On a supercongruence conjecture of Z.-W. Sun

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Abstract. In this paper, we partly prove a supercongruence conjectured by Z.-W. Sun in 2013. Let p be an odd prime and let $a\in\mathbb{Z}^+$. Then if $p\equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{\lfloor\frac{5}{6}p^a\rfloor}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p^a}\right)\pmod{p^2},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

Keywords: Supercongruences; Binomial coefficients; Fermat quotient; Jacobi symbol.

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1. Introduction

In the past years, congruences for sums of binomial coefficients have attracted the attention of many researchers (see, for instance, [2, 4, 5, 10, 12, 16, 17, 19]). In 2011, Sun [17] proved that for any odd prime p and $a\in\mathbb{Z}^+$,

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2}.$$

Liu and Petrov [7] showed some congruences on sums of q-binomial coefficients.

In 2006, Adamchuk [1] conjectured that for any prime $p\equiv 1 \pmod 3$,

$$\sum_{k=1}^{\frac{2}{3}(p-1)}\binom{2k}{k}\equiv 0\pmod{p^2}.$$

Recently, Mao [9] confirmed this conjecture.
Pan and Sun [13] proved that for any prime $p\equiv 1 \pmod 4$ or $1<a\in\mathbb{Z}^+$,

$$\sum_{k=0}^{\lfloor\frac{3}{4}p^a\rfloor}\frac{\binom{2k}{k}}{(-4)^k}\equiv\left(\frac{2}{p^a}\right)\pmod{p^2}.$$

In 2017, Mao and Sun [11] showed that for any prime $p \equiv 1 \pmod 4$ or $1 < a \in \mathbb{Z}^+$,

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{(16)^k} \equiv \left(\frac{-1}{p^a}\right) \pmod{p^3}.$$

Sun [15] proved that for any odd prime p and $a \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}. \tag{1.1}$$

In this paper, we partly prove Sun's conjecture [15, Conjecture 1.2(i)].

Theorem 1.1. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

We shall prove Theorem 1.1 in Section 2.

2. Proof of Theorem 1.1

Lemma 2.1. ([6]). *For any prime $p > 3$, we have the following congruences modulo p*

$$H_{\lfloor p/2 \rfloor} \equiv -2q_p(2),\ H_{\lfloor p/3 \rfloor} \equiv -\frac{3}{2}q_p(3),\ H_{\lfloor p/6 \rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3).$$

Proof of Theorem 1.1. In view of (1.1), we just need to verify that

$$\sum_{k=(p^a+1)/2}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv 0 \pmod{p^2}. \tag{2.1}$$

Let k and l be positive integers with $k + l = p^a$ and $0 < l < p^a/2$. In view of [13], we have

$$\frac{l}{2}\binom{2l}{l} = \frac{(2l-1)!}{(l-1)!^2} \not\equiv 0 \pmod{p^a} \tag{2.2}$$

and

$$\binom{2k}{k} \equiv -p^a \frac{(l-1)!^2}{(2l-1)!} = -\frac{2p^a}{l\binom{2l}{l}} \pmod{p^2}. \tag{2.3}$$

So we have

$$\sum_{k=(p^a+1)/2}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \sum_{k=(p^a+1)/2}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{-2p^a}{(p^a-k)\binom{2p^a-2k}{p^a-k}16^k} = \frac{-2p^a}{16^{p^a}} \sum_{k=\lfloor \frac{p^a}{6} \rfloor + 1}^{(p^a-1)/2} \frac{16^k}{k\binom{2k}{k}} \pmod{p^2}.$$

It is easy to see that for $k = 1, 2, \ldots, (p^a - 1)/2$,

$$\frac{\binom{(p^a-1)/2}{k}}{\binom{2k}{k}/(-4)^k} = \frac{\binom{(p^a-1)/2}{k}}{\binom{1/2}{k}} = \prod_{j=0}^{k-1} \frac{(p^a-1)/2 - j}{-1/2 - j} = \prod_{j=0}^{k-1}\left(1 - \frac{p^a}{2j+1}\right) \equiv 1 \pmod{p}. \quad (2.4)$$

This, with Fermat little theorem yields that

$$\sum_{k=(p^a+1)/2}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv -\frac{p^a}{8} \sum_{k=\lfloor \frac{p^a}{6} \rfloor + 1}^{(p^a-1)/2} \frac{(-4)^k}{k\binom{(p^a-1)/2}{k}} \equiv -p^a \sum_{k=\lfloor \frac{p^a}{6} \rfloor}^{(p^a-3)/2} \frac{(-4)^k}{\binom{(p^a-3)/2}{k}} \pmod{p^2}.$$

Thus, by (2.1) we only need to show that

$$p^{a-1} \sum_{k=\lfloor \frac{p^a}{6} \rfloor}^{(p^a-3)/2} \frac{(-4)^k}{\binom{(p^a-3)/2}{k}} \equiv 0 \pmod{p}. \quad (2.5)$$

Now we set $n = (p^a - 1)/2, m = \lfloor \frac{p^a}{6} \rfloor, \lambda = -4$, then we only need to prove that

$$p^{a-1} \sum_{k=m}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} \equiv 0 \pmod{p}. \quad (2.6)$$

In view of [18], we have

$$\sum_{k=m}^{n-1} \frac{\lambda^k}{\binom{n-1}{k}} = n \sum_{k=0}^{n-1-m} \frac{\lambda^{m+k}}{(\lambda+1)^{k+1}} \sum_{i=0}^{n-1-m-k} \frac{(-1)^i \binom{n-1-m-k}{i}}{m+i+1} + \frac{n\lambda^n}{(\lambda+1)^{n+1}} \sum_{k=m}^{n-1} \frac{(\lambda+1)^{k+1}}{k+1}.$$

It is easy to check that for each $0 \leq k \leq n-1-m$

$$\begin{aligned}\sum_{i=0}^{n-1-m-k} \binom{n-1-m-k}{i} \frac{(-1)^i}{m+i+1} &= \int_0^1 \sum_{i=0}^{n-1-m-k} \binom{n-1-m-k}{i} (-x)^i x^m dx \\ &= \int_0^1 x^m (1-x)^{n-1-m-k} dx = B(m+1, n-m-k),\end{aligned}$$

where $B(P,Q)$ stands for the beta function. It is well known that the beta function relate to gamma function:

$$B(P,Q) = \frac{\Gamma(P)\Gamma(Q)}{\Gamma(P+Q)}.$$

So

$$B(m+1, n-m-k) = \frac{\Gamma(m+1)\Gamma(n-m-k)}{\Gamma(n-k+1)} = \frac{m!(n-m-k-1)!}{(n-k)!} = \frac{1}{(m+1)\binom{n-k}{m+1}}.$$

Therefore

$$\begin{aligned}\sum_{k=m}^{n-1}\frac{\lambda^k}{\binom{n-1}{k}} &= \frac{n}{m+1}\sum_{k=0}^{n-1-m}\frac{\lambda^{m+k}}{(\lambda+1)^{k+1}\binom{n-k}{m+1}}+\frac{n\lambda^n}{(\lambda+1)^{n+1}}\sum_{k=m}^{n-1}\frac{(\lambda+1)^{k+1}}{k+1}\\ &= \frac{n}{m+1}\sum_{k=m+1}^{n}\frac{\lambda^{m+n-k}}{(\lambda+1)^{n-k+1}\binom{k}{m+1}}+\frac{n\lambda^n}{(\lambda+1)^{n+1}}\sum_{k=m+1}^{n}\frac{(\lambda+1)^{k}}{k}\\ &= \frac{n\lambda^n}{(\lambda+1)^{n+1}}\left(\frac{\lambda^m}{m+1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}}+\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k}\right).\end{aligned}$$

By (2.6), we just need to show that

$$p^{a-1}\frac{\lambda^m}{m+1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}} \equiv -p^{a-1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k} \pmod p. \tag{2.7}$$

It is obvious that

$$\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}} = \sum_{k=m+1}^{n}\frac{1}{\binom{k}{m+1}}\left(\frac{3}{4}\right)^k = \sum_{k=m+1}^{n}\frac{1}{\binom{k}{m+1}}\sum_{j=0}^{k}\frac{\binom{k}{j}}{(-4)^j} = \mathfrak{B}+\mathfrak{C},$$

where

$$\mathfrak{B} = \sum_{j=m+1}^{n}\frac{1}{(-4)^j}\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{k}{m+1}}, \qquad \mathfrak{C} = \sum_{j=0}^{m}\frac{1}{(-4)^j}\sum_{k=m+1}^{n}\frac{\binom{k}{j}}{\binom{k}{m+1}}.$$

By the following transformation

$$\frac{\binom{k}{j}}{\binom{k}{m+1}} = \frac{k!(m+1)!(k-m-1)!}{j!(k-j)!k!} = \frac{(m+1)!(k-m-1)!(j-m-1)!}{j!(k-j)!(j-m-1)!} = \frac{\binom{k-m-1}{j-m-1}}{\binom{j}{m+1}}.$$

We have

$$\mathfrak{B} = \sum_{j=m+1}^{n}\frac{1}{(-4)^j}\sum_{k=j}^{n}\frac{\binom{k-m-1}{j-m-1}}{\binom{j}{m+1}} = \sum_{j=m+1}^{n}\frac{1}{(-4)^j\binom{j}{m+1}}\sum_{k=0}^{n-j}\binom{k+j-m-1}{j-m-1}.$$

By [3, (1.48)], we have

$$\mathfrak{B} = \sum_{j=m+1}^{n}\frac{1}{(-4)^j\binom{j}{m+1}}\binom{n-m}{j-m}.$$

It is easy to show that

$$\frac{\binom{n-m}{j-m}}{\binom{j}{m+1}} = \frac{(n-m)!(m+1)!(j-m-1)!}{j!(n-j)!(j-m)!} = \frac{n+1}{j-m}\frac{\binom{n}{j}}{\binom{n+1}{m+1}}.$$

Thus,

$$\mathfrak{B} = \frac{n+1}{\binom{n+1}{m+1}} \sum_{j=m+1}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j}.$$

Now we calculate $\mathfrak{C}$. First we have the following transformation

$$\frac{\binom{k}{j}}{\binom{k}{m+1}} = \frac{k!(m+1)!(k-m-1)!}{j!(k-j)!k!} = \frac{(m+1)!(k-m-1)!(m-j+1)!}{j!(k-j)!(m-j+1)!} = \frac{\binom{m+1}{j}}{\binom{k-j}{m-j+1}}.$$

Thus,

$$\mathfrak{C} = \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{(-4)^j} \sum_{k=m+1}^{n} \frac{1}{\binom{k-j}{m-j+1}} = \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{(-4)^j} \sum_{k=0}^{n-m-1} \frac{1}{\binom{k+m+1-j}{m-j+1}}.$$

By using package `Sigma`, we find the following identity,

$$\sum_{k=0}^{N} \frac{1}{\binom{k+i}{i}} = \frac{i}{i-1} - \frac{N+1}{(i-1)\binom{N+i}{N}}.$$

Substituting $N = n-m-1, i = m+1-j$ into the above identity, we have

$$\mathfrak{C} = \sum_{j=0}^{m-1} \binom{m+1}{j} \frac{1}{(-4)^j} \left(\frac{m+1-j}{m-j} - \frac{n-m}{(m-j)\binom{n-j}{n-m-1}} \right) + (m+1)\left(-\frac{1}{4}\right)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

It is easy to check that

$$\frac{(n-m)\binom{m+1}{j}}{\binom{n-j}{n-m-1}} = \frac{(m+1)!((n-m)!(m+1-j)!}{j!(n-j)!(m+1-j)!} = \frac{(m+1)!((n-m)!}{j!(n-j)!} = \frac{(n+1)\binom{n}{j}}{\binom{n+1}{m+1}}.$$

Therefore

$$\mathfrak{C} = (m+1) \sum_{j=0}^{m-1} \frac{\binom{m}{j}}{(m-j)(-4)^j} - \frac{n+1}{\binom{n+1}{m+1}} \sum_{j=0}^{m-1} \frac{\binom{n}{j}}{(m-j)(-4)^j} + (m+1)\left(-\frac{1}{4}\right)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

Hence

$$\mathfrak{B} + \mathfrak{C} = (m+1) \sum_{j=0}^{m-1} \frac{\binom{m}{j}}{(m-j)(-4)^j} + \frac{n+1}{\binom{n+1}{m+1}} \sum_{\substack{j=0 \\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} + (m+1)\left(-\frac{1}{4}\right)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

That is

$$\frac{\lambda^m}{m+1}(\mathfrak{B} + \mathfrak{C}) = \lambda^m \sum_{j=0}^{m-1} \frac{\binom{m}{j}}{(m-j)(-4)^j} + \frac{\lambda^m}{\binom{n}{m}} \sum_{\substack{j=0 \\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} + H_{n-m}. \tag{2.8}$$

It is obvious that

$$\sum_{k=1}^{n}\frac{(-3)^k}{k}=\int_0^1\sum_{k=1}^{n}(-3)^k x^{k-1}dx=-3\int_0^1\sum_{k=0}^{n-1}(-3x)^k dx=-3\int_0^1\frac{1-(-3x)^n}{1+3x}dx$$
$$=3\int_0^1\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1+3x)^{k-1}dx=\int_1^4\sum_{k=1}^{n}(-1)^k y^{k-1}dy=\sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{4^k-1}{k}$$

and

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}=\int_0^1\sum_{k=1}^{n}\binom{n}{k}(-1)^k x^{k-1}dx=\int_0^1\frac{(1-x)^n-1}{x}dx=\int_0^1\frac{y^n-1}{1-y}dy$$
$$=-\int_0^1\sum_{k=0}^{n-1}y^k dy=-\sum_{k=0}^{n-1}\frac{1}{k+1}=-\sum_{k=1}^{n}\frac{1}{k}.$$

This, with [16, (1.20)] yields that

$$\sum_{k=1}^{n}\frac{(\lambda+1)^k}{k}=\sum_{k=1}^{n}\frac{(-3)^k}{k}=\sum_{k=1}^{n}\binom{n}{k}\frac{(-4)^k}{k}+H_n.$$

On the other hand, by [3, (1.48)] we have

$$\sum_{k=1}^{m}\frac{(\lambda+1)^k-1}{k}=\sum_{k=1}^{m}\frac{(-3)^k-1}{k}=\sum_{k=1}^{m}\frac{1}{k}\sum_{j=1}^{k}\binom{k}{j}(-4)^j=\sum_{j=1}^{m}\frac{(-4)^j}{j}\sum_{k=j}^{m}\binom{k-1}{j-1}$$
$$=\sum_{j=1}^{m}\frac{(-4)^j}{j}\binom{m}{j}=(-4)^m\sum_{j=0}^{m-1}\frac{1}{(m-j)(-4)^j}\binom{m}{j}.$$

Hence

$$\sum_{k=1}^{m}\frac{(\lambda+1)^k}{k}=(-4)^m\sum_{j=0}^{m-1}\frac{\binom{m}{j}}{(m-j)(-4)^j}+H_m.$$

So

$$\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k}=\sum_{k=1}^{n}\binom{n}{k}\frac{(-4)^k}{k}+H_n-\lambda^m\sum_{j=0}^{m-1}\frac{\binom{m}{j}}{(m-j)(-4)^j}-H_m. \tag{2.9}$$

In view of [16, (1.20)], and by (2.2), (2.3) and (2.4) we have

$$p^{a-1}\sum_{k=1}^{n}\binom{n}{k}\frac{(-4)^k}{k}\equiv p^{a-1}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\equiv p^{a-1}\sum_{k=1}^{p^a-1}\frac{\binom{2k}{k}}{k}\equiv 0 \pmod p. \tag{2.10}$$

It is obvious that

$$p^{a-1}H_n=p^{a-1}\sum_{k=1}^{n}\frac{1}{k}\equiv p^{a-1}\sum_{j=1}^{(p-1)/2}\frac{1}{jp^{a-1}}=H_{(p-1)/2} \pmod p$$

and $p^{a-1}H_m \equiv H_{\lfloor p/6\rfloor} \pmod p$, $p^{a-1}H_{n-m} \equiv H_{\lfloor p/3\rfloor} \pmod p$.
Now $p \equiv 1 \pmod 3$, so by [8, Lemma 17(2)], we have $\binom{n}{m} \not\equiv 0 \pmod p$. These, with (2.7)-(2.10) yield that we only need to prove that

$$p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} \equiv 0 \pmod p. \tag{2.11}$$

Now $n=(p^a-1)/2, m=(p^a-1)/6$. So by Fermat little theorem we have

$$p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} \equiv -3(-1)^{(p^a-1)/2}p^{a-1}\sum_{\substack{j=0\\ j\neq n-m}}^{n} \frac{\binom{n}{j}(-4)^j}{3j+1} \pmod p.$$

There are only the items $3j+1=p^{a-1}(3k+1)$ with $k=0,1,\ldots,(p-1)/2$ and $k\neq (p-1)/3$, so by [9, Theorem 1.2] we have

$$\begin{aligned} p^{a-1}\sum_{\substack{j=0\\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} &\equiv -3(-1)^{\frac{p^a-1}{2}}\sum_{\substack{k=0\\ k\neq (p-1)/3}}^{(p-1)/2} \frac{\binom{n}{kp^{a-1}+\frac{p^{a-1}-1}{3}}(-4)^{kp^{a-1}+\frac{p^{a-1}-1}{3}}}{3k+1} \\ &\equiv -3(-1)^{\frac{p^a-1}{2}}(-4)^{\frac{p^{a-1}-1}{3}}\binom{\frac{p^{a-1}-1}{2}}{\frac{p^{a-1}-1}{3}}\sum_{\substack{k=0\\ k\neq (p-1)/3}}^{(p-1)/2} \frac{\binom{n}{k}(-4)^k}{3k+1} \equiv 0 \pmod p. \end{aligned}$$

Therefore the proof of Theorem 1.1 is complete. □

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