PARALLEL COMPUTATION OF TROPICAL VARIETIES, THEIR POSITIVE PART, AND TROPICAL GRASSMANNIANS

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ABSTRACT. In this article, we present a massively parallel framework for computing tropicalizations of algebraic varieties which can make use of finite symmetries. We compute the tropical Grassmannian $TGr_0(3,8)$, and show that it refines the 15-dimensional skeleton of the Dressian Dr(3,8) with the exception of 23 special cones for which we construct explicit obstructions to the realizability of their tropical linear spaces. Moreover, we propose algorithms for identifying maximal-dimensional tropical cones which belong to the positive tropicalization. These algorithms exploit symmetries of the tropical variety even though the positive tropicalization need not be symmetric. We compute the maximal-dimensional cones of the positive Grassmannian $TGr^+(3,8)$ and compare them to the cluster complex of the classical Grassmannian Gr(3,8).

1. Introduction

Tropical geometry studies combinatorial objects arising from systems of polynomial equations. These so-called *tropical varieties* arise naturally in many areas within and beyond mathematics, such as algebraic geometry [Mik05], combinatorics [AK06], as well as optimization [ABGJ18], biology [SS04; YZZ19], economics [TY19; BK19], and physics [HM06; HJ11]. Wherever they emerge, tropical varieties are often tied to concrete computational problems, which is why their computation is of key importance for many applications.

The process of tropicalization associates to any algebraic variety a tropical variety. In this paper, we present an implementation for computing tropicalizations in parallel. It builds on the approach originally developed by Bogart, Jensen, Speyer, Sturmfels, and Thomas [BJSST07] and implemented by Jensen in GFAN [Jen17]. It is also available in the computer algebra system Singular [DGPS19], see [MR19]. The method is based on the fact that the tropicalization of an irreducible algebraic variety is the support of a polyhedral cell complex, which is connected in

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codimension 1. The tropical variety is determined via a graph traversal with the nodes corresponding to the maximal polyhedra and edges corresponding to their one-codimensional facets. To pass from one maximal cone to its neighboring cones, we compute tropical links using an algorithm described in [HR18], which relies on triangular decomposition and Puiseux expansions. Our algorithm is implemented in a framework for massively parallel computations in computer algebra [Bh+18], which is based on the idea of separation of computation and coordination. For the coordination layer, it relies on the workflow management system GPI-SPACE [PR19] to model parallel algorithms in terms of Petri nets, while the computational layer is built on SINGULAR.

Furthermore, we develop a first algorithm towards computing *positive* tropicalizations as studied by Speyer and Williams [SW05]. While tropical varieties arise from solutions of systems of polynomial equations, positive tropical varieties arise from positive real solutions of systems of polynomial equations. They relate to graphical models in algebraic statistics [PS04] and, more recently, the tropicalization of semialgebraic sets in non-archimedian semidefinite programming [AGS20; JSY18]. It is also conjectured [SW05, Conjecture 8.1] that they encode the combinatorics of cluster algebras of finite type. This was proven recently by Brodsky and Stump for many important cases [BS18].

One class of tropical varieties of particular interest and the main example in this article are tropical Grassmannians $\mathrm{TGr}_0(k,n)$. In algebraic geometry, Grassmannians $\mathrm{Gr}(k,n)$ parametrize all k-dimensional linear spaces in K^n for a given field K. In tropical geometry, their tropicalizations $\mathrm{TGr}_0(k,n)$ parametrize all k-dimensional tropical linear spaces in \mathbb{R}^n that are realizable over K. In both algebraic and tropical geometry, Grassmannians form one of the simplest moduli spaces and offer a strong basis for the understanding of general (tropical) varieties. In more applied context, the real points on $\mathrm{Gr}(k,n)$ and their tropicalization on $\mathrm{TGr}_0(k,n)$ are linked to the soliton solutions of the Kadomtsev-Petviashvili equation [KW13a] with positivity corresponding to regularity at all times [KW13b; KW14].

We employ our algorithms to compute all maximal-dimensional cones in $TGr^+(3, 8)$ and compare them to the cluster complex of Gr(3, 8). We verify that [SW05, Conjecture 8.1] holds, which is proven to be true in [BS18]. This serves as a verification of our computations, and as an alternative proof of the conjecture in this specific case. Recently, two groups of researchers, Arkani-Hamed, Lam and Spradlin [ALS20] as well as Speyer and Williams [SW20], have independently proven that on the positive part the tropical Grassmannian $TGr_0(k, n)$ equals the positive Dressian $Dr^+(k, n)$.

This article is organized as follows: In Section 2, we fix our notation on tropical geometry, and give some background material required in the subsequent sections.

In Section 3, we present our massively parallel algorithm for computing tropical varieties with symmetry, which is described in terms of a Petri net. We discuss our implementation in the SINGULAR/GPI-SPACE framework.

Continuing the work of [SS04] and [HJJS09] we use our implementation to compute the tropical Grassmannian $TGr_0(3,8)$, and analyze its natural fan structures using the software polymake [GJ00]. Thus we give a positive answer to Question 37 in [HJS14] whether it is feasible to compute $TGr_0(3,8)$. Furthermore, we compare it to the *Dressian* Dr(3,8) as described in [HJS14]. The Dressian Dr(k,n) parametrizes all k-dimensional tropical linear spaces in \mathbb{R}^n , also known as valuated-matroids, independent of their realizability and is generally of higher dimension than the tropical Grassmannian $TGr_p(k,n)$ it contains. We show that $TGr_0(3,8)$ refines the 16-dimensional skeleton of Dr(3,8) with exception of 23 extended Fano cones for which explicit obstructions for the realizability of tropical linear spaces are presented.

In Section 5, we propose general algorithms for computing all maximal-dimensional cones in a tropical variety Trop(I) which belong to the positive tropicalization $\text{Trop}^+(I)$. These algorithms exploit the symmetry of Trop(I) even though $\text{Trop}^+(I)$ itself need not be entirely symmetric.

In Section 6, we compute all maximal-dimensional cones in $TGr^+(3,8)$ and compare them to the cluster complex of Gr(3,8), verifying that [SW05, Conjecture 8.1] holds.

In Section 7, we discuss three open questions beyond the scope of this article. These are the coarsest structure on tropical varieties, positive tropicalizations and cluster complexes of infinite type, and real tropicalizations and the topology of real algebraic varieties.

All data and other auxiliary materials will be available shortly and hosted on https://polymake.org. All polynomial data will be uploaded in SINGULAR format, while all tropical data are directly available via POLYMAKE using POLYDB.

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2. Background

In this section, we fix a few notations by briefly going over some basic concepts of immediate relevance to us. Our notation is largely compatible with that of [MS15],

with the exception that we will use polynomials instead of Laurent polynomials and the max-convention instead of the min-convention. This is because the software that we will be presenting in the latter sections is built on infrastructure using polynomials and the max-convention.

Convention 2.1 For the remainder of the section, let K be an algebraically closed field with a non-trivial valuation $\nu \colon K^* \to \mathbb{R}$, ring of integers R, and residue field \mathfrak{K} . We fix a splitting $\mu \colon (\nu(K^*), +) \to (K^*, \cdot)$ and abbreviate $t^a \coloneqq \mu(a)$ for $a \in K^*$. We use $\overline{(\cdot)}$ to denote the canonical projection $R \to \mathfrak{K}$, and we fix a multivariate polynomial ring $K[x] \coloneqq K[x_1, \ldots, x_n]$.

Definition 2.2 The *initial form* of a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} x^{\alpha} \in K[x]$ with respect to a weight vector $w \in \mathbb{R}^n$ is given by

$$\operatorname{in}_w(f) \coloneqq \sum_{w \cdot \alpha - \nu(c_\alpha) \text{ maximal}} \overline{t^{-\nu(c_\alpha)} c_\alpha} \cdot x^\alpha \in \mathfrak{K}[x],$$

whereas the *initial ideal* of an ideal $I \subseteq K[x]$ with respect to $w \in \mathbb{R}^n$ is given by

$$\operatorname{in}_w(I) := \langle \operatorname{in}_w(g) \mid g \in I \rangle \subseteq \mathfrak{R}[x].$$

The following two equivalent definitions for tropical variety is part of the *Fundamental Theorem of Tropical Algebraic Geometry* [MS15, Theorem 3.2.5].

Definition 2.3 Let $I \subseteq K[x]$ be an ideal and $V(I) \subseteq K^n$ its corresponding affine variety. The *tropical variety* of I is defined to be

$$\operatorname{Trop}(I) := \operatorname{cl}\left(\left\{(-\nu(z_1), \dots, -\nu(z_n)) \in \mathbb{R}^n \mid (z_1, \dots, z_n) \in V(I) \cap (K^*)^n\right\}\right)$$
$$= \left\{w \in \mathbb{R}^n \mid \operatorname{in}_w(I) \text{ contains no monomial}\right\}.$$

where $cl(\cdot)$ denotes the closure in the euclidean topology.

Example 2.4 Let $K = \mathbb{C}\{\{t\}\}$ be the field of complex Puiseux series and ν its natural valuation. Consider the linear ideal $I = \langle x+y+1 \rangle \subseteq K[x,y]$. Figure 1 shows Trop(I) using both definitions, with valuations resp. weight vectors highlighted.

Tropical geometry usually involves Laurent polynomials $K[x^{\pm}]$ and takes place in the algebraic torus $(K^*)^n$. When working with polynomials, it is therefore important to assume all ideals to be saturated with respect to the product of all variables, or saturated in short.

Theorem 2.5 (Structure Theorem for Tropical Varieties [MS15, Theorem 3.3.6]) Let $I \subseteq K[x]$ be saturated and prime of dimension d. Then Trop(I) is the support of a balanced polyhedral complex, pure of dimension d, connected in codimension 1.

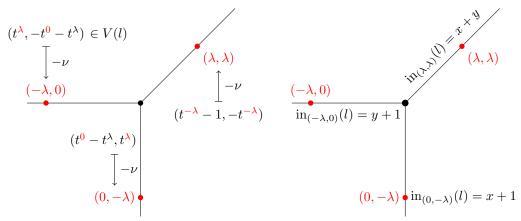


FIGURE 1. Trop($\langle l \rangle$) defined two ways with l = x + y + 1.

Definition 2.6 We define the *Gröbner polyhedron* of a homogeneous ideal $I \subseteq K[x]$ around a weight vector $w \in \mathbb{R}^n$ to be

$$C_w(I) := \operatorname{cl}\Big(\{v \in \mathbb{R}^n \mid \operatorname{in}_v(I) = \operatorname{in}_w(I)\}\Big).$$

The Gröbner complex $\Sigma(I)$ is the set of all Gröbner polyhedra of I.

Proposition 2.7 ([MS15, Theorem 2.5.3]) Let $I \subseteq K[x]$ be a homogeneous ideal. Then $C_w(I)$ is a closed convex polyhedron for all $w \in \mathbb{R}^n$, and $\Sigma(I)$ is a finite polyhedral complex. In particular, Trop(I) is the support of the subcomplex of the Gröbner fan consisting of all Gröbner polyhedra whose initial ideals contain no monomials.

For the sake of simplicity, we will restrict ourselves to what is commonly called the *constant coefficient case*. While the parallel framework described in Section 3 works in full generality, the rest of the paper falls in this very special case.

Convention 2.8 From now on, assume that the field K is either $\overline{\mathbb{Q}}\{\{t\}\}$, $\mathbb{C}\{\{t\}\}$ or $\overline{\mathbb{F}}_p\{\{t\}\}$ for some prime p, and that all ideals are both homogeneous and generated by polynomials with coefficients in $\overline{\mathbb{Q}}$, \mathbb{C} or $\overline{\mathbb{F}}_p$.

In particular, all Gröbner polyhedra will be invariant under multiplication by positive real numbers, and we will be refer to Gröbner polyhedra and Gröbner complexes as *Gröbner cones* and *Gröbner fans* instead.

Additionally, all Gröbner cones in this paper will be inside the tropical variety unless explicitly specified otherwise. This means that maximal or maximal-dimensional Gröbner cones will only be (inclusion) maximal or maximal-dimensional among the Gröbner cones on the tropical variety, i.e., a maximal-dimensional Gröbner cone is a Gröbner cone $C_w(I) \subseteq \text{Trop}(I)$ with $\dim C_w(I) = \dim \text{Trop}(I)$.

Moreover, we will use Trop(I) to denote both the set in Definition 2.3 and the subfan of the Gröbner fan covering it by Proposition 2.7. In written text, we will refer to the latter as the $Gr\ddot{o}bner\ structure$ on Trop(I).

Finally, let us recall the definition of Grassmannians.

Definition 2.9 Let $1 \le k \le n$. In the following, we will abbreviate the *n*-element set $\{1, \ldots, n\}$ by [n] and the set of all *k*-element subsets of [n] by $\binom{[n]}{k}$. The *Grass-mannian* Gr(k, n) is the variety defined by the ideal

$$\mathcal{I}_{k,n} := \left\langle \mathcal{P}_{I,J} \mid I \in \binom{[n]}{k-1}, J \in \binom{[n]}{k+1} \right\rangle \subseteq K \left[p_L \mid L \in \binom{[n]}{k} \right],$$

where

$$\mathcal{P}_{I,J} := \sum_{j \in J \setminus I} (-1)^{|\{i \in I \mid i < j\}| + |\{j' \in J \mid j' > j\}|} \cdot p_{I \cup j} \cdot p_{J \setminus j}.$$

The ideal $\mathcal{I}_{k,n}$ is commonly referred to as $Pl\ddot{u}cker\ ideal$, while the $\mathcal{P}_{I,J}$ are commonly called $Pl\ddot{u}cker\ relations$. Note that $\mathcal{P}_{I,J}$ is a trinomial if and only if $|I \cap J| = k - 2$, in which case we will refer to them as 3-term $Pl\ddot{u}cker\ relations$. The 3-term $Pl\ddot{u}cker\ relations$ do not generate the $Pl\ddot{u}cker\ ideal$ if $n \geq d + 3 \geq 6$, but they always generated the $Pl\ddot{u}cker\ ideal$ up to saturation, see [HJJS09, Section 2].

The tropical Grassmannian is the tropicalization of the Plücker ideal, and we will denote it by $\mathrm{TGr}_p(k,n) := \mathrm{Trop}(\mathcal{I}_{k,n})$, where p is the characteristic of the field K. This is well-defined, as the tropical Grassmannian only depends on the characteristic of K, since the coefficients of the Plücker relations are integers.

Similarly to the classical Grassmannian, its tropicalization $\mathrm{TGr}_p(k,n)$ is the easiest example of a non-trivial moduli space. Each point on $\mathrm{TGr}_p(k,n)$ corresponds to the tropicalization of a k-dimensional linear space in the projective space \mathbb{P}^{n-1} .

In this article, we will mainly focus on the case p = 0, k = 3, and n = 8. This is a continuation of the two articles [SS04] and [HJJS09], which discuss the tropical Grassmannians $TGr_p(2, n)$, $TGr_p(3, 6)$ and $TGr_p(3, 7)$.

3. Parallel computation of tropical varieties

In this section, we discuss the realization of our algorithm for computing tropical varieties in terms of a Petri net and the technical background on massive parallelization. We begin by recalling the algorithms for computing tropical varieties.

3.1. Computing tropical varieties. The general framework for computing tropical varieties of polynomial ideals has remained unchanged since its initial conception in [BJSST07] and implementation in GFAN [Jen17]. Tropical varieties are computed by traversing all maximal Gröbner cones which lie on them, see Figure 2 for an illustration. The traversal starts at a pre-computed starting point on the tropical variety. The inequalities and equations of each Gröbner cone are uniquely determined by a Gröbner basis with respect to a weighted ordering given by any of its relative interior points. The approach consists of two key sub-steps:

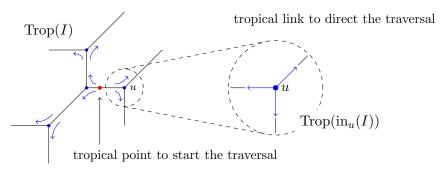


FIGURE 2. Traversing the tropical variety.

Gröbner walk: The Gröbner walk [CKM97; FJLT07] is a well-established technique for transforming Gröbner bases with respect to one ordering to that of another for a fixed ideal. To determine a tropical variety, it is used to compute Gröbner bases with respect to the different orderings associated to the different Gröbner cones of full dimension on the tropical variety.

Tropical link: Tropical links describe tropical varieties locally around a point. They are tropical varieties of a combinatorially simpler type and can therefore be computed using specialized algorithms. When computing tropical varieties, they are used to identify the directions pointing towards the maximal cones neighboring a facet (of codimension one). The computation of tropical links has been the bottleneck of the algorithm for a long time. This issue has been resolved through newer approaches using projections [Cha13] or root approximations [HR18]. Newer versions of GFAN rely on [Cha13], while our implementation relies on [HR18].

3.2. Massive parallelization in computer algebra. Our implementation builds on a framework for massively parallel computations in computer algebra [Bh+18; Ris19], which combines the computer algebra system SINGULAR with the workflow management system GPI-SPACE. This framework originated in work on a parallel smoothness criterion for algebraic varieties [Bh+18; Ris19], and has been used in [Rei18; BFRR20] to realize a massively parallel fan traversal for computing GIT-fans. For an overview and more applications see [BFR18] and [Ben+20]. The results of the current section extend the traversal of complete fans developed for the GIT-fan algorithm. More details can be found in the thesis of the first author [Ben18].

The workflow management system GPI-SPACE is based on the idea of separation of coordination and computation [GC92]. In the coordination layer it uses the language of Petri nets [Pet62] to model a computer program in the form of a concurrent system. It allows for running computations on large scientific computing clusters, and consists of the following three main components [Bh+18, Section 4]:

- (1) a distributed runtime system managing available resources and assigning jobs to resources,
- (2) a virtual memory layer allowing processes to communicate and share data,

(3) a workflow manager tracking the global structure and state, formulated as a so-called Petri net.

Definition 3.1 A Petri net is a finite bipartite directed graph N = (P, T, F), where P and T are disjoint vertex sets called places and transitions respectively, and where the set of edges $F \subseteq (P \times T) \cup (T \times P)$ is called the set of flow relations. Given $p \in P$ and $t \in T$, we call p an input to t if $(p, t) \in F$ and p an output of t if $(t, p) \in F$.

Petri nets depict a static model of the algorithm, with transitions representing processes and places representing data passed between them. The dynamics of the algorithm, i.e., its execution, is described via the notion of markings:

Definition 3.2 Let (P, T, F) be a Petri net. A marking M is a map $M: P \to \mathbb{N}_{\geq 0}$, and we say a place $p \in P$ holds k tokens under M if M(p) = k.

We call a transition $t \in T$ enabled if M(p) > 0 for all $p \in P$ with $(p, t) \in F$. In this case, the transition t may be fired by consuming one token of each input and returning one token in each output, which leads to a new marking M' given by

$$M'(p) := M(p) - |\{(p,t)\} \cap F| + |\{(t,p)\} \cap F|$$

for all $p \in P$. We denote the firing process by $M \xrightarrow{t} M'$.

One important principle to adhere in modelling the state of algorithms via markings in GPI-SPACE is locality: Firing enabled places should be a local process and not block other enabled transitions from firing simultaneously. To ensure locality, GPI-SPACE requires the user to impose restrictions on places such that any token that is in an input to multiple transitions can only be consumed by a single well-defined transition.

Example 3.3 Figure 3 shows an example which unwraps a list of tokens. In it, both split and consume empty have input ℓ . However, split only consumes non-empty lists from ℓ , while consume empty only consumes empty lists. Thus there is always a single well-defined transition that can fire.

As illustrated, we will usually give conditions in the form "if (not) condition". As a consequence of locality, enabled transitions may fire simultaneously. This includes single enabled transitions with enough input tokens to fire multiple times (Figure 4 top), which is called data parallelism, and multiple enabled transitions with separate input tokens (Figure 4 bottom), which is called task parallelism.

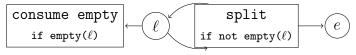


FIGURE 3. Petri net unwrapping a list

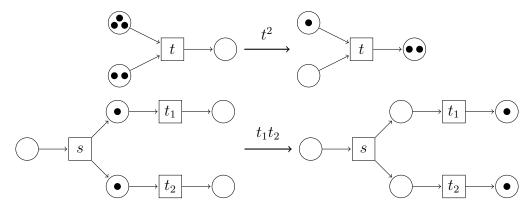


FIGURE 4. Data parallelism (top) and task parallelism (bottom).

Markings are only a very basic tool for describing the dynamics of algorithms. There are many extensions of Petri nets realized in GPI-SPACE such as allowing tokens to carry data (*colored* Petri nets), and to allow for transitions to take some time to execute (*timed* Petri nets).

3.3. Parallel traversals of tropical varieties. Figure 5 shows a Petri net modelling the traversal of the tropical variety as mentioned in Section 3.1. The computation of the facets requires Gröbner bases, while the computation of the neighbors requires tropical links. They are separated for finer control over job sizes, see [Ben18] for more details. The Petri net consists of the following places and transitions:

Place I: read-only input data containing ideal and symmetries.

Transition starting cone: computes a random maximal Gröbner cone on the tropical variety and places it into m.

Place m: holds maximal Gröbner cones on the tropical variety.

Transition store cone: takes a Gröbner cone and inserts it into the external storage, discarding cones that are already known. Afterwards, replaces the boolean

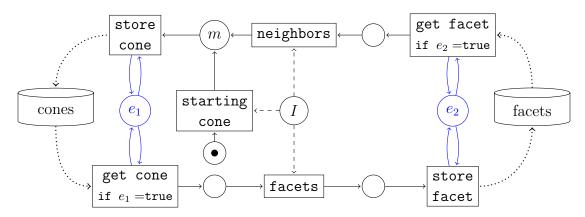


FIGURE 5. Petri net modelling the traversal of a tropical variety

token in e_1 by true or false depending on whether unprocessed cones remain in the storage.

Place e_1 : holds true if storage contains unprocessed cones, false otherwise.

Transition get cone: retrieves an unprocessed Gröbner cone and marks it as processed, provided that e_1 holds a true-token.

Transition facets: computes the facets of a Gröbner cone.

Transitions store facet and get facet: analogous to store cone and get cone. Place e_2 : analogous to e_1 .

Transition neighbors: takes a facet and computes all incident maximal Gröbner cones on the tropical variety.

The algorithm terminates if e_1 and e_2 hold a false-token, i.e. no unprocessed cones and facets remain in storage, and if the input places to store cones and store facet are empty.

3.4. Timings for the tropical Grassmannian. Table 1 shows the timings to compute the tropical Grassmannians $TGr_0(3,7)$ and $TGr_0(3,8)$ in relation to the number of CPU cores used. Additionally, speedup lists the ratios between the single-and multi-core computation times, while efficiency further divides that number by the number of cores. Note that, due to the size of $TGr_0(3,8)$, no single core computation could be carried out. All efficiency numbers instead are based on the 15 core timing (marked with * in Table 1).

All computations were run on a cluster at the Fraunhofer ITWM [Fra]. Each node is fitted with two Intel Xeon E5-2670 processors and 64 GB of memory, amounting to 16 CPU cores per node at a base clock of 2,6 Ghz. The computations of $TGr_0(3,8)$ were done with a fixed configuration of 15 compute jobs and one storage interface job per node.

$\mathrm{TGr}_0(3,7)$						$\mathrm{TGr}_0(3,8)$				
nodes	cores	time [s]	speedup	eff.	nodes	cores	time [s]	speedup	eff.	
1	1	792.8	1.000	1.000	1	15	98926.1	*15.000	*1.000	
1	2	382.8	2.070	1.035	2	30	37398.7	39.675	1.322	
1	4	191.1	4.147	1.037	4	60	14486.3	102.435	1.707	
1	8	98.1	8.080	1.001	8	120	6597.3	224.925	1.874	
1	12	74.1	10.691	0.891	16	240	3297.9	449.955	1.874	
1	16	58.0	13.653	0.853	24	360	2506.0	592.125	1.645	
2	24	42.8	18.522	0.772	32	480	2001.7	741.285	1.544	
2	32	39.7	19.942	0.623	40	600	1509.6	982.935	1.638	
		•	,		48	720	1267.3	1170.855	1.626	
					56	840	1188.2	1248.825	1.487	

Table 1. Timings for computing the tropical Grassmannians $TGr_0(3,7)$ and $TGr_0(3,8)$ in parallel

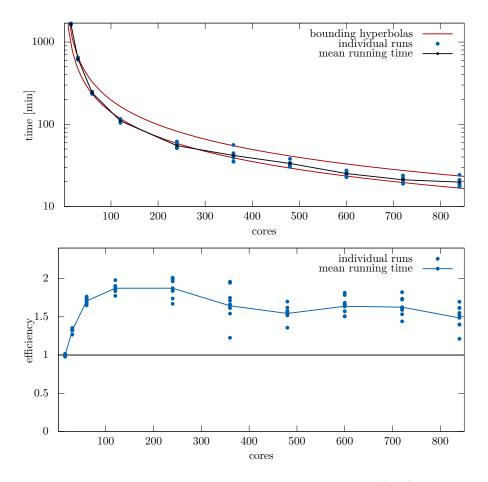


FIGURE 6. Timings and efficiency of $TGr_0(3,8)$

As shown in Table 1, computing $TGr_0(3,7)$ scales favorably up to around 12 CPU cores, after which a noticeable drop in efficiency can be observed. This is expected, however, as $TGr_0(3,7)$ is up to symmetry covered by only 125 cones, and the number of cores exceed the maximum queue size.

Figure 6 shows the timings and the efficiency graph for $TGr_0(3,8)$. The timings scale very well, with no significant drop in efficiency, even to more than 800 cores. We do not encounter the queue size problem due to the size of $TGr_0(3,8)$, and there is no visible decrease in efficiency for increasing core count, as one could expect due to increased overhead and communication.

We observe a surprising surge in efficiency around 60 cores. At the time of writing, no comprehensive explanation has been found for this behaviour. We suspect that this is partly due to technical effects of the cluster hardware, e.g. with regard to the memory bandwidth. In [Bh+18], experiments with a different algorithm on the same cluster have shown that distributing the number of used processor cores over more machines can lead to a speedup. This might indicate a memory bottleneck, and could partially explain the unexpected behaviour of the efficiency graph. Moreover,

in [Bh+18] it was observed that massively parallel implementations can lead to a superlinear speedup by allowing the randomized algorithm to find a faster path to the final result. In our setting this amounts to different possible expansions of the tropical variety leading to different computation times.

4. Tropical Grassmannians and Dressians

In this section, we compare the tropical Grassmannian $TGr_0(3,8)$ to the Dressian Dr(3,8) described in [HJS14], i.e., we compare the moduli of realizable tropical linear spaces or valuated matroids with the moduli of all tropical linear spaces or valuated matroids. The main difficulty stems from the fact that both are covered by thousands of cone orbits with respect to the sizeable group S_8 . We begin with a brief introduction and some formal definitions.

Definition 4.1 Let $1 \le k \le n$ and recall 3-term Plücker relations $\mathcal{P}_{I,J}$ from Definition 2.9. The *Dressian* Dr(k,n) is the intersection of their tropical hypersurfaces:

$$Dr(k, n) := \bigcap_{I, J} Trop(P_{I, J}).$$

where the intersection is taken over all sets $I \in \binom{[n]}{k-1}$, $J \in \binom{[n]}{k+1}$ with $|I \cap J| = k-2$. By definition, the Dressian is the support of the common refinement of the Gröbner subfans covering $\text{Trop}(P_{I,J})$, and we will use Dr(k,n) to denote both the set and the polyhedral fan covering the set.

The Dressian is a tropical prevariety, i.e., it is the intersection of the tropical hypersurfaces of a finite generating set. Usually, tropical prevarieties and tropical varieties have little in common besides one being trivially contained in the other. In fact, merely testing both objects for equality is a hard task [The06; GRS19], and it is unclear what distinguishes a generating set for whom equality holds, commonly called a tropical basis [JS18]. Nevertheless, the Dressian is interesting for many reasons:

- The Dressian is the moduli space of all tropical linear spaces, also known as valuated matroids. Similar to the Grassmannian in algebraic geometry, the Dressian can be regarded as one of the simplest moduli spaces in tropical geometry.
- The hypersimplex $\Delta(k, n)$ is the moment polytope for the torus action on the complex Grassmannian. The Dressian Dr(k, n) is the subfan of the secondary fan of the $\Delta(k, n)$ consisting of all of matroid subdivisions [GGMS87; MS15].
- Recent work of Huh and Brändén [BH19, Theorem 8.7] regard the Dressian Dr(d, n) as the tropicalization of the space of Lorentzian polynomials supported on $\Delta(d, n)$, i.e., on the basis of the uniform matroid of rank d on n elements.

As mentioned above, the Dressian is a subfan of the secondary fan of the hypersimplex and depends only on two parameters. We will consider the Dressian equipped with this fan structure which is also known as *Plücker structure*; see [APS19]. This

\overline{d}	n	char.	stru.	# rays	# orbits	# max. cones	# orbits
3	6	any	G	65	3	1035	7
3	6	any	D	65	3	1005	7
3	7	p = 2	G	751	7	252420	125
3	7	$p \neq 2$	G	721	6	252000	125
3	7	$p \neq 2$	D	616	5	211365	94

TABLE 2. The number of rays and maximal cones of the tropical Grassmannians $TGr_p(3,6)$ and $TGr_p(3,6)$ with the Gröbner structure (G) inherited from the Gröbner fan or the coarsest Plücker structure (D) inherited from the Dressian.

is the polyhedral structure that we obtain from the intersection of the tropical hypersurfaces of the 3-term Plücker relations. This structure is the coarsest structure, as for any two vectors that lie in distinct maximal cones there is a tropical 3-term Plücker relation whose maximum is attained twice, but on different terms. Thus a positive combination of these vectors attains the maximum only at a single term.

The tropical Grassmannian $\mathrm{TGr}_p(d,n)$ depends on the characteristic p of the underling field though almost all tropical Grassmannians agree with $\mathrm{TGr}_0(d,n)$. As a tropical variety it is naturally a subfan of a Gröbner fan. In contrast to the Plücker structure the Gröbner structure typically is not a coarsest structure.

There are many explicit computational results on the Dressian and the tropical Grassmannian inside it. In particular, all existing computations verify that the Plücker structure of the Dressian coarsens the Gröbner fan structure of the tropical Grassmannian in characteristic p=0, i.e., that there is a subfan of the Dressian supported on the tropical Grassmannian. This subfan is often strictly coarser than the Gröbner fan restricted to it. The following remarks and Table 2 summarize these results by comparing tropical Grassmannians with the Gröbner and the Plücker structures for all known (k, n) and p using their rays and maximal cones, the bold numbers representing the number of orbits with respect to the natural S_n -action on the coordinates.

Remark 4.2 The tropical Grassmannian $TGr_p(2, n)$ with the Gröbner structure is independent of the characteristic, has $2^{n-1} - n - 1$ rays in $\lceil \frac{n-3}{2} \rceil$ orbits and (2n-5)!! maximal cones. The number of S_n -symmetry classes equals the number of trivalent trees with n leaves [OEIS19, A000672]. It is the moduli spaces of tropical lines, phylogenetic trees with n labeled leaves, and tropical rational curves of genus 0 with n marked points. This structure is the coarsest fan structure. The tropical Grassmannian and Dressian agree. The rays correspond to split hyperplanes and the tropical Grassmannian is isomorphic to the split complex. The connected matroids

in the corresponding matroid subdivisions are sparse matroids. See [SS04], [HJ08] and [JS17] for further details.

Remark 4.3 The tropical Grassmannian $TGr_p(3,6)$ is independent of the characteristic, the Dressian and tropical Grassmannian have the same support, but the Gröbner structure is a refinement of the Dressian. To be precise, there is a cone over an three dimensional bipyramid in the Plücker structure which the Gröbner structure refines into three cones over tetrahedra. The tropical Grassmannian $TGr_p(3,6)$ with the Gröbner structure is a simplicial fan.

The tropical Grassmannian $\mathrm{TGr}_p(3,7)$ depends on the characteristic. The Gröbner structure on $\mathrm{TGr}_2(3,7)$ is coarse, but not a subfan of the Dressian $\mathrm{Dr}(3,7)$. The Gröbner structure on $\mathrm{TGr}_p(3,7)$ is a refinement of the Plücker structure if $p \neq 2$. The number of rays and maximal cones is summarized in Table 2. Further details can be derived from [SS04] and [HJJS09].

We extend the previous result by combining our computation in Section 3 with the following computational result in [HJS14] describing the Dressian Dr(3,8).

Proposition 4.4 ([HJS14, Theorem 31]¹) The Dressian Dr(3,8) is a non-pure 17-dimensional fan with a 8-dimensional lineality space, it consists of 15 470 rays in twelve S_8 -orbits and 117 595 485 cones of dimension 16 in 4789 S_8 -orbits.

The following theorem is a summary of a elaborate computation based on SIN-GULAR, POLYMAKE and the framework described in Section 3.

Theorem 4.5 The Gröbner subfan supported on the tropical Grassmannian $TGr_0(3, 8)$ is a 16-dimensional fan with a 8-dimensional lineality space, it consists of 732 725 rays in 95 S_8 -orbits and 278 576 760 maximal cones in 14 763 S_8 -orbits.

Moreover, the coarsest fan structure supported on $TGr_0(3,8)$ is a subfan of the Dressian consisting of 15 470 rays in twelve S_8 -orbits and 117 445 125 maximal cones in 4766 S_8 -orbits.

Proof. We computed the tropical Grassmannian $TGr_0(3,8)$ with the Gröbner structure on with the methods of Section 3.

In order to confirm that Plücker structure on $TGr_0(3,8)$ is well-defined, we tested that the relative interior of any 16-dimensional Dressian cone is either contained in or disjoint to $TGr_0(3,8)$. For that, we verified that any maximal Gröbner cone on $TGr_0(3,8)$ is contained in a 16-dimensional Dressian cone, and that every 15-dimensional Gröbner cone on $TGr_0(3,8)$ intersecting the relative interior of a 16-dimensional Dressian cone is contained in exactly two maximal Gröbner cones. \square

¹Our computation showed that the data of [HJS14] misses a 16-dimensional simplicial cone of orbit size 840. The orbit is represented by a cone containing the corank vector of the sparse-paving matroid with non-bases 015, 024, 067, 126, 137, 235, 346, 457. It is a cone whose eight rays correspond to vertex splits all lying in the same S_8 -orbit.

Remark 4.6 There are 23 S_8 -orbits of 16-dimensional cones in the Dressian Dr(3, 8) whose relative interior does not intersect the tropical Grassmannian $TGr_0(3, 8)$. Remarkably, the following single polynomial is a witness for all Dressian orbits:

$$f \coloneqq 2 \, p_{123} p_{467} p_{567} - p_{367} p_{567} p_{124} - p_{167} p_{467} p_{235} - p_{127} p_{567} p_{346} - p_{126} p_{367} p_{457}$$
$$- p_{237} p_{467} p_{156} + p_{134} p_{567} p_{267} + p_{246} p_{567} p_{137} + p_{136} p_{267} p_{457} \in \mathcal{I}_{3,7} \subset \mathcal{I}_{3,8}.$$

To be precise, for any of the aforementioned 23 orbits $\mathcal{D} \subseteq \operatorname{Dr}(3,8)$ there exists a Dressian cone $\sigma \in \mathcal{D}$ such that $\operatorname{in}_w(f) = 2 \cdot p_{123} p_{467} p_{567}$ for all relative interior points $w \in \operatorname{Relint}(\sigma)$.

This observation agrees with [HJS19, Proposition 5.5], which states that regular matroid subdivisions that contain matroid polytopes of extensions of the Fano matroid lead to points in the Dressian that are not on the tropical Grassmannian. Our witness polynomial f is in fact a witness of the higher-dimensional Fano cone, whose 16-dimensional faces generate the 23 S_8 -orbits.

As an immediate consequence, we get:

Theorem 4.7 The quadratic Plücker relations and the cubic polynomials in the S_8 -orbit of f are a tropical basis of the Plücker ideal $\mathcal{I}_{3,8}$.

Proof. By definition, the Plücker relations generate the Plücker ideal $\mathcal{I}_{3,8}$. Recall that Dr(3,8) is 17-dimensional. By [HJS19, Remark 5.4], the polynomials in $S_8 \cdot f$ are witnesses for all 17-dimensional cones of Dr(3,8), i.e., for every point w inside a 17-dimensional cone of Dr(3,8) there is a $\sigma \in S_8$ with $w \notin Trop(\sigma \cdot f)$. In Theorem 4.5 we verified that any 16-dimensional cone of Dr(3,8) either lies on $TGr_0(3,8)$ or has a relative interior disjoint to it. In Remark 4.6, we verified that the polynomials in $S_8 \cdot f$ are witnesses for the latter.

During our computations, we also encountered the following phenomena, which will be relevant for Section 6.2. We conjecture for them to hold for arbitrary k and n in characteristic 0:

Theorem 4.8 For any $w, v \in TGr_0(3, 8)$ we have

w and v lie on the same cone of $Dr(3,8) \iff in_w(\mathcal{I}_{3,8}) : p^{\infty} = in_v(\mathcal{I}_{3,8}) : p^{\infty},$ where $(\cdot) : p^{\infty}$ denotes the saturation on the product of all Plücker variables.

Proof. The statement was proven through explicit computations in SINGULAR. Note that it suffices to verify that weight vectors in the same Dressian cone have the same saturated initial ideal, because weight vectors in different Dressian cones have different saturated initial ideals:

Let $w, v \in \mathrm{TGr}_p(k, n)$ be in two distinct Dressian cones, which means there exist a three term Plücker relation $\mathcal{P} = s_0 + s_1 + s_2$ such that $\mathrm{in}_w(\mathcal{P}) \neq \mathrm{in}_v(\mathcal{P})$, and assume that $\mathrm{in}_w(\mathcal{I}_{k,n}) : p^{\infty} = \mathrm{in}_v(\mathcal{I}_{k,n}) : p^{\infty}$. There are two cases that might appear. In the

first case $\operatorname{in}_w(\mathcal{P}) \neq \mathcal{P}$ and $\operatorname{in}_v(P) \neq \mathcal{P}$. Let us assume $\operatorname{in}_w(\mathcal{P}) = s_0 + s_1$ and $\operatorname{in}_v(P) = s_0 + s_2$, then $s_0 \in \operatorname{in}_w(\mathcal{I}_{k,n}) : p^{\infty} = \operatorname{in}_v(\mathcal{I}_{k,n}) : p^{\infty}$ contradicting that $\operatorname{in}_w(\mathcal{I}_{k,n})$ and $\operatorname{in}_v(\mathcal{I}_{k,n})$ are monomial free. The second case is $\operatorname{in}_w(\mathcal{P}) = \mathcal{P}$ or $\operatorname{in}_v(P) = \mathcal{P}$, say $\operatorname{in}_w(\mathcal{P}) = s_0 + s_1 + s_2$ and $\operatorname{in}_v(P) = s_0 + s_2$, then $s_1 \in \operatorname{in}_w(\mathcal{I}_{k,n}) : p^{\infty} = \operatorname{in}_v(\mathcal{I}_{k,n}) : p^{\infty}$ again contradicting that $\operatorname{in}_w(\mathcal{I}_{k,n})$ and $\operatorname{in}_v(\mathcal{I}_{k,n})$ are monomial free.

In general, the Gröbner structure on a tropical variety is far from being as coarse as possible, which incurs many iterations in the traversal of the tropical variety that might seem unnecessary. For example, there is a maximal cone in the tropical Grassmannian $TGr_0(3,8)$ equipped with the Plücker structure that is refined into 2620 maximal cones with the Gröbner structure. Therefore, an important open question is whether the tropical variety can be equipped with a natural polyhedral structure that is coarser than that of the Gröbner fan.

Theorem 4.8 states that saturated initial ideals provides such a structure for the tropical Grassmannian $TGr_0(k, n)$ for $k \leq 3$ and $n \leq 8$. While it is unknown whether it holds for all tropical Grassmannians over fields of characteristic 0, the result does not generalize to arbitrary tropical varieties, as neither the set of all weight vectors which share the same saturated initial ideal nor its euclidean closure need be convex:

Example 4.9 Consider the homogeneous ideal

$$I \coloneqq \langle x(a-b) - y(c-d), x(c-d) - y(a-b) \rangle \subseteq \mathbb{C}\{\{t\}\}[a,b,c,d,x,y].$$

It is not hard to see that its tropical variety $\operatorname{Trop}(I)$ in \mathbb{R}^6 is four-dimensional with a two-dimensional lineality space generated by (1,1,1,1,0,0) and (0,0,0,0,1,1). Figure 7 contains a quick $\operatorname{SINGULAR}$ computation which shows that the Gröbner subfan supported on $\operatorname{Trop}(I)$ consists of 12 rays and 16 maximal cones. Figure 8 furthermore illustrates the combinatorial structure of $\operatorname{Trop}(I)$. It shows the intersection $\operatorname{Trop}(I) \cap \operatorname{Lin}(e_1, e_2, e_3, e_5)$, which removes the two-dimensional lineality space and whose maximal cones are contained in either $\operatorname{Lin}(e_1, e_2, e_3)$ or $\operatorname{Lin}(e_1 + e_2, e_5)$. In other words, the following equality holds both in terms of sets and polyhedral complexes:

$$\operatorname{Trop}(I) = \left(\left(\operatorname{Trop}(I) \cap \operatorname{Lin}(e_1, e_2, e_3) \right) \cup \left(\operatorname{Trop}(I) \cap \operatorname{Lin}(e_1 + e_2, e_5) \right) \right) + \operatorname{Lin}(e_1 + e_2 + e_3 + e_4, e_5 + e_6).$$

The intersection $\operatorname{Trop}(I) \cap \operatorname{Lin}(e_1, e_2, e_3)$ resembles the standard tropical plane in \mathbb{R}^3 with two opposing maximal cones barycentric subdivided, while the intersection $\operatorname{Trop}(I) \cap \operatorname{Lin}(e_1 + e_2, e_5)$ resembles \mathbb{R}^2 divided into octants. In the first intersection all saturated initial ideals with respect to relative interior points of maximal cones are generated by a - b and c - d, whereas in the second intersection the saturated initial ideals with respect to relative interior points of maximal cones are distinct.

```
> ring r = 0,(a,b,c,d,x,y),dp;
> ideal I = x*(a-b)-y*(c-d), x*(c-d)-y*(a-b);
> LIB "tropical.lib";
> tropicalVariety(I);
                                   MAXIMAL_CONES
                                   \{0\ 2\} # Dimension 4
       0
               0 0 # 0
       0
          0
              -1 0 # 1
                                   {1 2}
                                   {2 3}
          0
                                   {0 7}
       0
          0
                                   {0 11}
                                   {1 4}
                                   {3 5}
          0
                                   {2 6}
          0
                                   {4 8}
                                   {5 10}
          0
                                   {6 7}
               1 0 # 10
                                   {6 11}
                                   {7 9}
LINEALITY_SPACE
                                   {8 9}
      1 1
               0 0 # 0
                                   {9 10}
          0
                                   {9 11}
```

FIGURE 7. SINGULAR code for Example 4.9 (output cleaned up for clarity)

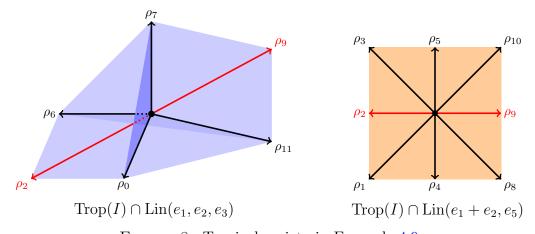


Figure 8. Tropical variety in Example 4.9

5. Computing positive tropicalizations

In this section, we recall the notion of positive tropicalization by Speyer and Williams [SW05], and we introduce algorithms for testing which maximal-dimensional Gröbner cones lie in the positive tropicalization. These algorithms exploit the symmetry of the tropical variety even though the positive tropical variety inside of it may not be symmetric with respect to it.

We distinguish between cones whose initial ideals are binomial and cones whose initial ideals are not. For binomial cones, we state a simple combinatorial algorithm. For non-binomial cones, we reduce the problem to dimension zero which can then be tackled symbolically, numerically or with a mix of both.

Convention 5.1 For the remainder of the section, let $K := \mathbb{C}\{\{t\}\}$ be the field of complex Puiseux series and fix an ideal $I \subseteq K[x] := K[x_1, \ldots, x_n]$ that is generated over the subfield $R \subseteq K$ of Puiseux series whose lowest coefficient is real:

$$R := \left\{ \sum_{\alpha > \lambda} c_{\alpha} t^{\alpha} \in K \mid 0 \neq c_{\lambda} \in \mathbb{R} \right\} \cup \{0\}.$$

In particular, any Gröbner basis of I will consist of polynomials with coefficients in R and any initial ideal of I will be generated over $\mathbb{R}[x]$. We denote by $R_{>0}$ the set of complex Puiseux series whose lowest term is real and positive,

Suppose I is invariant under a group S acting on K[x] via signed permutation of the variables, i.e., for each group element $\sigma \in S$ and all variables $x_i \in K[x]$ there is a permutation $|\sigma| \in S_n$ and a sign $u_i \in \{\pm 1\}$ with $\sigma \cdot x_i = u_i \cdot x_{|\sigma|(i)}$. This implies that V(I) is invariant under S acting on K^n via signed permutation of the components, and that Trop(I) is invariant under S acting on \mathbb{R}^n via unsigned permutation of the components.

Definition 5.2 We define the *positive tropicalization* of an ideal $I \subseteq K[x]$ to be

$$\operatorname{Trop}^+(I) := \operatorname{cl}\Big(\nu\big(V(I) \cap (R_{>0})^n\big)\Big) \subseteq \mathbb{R}^n,$$

where again $\nu(\cdot)$ denotes componentwise valuation and $cl(\cdot)$ denotes the closure in the euclidean topology.

For the sake of convenience, we call a weight vector $w \in \mathbb{R}^n$, an initial ideal $\operatorname{in}_w(I) \subseteq \mathbb{C}[x]$, and a Gröbner cone $C_w(I) \subseteq \operatorname{Trop}(I)$ positive if $w \in \operatorname{Trop}^+(I)$.

Note that under the Fundamental Theorem of Tropical Geometry, positive tropical varieties also admit an algebraic description:

Proposition 5.3 ([SW05, Proposition 2.2]) Let $I \subseteq K[x]$ be an ideal. Then

$$\operatorname{Trop}^+(I) = \Big\{ w \in \mathbb{R}^n \mid \operatorname{in}_w(I) \text{ monomial free and } \operatorname{in}_w(I) \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle \Big\}.$$

In particular, $Trop^+(I)$ is covered by all positive Gröbner cones if I is homogeneous.

As an easy corollary, we get that positivity only depends on the saturated initial ideals, which will be relevant for Section 6.2.

Corollary 5.4 Let $I \subseteq K[x] = K[x_1, ..., x_n]$ be an ideal and let $w, v \in \mathbb{R}^n$ be two weight vectors with $\operatorname{in}_w(I) : (\prod_{i=1}^n x_i)^{\infty} = \operatorname{in}_v(I) : (\prod_{i=1}^n x_i)^{\infty}$. Then

$$w \in \operatorname{Trop}^+(I) \iff v \in \operatorname{Trop}^+(I)$$

Proof. The statement follows directly from the following two facts:

- $\operatorname{in}_w(I)$ is monomial free if and only if $\operatorname{in}_w(I): (\prod_{i=1}^n x_i)^\infty \neq \langle 1 \rangle$, $\operatorname{in}_w(I) \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle$ if and only if $\operatorname{in}_w(I): (\prod_{i=1}^n x_i)^\infty \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle$.

5.1. Binomial cones. We decide positivity of binomial cones using the description of Proposition 5.3. We begin by recalling a well-known result on the Gröbner bases of binomial ideals, and derive an easy test for positivity of binomial ideals from it.

Proposition 5.5 ([ES96, Proposition 1.1]) Any reduced Gröbner basis of a binomial ideal consists solely of binomials.

Lemma 5.6 Let $J \subseteq \mathbb{R}[x]$ be a non-trivial binomial ideal, $G \subseteq J$ a reduced Gröbner basis with respect to any ordering >. Then

$$J \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle \iff G \cap \mathbb{R}_{\geq 0}[x] = \emptyset.$$

Proof. \Rightarrow : Trivial, as $G \subseteq J$ and $0 \notin G$.

 \Leftarrow : By Proposition 5.5, the Gröbner basis G consists solely of binomials. And by definition, each element of G is normalized. So let G contain only normalized binomials whose non-leading coefficient is negative. Then the S-polynomial of any polynomial $f \in \mathbb{R}[x]$ with respect to a Gröbner basis element $g \in G$,

$$\operatorname{spoly_>}(f,g) \coloneqq \frac{\operatorname{lcm}(\operatorname{LM_>}(f),\operatorname{LM_>}(g))}{\operatorname{LM_>}(f)} \cdot f - \frac{\operatorname{lcm}(\operatorname{LM_>}(f),\operatorname{LM_>}(g))}{\operatorname{LM_>}(g)} \cdot g,$$

will preserve the parity of f, i.e., if $f \in \mathbb{R}_{>0}[x]$ then also spoly_> $(f,g) \in \mathbb{R}_{>0}[x]$, possibly spoly (f, g) = 0.

Now assume there is a non-zero polynomial $f \in J \cap \mathbb{R}_{>0}[x]$. As G is a Gröbner basis, dividing f with respect to G will yield remainder 0. However, the division with respect to G is merely a nested chain of S-polynomials of f with respect to a sequence (g_1, \ldots, g_r) of possibly repeating elements $g_i \in G$:

$$\operatorname{spoly}_{>}(\underbrace{\operatorname{spoly}_{>}(\dots \operatorname{spoly}_{>}(f, g_1) \dots, g_{r-1})}_{=:f_r \neq 0}, g_r) = 0.$$

Abbreviating the penultimate non-zero S-polynomial with f_r , this implies two things: First, as spoly_> $(f_r, g_r) = 0$, f_r must be a multiple of g_r . Second, because spoly preserves the parity of f and $f \in \mathbb{R}_{\geq 0}[x]$, we also have $f_r \in \mathbb{R}_{\geq 0}[x]$. Both together contradict that G contains only binomials with a positive and a negative coefficient.

Proposition 5.7 Let $C_w(I) \subseteq \text{Trop}(I)$ be a maximal cone with $\text{in}_w(I)$ binomial, and let $G \subseteq \text{in}_w(I)$ be a reduced Gröbner basis with respect to any ordering >. Then for any element $\sigma \in S$ we have

$$\sigma \cdot C_w(I) \subseteq \operatorname{Trop}^+(I) \quad \Longleftrightarrow \quad \sigma \cdot G \cap \mathbb{R}_{\geq 0}[x] = \emptyset \ and \ \sigma \cdot G \cap \mathbb{R}_{\leq 0}[x] = \emptyset.$$

Proof. As $G \subseteq \operatorname{in}_w(I)$ is a reduced Gröbner basis with respect to the ordering >, $\sigma \cdot G \subseteq \operatorname{in}_{\sigma \cdot w}(I)$ will be a Gröbner basis with respect to the ordering $>_{\sigma}$ defined by

$$x^{\alpha} >_{\sigma} x^{\beta} : \iff x^{\sigma \cdot \alpha} > x^{\sigma \cdot \beta}.$$

where σ acts on the exponent vectors as it does on the weight space \mathbb{R}^n .

By Proposition 5.5, G consists solely of binomials and hence so does $\sigma \cdot G$. Moreover, $\sigma \cdot G$ is reduced up to normalization. The claim then follows from Lemma 5.6.

Example 5.8 Consider the Grassmannian Gr(2,5), whose Plücker ideal I is generated by three 3-term Plücker relations:

$$I := \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}, \ p_{02}p_{34} - p_{03}p_{24} + p_{04}p_{23}, \ p_{01}p_{34} - p_{03}p_{14} + p_{04}p_{13},$$
$$p_{01}p_{24} - p_{02}p_{14} + p_{04}p_{12}, \ p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} \rangle \leq \mathbb{C}\{\{t\}\}[p_{ij} \mid 0 \leq i < j \leq 4].$$

The Petersen Graph in Figure 9 illustrates the combinatorics of the tropical Grassmannian $TGr_0(2,5)$ modulo its 5-dimensional lineality space generated by

$$\sum_{\substack{0 \le i < j \le 4 \\ i \ne k \ne j}} e_{ij} \quad \text{for } k = 0, \dots, 4,$$

where e_{ij} denotes the unit vector in direction of p_{ij} in the weight space. Each vertex denotes a ray generated by the negative of the inscribed unit vector, and each edge denotes a maximal cone spanned by two rays. The edges in red are the maximal cones inside the positive tropical Grassmannian $TGr^+(2,5)$.

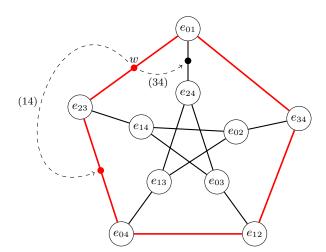


FIGURE 9. The tropical Grassmannian $TGr^+(2,5)$ and its positive cones.

The weight vector $w := e_{01} + e_{23}$ lies in the interior of a maximal cone $C_w(I)$. Its corresponding initial ideal $\operatorname{in}_w(I)$ is generated by the following binomial reduced Gröbner basis:

$$G \coloneqq \{p_{02}p_{13} - p_{12}p_{03}, \ p_{02}p_{14} - p_{12}p_{04}, \ p_{02}p_{34} - p_{03}p_{24}, \ p_{03}p_{14} - p_{13}p_{04}, \ p_{12}p_{34} - p_{13}p_{24}\}.$$

Thus, according to Lemma 5.6, w is contained in $TGr^+(2,5)$.

Moreover, consider the two transpositions $(14), (34) \in S_5$, which act on the coordinate ring as follows:

$$(14): \begin{cases} p_{01} \mapsto p_{04}, & p_{13} \mapsto -p_{34}, \\ p_{02} \mapsto p_{02}, & p_{14} \mapsto -p_{14}, \\ p_{03} \mapsto p_{03}, & p_{23} \mapsto p_{23}, \\ p_{04} \mapsto p_{01}, & p_{24} \mapsto -p_{12}, \\ p_{12} \mapsto -p_{24}, & p_{34} \mapsto -p_{13}, \end{cases} \text{ and } (34): \begin{cases} p_{01} \mapsto p_{01}, & p_{13} \mapsto p_{14}, \\ p_{02} \mapsto p_{02}, & p_{14} \mapsto p_{13}, \\ p_{03} \mapsto p_{04}, & p_{23} \mapsto p_{24}, \\ p_{04} \mapsto p_{03}, & p_{24} \mapsto p_{23}, \\ p_{12} \mapsto p_{12}, & p_{34} \mapsto -p_{34}. \end{cases}$$

Applying them to G yields

$$(14) \cdot G = \left\{ \begin{array}{l} -p_{02}p_{34} + p_{24}p_{03}, \\ -p_{02}p_{14} + p_{24}p_{01}, \\ -p_{02}p_{13} + p_{03}p_{12}, \\ -p_{03}p_{14} + p_{34}p_{01}, \\ p_{24}p_{13} - p_{34}p_{12} \end{array} \right\} \quad \text{and} \quad (34) \cdot G = \left\{ \begin{array}{l} p_{02}p_{14} - p_{12}p_{04}, \\ p_{02}p_{13} - p_{12}p_{03}, \\ -p_{02}p_{34} - p_{04}p_{23}, \\ p_{04}p_{13} - p_{14}p_{03}, \\ -p_{12}p_{34} - p_{14}p_{23} \end{array} \right\}$$

Hence, by Proposition 5.7, $(14) \cdot w = e_{04} + e_{23}$ lies on the positive tropical Grassmannian $TGr^+(2,5)$, whereas $(34) \cdot w = e_{01} + e_{24}$ does not.

From Proposition 5.7, we obtain the following simple algorithm:

Algorithm 5.9 (Positivity of binomial cones)

Input: (G, S), where

- $G \subseteq \operatorname{in}_w(I) \subseteq \mathbb{C}[x]$, a reduced Gröbner basis of a binomial initial ideal $\operatorname{in}_w(I)$,
- S, a group as in Convention 5.1.

Output: $S_w^+(I) := \{ \sigma \in S \mid \sigma \cdot C_w(I) \subseteq \operatorname{Trop}^+(I) \}$, a set of symmetries which map $C_w(I)$ into $\operatorname{Trop}^+(I)$.

1: **return** $\bigcap_{g \in G} \{ \sigma \in S \mid \sigma \cdot g \text{ has coefficients with mixed parity} \}$

Remark 5.10 By [BLMM17, Proof of Lemma 1], any maximal $C_w(I) \subseteq \text{Trop}(I)$ of multiplicity one has a primary initial ideal $\text{in}_w(I)$ and a binomial radical $\sqrt{\text{in}_w(I)}$. And since

And since
$$\left(\operatorname{in}_w(I) \text{ positive} \iff \sqrt{\operatorname{in}_w(I)} \text{ positive}\right)$$
 and $\sigma \cdot \sqrt{\operatorname{in}_w(I)} = \sqrt{\operatorname{in}_{\sigma \cdot w}(I)} \ \forall \sigma \in S$, we can use Algorithm 5.9 to test positivity within their orbit.

5.2. **Algorithm for general cones.** We decide positivity of general maximal Gröbner cones using Definition 5.2. The idea is to reduce the problem to dimension zero, for which we can explicitly compute the signs of the finite number of roots.

We begin with recalling a central lemma for the proof of the Fundamental Theorem of Tropical Geometry, which allows us to read off positivity from the zeroes of the initial ideal.

Lemma 5.11 ([MS15, Proposition 3.2.11]) Let $w = (w_1, \ldots, w_n) \in \text{Trop}(I)$, then

$$V(\operatorname{in}_w(I)) \cap (\mathbb{C}^*)^n = \{(\overline{t^{-w_1}z_1}, \dots, \overline{t^{-w_n}z_n}) \mid (z_1, \dots, z_n) \in V(I) \cap (\mathbb{C}^*)^n \text{ with } \nu(z_i) = w_i\}.$$

In particular, $C_w(I) \subseteq \operatorname{Trop}^+(I)$ if and only if $V(\operatorname{in}_w(I)) \cap \mathbb{R}_{>0}^n \neq \emptyset$.

Proof. \subseteq : This is [MS15, Proposition 3.2.11].

 \supseteq : Let $z := (z_1, \ldots, z_n) \in V(I)$ with $\nu(z) = w$. Then for any $f \in I$ we have f(z) = 0 by definition, which necessarily implies $\operatorname{in}_w(f)(\overline{t^{-w_1}z_1}, \ldots, \overline{t^{-w_n}z_n}) = 0$. Hence $z \in V(\operatorname{in}_w(I))$.

The next lemma allows us to reduce the problem to dimension zero.

Lemma 5.12 Let $J \subseteq \mathbb{R}[x]$ be weighted homogeneous with respect to a weight vector $0 \neq w = (w_1, \dots, w_n) \in \mathbb{Z}^n$, say $w_i \neq 0$. Then

$$V(J) \cap (\mathbb{R}_{>0})^n \neq \emptyset \iff V(J + \langle x_i - 1 \rangle) \cap (\mathbb{R}_{>0})^n \neq \emptyset$$

and moreover $\dim(J + \langle x_i - 1 \rangle) = \dim(J) - 1$.

Proof. \Leftarrow : Clear, as $V(J + \langle x_i - 1 \rangle) \subseteq V(J)$.

 \Rightarrow : Note that the weighted homogeneity of J induces a torus action

$$\mathbb{C}^* \times V(J) \longrightarrow V(J), \quad (a, (z_1, \dots, z_n)) \longmapsto (a^{-w_1} z_1, \dots, a^{-w_n} z_n),$$

with $w_i \neq 0$. Hence for any $z \in V(J) \cap (\mathbb{R}_{>0})^n$ there exists an $a \in \mathbb{C}^*$ with $a^{-w_i} \cdot z_i = 1$.

Algorithm 5.13 (Positivity reduced to dimension 0)

Input: $G \subseteq \operatorname{in}_w(I)$, a reduced Gröbner basis for a maximal cone $C_w(I) \subseteq \operatorname{Trop}(I)$. **Output:** H, generators of a zero-dimensional ideal $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ such that

$$C_w(I) \subseteq \operatorname{Trop}^+(I) \iff J \cap (\mathbb{R}_{>0})^n \neq \emptyset.$$

1: Compute a basis $b_1, \ldots, b_d \in \mathbb{R}^n$ of the d-dimensional vector subspace

$$C_0(\operatorname{in}_w(I)) = \{ v \in \mathbb{R}^n \mid \operatorname{in}_v(g) = g \text{ for all } g \in G \} \subseteq \mathbb{R}^n$$

such that the matrix $B \in \mathbb{R}^{d \times n}$ with rows b_1, \dots, b_d is in row-echelon form.

- 2: Let $\Lambda \subseteq \{1, \ldots, n\}$ denote the column-indices of the pivots of B.
- 3: **return** $H := G \cup \{x_i 1 \mid i \in \Lambda\}.$

Proof of correctness. By Lemma 5.11, we have $C_w(I) \subseteq \operatorname{Trop}^+(I)$ if and only if $V(\operatorname{in}_w(I)) \cap (\mathbb{R}_{>0})^n \neq \emptyset$.

By [BJSST07, Proposition 2.4], $C_0(\operatorname{in}_w(I))$ is the set of all vectors with respect to whom $\operatorname{in}_w(I)$ is weighted homogeneous. We can thus apply Lemma 5.12 iteratively d times to obtain $V(\operatorname{in}_w(I)) \cap (\mathbb{R}_{>0})^n \neq \emptyset$ if and only if $V(J) \cap (\mathbb{R}_{>0})^n \neq \emptyset$.

Additionally, we require an algorithm for computing the signs of the roots of a zero-dimensional ideal. We will treat this part as a black box, and discuss various possibilities in Remark 5.16.

Algorithm 5.14 (Black box algorithm for determining sign)

Input: $H \subseteq J$, a generating set of a zero-dimensional ideal $J \subseteq \mathbb{C}[x]$.

Output: $R \subseteq \{\pm 1\}^n$, such that

$$R = \begin{cases} \operatorname{sgn}(V(H)) & \text{if } V(H) \subseteq (\mathbb{R}^*)^n, \\ \emptyset & \text{otherwise} \end{cases}$$

where $sgn(\cdot)$ denotes the map that is componentwise

$$\mathbb{R}^* \longrightarrow \{\pm 1\}, \quad z \longmapsto \begin{cases} +1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \end{cases}$$

Combining Algorithms 5.13 and 5.14, we obtain Algorithm 5.15 for positivity within orbits of maximal cones.

Algorithm 5.15 (Positivity of maximal-dimensional cones)

Input: (G, S), where

- $G \subseteq \operatorname{in}_w(I)$, a reduced Gröbner basis of a maximal cone $C_w(I) \subseteq \operatorname{Trop}(I)$,
- S, a group as in Convention 5.1.

Output: $S_w^+(I) := \{ \sigma \in S \mid \sigma \cdot C_w(I) \subseteq \operatorname{Trop}^+(I) \}$, a set of symmetries which map $C_w(I)$ onto $\operatorname{Trop}^+(I)$.

1: Apply Algorithm 5.13:

$$H := \text{positivityReduction}(G) \subseteq K[x].$$

2: Apply Algorithm 5.14

$$R := \text{positivitySigns}(H) \subset \{\pm 1\}^n$$
.

3: Construct

$$P := \bigcap_{r \in R} \{ \sigma \in S \mid \sigma \cdot r \ge 0 \},$$

where S acts on $\{\pm 1\}^n$ as it acts on K^n .

4: return P

Remark 5.16 Computing the signs of a finite set of points $V(J) \subseteq \mathbb{C}^n$ for a zero-dimensional ideal $J \preceq \mathbb{C}[x]$ as in Algorithm 5.14 can be done symbolically, numerically or with a mix of both.

One conceptually straightforward option is to approximate V(J) using numerical algebraic geometry. Once a point in V(J) is known with sufficient precision, there are algorithms for certifying reality [HS12] and its sign can simply be read off.

Alternatively, one can symbolically compute a triangular decomposition of J into factors of the form

$$\langle p_1(x_1), x_2^{d_2} - p_2(x_1), \dots, x_n^{d_n} - p_n(x_1) \rangle$$
, p_i univariate polynomials,

from which one can proceed using numerical algorithms for the univariate case.

6. The maximal-dimensional cones of $TGr^+(3,8)$

In this section, we verify [SW05, Conjecture 8.1] for the Grassmannian $Gr_0(3,8)$, which relates the combinatorial structure of the positive tropicalization with the combinatorial structure of a cluster algebra. This serves as a test for the correctness of our computations, as the conjecture has been proven for $Gr_0(3,8)$ by Brodsky and Stump [BS18, Remark 2.23].

Cluster algebras are algebras with a remarkable hidden combinatorial structure. First introduced in [FZ02] by Fomin and Zelevinsky, cluster algebras are subrings of rational function fields $K(x_1, \ldots, x_n)$ generated by a union of overlapping algebraically independent n-subsets. These so-called clusters are connected through mutations, rules which transform one cluster to another, and together they form a simplicial complex called the cluster complex. In [FZ03], Fomin and Zelevinsky completely classify all cluster algebras of finite type, i.e., cluster algebras with finite cluster complexes. Similar to the Cartan-Killing classification of semisimple Lie algebras, their classification associates any finite type cluster algebra a Dynkin graph. One prominent family of Cluster algebras are Grassmannians $Gr_0(k, n)$, initially shown by Fomin and Zelevinsky for k = 2, later fully proven by Scott [Sco06].

The conjecture of Speyer and Williams is based on observations on the Grassmannians $Gr_0(2, n)$, $Gr_0(3, 6)$, and $Gr_0(3, 7)$. By [Sco06], this makes $Gr_0(3, 8)$ the only remaining Grassmannian whose cluster algebra is of finite type, i.e., whose cluster complex is finite.

Conjecture 6.1 ([SW05, Conjecture 8.1]) Let \mathcal{A} be a cluster algebra of finite type and $\mathcal{S}(\mathcal{A})$ its associated cluster complex. If the lineality space of Trop⁺ Spec \mathcal{A} has dimension |C|, then Trop⁺ Spec \mathcal{A} is abstractly isomorphic to the cone over $\mathcal{S}(\mathcal{A})$. If the condition on the lineality space does not hold, the resulting fan is a coarsening of the cone over $\mathcal{S}(\mathcal{A})$.

The conjecture was proven by Brodsky and Stump [BS18] for cluster algebras of type A and of all types of to rank at most 8, which includes $Gr_0(3,8)$.

6.1. Computing the cluster complex $S(Gr_0(3,8))$. Thanks to an implementation by Stump, SAGE [Sage19] features functions for computing and analyzing cluster complexes. The algorithm is based on a work of Ceballos, Labbé, and Stump [CLS14], and requires the root system of the cluster algebra. The root system for $Gr_0(3,8)$ is the exceptional group E_8 [Sco06, Theorem 5]:

SAGE returns an object of type cluster complex, whose maximal cells can be seen via

6.2. Computing the positive tropicalization $TGr^+(3,8)$. Using the algorithms in Section 5 on the computational results in Section 4, we obtain:

Proposition 6.2 There is a Dressian subfan supported on the positive tropical Grassmannian $TGr^+(3,8)$. It is a pure 16-dimensional subfan of the Dressian Dr(3,8) in \mathbb{R}^{56} with an 8-dimensional lineality space and f-vector (120, 2072, 14088, 48544, 93104, 100852, 57768, 13612).

Proof. By Corollary 5.4, positivity only depends on the saturated initial ideals², and, by Theorem 4.8, the saturated initial ideals of $TGr_0(3,8)$ only depend on the Dressian cones. It therefore suffices to check the 4766 S_8 -orbits of Dressian cones in Theorem 4.5 instead of the 14763 S_8 -orbits of Gröbner cones.

Of the 4766 saturated initial ideals of $Gr_0(3,8)$ all but one are binomial and thus admissible for Algorithm 5.9. The unique non-binomial saturated initial ideal arises from the Dressian orbit containing $-e_{015} - e_{024} - e_{067} - e_{126} - e_{137} - e_{235} - e_{346} - e_{457}$, and it has no positive cone in its orbit by Algorithm 5.15. To be more specific, it is not hard to see that the resulting ideal of Algorithm 5.13 has no real solution at all, for example by eliminating all but one variable.

Note that [SW05] considers positive tropicalizations with the coarsest structure refined by the individual Gröbner fans of all cluster variables. For the cluster variables of $Gr_0(3,8)$, recall the following result from [Sc006]:

Theorem 6.3 [Sco06, Theorem 8] The cluster algebra of $Gr_0(3,8)$ possesses 128 cluster variables:

48: Plücker variables p_{ijk} where $\{i, j, k\} \neq \{i, i+1, i+2\} \mod 8$.

56: quadratic Laurent binomials with positive coefficients, inherited from $Gr_0(3,6)$, describing six points in a special position:

$$Y^{123456} = (p_{346})^{-1} \cdot \left(p_{146} p_{236} p_{345} + p_{136} p_{234} p_{456} \right) \quad and \quad X^{123456} = Y^{612345}$$

and their D_8 -translates.

²Instead of Corollary 5.4, we could also rely on the recent results of [ALS20] and [SW20] that the positive tropical Grassmannian $TGr_0^+(k,n)$ equals the positive Dressian $Dr^+(k,n)$, which implies that the Plücker structure on $TGr_0^+(k,n)$ is a coarsening of the Gröbner structure.

24: cubic Laurent trinomials with positive coefficients describing eight points in a special position:

$$A = (p_{578})^{-1} \cdot \left(p_{178} p_{567} \cdot X^{123458} + p_{158} p_{678} \cdot X^{123457} \right) \quad and$$
$$B = (p_{158})^{-1} \cdot \left(p_{128} p_{567} \cdot X^{123458} + p_{258} \cdot A \right)$$

and their D_8 -translates.

Since the Gröbner fan of the Plücker variables consist of a single cone that is the whole space, refining with them does not change anything. Hence, it only remains the 80 polynomials X, Y, A and B.

Theorem 6.4 The positive tropical Grassmannian $TGr^+(3,8)$ endowed with the Plücker structure and refined by the Gröbner fans of all 120 cluster variables of Gr(3,8) is a 16-dimensional pure simplicial fan in \mathbb{R}^{56} with a 8-dimensional lineality space and f-vector (128, 2408, 17936, 67488, 140448, 163856, 100320, 25080). As an abstract simplicial complex, it is isomorphic to the cluster complex S(Gr(3,8)).

Proof. The refinement was straightforwardly computed by intersecting all maximal Dressian cones on $TGr^+(3,8)$ with the maximal cones of the Gröbner fans of the cluster variables.

The isomorphism of the two simplicial complexes was tested using NAUTY [MP14] by McKay, which was called in Polymake through the function fan::isomorphic. The function takes two objects of type IncidenceMatrix, in our case:

- (1) the output of SAGE's C.facets(), C being the cluster complex of type E_8 ,
- (2) the output of POLYMAKE's \$F→MAXIMAL_CONES, \$F being the polyhedral fan supported on TGr⁺(3,8) described above.

7. Open questions

In this section, we briefly discuss some open questions beyond the scope of our article.

7.1. Coarsest structures on tropical varieties. A frequently arising question on the geometry of tropical varieties is whether they have a natural coarsest structure, i.e., whether there is a natural coarsest polyhedral complex supported on them. While it is long known that there is no unique coarsest structure [ST08, Ex. 5.2] and that natural coarsenings of the Gröbner fan exist [Car12], the question remains largely open.

For tropical Grassmannians in characteristic 0 specifically, the question boils down to the following conjecture which all computations up to and including ours support:

Conjecture 7.1 For any $w, v \in \mathrm{TGr}_0(k, n)$ in the relative interior of a maximal Gröbner cone we have

w and v lie on the same cone of $\operatorname{Dr}(k,n) \iff \operatorname{in}_w(\mathcal{I}_{n,k}): p^{\infty} = \operatorname{in}_v(\mathcal{I}_{n,k}): p^{\infty}$,

where $\mathcal{I}_{n,k}$ is the Plücker ideal and (\cdot) : p^{∞} denotes the saturation at the product of all Plücker variables. In particular, there is a subfan of the Dressian Dr(n,k) which coarsens the Gröbner subfan supported on $TGr_0(n,k)$.

In addition to any theoretical insight that such a coarsest structure could offer, the question is of direct relevance for two practical reasons.

On the one hand, it will improve our understanding for the complexity of tropical varieties and consequently also the feasibility of computations in tropical geometry, especially with a view towards applications [LY19]. Current bounds on the f-vector of general tropical varieties are derived from universal Gröbner bases [JS18], and are thus expected to be far from optimal.

On the other hand, it will help with concrete large scale computations. For $TGr_2(4,8)$, our implementation in Section 3 gets stuck on a handful of isolated Gröbner bases containing polynomials of degree 15, for whom simple division with remainder takes several days. Having a coarser structure might allow us to skip those problematic Gröbner cones which are still few and far in between.

7.2. Positive tropicalizations and cluster complexes. Section 5 contains algorithms for testing whether a maximal Gröbner cone $C_w(I) \subseteq \text{Trop}(I)$ is contained in the positive tropicalization $\text{Trop}^+(I)$. It is currently unclear whether this is sufficient to compute $\text{Trop}^+(I)$ as not much is known about its structure. If I is prime, is $\text{Trop}^+(I)$ pure? If $\text{Trop}^+(I) \neq \emptyset$, is $\dim(\text{Trop}^+(I)) = \dim(\text{Trop}(I))$?

In [SW05, Section 8] Speyer and Williams moreover suspect an analogue of [SW05, Conjecture 8.1] to hold for infinite type cluster algebras. For Grassmannians specifically, this means:

Conjecture 7.2 Let $S_{k,n}$ denote the (possibly infinite) cluster complex of the Grassmannian Gr(k,n). Then

$$S_{k,n} = \varprojlim \mathrm{TGr}_0^+(k,n),$$

where the inverse limit is taken over all reembeddings of $\mathrm{TGr}_0(k,n)$ into $\mathbb{R}^{|\Lambda|}$, where Λ is any finite set of cluster variables containing the Plücker variables.

It would be interesting to test the conjecture for $TGr_0^+(4,8)$ once it can be computed.

7.3. Real tropicalizations and the topology of real algebraic varieties. One motivation for tropical geometry stems from the fact that tropical varieties are capable of faithfully capturing the topology of their complex algebraic counterparts. Under special circumstances, the same holds true for real algebraic varieties, as can been seen in Viro's patchworking [Vir06] and a series of works on polynomial systems with many real solutions [ElH18; BSS18; BDIM19]. Currently, computing

the topology of real algebraic varieties remains by and large a challenging open problem with promising solutions only for special cases [BHR18].

Our algorithms in Section 5 can easily be modified to test maximal cones for inclusion in the following notion of real tropicalization:

Definition 7.3 We define the *real tropicalization* of an ideal $I \subseteq K[x]$ to be

$$\operatorname{Trop}_R(I) \coloneqq \operatorname{cl}\Bigl(\nu\bigl(V(I)\cap (R^*)^n\bigr)\Bigr) \subseteq \mathbb{R}^n,$$

where R denotes the subset of complex Puiseux series whose lowest coefficient is real and $cl(\cdot)$ the closure in the euclidean topology.

In the light of recent works on the topic of real tropicalizations [JSY18], albeit slightly different to Definition 7.3, a natural question is whether the real tropicalization as in Definition 7.3 be used to compute the topology of real algebraic varieties in practice.

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