k.

• $k = 2$ (The usual Collatz conjecture): $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to \mathbf{4} \to 2 \to 1 \to \mathbf{4}\ \ldots$

• $k = 6$ (There is a cycle of length 8): $17 \to 64 \to 30 \to 13 \to 52 \to 24 \to 10 \to 3 \to 22 \to 9 \to 40 \to 18 \to 7 \to \mathbf{34} \to 15 \to 58 \to 27 \to 94 \to 45 \to 148 \to 72 \to \mathbf{34}\ \ldots$

• $k = 1700$ (There is a cycle of length 1124): $17 \to 5146 \to 1724 \to 13 \to 513 \to 1718 \to 10 \to -844 \to -1271 \to 1282 \to^{(20 \overset{steps)}{\ldots}} \to \mathbf{3730} \to^{(1123 \overset{steps)}{\ldots}} \to \mathbf{3730}\ \ldots$

- $k = 1$ (The orbit diverges): $17 \to 9 \to 5 \to 3 \to 2 \to 4 \to 10 \to 28 \to 82 \to 244 \to 730 \to 2188 \to 6562\ \ldots$

- $k = 5$ (The orbit diverges): $17 \to 7 \to 2 \to 16 \to 58 \to 184 \to 562 \to 1696 \to 5098 \to 15304 \to 45922\ \ldots$

- $k = 17$ (The orbit diverges): $17 \to 1 \to -7 \to -11 \to -13 \to -14 \to 4 \to 58 \to 220 \to 706 \to 2164\ \ldots$

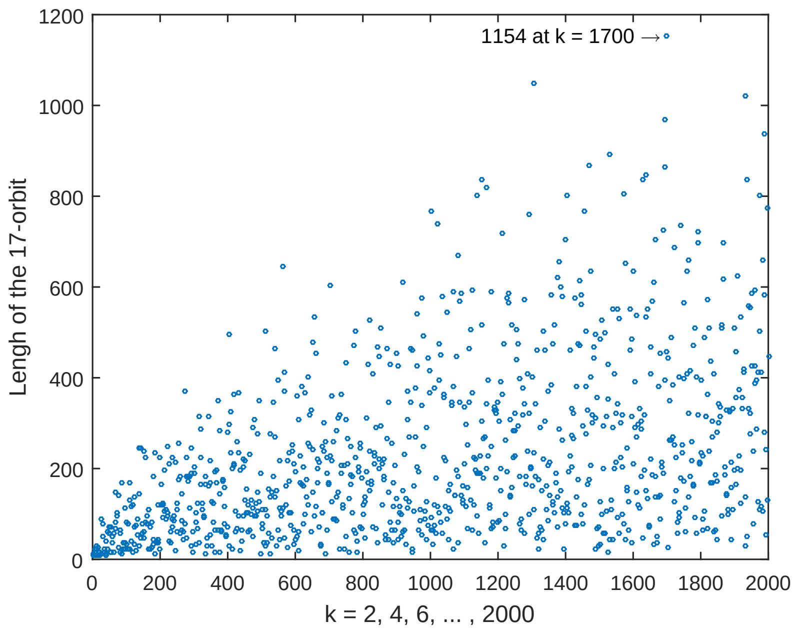

Figure 1: Length of the 17-orbit ($k \in E$).

The previous figure represents the length of the 17-orbit when $k \in E$. We can see that the length, depending of $k \in E$, is not trivial.
None of these sequences appear in OEIS [12].
Example (22) shows that if $k \in O$, the $n - orbit$ diverges. This result is easy to prove:
If $k \in O$ and $n_1 \in E$, then $2 \nmid n_1$ $(arith\ k)$; see (6). Suppose $n_1 = 2 \cdot a$ and $k = 2 \cdot b + 1$; then, $f_k(n_1) = n_1 \odot_k 3 + 1 = 6a + 6b - 2 = n_2 \in E$. Hence, $f_k(n_2) = n_2 \odot_k 3 + 1 \in E$. The $n_1 - orbit$ will then be:

$$\underbrace{n_1}_{\in E} \to \underbrace{n_1 \odot_k 3 + 1}_{n_2 \in E} \to \underbrace{n_2 \odot_k 3 + 1}_{n_3 \in E} \to \ldots \tag{7}$$

Clearly, the sequence diverges. We have only one exception: $f_k(n_s) = f_k(n_{s+1})$ if $n_s = \frac{5}{2} - \frac{3}{2}k = n_{s+1} = n_1$. In this case, $f_k(\frac{5}{2} - \frac{3}{2}k) = \frac{5}{2} - \frac{3}{2}k$.

Thus, if $k \in O$, $n \in E$ and $n \neq \frac{5}{2} - \frac{3}{2}k$, then the $n-orbit$ diverges.
If $k, n_1 \in O$, then $f_k(n_1) = n_1 \oslash_k 2 = \frac{1}{2}n_1 - \frac{1}{2}k + 1 = n_2$. The only possibility for the sequence to converge is that $n_2 \in O$. Then, $f_k(n_2) = n_2 \oslash_k 2 = \frac{1}{4}n_1 - \frac{3}{4}k + \frac{3}{2} = n_3$. Similarly to n_2, the only possibility for the sequence to converge is that $n_3 \in O$. In s-steps, we have:

$$f_k(n_s) = n_s \oslash_k 2 = \frac{1}{2^s}n_1 - \frac{2^s - 1}{2^s}k + \frac{2^s - 1}{2^{s-1}}. \tag{8}$$

We observe that the sequence arrives to $2 - k$ in an infinite number of steps ($s \to \infty$). However, $(2-k) \oslash_k 2 = 2 - k$, then the cycle approaches $2 - k$ with an infinite sequence of odd numbers, which is impossible. Then, the sequence goes through an even number; hence, the $n_1 - orbit$ diverges. Similar to the previous case, we have an exception: $f_k(n_s) = f_k(n_{s+1})$ if $n_s = 2 - k = n_{s+1} = n_1$. In this case, $f_k(2-k) = 2 - k$.
Therefore, if $k, n \in O$ and $n \neq 2 - k$, then the $n - orbit$ diverges.
Example (22) also suggests a conjecture:

Conjecture 23. If $k \in E$ and n is an integer, the $n - orbit$ is periodic.

The careful analysis in this section indicates that there are fundamental number properties P that are equivalent on $\mathcal{Z}_k,\ k \in E$ and false on $\mathcal{Z}_k,\ k \in O$. A small modification in the definition of the product in Peano arithmetic, see Proposition (3), leads to this suggestion, which should be studied with caution.

6 Extension of the main idea

In this paper, we have seen that the usual product can be generated by the sequence $(a_n) = 2, 2, \ldots, 2$. With the same idea, we have considered other product mappings $\odot_k$ generated by the sequences $(a_n) = k, k, \ldots, k,\ k \in \mathbb{Z}$. Following the same steps as in the previous sections, given an integer sequence $(a_n) = a_1, a_2, \ldots, a_n$, we can define the product generated as follows:

Definition 24 (*Product $\odot_{a_n}$, generated by (a_n)*)**.** Given $m \in \mathbb{Z}$ and an integer sequence $(a_n) = a_1, a_2, \ldots, a_n$, for all positive integer n, we define the expression

$$m \odot_{a_n} n = (m - n + 1) + (m - n + 1 + a_1) + \ldots + (m - n + 1 + a_1 + \ldots + a_{n-1}) \tag{9}$$

as the product associated with (a_n).

Clearly, $m \odot_{a_n} n = (m - n + 1)n + (n-1)a_1 + (n-2)a_2 + \ldots + 1 \cdot a_{n-1}$. Thus, we can write Definition (24) as follows:

$$m \odot_{a_n} n = (m - n + 1)n + \sum_{i=1}^{n-1} (n - i) \cdot a_i. \tag{10}$$

Initially, the definition applies to only positive integer n. However, if we can obtain a formula for $\sum_{i=1}^{n-1}(n-i)\cdot a_i$, we can easily extend the product for all integers, just like we did in (2).

Similarly, we can define the divisor of an integer and the quotient generated by a sequence (a_n).

Definition 25 (*Divisors of an integer generated by* (a_n))**.** An integer $d>0$ is called a divisor of an integer a, generated by an integer sequence (a_n) or, simply, d is a divisor of a $(arith\ a_n)$, if there exists some integer b such that $a = b \odot_{a_n} d$. We can write:

$$d \mid a\ (arith\ a_n) \ \Leftrightarrow\ \exists b \in \mathbb{Z}\ such\ that\ b \odot_{a_n} d = a.$$

Definition 26 (*Quotient* $\oslash_{a_n}$*, generated by* (a_n))**.** Given $a, b \in \mathbb{Z}$ $(b \neq 0)$, an integer c is called the quotient of a divided by b generated by an integer sequence (a_n) if and only if $c \odot_{a_n} b = a$. We write:

$$a \oslash_{a_n} b = c \Leftrightarrow c \odot_{a_n} b = a.$$

As in Proposition (9), we can study the new quotient with the usual quotient.

Proposition 27. *Given $a, k \in \mathbb{Z}$, an integer sequence (a_n) and a positive integer b,*

$$a \oslash_{a_n} b = \frac{1}{b}\cdot(a - \sum_{i=1}^{b-1}(b-i)a_i) + b - 1. \tag{11}$$

Proof. The proof is similar to that of Proposition (9):

$a \oslash_{a_n} b = c \Leftrightarrow c \odot_{a_n} b = a \Leftrightarrow (c-b+1)b + \sum_{i=1}^{b-1}(b-i)a_i = a \Leftrightarrow c = \frac{1}{b}\cdot(a - \sum_{i=1}^{b-1}(b-i)a_i) + b - 1$ $\square$

Now, similarly to (11), we obtain the following corollary.

Corollary 28. *$a \oslash_{a_n} b$ is an integer $\Leftrightarrow$ b is a divisor of a $(arith\ a_n)$.*

Finally, we obtain the following definition.

Definition 29 (*Prime generated by* (a_n))**.** An integer $p > 1$ is called a prime $(arith\ a_n)$, or simply a $(a_n)-prime$, if it has only two divisors $(arith\ a_n)$. An integer greater than 1 that is not a prime $(arith\ a_n)$ is termed composite $(arith\ a_n)$.

As shown in the previous sections, in any $k-arithmetic$, an integer has at least two divisors $(arith\ k)$. Now, there may be a sequence that generate an arithmetic where all numbers $a > 1$ have at least three divisors $(arith\ a_n)$. Therefore, in this case, it is interesting to consider the set of integers $p > 1$ with exactly three divisors $(arith\ a_n)$.

In the following lines, we evaluate, via an algebraic computation system, the previous results. We offer some interesting examples of arithmetics generated by sequences and attempt to study the primes, the squares and an analogue of Formula (5) of Theorem (17) in the simplest cases.

Example 30. We start with an arithmetic progression whose first term is a and whose difference is b: $a,\ a+b,\ a+2b,\ \ldots$. In this case, we can obtain an explicit formula for the product $m \odot_{a_n} n$:

$$m \odot_{a_n} n := (m-n+1)\cdot n + \frac{n\cdot(n-1)\cdot a}{2} + \frac{n\cdot(n-1)\cdot(n-2)\cdot b}{6} \tag{12}$$

When the sequence (a_n) is generated by a polynomial, we can always find a formula similar to (12). We can even consider the Euler-Maclaurin summation formula in (10) and study the arithmetics generated by functions. However, we leave this approach for another time. Varying a and b, we obtain the following results:

1. If $a \in O$ and $b \equiv 0 \pmod 3$, then the primes ($arith\ a_n$) are:

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,\ 1024,\ \ldots$$

The powers of 2. A000079.

2. If $a \in O$ and $b \equiv 1 \pmod 3$, then the primes ($arith\ a_n$) are:

$$2,\ 6,\ 8,\ 18,\ 24,\ 32,\ 54,\ 72,\ 96,\ 128,\ 162,\ 216,\ 288,\ 384,\ \ldots \tag{13}$$

The sequence completes the following set: $\{2^{2s-1}\cdot 3^{t-1} : s,t \in \mathbb{N}\}$.

3. If $a \in O$ and $b \equiv 2 \pmod 3$, then the primes ($arith\ a_n$) are: $\{\emptyset\}$.
All integer $p > 1$ is composite ($arith\ a_n$).

4. If $a \in E$ and $b \equiv 0 \pmod 3$, then the primes ($arith\ a_n$) are:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ \ldots$$

The usual primes. A000040.

5. If $a \in E$ and $b \equiv 1 \pmod 3$, then the primes ($arith\ a_n$) are:

$$3,\ 9,\ 27,\ 81,\ 243,\ 729,\ 2187,\ 6561,\ 19683,\ \ldots$$

The powers of 3. A000244.

6. If $a \in E$ and $b \equiv 2 \pmod 3$, then the primes ($arith\ a_n$) are:

$$7,\ 13,\ 19,\ 21,\ 31,\ 37,\ 39,\ 43,\ 57,\ 61,\ 63,\ 67,\ 73,\ 79,\ \ldots \tag{14}$$

The sequence completes the following set: $\{3^{s-1}\cdot p : s \in \mathbb{N},\ p$ usual prime of the form $6s+1\}$.

In 3., $a \in O$ and $b \equiv 2 \pmod 3$, all numbers have at least three divisors ($arith\ a_n$): if $n = 2^s \cdot h$ (h odd, $s = 0, 1, 2, \ldots$) $\Rightarrow 1,\ 2^{s+1},\ 6n$, are divisors of n ($arith\ a_n$). If we consider the set of integers greater than one with exactly three divisors, we obtain:

$$3,\ 4,\ 9,\ 12,\ 16,\ 27,\ 36,\ 48,\ 64,\ 81,\ 108,\ 144,\ 192,\ 243,\ \dots \tag{15}$$

The sequence completes the following set: $\{3^i \cdot 4^j > 1 : i, j \in \mathbb{N}\}$. See A025613.

Now, we can consider an analogue of Formula (5) of Theorem (17) in each case.

We use the notation $\bigcup_p \mathbb{Z} \odot p$, where p is prime ($arith\ a_n$).

1. If $a \in O$ and $b \equiv 0 \pmod 3$: $\displaystyle\bigcup_p \mathbb{Z} \odot p \ = \ \mathbb{Z} \setminus \{0\}$.

2. If $a \in O$ and $b \equiv 1 \pmod 3$:

$$\begin{aligned} \bigcup_p \mathbb{Z} \odot p \ &= \ \mathbb{Z} \setminus \{\dots, -18, -14, -10, -8, -6, -2, 0, 2, 6, 8, 10, 14, 18 \dots\} = \\ &= \ \mathbb{Z} \setminus \big\{\{2^{2s-1} \cdot (2t+1) : s \in \mathbb{N},\ t \in \mathbb{Z}\} \ \cup \ \{0\}\big\}. \end{aligned} \tag{16}$$

Related with the numbers whose binary representation ends in an odd number of zeros. See A036554.

3. If $a \in O$ and $b \equiv 2 \pmod 3$: $\displaystyle\bigcup_p \mathbb{Z} \odot p \ = \ \{\emptyset\}$.

4. If $a \in E$ and $b \equiv 0 \pmod 3$: $\displaystyle\bigcup_p \mathbb{Z} \odot p \ = \ \mathbb{Z} \setminus \{-1, 1\}$.

5. If $a \in E$ and $b \equiv 1 \pmod 3$:

$$\begin{aligned} \bigcup_p \mathbb{Z} \odot p \ &= \mathbb{Z} \setminus \{\dots, -10, -9, -7, -4, -3, -1, 0, 2, 5, 6, 8, 11, 14, \dots\} = \\ &= \mathbb{Z} \setminus \big\{\{3^{s-1} \cdot (3t+2) : s \in \mathbb{N},\ t \in \mathbb{Z}\} \ \cup \ \{0\}\big\}. \end{aligned} \tag{17}$$

6. If $a \in E$ and $b \equiv 2 \pmod 3$:

$$\begin{aligned} \bigcup_p \mathbb{Z} \odot p &= \mathbb{Z} \setminus \{\dots, -8, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 8, \dots\} = \\ &= \{t \cdot p : t \in \mathbb{Z},\ p \text{ usual prime of the form } 6s+1\}. \end{aligned} \tag{18}$$

Related with A230780.

The next step is to consider the arithmetic generated by a 2-degree polynomial. We could classify the results, as in Example (30), and finally, we could try to find a general theorem for an arithmetic generated by any polynomial. We postpone this work to another time. Notably, there are some very interesting connections. For instance, if we consider the sequence, $(a_n) = 1, 6, 21, \ldots$, generated by the polynomial, $p(x) = 1 + 5x^2$, (see A212656), then the primes ($arith\ a_n$) are:

$$2,\ 4,\ 6,\ 12,\ 18,\ 36,\ 54,\ 108,\ 162,\ 324,\ \ldots \tag{19}$$

which could also have been obtained with a chessboard. See A068911.

To finish the paper, motivated by Theorem (4), we study the squares of some arithmetics generated by sequences.

Example 31. In the following cases, we compute $S_{a_n} = \{a \odot_{a_n} a : a \in \mathbb{N}\}$.

1. $(a_n) = 0, 1, 2, 3, 4, \ldots$ (*A001477*) $\Rightarrow$ $S_{a_n} = \{1, 2, 4, 8, 15, 26, 42, 64, 93, 130 \ldots\}$. The "cake numbers" appear. See A000125.

2. $(a_n) = 1, 2, 3, 4, \ldots$ (*A000027*) $\Rightarrow$ $S_{a_n} = \{1, 3, 7, 14, 25, 41, 63, 92, 129, \ldots\}$. The 3-dimensional analogue of centered polygonal numbers appear. This sequence is very interesting. See A004006.

Now, we can say that the "cake numbers" are the 3-dimensional analogues of centered polygonal numbers plus one.

3. $(a_n) = 1, 2, 4, 8, \ldots$ (*A000079*) $\Rightarrow$ $S_{a_n} = \{1, 3, 7, 15, 31, 63, \ldots\}$. Mersenne numbers appear. See A000225. In this concrete arithmetic, none of the Mersenne numbers is prime ($arith\ a_n$).

4. $(a_n) = 2, 3, 5, 7, \ldots$ (usual primes, *A000040*) $\Rightarrow$ $S_{a_n} = \{1, 4, 10, 21, 39, 68, \ldots\}$. Convolution of natural numbers with $(1, p(1), p(2), \ldots)$, where $p(s)$ is the s-th prime. See A023538.

5. $(a_n) = 1, -1, 1, -1, \ldots$ (*A033999*) $\Rightarrow$ $S_{a_n} = \{1, 3, 4, 6, 7, 9, 10, 12 \ldots\}$. Numbers that are congruent to 0 or 1 (mod 3). See A032766.

In this example is interesting to consider the set of cubes $C_{a_n} = \{(a \odot_{a_n} a) \odot_{a_n} a :\ a \in \mathbb{N}\}$.

$C_{a_n} = \{1, 5, 7, 14, 17, 27, 31, 44, 49, 65, \ldots\}$.

Maximum number of intersections in self-intersecting n-gon. See A105638.

6. $(a_n) = 0, 1, 0, 1, 0, 1, \ldots$ (*A000035*) $\Rightarrow$ $S_{a_n} = \{1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, \ldots\}$. Quarter-squares. See A002620.

$C_{a_n} = \{1, 2, 7, 14, 29, 48, 79, 116, 169, 230, \ldots\}$. Number of paraffins. See A005998.

7. $(a_n) = 1,\ -1,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1,\ldots$
The number of zeros in the sequence follow the pattern $2^n - 1$.

$$S_{a_n} = \{1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, \ldots\}.$$

This sequence coincides with the "pancake numbers". See A058986. This coincidence deserves attention.

As we can see, the OEIS [12] has been fundamental in this part of the work.

7 Conclusion

In this paper, we have generalized the Peano arithmetic usual product to $\odot_k$. This fact has allowed us to extend Furstenberg's theorem of the infinitude of primes. The study of the arithmetic generated by $\odot_k$ is interesting. For instance, the fundamental *k-arithmetic* theorem makes the powers of two appear to be primes of other arithmetics. Additionally, new versions of the classical conjectures of number theory are obtained that are connected with the usual ones. This point deserves attention. Finally, the extension of the main idea invites us to consider the arithmetic generated by any integer sequence. The number of new sequences and connections that appear is enormous; hence, more work related to this topic is necessary.

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(Concerned with sequences A000027 , A000035, A000040, A000079, A000125, A000225, A000244, A001477, A002620, A004006, A005998, A023538, A025613, A032766, A033999, A036554, A049988, A058986, A068911, A105638, A212656, and A230780.)

Notes for helping the referee.

With these programs, you can check the paper's results. Please, copy and paste in a Maple worksheet or download the maple document "An extension of Furstenberg T. Highlights.mw" from the Google site https://sites.google.com/site/extensionfurstenbergprimes/

2. Basic definitions and properties.

Theorem 4. (k-arithmetic polygonal theorem).

```
> restart:
```

The *k-arithmetic* product generalizes the usual product in the following way: the operation *square of a number* is a square (polygon) in the usual arithmetic but it is a triangle in $1-arithmetic$, a pentagon in $3-arithmetic$, a hexagon in $4-arithmetic$ etc.

```
> multi:=proc(m,n,k)
> global p:
> p:=(m-n+1)*n+n*(n-1)*k/2:
> end:
```

Multi program: This program evaluates the $k-arithmetic$ product, $\odot_k$. For instance, if you want to calculate $3 \odot_1 5$, you must execute: **multi(3,5,1): p;**

```
> multi(3,5,1):  p;
```

$$5$$

Theorem 4. (*k-arithmetic polygonal theorem*).
The product $n \odot_k n$ *is the nth* $(k+2)$*-gonal number.*

```
> multi(5,5,1):  p;
```

$$15$$

$5 \odot_1 5 = 15$. That is to say, the 5th triangular number is 15.

```
> multi(4,4,3):  p;
```

$$22$$

$4 \odot_3 4 = 22$. That is to say, the 4th pentagonal number is 22.

Proposition 5.

Given $a, b, c, d, k \in \mathbb{Z}$, the following properties are satisfied:

1. *The $k-$arithmetic product is not associative in general.*

```
> multi(2,3,1): multi(p,4,1): p;
```

$$6$$

In the previous line, we have evaluated $(2 \odot_1 3) \odot_1 4 = 6$.

```
> multi(3,4,1): multi(2,p,1): p;
```

$$-3$$

In the previous line, we have evaluated $2 \odot_1 (3 \odot_1 4) = -3$.

2. *The k-arithmetic product is not commutative in general but* $a \odot_k (1-a) = (1-a) \odot_k a$.

```
> multi(a,1-a,k): simplify(p);
```

$$-2\,a^2 + 2\,a + 1/2\,a^2k - 1/2\,ak$$

```
> multi(1-a,a,k): simplify(p);
```

$$-2\,a^2 + 2\,a + 1/2\,a^2k - 1/2\,ak$$

3. $(a-b) \cdot (c+d) = a \odot_k c + a \odot_k d - b \odot_k c - b \odot_k d$.

```
> multi(a,c,k): p1:=p:
> multi(a,d,k): p2:=p:
> multi(b,c,k): p3:=p:
> multi(b,d,k): p4:=p:
> simplify(p1+p2-p3-p4);
```

$$ac + ad - bc - bd$$

4. $(a+b) \odot_k (a+b) = a \odot_k a + b \odot_k b + k \cdot a \cdot b$.

```
> multi(a+b,a+b,k): simplify(p);
```

$$a + b + 1/2\,a^2k + abk + 1/2\,b^2k - 1/2\,ak - 1/2\,bk$$

```
> multi(a,a,k): p1:=p:
> multi(b,b,k): p2:=p:
> simplify(p1+p2+k*a*b);
```

$$a + b + 1/2\,a^2k + abk + 1/2\,b^2k - 1/2\,ak - 1/2\,bk$$

5. $(a-b)^2 = a \odot_k a + b \odot_k b - b \odot_k a - a \odot_k b$.

```
> multi(a,a,k):  p1:=p:
> multi(b,b,k):  p2:=p:
> multi(b,a,k):  p3:=p:
> multi(a,b,k):  p4:=p:
> simplify(p1+p2-p3-p4);
```

$$a^2 - 2\,ab + b^2$$

6. $a \odot_k (-b) = (k-2-a) \odot_k b$.

```
> multi(a,-b,k):  simplify(p);
```

$$-ab - b^2 - b + 1/2\,b^2k + 1/2\,bk$$

```
> multi(k-2-a,b,k):  simplify(p);
```

$$-ab - b^2 - b + 1/2\,b^2k + 1/2\,bk$$

3. Divisors and primes.

Lemma 13 & ***Theorem 16.***

```
> quotient:=proc(a,b,k)
> global q:
> q:=a/b + (b-1)*(1-(k/2)):
> end:
```

Quotient program: This program evaluates the $k-arithmetic$ quotient, $\oslash_k$. For instance, if you want to calculate $17 \oslash_1 2$, you must execute: **quotient(17,2,1): q;**

```
> quotient(17,2,1):  q;
```

$$9$$

```
> kdivisors := proc (n, k)
> local i, j, aux:
> global numkdiv, KD:
> numkdiv:=0:
> for i to (2*n) do
> quotient(n, i, k):
> if type(q, integer) then
> numkdiv:=numkdiv+1:  aux(numkdiv):= i:
> fi
> od:
> KD := array(1 ..  numkdiv):
> for j to numkdiv do KD[j] := aux(j):
> od:
> end:
```

Kdivisors program: This program calculates the number of divisors (*arith* k). For instance, if we want to calculate the divisors of 20 (*arith* 3), we must execute:
kdivisors(20,3): print(KD): numkdiv;

```
> kdivisors(20,3):  print(KD): numkdiv;
```

$$\begin{bmatrix} 1 & 5 & 8 & 40 \end{bmatrix}$$
$$4$$

Lemma 13. *Given a k-arithmetic and $a \in \mathbb{Z}$, the divisors of a (arith k) are:*
1. The usual divisors of a if $k \in E$.

```
> kdivisors(40,4):  print(KD): numkdiv;
```

$$\begin{bmatrix} 1 & 2 & 4 & 5 & 8 & 10 & 20 & 40 \end{bmatrix}$$
$$8$$

2. The usual divisors of $2a$ except the even usual divisors of a if $k \in O$.

```
> kdivisors(40,3):  print(KD): numkdiv;
```

$$\begin{bmatrix} 1 & 5 & 16 & 80 \end{bmatrix}$$
$$4$$

```
> iskprime:=proc(n,k)
> global isp:
> isp := 0:
> kdivisors(n,k):
> if numkdiv = 2 then isp:=1:
> fi:
> if n=1 then isp:=0:
> fi:
> end:
```

Iskprime program: This program evaluates whether a number is prime (*arith* k) or not. For instance, if we want to know if 17 is prime (*arith* 1), we must execute:
iskprime(17,1): isp;
If the value is 1, the number is prime (*arith* k). If the value is 0, the number is not prime (*arith* k).

```
> iskprime(17,1):  isp;
```

$$0$$

```
> kprimesless := proc (n,k)
> local i, j, t, Q;
> global P,npl:
> j := 1:
> npl:=0:
> for i to n-1 do
> iskprime(i,k):
> if isp=1 then Q(j):=i:  j:=j+1:
> fi:
> od:
> P := array(1 ..  (j-1)):
> for t to (j-1) do
> P[t]:=Q(t):
> npl:=j-1:
> od:
> end:
```

Kprimesless program: This program calculates the number of primes (*arith* k) less than a number. For instance, if we want to know the primes less than 85 (*arith* 2), that is, the usual primes, we must execute:
kprimesless(85,2): print(P); npl;

```
> kprimesless(85,2):  print(P); npl;
```

$$[\ 2\ \ 3\ \ 5\ \ 7\ \ 11\ \ 13\ \ 17\ \ 19\ \ 23\ \ 29\ \ 31\ \ 37\ \ 41\ \ 43\ \ 47\ \ 53\ \ 59\ \ 61\ \ 67\ \ 71\ \ 73\ \ 79\ \ 83\ ]$$

$$23$$

Theorem 16. *(Fundamental k-arithmetic theorem).*
Given a k-arithmetic the primes (arith k) are:

1. *The usual primes if* $k \in E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$.
2. *The powers of two if* $k \in O = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$.

```
> kprimesless(85,4): print(P); npl;
```

$$[\; 2 \;\; 3 \;\; 5 \;\; 7 \;\; 11 \;\; 13 \;\; 17 \;\; 19 \;\; 23 \;\; 29 \;\; 31 \;\; 37 \;\; 41 \;\; 43 \;\; 47 \;\; 53 \;\; 59 \;\; 61 \;\; 67 \;\; 71 \;\; 73 \;\; 79 \;\; 83 \;]$$

$$23$$

```
> kprimesless(85,1): print(P); npl;
```

$$[\; 2 \;\; 4 \;\; 8 \;\; 16 \;\; 32 \;\; 64 \;]$$

$$6$$

4. The extension of Furstenberg's theorem.

```
> S:=proc(a,b,k)
> global M1:=array(1..21):
> local i:
> for i to 21 do
> multi(i-11,a,k):
> M1[i]:=p+b:
> od:
> end:
```

S program: This program allows us to study the set $S_k(a, b)$. It shows us:

$$\{n \odot_k a + b : n = -10, -9, -8, \ldots, -1, 0, 1, 2, \ldots 10\}.$$

If you want to calculate $S_1(3, 2)$, you must execute:
S(3,2,1): print(M1);

```
> S(3,2,1): print(M1);
```

$$[\; -28 \;\; -25 \;\; -22 \;\; -19 \;\; -16 \;\; -13 \;\; -10 \;\; -7 \;\; -4 \;\; -1 \;\; 2 \;\; 5 \;\; 8 \;\; 11 \;\; 14 \;\; 17 \;\; 20 \;\; 23 \;\; 26 \;]$$

Theorem 17. *For all integer k, there are infinitely many primes in $\mathcal{Z}_k$.*

$$k \in E \Rightarrow \bigcup_{p\,\text{prime (arith k)}} S_k(p,0) = \mathbb{Z} \setminus \{-1,\ 1\}$$

If $k \in E$, $S_k(p,0) = S_2(p,0)$.

```
> S(5,0,2): print(M1);
```

$$[\ -45\ \ -40\ \ -35\ \ -30\ \ -25\ \ -20\ \ -15\ \ -10\ \ -5\ \ 0\ \ 5\ \ 10\ \ 15\ \ 20\ \ 25\ \ 30\ \ 35\ \ 40\ \ 45\ ]$$

```
> S(5,0,4): print(M1);
```

$$[\ -30\ \ -25\ \ -20\ \ -15\ \ -10\ \ -5\ \ 0\ \ 5\ \ 10\ \ 15\ \ 20\ \ 25\ \ 30\ \ 35\ \ 40\ \ 45\ \ 50\ \ 55\ \ 60\ \ 65\ \ 70\ ]$$

$$k \in O \Rightarrow \bigcup_{p\,\text{prime (arith k)}} S_k(p,0) = \mathbb{Z} \setminus \{0\}$$

$k \in O$. If $h = \pm\, 2^s \cdot m$, (m *odd*, $s \in \{0,1,\ldots\}$) then $h \in S_k(2^{s+1},0)$.

For example: imagine that we are working on $\mathcal{Z}_1$ and we want to know wich $S_1(2^t,0)$ contains the number 7.
$7 = 2^0 \cdot 7$, hence $7 \in S_1(2^1,0)$.

```
> S(2,0,1): print(M1);
```

$$[\ -21\ \ -19\ \ -17\ \ -15\ \ -13\ \ -11\ \ -9\ \ -7\ \ -5\ \ -3\ \ -1\ \ 1\ \ 3\ \ 5\ \ 7\ \ 9\ \ 11\ \ 13\ \ 15\ \ 17\ \ 19\ ]$$

If we want to calculate where is the number 40: $40 = 2^3 \cdot 5 \Rightarrow 40 \in S_1(2^4,0)$.

```
> S(16,0,1): print(M1);
```

$$[\ -200\ \ -184\ \ -168\ \ -152\ \ -136\ \ -120\ \ -104\ \ -88\ \ -72\ \ -56\ \ -40\ \ -24\ \ -8\ \ 8\ \ 24\ \ 40\ ]$$

5. Classic problem revisited.

Formula (6) of Definition (20).

If $k \in E$, $\{a \odot_k 2 : a \in \mathbb{Z}\} = \{\ldots,-4,-2,0,2,4,\ldots\}$.

If $k \in O$, $\{a \odot_k 2 : a \in \mathbb{Z}\} = \{\ldots,-3,-1,1,3,\ldots\}$.

```
> S(2,0,2*s): print(M1);
```

$$[\ -10+2s\ \ -8+2s\ \ -6+2s\ \ -4+2s\ \ -2+2s\ \ 2s\ \ 2+2s\ \ 4+2s\ \ 6+2s\ \ 8+2s\ ]$$

```
> S(2,0,2*s+1): print(M1);
```

$$[\ -9+2s\ \ -7+2s\ \ -5+2s\ \ -3+2s\ \ -1+2s\ \ 2s+1\ \ 3+2s\ \ 5+2s\ \ 7+2s\ \ 9+2s\ ]$$

Collatz Conjecture. Example 22.

```
> collatz:=proc(a,k)
> local collatz, M1, M2, aux, i, j, bound:
> global ns, kcollatzseq:
> bound:=500000:
> M1:=array(1..(2*bound+1)):    M2:=array(1..(2*bound+1)):
> for i to (2*bound+1) do
> M1[i]:=1:
> od:
> ns:=0:
> collatz:=a:
> aux:=1:
> j:=1:
> while aux<>0 do
> if abs(collatz)<bound then
> M2[j]:=collatz:
> if M1[collatz+(bound+1)]=0 then aux:=0:  fi:
> M1[collatz+(bound+1)]:=0:
> quotient(collatz,2,k):
> if type(q,integer) then collatz:=q:
> else multi(collatz,3,k):  collatz:=p+1:
> fi:
> j:=j+1:
> else aux:=0:  print(The_sequence_exceeds_the_bound);
> fi:
> end:
> ns:=j-2:
> kcollatzseq:=array(1..(ns+1)):
> for i to (ns+1) do kcollatzseq[i]:=M2[i]:  od:
> end:
```

Collatz program: This program calculates the Collatz orbit of a number on $\mathcal{Z}_k$. Also it calculates the number of steps of the sequence. For example: if you want to calculate the 6-orbit on usual arithmetic, you must be execute:
collatz(17,2): print(kcollatzseq); ns;

Example 22. In this example, we consider the 17-orbit varying k.

```
> collatz(17,2):  print(kcollatzseq); ns;
```

$$[\ 17\ \ 52\ \ 26\ \ 13\ \ 40\ \ 20\ \ 10\ \ 5\ \ 16\ \ 8\ \ 4\ \ 2\ \ 1\ \ 4\ ]$$
$$13$$

```
> collatz(17,6):  print(kcollatzseq); ns;
```

$$[\ 17\ \ 64\ \ 30\ \ 13\ \ 52\ \ 24\ \ 10\ \ 3\ \ 22\ \ 9\ \ 40\ \ 18\ \ 7\ \ 34\ \ 15\ \ 58\ \ 27\ \ 94\ \ 45\ \ 148\ \ 72\ \ 34\ ]$$
$$21$$

In order to execute collatz(17,1700) we need to increase the 'bound' to 5000000.

```
> collatz(17,1700):  ns;
```

$$1154$$

```
> collatz(17,1):  print(kcollatzseq);
```

$$\mathit{The\_sequence\_exceeds\_the\_bound}$$

$$\left[\begin{array}{cccccccccccccccc} 17 & 9 & 5 & 3 & 2 & 4 & 10 & 28 & 82 & 244 & 730 & 2188 & 6562 & 19684 & 59050 & 177148 \end{array}\right]$$

```
> collatz(17,5):  print(kcollatzseq);
```

$$\mathit{The\_sequence\_exceeds\_the\_bound}$$

$$\left[\begin{array}{ccccccccccccc} 17 & 7 & 2 & 16 & 58 & 184 & 562 & 1696 & 5098 & 15304 & 45922 & 137776 & 413338 \end{array}\right]$$

```
> collatz(17,17):  print(kcollatzseq);
```

$$\mathit{The\_sequence\_exceeds\_the\_bound}$$

$$\left[\begin{array}{ccccccccccccccc} 17 & 1 & -7 & -11 & -13 & -14 & 4 & 58 & 220 & 706 & 2164 & 6538 & 19660 & 59026 & 177124 \end{array}\right]$$

The last example allows us to visualize that if $k \in O$, the orbit diverges:
The $17 - \mathit{cycle}$ is approaching more and more to $2 - k = 2 - 17 = -15$ with a infinite sequence of odd numbers. It is impossible, then the sequence goes through an even number, hence the $17 - \mathit{orbit}$ (arith 17) diverge.

The following plot is the length of the $17-\mathit{orbit}$ with $k \in \{2, 4, ..., 100\}$. We can see that the length, depending of $k \in E$, is not trivial. (Figure 1 on the paper, was made with Matlab).

```
> M3:=array(1..50):  M4:=array(1..50):
> for i to 50 do
> collatz(17,2*i):
> M3[i]:=2*i:
> M4[i]:=ns:
> od:
> plot(Vector(M3), Vector(M4), style=point, symbol=asterisk, color=redmaple);
```

Also, if $k \in O$, we can check the two exceptions:

```
> multi(5/2-3/2*k,3,k):  p+1;
```

$$5/2 - 3/2\,k$$

```
> quotient(2-k,2,k):  q;
```

$$2 - k$$

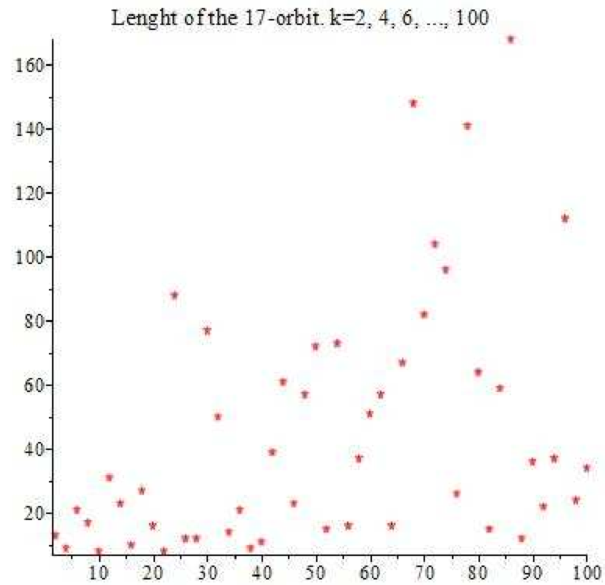

Figure 2: Length of the 17-orbit ($k \in E$).

6. Extension of the main idea.

Example 30* & *31.

```
> restart:

> productsequence := proc (m, n)
> local i, sum:
> global p:
> sum := 0;
> for i to n-1 do
> sum := sum+(n-i)*s[i]:
> od:
> p := (m-n+1)*n+sum:
> end:
```

Productsequence program: This program calculates the product $\odot_{a_n}$, generated by a sequence (a_n):

$$m \odot_{a_n} n = (m-n+1)n + \sum_{i=1}^{n-1}(n-i) \cdot a_i$$

You must first run the sequence and then, execute the program.

```
> apsequence := proc (n,a1,d)
> local i:
> global s:=array(1..n):
> s[1]:=a1:
> for i to (n-1) do s[i+1] := s[i]+d:
> od:
> end:
```

Apsequence program: This program calculates the arithmetic progression whose first term is a_1 and whose difference is d. For instance, if we want to calculate the first twenty terms of an arithmetic progression whose first term is 1 and whose difference is 3 we must execute: **apsequence(20,1,3): print(s);**

```
> apsequence(20,1,3):  print(s);
```

$$[\,1 \quad 4 \quad 7 \quad 10 \quad 13 \quad 16 \quad 19 \quad 22 \quad 25 \quad 28 \quad 31 \quad 34 \quad 37 \quad 40 \quad 43 \quad 46 \quad 49 \quad 52 \quad 55 \quad 58\,]$$

Now, we can use the '**productsequence**' program. First we run the sequence, for example 1000 terms, then we execute the product.
For instance, if we want to calculate $5 \odot_{a_n} 8$ where $(a_n) = 3, 7, 11, 15, \ldots$, we must execute: **apsequence(1000,3,4): productsequence(5,8): p;**

```
> apsequence(1000,3,4):  productsequence(5,8):  p;
```

$$292$$

```
> quotientsequence:=proc(a,b)
> global q:
> local i, sum:
> sum:=0:
> for i to b-1 do
> sum:=sum+(b-i)*s[i]:
> od:
> q:=(a-sum)/(b)+b-1:
> end:
```

Quotientsequence program: This program calculates the quotient $\oslash_{a_n}$, generated by a sequence (a_n).

$$a \oslash_{a_n} b = \frac{1}{b} \cdot (a - \sum_{i=1}^{b-1} (b-i)a_i) + b - 1.$$

You must first run the sequence and then, execute the program.
For instance, if we want to calculate $292 \oslash_{a_n} 8$ where $(a_n) = 3, 7, 11, 15, \ldots$, we must execute: **apsequence(1000,3,4): quotientsequence(292,8): q;**

```
> apsequence(1000,3,4):  quotientsequence(292,8):  q;
```

```
> sequencedivisors := proc (n)
> local i,j,saux:
> global sdivisors, snumdivisors:
> snumdivisors := 0:
> for i to 6*n do  #if the sequence is an arithmetic progression,
> #a divisor of n (arith a_n) is never greater than 6n
> quotientsequence(n, i):
> if type(q, integer) then
> snumdivisors := snumdivisors+1:
> saux(snumdivisors):=i:
> fi:
> od:
> sdivisors:=array(1..snumdivisors):
> for j to snumdivisors do
> sdivisors[j]:=saux(j):
> od:
> end:
```

Sequencedivisors program: This program calculates the number of divisors ($arith\ a_n$). You must first run the sequence and then, execute the program.
For instance, if we want to calculate the divisors of 20 ($arith\ a_n$) where $(a_n) = 1, 3, 5, 7, \ldots$, we must execute:
apsequence(10000,1,2): sequencedivisors(20): print(sdivisors); snumdivisors;

```
> apsequence(10000,1,2):  sequencedivisors(20):
> print(sdivisors); snumdivisors;
```

$$\begin{bmatrix} 1 & 3 & 5 & 8 & 40 & 120 \end{bmatrix}$$

$$6$$

```
> issequenceprime := proc (t)
> global isp:
> isp := 0:
> sequencedivisors(t):
> if snumdivisors = 2 then isp := 1:
> fi:
> if t=1 then isp:=0:
> fi:
> end:
```

Issequenceprime program: This program evaluates whether a number is prime ($arith\ a_n$) or not. You must first run the sequence and then, execute the program.

For instance, if we want to know if 17 is prime ($arith\ a_n$) where $(a_n) = 1, 3, 5, 7, \ldots$, we must execute:
apsequence(10000,1,2): issequenceprime(17): print(isp);
If the value is 1, the number is prime ($arith\ a_n$). If the value is 0, the number is not prime ($arith\ a_n$).

```
> apsequence(10000,1,2):  issequenceprime(17):  print(isp);
```

$$0$$

```
> primesless := proc (t)
> local i, j, l, Q;
> global P,npl:
> j := 1:
> npl:=0:
> for i to t-1 do
> issequenceprime(i):
> if isp = 1 then Q(j):= i:  j:=j+1:
> fi:
> od:
> P := array(1 ..  (j-1)):
> for l to (j-1) do
> P[l]:=Q(l):
> npl:=j-1:
> od:
> end:
```

Primesless program: This program calculates the number of primes ($arith\ a_n$) less than a number. You must first run the sequence and then, execute the program.
For instance, if we want to know the primes less than 100 ($arith\ a_n$) where $(a_n) = 1, 1, 1, 1, \ldots$, we must execute:
apsequence(10000,1,0): primesless(100): print(P); npl;

```
> apsequence(10000,1,0):  primesless(100):  print(P); npl;
```

$$\begin{bmatrix} 2 & 4 & 8 & 16 & 32 & 64 \end{bmatrix}$$

$$6$$

Example 30. Primes (*arith* a_n)**:**
$(a_n) = a,\ a+b,\ a+2b,\ \ldots$. Sequence generated by the polynomial $ax+b$.

1. If $a \in O$ and $b \equiv 0 \pmod 3$ then, primes (*arith* a_n) are:
$\{2,4,8,16,\ldots\} = \{2^s : s \in \mathbb{N}\}$.

```
> apsequence(10000,1,3):  primesless(100):  print(P); npl;
```

$$[\ 2\ \ 4\ \ 8\ \ 16\ \ 32\ \ 64\ ]$$
$$6$$

2. If $a \in O$ and $b \equiv 1 \pmod 3$ then, primes (*arith* a_n) are:
$\{2,6,8,18,24,32,54\ldots\} = \{2^{2s-1} \cdot 3^{t-1} : s,t \in \mathbb{N}\}$.

```
> apsequence(10000,1,1):  primesless(100):  print(P); npl;
```

$$[\ 2\ \ 6\ \ 8\ \ 18\ \ 24\ \ 32\ \ 54\ \ 72\ \ 96\ ]$$
$$9$$

3. If $a \in O$ and $b \equiv 2 \pmod 3$ then, primes (*arith* a_n) are:$\{\emptyset\}$.
All integer $p > 1$ is composite (*arith* a_n).

```
> apsequence(10000,1,2):  primesless(100):  print(P); npl;
```

$$[\,]$$
$$0$$

4. If $a \in E$ and $b \equiv 0 \pmod 3$ then, primes (*arith* a_n) are:
$\{2,3,5,7,11,13,17,19,23,\ldots\}$ (usual primes).

```
> apsequence(10000,2,3):  primesless(85):  print(P); npl;
```

$$[\ 2\ \ 3\ \ 5\ \ 7\ \ 11\ \ 13\ \ 17\ \ 19\ \ 23\ \ 29\ \ 31\ \ 37\ \ 41\ \ 43\ \ 47\ \ 53\ \ 59\ \ 61\ \ 67\ \ 71\ \ 73\ \ 79\ \ 83\ ]$$
$$23$$

5. If $a \in E$ and $b \equiv 1 \pmod 3$ then, primes (*arith* a_n) are:
$\{3,9,27,81,\ldots\} = \{3^s : s \in \mathbb{N}\}$.

```
> apsequence(10000,2,1):  primesless(100):  print(P); npl;
```

$$[\ 3\ \ 9\ \ 27\ \ 81\ ]$$
$$4$$

6. If $a \in E$ and $b \equiv 2 \pmod 3$ then, primes $(arith\ a_n)$ are:
$\{7, 13, 19, 21, 31, 37, 39, 43, 57, 61 \ldots\} = \{3^{s-1} \cdot p : s \in \mathbb{N},\ p$ usual prime of the form $6s+1\}$.

```
> apsequence(10000,2,2):  primesless(100):  print(P); npl;
```

$$[\ 7\ \ 13\ \ 19\ \ 21\ \ 31\ \ 37\ \ 39\ \ 43\ \ 57\ \ 61\ \ 63\ \ 67\ \ 73\ \ 79\ \ 93\ \ 97\ ]$$

$$16$$

```
> furstenbergextension := proc (a1,d)
> local i, j, l, h, zeropos:
> global vectorfursten:=array(1..121):
> apsequence(10000,a1,d):  primesless(200):
> #We initialize the array vectorfursten = [-60,-59, ...,-1,0,1, ..., 60]
> for i to 121 do
> vectorfursten[i]:=i-61:
> od:
> for j to npl do
> h:=P[j]:
> #For cancel in the proper way,
> #zeropos is the number such that (zeropos)  *  p=0
> zeropos:=round(h-1-(1/2)*a1*h+(1/2)*a1-(1/6)*h^2*d+(1/2)*h*d-(1/3)*d):
> #We use the "_" symbol for cancel.
> for l to 61 do
> productsequence(zeropos-31+l,h):
> fursten(l):=p:
> if abs(p)<61 then
> vectorfursten[p+61]:=_:
> fi:
> od:
> od:
> end:
```

Furstenbergextension program: With this program we can check the extension of Furstenberg's theorem when the sequence is an arithmetic progression whose firs term is a_1 and whose difference is d.
The program starts with the array $[-60, -59, \ldots, -1, 0, 1, \ldots, 59, 60]$ and then, it cancels the numbers $\{n \odot_{a_n} p\ :\ n \in \mathbb{Z},\ p\ is\ prime\ (arith\ a_n)\}$.

Example 30. Analogue of Formula (5) of Theorem 17.

1. If $a \in O$ and $b \equiv 0 \pmod 3$:

$$\bigcup_p \mathbb{Z} \odot p \;=\; \mathbb{Z} \setminus \{0\}.$$

```
> furstenbergextension(1,3):  print(vectorfursten);
```

$[\; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; 0 \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \;]$

For a better print of this "pdf document", we abbreviate the previous expression as follow (we do the same in the following points of this example).

$$[\; 0 \;]$$

2. If $a \in O$ and $b \equiv 1 \pmod 3$:

$$\bigcup_p \mathbb{Z} \odot p \;=\; \mathbb{Z} \setminus \big\{\{2^{2s-1} \cdot (2t+1) : s \in \mathbb{N},\; t \in \mathbb{Z}\} \;\cup\; \{0\}\big\}.$$

```
> furstenbergextension(1,1):  print(vectorfursten);
```

$[\; -26 \;\; -24 \;\; -22 \;\; -18 \;\; -14 \;\; -10 \;\; -8 \;\; -6 \;\; -2 \;\; 0 \;\; 2 \;\; 6 \;\; 8 \;\; 10 \;\; 14 \;\; 18 \;\; 22 \;\; 24 \;\; 26 \;\; 30 \;]$

The previous sequence is related with the numbers whose binary representation ends in an odd number of zeros. See A036554.

3. If $a \in O$ and $b \equiv 2 \pmod 3$:

$$\bigcup_p \mathbb{Z} \odot p \;=\; \{\emptyset\}.$$

```
> furstenbergextension(1,2):  print(vectorfursten);
```

$[\; -10 \;\; -9 \;\; -8 \;\; -7 \;\; -6 \;\; -5 \;\; -4 \;\; -3 \;\; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\; 9 \;\; 10 \;]$

4. If $a \in E$ and $b \equiv 0 \pmod 3$:

$$\bigcup_p \mathbb{Z} \odot p \;=\; \mathbb{Z} \setminus \{-1, 1\}.$$

```
> furstenbergextension(2,3):  print(vectorfursten);
```

$$[\; -1 \;\; 1 \;]$$

5. If $a \in E$ and $b \equiv 1 \pmod 3$:

$$\bigcup_p \mathbb{Z} \odot p \;=\; \mathbb{Z} \setminus \big\{\{3^{s-1} \cdot (3t+2) : s \in \mathbb{N},\; t \in \mathbb{Z}\} \;\cup\; \{0\}\big\}.$$

```
> furstenbergextension(2,1):  print(vectorfursten);
```

$$\begin{bmatrix} -16 & -13 & -12 & -10 & -9 & -7 & -4 & -3 & -1 & 0 & 2 & 5 & 6 & 8 & 11 & 14 & 15 & 17 & 18 & 20 \end{bmatrix}$$

6. If $a \in E$ and $b \equiv 2 \pmod 3$:

$$\bigcup_p \mathbb{Z} \odot p = \mathbb{Z} \setminus \{\dots, -8, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, \dots\} =$$
$$= \{t \cdot p : t \in \mathbb{Z},\; p \text{ usual prime of the form } 6s+1\}.$$

Related with A230780.

```
> furstenbergextension(2,2):  print(vectorfursten);
```

$$\begin{bmatrix} -11 & -10 & -9 & -8 & -6 & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 11 \end{bmatrix}$$

```
> squaressequence := proc (t)
> local i:
> global SQ:=array(1..t):
> for i to t do
> productsequence(i,i):
> SQ[i]:=p:
> od:
> end:
```

Squaressequence program: This program calculates the first squares ($arith\ a_n$). You must first run the sequence and then, execute the program.
For instance, if we want to calculate the first ten squares ($arith\ a_n$) where $(a_n) = 2, 5, 8, 11, \dots$ we must execute:
apsequence(10000,2,3): squaressequence(10): print(SQ);

```
> apsequence(10000,2,3):  squaressequence(10):  print(SQ);
```

$$\begin{bmatrix} 1 & 4 & 12 & 28 & 55 & 96 & 154 & 232 & 333 & 460 \end{bmatrix}$$

Example 31. Squares ($arith\ a_n$)**:**

1. $(a_n) = 0, 1, 2, 3, 4, \ldots \Rightarrow S_{a_n} = \{1, 2, 4, 8, 15, 26, 42, 64, 93, 130 \ldots\}$.
The "cake numbers" appear. See A000125.

```
> apsequence(10000,0,1):  squaressequence(10):  print(SQ);
```

$$\left[\, 1 \quad 2 \quad 4 \quad 8 \quad 15 \quad 26 \quad 42 \quad 64 \quad 93 \quad 130 \,\right]$$

2. $(a_n) = 1, 2, 3, 4, \ldots \Rightarrow S_{a_n} = \{1, 3, 7, 14, 25, 41, 63, 92, 129, \ldots\}$.
The 3-dimensional analogue of centered polygonal numbers appear. This sequence is very interesting. See A004006.
Now we can say that the "cake numbers" are the 3-dimensional analogue of centered polygonal numbers plus one.

```
> apsequence(10000,1,1):  squaressequence(10):  print(SQ);
```

$$\left[\, 1 \quad 3 \quad 7 \quad 14 \quad 25 \quad 41 \quad 63 \quad 92 \quad 129 \quad 175 \,\right]$$

```
> gpsequence := proc (n,a1,r)
> local i:
> global s:=array(1..n):
> s[1]:=a1:
> for i to (n-1) do s[i+1] := s[i]*r:
> od:
> end:
```

Gpsequence program: This program calculates the geometric progression whose first term is a_1 and whose ratio is r. For instance, if we want to calculate the first ten terms of a geometric progression whose first term is 1 and whose ratio is 3 we must execute: **gpsequence(10,1,3): print(s);**

```
> gpsequence(10,1,3):  print(s);
```

$$\left[\, 1 \quad 3 \quad 9 \quad 27 \quad 81 \quad 243 \quad 729 \quad 2187 \quad 6561 \quad 19683 \,\right]$$

---

3. $(a_n) = 1, 2, 4, 8, \ldots \Rightarrow S_{a_n} = \{1, 3, 7, 15, 31, 63, \ldots\}$.
Mersenne numbers appear.

```
> gpsequence(100,1,2):  squaressequence(10):  print(SQ);
```

$$\left[\, 1 \quad 3 \quad 7 \quad 15 \quad 31 \quad 63 \quad 127 \quad 255 \quad 511 \quad 1023 \,\right]$$

```
> primessequence := proc (n)
> local i,aux:
> global s:=array(1..n):
> s[1] := 2:
> for i to n-1 do s[i+1] := nextprime(s[i]):
> od:
> end:
```

Primessequence program: This program calculates the prime numbers sequence. For instance, if we want to calculate the first twenty primes, we must execute: **primessequence(20): print(s);**

```
> primessequence(20):  print(s);
```

$$\begin{bmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 & 37 & 41 & 43 & 47 & 53 & 59 & 61 & 67 & 71 \end{bmatrix}$$

4. $(a_n) = 2, 3, 5, 7, \ldots \;\Rightarrow\; S_{a_n} = \{1, 4, 10, 21, 39, 68, \ldots\}$.
Convolution of natural numbers with $(1, p(1), p(2), \ldots)$, where $p(s)$ is the s-th prime. See A023538.

```
> primessequence(10000):  squaressequence(17):  print(SQ);
```

$$\begin{bmatrix} 1 & 4 & 10 & 21 & 39 & 68 & 110 & 169 & 247 & 348 & 478 & 639 & 837 & 1076 & 1358 & 1687 & 2069 \end{bmatrix}$$

```
> oneminusonesequence:=proc(n)
> global s:=array(1..n):
> local i,j:
> for i to n do
> s[i]:=(-1)^(i+1):
> od:
> end:
```

Oneminusonesequence program: This program calculates the $((-1)^n)_{n\geq 0}$ sequence. For instance, if we want to calculate the first twenty terms, we must execute: **oneminusonesequence(20): print(s);**

```
> oneminusonesequence(20):  print(s);
```

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

5. $(a_n) = 1, -1, 1, -1, \ldots \quad \Rightarrow \quad S_{a_n} = \{1, 3, 4, 6, 7, 9, 10, 12 \ldots\}$.
Numbers that are congruent to 0 or 1 (mod 3).

```
> oneminusonesequence(10000):  squaressequence(20):  print(SQ);
```

$$\left[\begin{array}{cccccccccccccccccccc} 1 & 3 & 4 & 6 & 7 & 9 & 10 & 12 & 13 & 15 & 16 & 18 & 19 & 21 & 22 & 24 & 25 & 27 & 28 & 30 \end{array}\right]$$

In this example is interesting to consider the set of cubes.
$C_{a_n} = \{(a \odot_{a_n} a) \odot_{a_n} a : \ a \in \mathbb{N}\}$.

```
> cubessequence := proc (t)
> local i:
> global CB:=array(1..t):
> for i to t do
> productsequence(i,i):
> productsequence(p,i):
> CB[i]:=p:
> od:
> end:
```

Cubessequence program: This program calculates the first cubes ($arith\ a_n$). You must first run the sequence and then, execute the program.
For instance, if we want to calculate the first twenty cubes ($arith\ a_n$) where $(a_n) = 1, -1, 1, -1, \ldots$, we must execute:
oneminusonesequence(10000): cubessequence(20): print(CB);

```
> oneminusonesequence(10000):  cubessequence(20):  print(CB);
```

$$\left[\begin{array}{cccccccccccccccccccc} 1 & 5 & 7 & 14 & 17 & 27 & 31 & 44 & 49 & 65 & 71 & 90 & 97 & 119 & 127 & 152 & 161 & 189 & 199 & 230 \end{array}\right]$$

Maximum number of intersections in self-intersecting n-gon. See A105638.

```
> zeroonesequence:=proc(n)
> global s:=array(1..n):
> local i,j:
> for i to n do
> if irem(i,2)=1 then s[i]:=0:  fi:
> if irem(i,2)=0 then s[i]:=1:  fi:
> od:
> end:
```

Zeroonesequence program: This program calculates the sequence $0, 1, 0, 1, \ldots$ For instance, if we want to calculate the first twenty terms, we must execute:
zeroonesequence(20): print(s);

```
> zeroonesequence(20):  print(s);
```

$$\left[\begin{array}{cccccccccccccccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

---

6. $(a_n) = 0, 1, 0, 1, 0, 1, \ldots \;\Rightarrow\; S_{a_n} = \{1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, \ldots\}$.
Quarter-squares. See A002620.

```
> zeroonesequence(10000): squaressequence(20): print(SQ);
```

$$\left[\begin{array}{cccccccccccccccccccc} 1 & 2 & 4 & 6 & 9 & 12 & 16 & 20 & 25 & 30 & 36 & 42 & 49 & 56 & 64 & 72 & 81 & 90 & 100 & 110 \end{array}\right]$$

$C_{a_n} = \{1, 2, 7, 14, 29, 48, 79, 116, 169, 230, \ldots\}$. Number of paraffins. See A005998.

```
> cubessequence(17): print(CB);
```

$$\left[\begin{array}{ccccccccccccccccc} 1 & 2 & 7 & 14 & 29 & 48 & 79 & 116 & 169 & 230 & 311 & 402 & 517 & 644 & 799 & 968 & 1169 \end{array}\right]$$

```
> fpsequence := proc (n)
> #We print this sequence in blocks: n is the number of blocks.
> #A block is (1,-1,0,...,0).
> local i, j, t, block, aux:
> global s, counter:
> block := 3: counter := 1:
> for i to n do
> for j to block do
> if irem(j, block) = 1 then aux(counter) := 1 fi:
> if irem(j, block) = 2 then aux(counter) := -1 fi:
> if irem(j, block) <> 1 and irem(j, block) <> 2 then aux(counter) := 0 fi:
> counter := counter+1:
> od:
> block := (2^(i+1)-1)+2:
> od:
> s:=array(1..(counter-1)):
> for t to (counter-1) do
> s[t]:=aux(t):
> od:
> end:
```

Fpsequence program: This program calculates the following sequence.
$1,\ -1,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1, \ldots$
The number of zeros in the sequence follow the pattern $2^n - 1$. We print this sequence in blocks. The number of blocks is n. A block is $(1, -1, 0, ..., 0)$. For instance, if we want to calculate the first four blocks, we must execute: **fpsequence(3): print(s);**

```
> fpsequence(3): print(s);
```

$$\begin{bmatrix} 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

---

7. $(a_n) = 1,\ -1,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1, \ldots$
The number of zeros in the sequence follow the pattern $2^n - 1$.

$S_{a_n} = \{1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, \ldots\}$.

```
> fpsequence(10):  squaressequence(20):  print(SQ);
```

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 11 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 23 & 24 \end{bmatrix}$$

This sequence coincides with the "pancake numbers". This coincidence deserves attention. See A058986.