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ON THE SET $\{\pi(kn): k = 1, 2, 3, ...\}$

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ABSTRACT. An open conjecture of Z.-W. Sun states that for any integer n > 1 there is a positive integer $k \leq n$ such that $\pi(kn)$ is prime, where $\pi(x)$ denotes the number of primes not exceeding x. In this paper, we show that for any positive integer n the set $\{\pi(kn) : k = 1, 2, 3, ...\}$ contains infinitely many P_2 -numbers which are products of at most two primes. We also prove that under the Bateman–Horn conjecture the set $\{\pi(4k) : k = 1, 2, 3, ...\}$ contains infinitely many primes.

1. INTRODUCTION

For $x \ge 0$, let $\pi(x)$ denote the number of primes not exceeding x. For the asymptotic behavior of the prime-counting function $\pi(x)$, by the Prime Number Theorem we have

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to +\infty.$$

Since there are no simple closed formula for the exact values of $\pi(x)$ with x > 0, it is difficult to obtain combinatorial properties of the prime-counting function $\pi(x)$.

In 1962, S. Golomb [2]] proved that for any integer m > 1 there is an integer n > 1 with $n/\pi(n) = m$ (i.e., $\pi(n) = n/m$). In 2017 Z.-W. Sun [6] obtained the following general result: For any $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we have

$$\pi(n) = \frac{n+a}{m}$$
 for some integer $n > 1$

if and only if $a \leq s_m$, where

$$s_m := \max\{km - p_k : k \in \mathbb{Z}^+\} = \max\{km - p_k : k = 1, 2, \dots, \lfloor e^{m+1} \rfloor\}$$

with p_k the k-th prime. This implies that for any integer m > 4 we have $\pi(mn) = m + n$ for some $n \in \mathbb{Z}^+$ (cf. [6, Corollary 1.2]).

On Feb. 9, 2014, Z.-W. Sun [4] made the following conjecture.

Conjecture 1.1. (Sun [5, Conjecture 2.1(i)]) For any integer n > 1, there is a positive integer $k \leq n$ such that $\pi(kn)$ is prime.

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This conjecture was verified by Sun for all $n = 2, 3, ..., 2 \times 10^7$ (cf. [4]). For n = 10, among the ten numbers

$$\pi(10) = 4, \ \pi(20) = 8, \ \pi(30) = 10, \ \pi(40) = 12, \ \pi(50) = 15,$$

 $\pi(60) = 17, \ \pi(70) = 19, \ \pi(80) = 22, \ \pi(90) = 24, \ \pi(100) = 25.$

only $\pi(60) = 17$ and $\pi(70) = 19$ are prime. Note also that among the 13 numbers

$$(13) = 6, \ \pi(2 \times 13) = 9, \ \pi(3 \times 13) = 12, \ \pi(4 \times 13) = 15, \ \pi(5 \times 13) = 18$$

$$\pi(6 \times 13) = 21, \ \pi(7 \times 13) = 24, \ \pi(8 \times 13) = 27, \ \pi(9 \times 13) = 30$$

$$\pi(10 \times 13) = 31, \ \pi(11 \times 13) = 34, \ \pi(12 \times 13) = 36, \ \pi(13 \times 13) = 39$$

only $\pi(10 \times 13) = 31$ is prime.

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Motivated by Conjecture 1.1, for any $n \in \mathbb{Z}^+$ we introduce the set

$$\mathcal{A}_n = \{ \pi(kn) : k \in \mathbb{Z}^+ \}.$$
(1.1)

Clearly, \mathcal{A}_1 coincides with $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathcal{A}_2 = \mathbb{Z}^+$ since $\pi(p_j + 1) = j$ for all $j = 2, 3, \ldots$. As $\lim_{k \to +\infty} \pi(3k) = +\infty$, and $\pi(3(k+1)) - \pi(3k) \in \{0, 1\}$ for all $k \in \mathbb{Z}^+$, we see that

 $\mathcal{A}_3 = \{ m \in \mathbb{Z}^+ : m \ge \pi(3) \} = \{ 2, 3, \ldots \}.$

It is not known whether \mathcal{A}_4 contains infinitely many primes.

Throughout this paper, for any $A \subseteq \mathbb{Z}^+$ and $x \ge 0$, we define

$$A(x) := \{ a \le x : \ a \in A \}.$$
(1.2)

Now we present our first theorem.

Theorem 1.1. Let $S \subseteq \mathbb{Z}^+$ with

$$\lim_{x \to +\infty} \frac{|S(x)|}{x/\log x} = +\infty$$

Then, for any $n \in \mathbb{Z}^+$ the set \mathcal{A}_n contains infinitely many elements of S.

If
$$S = \{a \in \mathbb{Z}^+ : a \equiv r \pmod{m}\}$$
 with $m, r \in \mathbb{Z}^+$, then
$$\lim_{x \to +\infty} \frac{|S(x)|}{x} = \frac{1}{m} \text{ and hence } \lim_{x \to +\infty} \frac{|S(x)|}{x/\log x} = +\infty$$

Thus Theorem 1.1 yields the following corollary.

Corollary 1.1. For any $n, m, r \in \mathbb{Z}^+$, there are infinitely many $a \in \mathcal{A}_n$ with $a \equiv r \pmod{m}$.

In contrast, Sun [5, Conjecture 2.2] conjectured that for each $n \in \mathbb{Z}^+$ we have $n \mid \pi(kn)$ for some $k = 1, \ldots, p_n$, and also $\{\pi(kn) : k = 1, \ldots, 2p_n\}$ contains a complete system of residues modulo n.

As usual, for any $r \in \mathbb{Z}^+$, we call $n \in \mathbb{Z}^+$ a P_r -number if it is a product of at most r primes. It is known (cf. [7, Theorem 6.4]) that the number of P_2 -numbers up to X is $\gg \frac{X}{\log X} \log \log X$ for X > 1. So Theorem 1.1 has the following consequence.

Corollary 1.2. For any $n \in \mathbb{Z}^+$, the set \mathcal{A}_n contains infinitely many P_2 -numbers.

The following conjecture extends a conjecture of Hardy and Littlewood concerning twin primes.

Conjecture 1.2 (P. T. Bateman and R. A. Horn [1]). For $N \in \mathbb{Z}^+$ let V(N) denote the number of positive integers $n \leq N$ with 4n + 1 and 4n + 3 twin prime. Then

$$V(N) = 4\mathfrak{S}\frac{N}{\log^2 N} \Big(1 + o(1)\Big) \quad as \ N \to +\infty,$$

where the twin prime constant \mathfrak{S} is given by

$$\mathfrak{S} = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) \approx 0.6601618$$

with p in the product runs over all odd primes.

Now we state our second theorem.

Theorem 1.2. Assuming the truth of Conjecture 1.2, there are infinitely many primes in \mathcal{A}_4 .

We are going to prove Theorems 1.1 and 1.2 in Sections 2 and 3 respectively.

2. Proof of Theorem 1.1

For $A \subseteq \mathbb{Z}^+$, we write A^c for $\mathbb{Z}^+ \setminus A$, the complement of A.

Lemma 2.1. For integers $K \geq 3$, we have

$$|\mathcal{A}_n^c(X)| \ll_n \frac{X}{\log X},\tag{2.1}$$

where $X = \pi(Kn)$ with $n \in \mathbb{Z}^+$.

Proof. Note that

$$|\mathcal{A}_{n}^{c}(X)| = \left| \left\{ a \in \mathbb{Z}^{+} : a \in \bigcup_{k=1}^{K} (\pi((k-1)n), \pi(kn)) \right\} \right|$$

=
$$\sum_{\substack{1 \le k \le K \\ \pi(kn) - \pi(kn-n) \ge 2}} (\pi(kn) - \pi(kn-n) - 1).$$
 (2.2)

Since $\pi(kn) - \pi(kn - n) - 1 \le n$, we have

$$|\mathcal{A}_n^c(X)| \le n|\mathcal{K}|,$$

where

$$\mathcal{K} = \{ 1 \le k \le K : \ \pi(kn) - \pi(kn - n) \ge 2 \}.$$
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For each $k \in \mathcal{K}$, there exist two primes p and q such that $kn - k and hence <math>2 \le q - p < n$. Thus

$$|\mathcal{K}| \le \sum_{2 \le h < n} |\{p \le Kn : p \text{ and } p + h \text{ are both prime}\}|.$$

It is well known (cf. [3, Theorem 6.7]) that

$$\pi_h(Kn) := |\{p \le Kn : p \text{ and } p+h \text{ are both prime}\}| \ll_h \frac{Kn}{\log^2(Kn)}$$

where the implied constant may depend on $h \ge 2$. Combining the above, we obtain

$$|\mathcal{A}_n^c(X)| \le n|\mathcal{K}| \ll_n \frac{K}{\log^2 K}.$$
(2.3)

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In view of the Prime Number Theorem,

$$X = \frac{Kn}{\log(Kn)} \Big(1 + o(1) \Big). \tag{2.4}$$

Now (2.1) follows from (2.3) and (2.4). This concludes the proof.

Proof of Theorem 1.1. By Lemma 2.1, there is a constant $C_n > 0$ such that for any $K \in \{3, 4, \ldots\}$ we have

$$|\mathcal{A}_n^c(X)| \le C_n \frac{X}{\log X},\tag{2.5}$$

where X = Kn. As $\lim_{x \to +\infty} |S(x)|/(x/\log x) = +\infty$, if $K \in \mathbb{Z}^+$ is large enough then

$$\frac{|S(X)|}{X/\log X} \ge 2C_n. \tag{2.6}$$

Let $K \in \{3, 4, \ldots\}$ be large enough so that (2.6) holds. Then, for $X = \pi(Kn)$ we have

$$\frac{|S_1(X)|}{X/\log X} + \frac{|S_2(X)|}{X/\log X} = \frac{|S(X)|}{X/\log X} \ge 2C_n,$$

where $S_1 = S \cap \mathcal{A}_n$ and $S_2 = S \cap \mathcal{A}_n^c$. As

$$\frac{|S_2(X)|}{X/\log X} \le \frac{|\mathcal{A}_n^c(X)|}{X/\log X} \le C_n$$

by (2.5), we obtain

$$|S_1(X)| \ge C_n \frac{X}{\log X}.$$

In view of the above, $\lim_{x\to+\infty} |S_1(x)| = +\infty$. So \mathcal{A}_n contains infinitely many elements of S. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $X = \pi(4K)$ with $K \in \{3, 4, \ldots\}$. Applying (2.2) with n = 4, we get

$$|\mathcal{A}_{4}^{c}(X)| = \sum_{\substack{1 \le k \le K \\ \pi(4k) - \pi(4k-4) \ge 2}} \left(\pi(4k) - \pi(4k-4) - 1\right).$$

For each integer k > 1, the interval (4k - 4, 4k] contains at most two primes. Note also that $\pi(4) - \pi(0) = 2$. So we have

$$|\mathcal{A}_4^c(X)| = 1 + |\mathcal{V}|,$$

where

$$\mathcal{V} = \{ 1 \le k < K : \ \pi(4k+4) - \pi(4k) = 2 \}.$$

For any k = 1, ..., K - 1, clearly $\pi(4k + 4) - \pi(4k) = 2$ if and only if both 4k + 1 and 4k + 3 are twin prime. Under Conjecture 1.2, we have

$$|\mathcal{V}| = V(K-1) = 4\mathfrak{S}\frac{K}{\log^2 K} \Big(1 + o(1)\Big)$$

and hence

$$|\mathcal{A}_4^c(X)| = 4\mathfrak{S}\frac{K}{\log^2 K} \Big(1 + o(1)\Big).$$

By the Prime Number Theorem,

$$X = \frac{4K}{\log K} \Big(1 + o(1) \Big) \text{ and } \pi(X) = \frac{X}{\log X} (1 + o(1)).$$

Thus

$$\pi(X) - |\mathcal{A}_4^c(X)| = (1 - \mathfrak{S}) \frac{X}{\log X} \Big(1 + o(1) \Big).$$

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Note that $\mathfrak{S} < 1$. By the above,

$$|\{p \le X : p \text{ is a prime in } \mathcal{A}_4\}| \ge \pi(X) - |\mathcal{A}_4^c(X)| \to +\infty$$

as $X = 4K \to +\infty$. So \mathcal{A}_4 contains infinitely many primes. This concludes the proof. \Box

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