

Points of increase of the sum of digits function of the base phi expansion

Michel Dekking

DIAM, Delft University of Technology, Faculty EEMCS,
P.O. Box 5031, 2600 GA Delft, The Netherlands.

Email: F.M.Dekking@TUDelft.nl

Abstract

We prove that the sequence of first differences of the points of increase of the sum of digits function of the phi expansions of the natural numbers is a morphic sequence. We also show that it is the union of two generalized Beatty sequences based on the golden mean. We give a similar expression for the sequence of first differences of the points of increase of the sum of digits function of the Zeckendorf expansions.

Keywords: Base phi; Wythoff sequence; Fibonacci word; generalized Beatty sequence; Zeckendorf expansion

1 Introduction

Base phi representations were introduced by George Bergman in 1957 ([2]). Here phi is the golden mean, $\varphi = (1 + \sqrt{5})/2$.

A natural number N is written in base phi if N has the form

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1 , and where $d_i d_{i+1} = 11$ is not allowed.

We write these expansions as

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

Ignoring leading and trailing 0's, the base phi representation of a number N is unique ([2]).

Let for $N \geq 0$

$$s_\beta(N) := \sum_{k=L}^{k=R} d_k(N)$$

be the sum of digits function of the base phi expansions. We have

$$(s_\beta(N)) = 0, 1, 2, 2, 3, 3, 3, 2, 3, 4, 4, 5, 4, 4, 4, 5, 4, 4, 2, 3, 4, 4, 5, 5, 4, 5, 6, 6, 7, 5, 5, 5, 6, \dots$$

In [3] asymptotic expressions as $x \rightarrow \infty$ for $\sum_{N < x} s_\beta(N)$ were obtained.

A number N is called a *point of increase* of the function s_β if

$$s_\beta(N + 1) > s_\beta(N).$$

Let I be the function listing the points of increase. We see that the first six points of increase are $I(1) = 0$, $I(2) = 1$, $I(3) = 3$, $I(4) = 7$, $I(5) = 8$, $I(6) = 10$.

Before we continue, we make a comparison with other expansions of the natural numbers. For the binary expansion it is simple to see that in this case the points of increase of the sum of digits function $0, 1, 1, 2, 1, 2, 2, 3, \dots$ are given by the even numbers. For the expansions in the Zeckendorf numeration system, which is related to base phi expansions, see Section 7.

Theorem 1 *The points of increase of the function s_β are given by the function I , which has $I(1) = 0$, and $(I(n+1) - I(n))$ is the fixed point of the morphism μ on the alphabet $\{1, 2, 4\}$ given by*

$$\mu(1) = 12, \mu(2) = 4, \mu(4) = 1244.$$

Another way to formulate this result is the following.

Theorem 2 *The values of I , the points of increase of the function s_β , are given by the union of the two generalized Beatty sequences $(\lfloor n\varphi \rfloor + 2n)_{n \geq 0}$, and $(4\lfloor n\varphi \rfloor + 3n + 1)_{n \geq 0}$.*

Here $\lfloor \cdot \rfloor$ denotes the floor function, and $(\lfloor n\varphi \rfloor)$ is the well known lower Wythoff sequence. These theorems will be proved in Section 6.

2 Generalized Beatty sequences

The two sequences occurring in Theorem 2 are sequences V of the type $V(n) = p\lfloor n\alpha \rfloor + qn + r$, $n \geq 1$, where α is a real number, and p, q , and r are integers. As in [1], we call them *generalized Beatty sequences*. If S is a sequence, we denote its sequence of first order differences as ΔS , i.e., ΔS is defined by

$$\Delta S(n) = S(n+1) - S(n), \quad \text{for } n = 1, 2, \dots$$

It is well known ([9]) that the sequence $\Delta(\lfloor n\varphi \rfloor)$ is equal to the Fibonacci word $x_{1,2} = 1211212112\dots$ on the alphabet $\{1, 2\}$. More generally, we have the following simple lemma.

Lemma 3 ([1]) *Let $V = (V(n))_{n \geq 1}$ be the generalized Beatty sequence defined by $V(n) = p\lfloor n\varphi \rfloor + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the Fibonacci word on the alphabet $\{2p+q, p+q\}$. Conversely, if $x_{a,b}$ is the Fibonacci word on the alphabet $\{a, b\}$, then any V with $\Delta V = x_{a,b}$ is a generalized Beatty sequence $V = ((a-b)\lfloor n\varphi \rfloor) + (2b-a)n + r$ for some integer r .*

3 The base phi expansion and morphisms

A morphism is a map from the set of infinite words over an alphabet to itself, respecting the concatenation operation. The canonical example is the Fibonacci morphism σ on the alphabet $\{0, 1\}$ given by

$$\sigma(0) = 01, \quad \sigma(1) = 0.$$

We recall some results from [5]. In that paper, four possibilities for the digits of $\beta(N)$ close to the ‘decimal’ point were considered, and coded by a ‘type’ from an alphabet of four letters $\{A, B, C, D\}$. The code is denoted T . It is defined by

$$\begin{aligned} T(N) = A & \text{ iff } d_1 d_0(N) = 10, & T(N) = B & \text{ iff } d_1 d_0 d_{-1}(N) = 000, \\ T(N) = C & \text{ iff } d_0(N) = 1, & T(N) = D & \text{ iff } d_1 d_0 d_{-1}(N) = 001. \end{aligned}$$

Let the morphism γ on the alphabet $\{A, B, C, D\}$ be given by

$$\gamma(A) = AB, \quad \gamma(B) = C, \quad \gamma(C) = D, \quad \gamma(D) = ABC.$$

Theorem 4 ([5]) *The sequence $(T(N))_{N \geq 2}$ is the unique fixed point of the morphism γ .*

Using Lemma 3, the following result can be deduced from Theorem 4.

Theorem 5 ([5]) *Let $\beta(N) = (d_i(N))$ be the base phi representation of a natural number N . Then:*

- A: $d_1 d_0(N) = 10$ if and only if $N = \lfloor n\varphi \rfloor + 2n - 1$ for some natural number $n \geq 1$,
- B: $d_1 d_0 d_{-1}(N) = 000$ if and only if $N = \lfloor n\varphi \rfloor + 2n$ for some natural number $n \geq 1$,
- C: $d_0(N) = 1$ if and only if $N = \lfloor n\varphi \rfloor + 2n + 1$ for some natural number $n \geq 0$,
- D: $d_1 d_0 d_{-1}(N) = 001$ if and only if $N = 3\lfloor n\varphi \rfloor + n + 1$ for some natural number $n \geq 1$.

The following table gives the relevant information for the first 24 natural numbers.

N	$\beta(N)$	$T(N)$	$s_\beta(N)$	N	$\beta(N)$	$T(N)$	$s_\beta(N)$	N	$\beta(N)$	$T(N)$	$s_\beta(N)$
1	1	C	1	9	10010 · 0101	A	4	17	101010 · 000001	A	4
2	10 · 01	A	2	10	10100 · 0101	B	4	18	1000000 · 000001	B	2
3	100 · 01	B	2	11	10101 · 0101	C	5	19	1000001 · 000001	C	3
4	101 · 01	C	3	12	100000 · 101001	D	4	20	1000010 · 010001	A	4
5	1000 · 1001	D	3	13	100010 · 001001	A	4	21	1000100 · 010001	B	4
6	1010 · 0001	A	3	14	100100 · 001001	B	4	22	1000101 · 010001	C	5
7	10000 · 0001	B	2	15	100101 · 001001	C	5	23	1001000 · 100101	D	5
8	10001 · 0001	C	3	16	101000 · 100001	D	4	24	1001010 · 000101	A	5

4 The Recursive Structure Theorem

The following result was anticipated in [7], [8], and [11], and proved in [6].

Theorem 6 [Recursive Structure Theorem]

I For all $n \geq 1$ and $k = 1, \dots, L_{2n-1}$ one has $\beta(L_{2n} + k) = \beta(L_{2n}) + \beta(k) = 10 \dots 0 \beta(k) 0 \dots 01$.

II For all $n \geq 2$ and $k = 1, \dots, L_{2n-2} - 1$

$$I_n : \beta(L_{2n+1} + k) = 1000(10)^{-1} \beta(L_{2n-1} + k) (01)^{-1} 1001,$$

$$K_n : \beta(L_{2n+1} + L_{2n-1} + k) = 1010(10)^{-1} \beta(L_{2n-1} + k) (01)^{-1} 0001.$$

Moreover, for all $n \geq 2$ and $k = 0, \dots, L_{2n-3}$

$$J_n : \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1} \beta(L_{2n-2} + k) (01)^{-1} 001001.$$

This theorem is useful in the proof of Lemma 7.

5 The types of points of increase

We say N is of type E if $d_2 d_1 d_0(N) = 001$. Note that this is a special case of type C. The first three type E numbers are given by $N = 1$, $N = 8$ and $N = 19$.

Lemma 7 If N has type E, i.e., $d_2 d_1 d_0(N) = 001$, then $d_2 d_1 d_0(N) \cdot d_{-1} d_{-2} d_{-3}(N) = 001 \cdot 000$.

Proof: The statement is true for $N = 1$ and $N = 8$. All other numbers of type E have an expansion of length at least 5 before and after the ‘decimal’ point. Since the relations in the Recursive Structure Theorem 6 only change at most two of the exterior digits, the claim follows by induction from this theorem. \square

As in [6] we give ourselves the freedom to write also non-admissible representations as $\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R$. For example, since $4 = 2 \times 2$ and $\beta(2) = 10 \cdot 01$, we will write $\beta(4) \doteq 20 \cdot 02$. Here the \doteq -sign indicates that we consider a non-admissible representation.

Proposition 8 A number N is a point of increase if and only if N is of type B or of type E.

Proof: If N is of type B: $\beta(N) = v00 \cdot 0w$ for two words v and w , then $\beta(N + 1) = v01 \cdot 0w$. So the number of 1’s in the expansion of $N + 1$ has increased by 1.

If N is of type E, then by Lemma 7, $\beta(N) = v001 \cdot 000w$. So

$$\beta(N + 1) \doteq v002 \cdot 000w \doteq 010 \cdot 010w.$$

So the number of 1’s in $N + 1$ has increased by 1. This ends the proof of one side of the proposition.

The proof for the other side of the proposition is more involved, as the action of adding 1 is no longer purely local as for type B and type E. We use that the natural numbers are partitioned in numbers with types from $\{A, B, C, D\}$, as implied by Theorem 4.

– Suppose N is of type A. Then $\beta(N) = v010 \cdot w$. So

$$\beta(N + 1) \doteq v011 \cdot w \doteq v100 \cdot 10w.$$

In general, this will not be the phi-expansion of $N + 1$: if v has suffix 1, then another golden mean shift takes place, and more might follow. However, all these operations will not increase the number of 1's.

– Suppose N is of type D. Then $\beta(N) = v00 \cdot 1w$. So

$$\beta(N + 1) \doteq v01 \cdot 1w \doteq v10 \cdot 0w.$$

In general, this will not be the phi-expansion of $N + 1$, but as above the number of 1's can not increase.

— Suppose N is of type C, but not of type E. Then there are three possibilities: $\beta(N) = v0101 \cdot 00w$, $\beta(N) = v0101 \cdot 0100w$, or $\beta(N) = v0101 \cdot 01010w$. In the first case we have

$$\beta(N + 1) \doteq v0102 \cdot 00w \doteq v0110 \cdot 01w \doteq v1000 \cdot 01w.$$

In general, this will not be the phi-expansion of $N + 1$, but as above the number of 1's can not increase.

In the second case we have

$$\beta(N + 1) \doteq v0102 \cdot 0100w \doteq v0110 \cdot 0200w \doteq v1000 \cdot 1001w.$$

In general, this will not be the phi-expansion of $N + 1$, but as above the number of 1's can not increase.

In the third case, where $\beta(N) = v0101 \cdot 01010w$, we have

$$\beta(N + 1) \doteq v0102 \cdot 01010w \doteq v0110 \cdot 02010w \doteq v1000 \cdot 10020w.$$

This will not be the phi-expansion of $N + 1$. On the left side in $v1000$, the number of 1's can not increase. On the right side we have the two subcases $w = 0x$ and $w = 10x$. In the first subcase

$$\beta(N + 1) \doteq v1000 \cdot 100200x \doteq v1000 \cdot 101001x.$$

Again, this might not be the phi-expansion of $N + 1$, but as before the number of 1's can not increase.

In the second subcase

$$\beta(N + 1) \doteq v1000 \cdot 1002010x \doteq v1000 \cdot 1010020x.$$

This behaves as $\beta(N + 1) \doteq v1000 \cdot 10020w$ above. We can therefore iterate, ending in the worst case in a $\beta(N)$ of the form $(10)^K 1$ for some positive integer K . The conclusion is that apart from the 1's in v and w , there will be maximally four 1's in $\beta(N + 1)$, the same corresponding number as in $\beta(N)$. For $\beta(N) = (10)^K 1$, one finds that the number of 1's in $\beta(N + 1)$ is even one less (respectively $K + 1$ and K).

After checking the result for the first 17 natural numbers, which might not be covered by the arguments above, the proposition is proved. \square

6 The positions of the points of increase

We first prove Theorem 2.

Proof of Theorem 2: There will be points of increase at the numbers $N = 0, 3, 7, 10, 14, 18, 21, \dots$ of type B given by

$$N = \lfloor n\varphi \rfloor + 2n, \quad n \geq 0$$

as follows directly from Proposition 8 and Theorem 5. The other points of increase are numbers N of type E. If N is such a number, then $\beta(N) = v001 \cdot 000w$ for some words v, w , and $\beta(N + 1) = v010 \cdot 010w$, as computed in the proof of Proposition 8. In the paper [5] it has been shown in Remark 6.3 that the numbers N with $d_{-2}(N) = 1$ occur in one of three generalized Beatty sequences given by

$$(4\lfloor n\varphi \rfloor + 3n + r) \text{ for } r = 2, 3 \text{ or } 4.$$

Since $N + 1$ is such a number, and N is not, it follows that $N = 4\lfloor n\varphi \rfloor + 3n + 1$ for some $n \geq 0$. \square

Proof of Theorem 1: This is based on Theorem 2. Let

$$I_B := (\lfloor n\varphi \rfloor + 2n)_{n \geq 0}, \quad I_E := (4\lfloor n\varphi \rfloor + 3n + 1)_{n \geq 0}.$$

By Lemma 3, the difference sequence of the sequence $(\lfloor n\varphi \rfloor + 2n, n \geq 1)$ is equal to the Fibonacci word $x_{4,3} = 4344344344\dots$ on the alphabet $\{4, 3\}$, and the difference sequence of the sequence $(4\lfloor n\varphi \rfloor + 3n + 1, n \geq 1)$ is the Fibonacci word $x_{11,7} = 11, 7, 11, 11, \dots$. However, in Theorem 2 the sequences start at $n = 0$, yielding the two difference sequences

$$\Delta I_B =: 3x_{3,4} = 34344344344\dots \quad \Delta I_E = 7x_{11,7} = 7, 11, 7, 11, 11, \dots$$

It is well-known (see, e.g., Lemma 12 in [4]) that the sequences $bx_{a,b}$ are fixed points of the morphisms $g_{a,b}$ given by

$$g_{a,b}(a) = ab, \quad g_{a,b}(b) = abb.$$

The return words of 3 in ΔI_B are 34 and 344. We code these words by the differences that they yield between successive occurrences of 3's, i.e., by the letters 7 and 11. Then, since

$$g_{3,4}(34) = 34344, \quad g_{3,4}(344) = 34344344,$$

the return words induce a derived morphism

$$7 \rightarrow 7, 11, \quad 11 \rightarrow 7, 11, 11.$$

This derived morphism happens to be equal to $g_{11,7}$, the morphism giving the sequence ΔI_E . This implies that to merge the two sequences I_B and I_E to obtain I , one has to replace the 3's in ΔI_B by 1,2. This so called decoration of ΔI_B , induces a morphism μ on the alphabet $\{1, 2, 4\}$ in the usual way (see, e.g., [4]), given by

$$\mu(1) = 12, \quad \mu(2) = 4, \quad \mu(4) = 1244.$$

This proves the theorem. □

7 The Zeckendorf numeration system

In the Zeckendorf numeration system the natural numbers are written as the 'greedy' sum of Fibonacci numbers as

$$N = \sum_{i \geq 0} d_i(N) F_i,$$

where the d_i are from $\{0, 1\}$, and $d_i d_{i-1} = 1$ is not allowed. Let $s_Z(N) = \sum_{i \geq 0} d_i(N)$ be the sum of digits of such an expansion¹, and let I_Z be the function of the points of increase of the function s_Z

Theorem 9 *The function I_Z , the points of increase of the function s_Z , is given² for $n = 1, 2, \dots$ by*

$$I_Z(n) = (\lfloor (n-1)\varphi \rfloor + n - 1).$$

Proof: The function s_Z as a sequence is a morphic sequence on an infinite alphabet, i.e., $(s_Z(N))$ is a letter to letter projection of a fixed point of a morphism τ . The alphabet is $\{0, 1, \dots, j, \dots\} \times \{0, 1\}$, and τ is the morphism given by

$$\begin{aligned} \tau((j, 0)) &= (j, 0)(j+1, 1), \\ \tau((j, 1)) &= (j, 0). \end{aligned}$$

The letter-to-letter map is given by the projection on the first coordinate: $(j, i) \rightarrow j$ for $i = 0, 1$. The fixed point of τ with initial symbol $(0, 0)$ projected on the first coordinate equals $(s_Z(N))$.

¹Sequence A007895 in [10].

²Sequence A026274 in [10].

See the Comments of sequence A007895 in [10] for a proof of this.

Projection on the second coordinate of τ yields the Fibonacci morphism σ given by

$$\sigma(0) = 01, \sigma(1) = 0.$$

Thus the second coordinate of the fixed point of τ equals the infinite Fibonacci word $x_F = 0100101001001\dots$. Obviously, the increase points of s_Z occur if and only if the word $(j, 0)(j+1, 1)$ occurs in the fixed point of τ if and only if the word 01 occurs in x_F . Since 11 does not occur in x_F , this means that we have to shift the positions of 1's in x_F by 1. It is well known that these positions are given by the upper Wythoff sequence $(\lfloor n\varphi^2 \rfloor) = (\lfloor n\varphi \rfloor + n)$. \square

References

- [1] J.-P. Allouche and F.M. Dekking, Generalized Beatty sequences and complementary triples, *Mosc. J. Comb. Number Theory* 8 (2019), 325–342. doi.org/10.2140/moscow.2019.8.325
- [2] G. Bergman, A number system with an irrational base, *Math. Mag.* 31 (1957), 98–110.
- [3] C. Cooper and R. E. Kennedy, The first moment of the number of 1's function in the beta-expansion of the positive integers, *Journal of Institute of Mathematics & Computer Sciences* 14 (2001), 69–77.
- [4] F.M. Dekking, Morphic words, Beatty sequences and integer images of the Fibonacci language, *Theoretical Computer Science* 809 (2020), 407–417. doi.org/10.1016/j.tcs.2019.12.036
- [5] F.M. Dekking, Base phi representations and golden mean beta-expansions, To appear in *Fibonacci Quart.* (2020).
- [6] F. Michel Dekking, How to add two natural numbers in base phi. To appear in *Fibonacci Quart.* (2020).
- [7] E. Hart, On Using Patterns in the Beta-Expansions To Study Fibonacci-Lucas Products, *Fibonacci Quart.* 36 (1998), 396–406.
- [8] E. Hart and L. Sanchis, On the occurrence of F_n in the Zeckendorf decomposition of nF_n , *Fibonacci Quart.* 37 (1999), 21–33.
- [9] M. Lothaire, *Algebraic combinatorics on words*, Cambridge University Press, 2002.
- [10] *On-Line Encyclopedia of Integer Sequences*, founded by N. J. A. Sloane, electronically available at <http://oeis.org>.
- [11] G.R. Sanchis and L.A. Sanchis, On the frequency of occurrence of α^i in the α -expansions of the positive integers, *Fibonacci Quart.* 39 (2001), 123–173.