ON THE VOLUME OF HYPERPLANE SECTIONS OF A d-CUBE

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ABSTRACT. We obtain an optimal upper bound for the normalised volume of a hyperplane section of an origin-symmetric *d*-dimensional cube. This confirms a conjecture posed by Imre Bárány and Péter Frankl.

1. Statement of the results

Let $\mathcal{C}^d = [-1/2, 1/2]^d$ be the *d*-dimensional unit cube. Throughout this paper we assume that $d \geq 2$. For a nonzero vector $\boldsymbol{v} \in \mathbb{R}^d$ we will denote by \boldsymbol{v}^{\perp} the hyperplane orthogonal to \boldsymbol{v} and consider the section $\mathcal{C}^d \cap \boldsymbol{v}^{\perp}$ of the cube \mathcal{C}^d . Let further $\|\cdot\|_1$ and $\|\cdot\|_2$ denote ℓ_1 and ℓ_2 norms, respectively. In this paper we will be interested in the quantity

(1)
$$V_d = \max_{\boldsymbol{v} \in \mathbb{R}^d} \frac{\|\boldsymbol{v}\|_1}{\|\boldsymbol{v}\|_2} \cdot \operatorname{vol}_{d-1}(\mathcal{C}^d \cap \boldsymbol{v}^{\perp}),$$

where $\operatorname{vol}_{d-1}(\cdot)$ stands for the (d-1)-volume. Imre Bárány and Péter Frankl [3] conjectured that the maximum in (1) is given by the vector $\boldsymbol{v} = \mathbf{1}_d := (1, \ldots, 1)$. Our main result confirms this conjecture.

Theorem 1. We have

(2)
$$V_d = \sqrt{d} \cdot \operatorname{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^{\perp}).$$

It is known that

$$\lim_{d \to \infty} \operatorname{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^{\perp}) = \sqrt{\frac{6}{\pi}}$$

(see [9], [11] and e. g. [5]). The expression (2) also allows finding the exact value of V_d in the following way. Let $s = (s_1, \ldots, s_d) \in \mathbb{S}^{d-1}$ be a unit vector. Then

(3)
$$\operatorname{vol}_{d-1}(\mathcal{C}^d \cap \boldsymbol{s}^{\perp}) = \frac{2}{\pi} \int_0^\infty \prod_{i=1}^d \frac{\sin s_i t}{s_i t} \, \mathrm{d}t$$

(see e. g. [2]). Consider the *sinc* integral [4]

$$\sigma_d = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^d \mathrm{d}t \; .$$

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In view of (2) and (3) we have

(4)
$$V_d = \frac{2\sqrt{d}}{\pi} \int_0^\infty \left(\frac{\sin\frac{t}{\sqrt{d}}}{\frac{t}{\sqrt{d}}}\right)^d \mathrm{d}t = d\,\sigma_d\,.$$

Further

$$\sigma_d = \frac{d}{2^{d-1}} \sum_{0 \le r < d/2, r \in \mathbb{Z}} \frac{(-1)^r (d-2r)^{d-1}}{r! (d-r)!}$$

(see e. g. [10]). The sequences of numerators and denominators of $\sigma_d/2$ can be found in [12]. Theorem 1 and (3) immediately imply the following lower bound for sinc integrals.

Corollary 2. For any unit vector $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{S}^{d-1}$

$$\frac{2||\boldsymbol{s}||_1}{\pi d} \int_0^\infty \prod_{i=1}^d \frac{\sin s_i t}{s_i t} \, \mathrm{d}t \le \sigma_d \, .$$

It is known that $0 < \sigma_{d+1}/\sigma_d < 1$ (see e. g. [1, Lemma 1]). Theorem 1 also implies the following lower bound for the ratio of consecutive sinc integrals.

Corollary 3. We have

$$\frac{d}{d+1} \le \frac{\sigma_{d+1}}{\sigma_d}.$$

2. Intersection body of \mathcal{C}^d

We can associate with each star body \mathcal{L} the distance function $f_{\mathcal{L}}(\boldsymbol{x}) = \inf\{\lambda > 0 : \boldsymbol{x} \in \lambda \mathcal{L}\}$. The intersection body $I\mathcal{L}$ of a star body $\mathcal{L} \subset \mathbb{R}^d$ (recall that we assume $d \geq 2$) is defined as the **0**-symmetric star body with distance function

$$f_{I\mathcal{L}}(\boldsymbol{x}) = rac{\|\boldsymbol{x}\|_2}{\operatorname{vol}_{d-1}(\mathcal{L} \cap \boldsymbol{x}^{\perp})}$$

The Busemann theorem (see e. g. [6], Chapter 8) states that if \mathcal{L} is **0**-symmetric and convex, then $I\mathcal{L}$ is a convex set. For more details on intersection bodies we refer the reader to [7, 8].

For convenience, in what follows we will work with normalised cube

$$\mathcal{Q}^d = \frac{1}{\operatorname{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^{\perp})^{1/(d-1)}} \cdot \mathcal{C}^d.$$

Then, in particular,

(5) $\operatorname{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{1}_d^{\perp}) = 1.$

Lemma 4. The affine hyperplane

$$\mathcal{H} = \{ \boldsymbol{x} \in \mathbb{R}^d : x_1 + \dots + x_d = \sqrt{d} \}$$

is a supporting hyperplane of IQ^d .

Proof. Let $f = f_{IQ^d}$ denote the distance function of IQ^d , so that $IQ^d = \{ \boldsymbol{x} \in \mathbb{R}^d : f(\boldsymbol{x}) \leq 1 \}$. By (5), for the point

(6)
$$\boldsymbol{h} := \frac{1}{\sqrt{d}} \boldsymbol{1}_d = \left(\frac{1}{\sqrt{d}}, \dots, \frac{1}{\sqrt{d}}\right)$$

we have $f(\mathbf{h}) = 1$. Therefore \mathbf{h} is on the boundary of IQ^d .

Suppose, to derive a contradiction, that \mathcal{H} is not a supporting hyperplane of $I\mathcal{Q}^d$. Observe that $\mathbf{h} \in \mathcal{H} \cap I\mathcal{Q}^d$. Hence for any $\epsilon > 0$ there exists a point $\mathbf{p} = (p_1, \ldots, p_d)$ in the interior of $I\mathcal{Q}^d$ with

$$\|\boldsymbol{h} - \boldsymbol{p}\|_2 < \epsilon$$

and $p_1 + \dots + p_d > \sqrt{d}$.

By (7) we may assume that $\boldsymbol{p} \in \mathbb{R}^d_{>0}$. Further, as the point \boldsymbol{p} is in the interior of $I\mathcal{Q}^d$ we may assume, for simplicity, that the entries of \boldsymbol{p} are pairwise distinct: $p_i \neq p_j$ for $i \neq j$. Consider d points

For each *i*, the section $\mathcal{Q}^d \cap \boldsymbol{p}_i^{\perp}$ is the image of the section $\mathcal{Q}^d \cap \boldsymbol{p}_1^{\perp}$ under an orthogonal transformation defined by a permutation matrix. Therefore $\boldsymbol{p}_i \in I\mathcal{Q}^d$. Set

$$oldsymbol{y} = rac{1}{d}(oldsymbol{p}_1 + \dots + oldsymbol{p}_d) = rac{\sum_{i=1}^d p_i}{\sqrt{d}}oldsymbol{h}$$
 .

By construction, \boldsymbol{y} is a convex combination of the points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d$. Since $I\mathcal{Q}^d$ is convex, $\boldsymbol{y} = (y_1, \ldots, y_d) \in I\mathcal{Q}^d$. Further

$$y_1 + \dots + y_d = \sum_{i=1}^d p_i > \sqrt{d}$$
.

Therefore the point h must be in the interior of IQ^d . The derived contradiction completes the proof.

3. Proof of Theorem 1

It is sufficient to show that for any unit vector $\boldsymbol{v} \in \mathbb{S}^{d-1}$ the inequality

(8)
$$\|\boldsymbol{v}\|_1 \cdot \operatorname{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{v}^{\perp}) \leq \sqrt{d} \cdot \operatorname{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{1}_d^{\perp}) = \sqrt{d}$$

holds.

In view of symmetry of \mathcal{Q}^d we may assume without loss of generality that $\boldsymbol{v} \in \mathbb{R}^d_{\geq 0}$. Consider the plane \mathcal{P} spanned by the vector \boldsymbol{h} , defined by (6), and the vector \boldsymbol{v} and let α be the angle between these two vectors with $\cos(\alpha) = \boldsymbol{h} \cdot \boldsymbol{v}$ (Figure 1). It is not difficult to see that $\cos(\alpha) \geq 1/\sqrt{d}$ and, consequently, $\alpha < \pi/2$.

Notice that h is orthogonal to the line $\mathcal{H} \cap \mathcal{P}$. Let u denote the intersection point of the line spanned by v and $\mathcal{H} \cap \mathcal{P}$. Further, let w be the orthogonal projection of v onto the line spanned by h.

Then we have

(9)
$$\cos(\alpha) = \|\boldsymbol{w}\|_2 = \frac{1}{\|\boldsymbol{u}\|_2}.$$

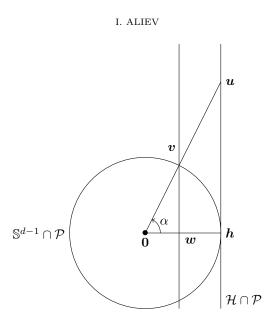


FIGURE 1. Geometric argument on the plane \mathcal{P}

Since $h \in \mathcal{H}$, all points x on the line passing through the points v and w have $x_1 + \cdots + x_d = \sqrt{d} \|w\|_2$. Therefore, we have $\|v\|_1 = \sqrt{d} \|w\|_2$. In was shown in Lemma 4 that \mathcal{H} is a supporting hyperplane of IQ^d . Hence we have

(10)
$$\operatorname{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{v}^{\perp}) \leq \|\boldsymbol{u}\|_2.$$

Finally, using (9) and (10), we have

$$\|\boldsymbol{v}\|_1 \cdot \operatorname{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{v}^\perp) \leq \sqrt{d} \, \|\boldsymbol{w}\|_2 \|\boldsymbol{u}\|_2 = \sqrt{d} \,,$$

that confirms (8).

4. Proof of Corollary 3

It was observed in [3] that the sequence $\{V_d\}_{d=1}^{\infty}$ is increasing: $V_d \leq V_{d+1}$ for all $d \geq 2$. It is now sufficient to note that, by Theorem 1 (see (4)), we can write $V_d = d\sigma_d$.

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