Catalan and Schröder permutations sortable by two restricted stacks

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Abstract

We investigate the permutation sorting problem with two restricted stacks in series by applying a right greedy process. Instead of σ-machine introduced recently by Cerbai *et al.*, we focus our study on (σ, τ)-machine where the first stack avoids two patterns of length three, σ and τ, and the second stack avoids the pattern 21. For $(\sigma, \tau) = (123, 132)$, we prove that sortable permutations are those avoiding some generalized patterns, which provides a new set enumerated by the Catalan numbers. When $(\sigma, \tau) = (132, 231)$, we prove that sortable permutations are counted by the Schröder numbers. Finally, we suspect that other pairs of length three patterns lead to sets enumerated by the Catalan numbers and leave them as open problems.

1 Introduction

For several decades, the problem of permutation sorting was widely studied in the literature. It consists in rearranging an arbitrary permutation into the identity one using a sequence of specific transformations. One of the most relevant references is probably *Sorting and Searching* [13], the third volume of Knuth's seminal monograph TAOCP, where the author studies the complexity of several sorting algorithms. In this paper we consider stack sorting problems which have been initiated in [12], the first volume of TAOCP. From an input permutation $\pi = \pi_1\pi_2\dots\pi_n$, the process consists in reading π from left to right, and for each entry π_i either *push* it onto the stack, or *pop* the top of the stack into the output. Knuth [12] proved that a permutation π is *sortable* (*i.e.* there is a sequence of push and pop operations such that the output is the identity permutation) if and only if there do not exist $i < j < k$ such that $\pi_k < \pi_i < \pi_j$, or equivalently π avoids the pattern 231; and sortable permutations are counted by Catalan numbers. This result is the birth of the theory of pattern containment, and it already led to many other variations of the stack sorting problem, see [6] for an early survey and the references therein. For instance, some studies consider relaxed rules for push and pop operations [1], whereas others deal with several stacks in series [3, 14, 17] or parallel [2, 19]. Today, the characterization of sortable permutations using two stacks in series remains an open problem. However, Pierrot and Rossin [15] give a polynomial time algorithm to decide whether a permutation is sortable with two stacks in series. West [20] studies the restricted case where permutations are sorted by two passes through a stack and provides a characterization of sortable permutations in terms of generalized patterns.

Recently, motivated by a better understanding of permutations sortable using two stacks in series, Cerbai *et al.* [8] investigate the sorting σ-*machine* where at each step of the process it performs the rightmost possible operation with respect to the following two conditions: (a) the first stack avoids the pattern σ; (b) the second stack avoids the pattern 21 (which is a necessary condition for the machine to sort permutations). The avoidance of a pattern in a stack means that the stack read from top to bottom avoids the pattern. Let $Sort(\sigma)$ be the set of sortable permutations by the σ-machine. The authors of [8] prove that $Sort(12)$ is equal to the set $Av(213)$ of permutations avoiding the pattern 213. Also, they characterize and enumerate length n permutations in $Sort(k(k-1)\cdots 21)$ and $Sort(123)$, and they leave open the characterisation for the remaining length three patterns. Some of these results were generalised by Cerbai in [9], where permutations and patterns with repeated elements are allowed.

In this paper, we consider the (σ,τ)-machine as described above where condition (a) is replaced by: *the first stack avoids* σ *and* τ. See Figure 1 for a description of the (σ,τ)-machine. For $(\sigma,\tau)=(123,132)$, we prove that sortable permutations are those avoiding some generalized patterns, which provides still another set enumerated by Catalan numbers. When $(\sigma,\tau)=(132,231)$, we prove that sortable permutations are counted by Schröder numbers. See Table 1 for an overview of these results. Finally, some conjectures and open questions are presented and discussed.

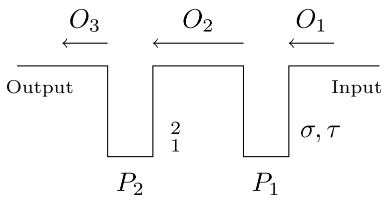

Figure 1: The (σ,τ)-machine consists of two stacks in series where the first stack P_1 avoids (from top to bottom) σ and τ while the second P_2 avoids the pattern 21. At each step of the process, we perform the rightmost possible operation among O_1, O_2, O_3, where O_1 pushes in P_1 the current entry of the input permutation, O_2 pops the top of P_1 and pushes it in P_2, and O_3 pops the top of P_2 and pushes it in the output permutation. For instance, if $\sigma=123$, $\tau=132$ then $\pi=35124$ is sortable by applying the following operations: $O_1, O_1, O_2, O_1, O_1, O_1, O_2, O_2, O_2, O_3, O_3, O_2, O_3, O_3, O_3$.

σ,τ	OEIS	Sequence $\lvert Sort_n(\sigma,\tau)\rvert$	Avoided patterns
123,132	A000108	$1,2,5,14,42,132,429,\ldots$ (Catalan numbers)	3214, 2314, 4213, $[24\bar{1}3$
132,231	A006318	$1,2,6,22,90,394,1806,\ldots$ (Schröder numbers)	1324, 2314

Table 1: Enumeration and characterization of sortable permutations with the (σ,τ)-machine.

To end this section, we present classical definitions about pattern avoidance in permutations (see [11] for instance). For two permutations σ and π, we say that π avoids the pattern σ whenever there does not exist a subsequence of π order isomorphic to σ. Whenever the occurrence of the pattern σ is constrained to start with the first entry of π we denote the vincular pattern by placing [in front the pattern; for instance an occurrence of [12 in π corresponds to $\pi_1\pi_i$ for some $i>1$ with $\pi_i>\pi_1$. A barred pattern σ is a one where some entries are barred. Let $\hat{\sigma}$ be the pattern obtained from σ by removing all barred entries and rescaling the rest to a permutation. A permutation π avoids σ if each occurrence of $\hat{\sigma}$ in π can be extended in an occurrence of σ (considered without bars). For instance, a permutation π avoids the pattern $[24\bar{1}3$ if any subsequence $\pi_1\pi_i\pi_j$, $1<i<j$, $\pi_1<\pi_j<\pi_i$, can be extended into an occurrence of the pattern 2413.

2 The $(123,132)$-machine

In this section we characterize and enumerate length n sortable permutations with the $(123,132)$-machine. We begin by stating general facts about the (σ,τ)-machine, which are direct consequences of the characterization of sortable permutations with an unrestricted stack obtained by Knuth [12].

Fact 1: During the sorting process by the (σ,τ)-machine,

– if u is in the stack P_2, and v above w in P_1 with $w<u<v$, *i.e.* uvw yields the pattern 231, then the permutation is not sortable;

– if u is above v which is in turn is above w in P_1 with $w<u<v$, then the permutation is not sortable.

Theorem 1. *A permutation π belongs to $Sort(123, 132)$ if and only if π avoids* 3214*,* 2314*,* 4213 *and* $[24\bar{1}3$.

Proof. First, let us prove that a sortable permutation π cannot contain an occurrence of one of the patterns 3214, 2314 and 4213. For a contradiction, let us assume that π contains such a pattern, and for an occurrence θ of it we pick a particular one: $\theta = \pi_i\pi_j\pi_k\pi_\ell$, $1 \leq i < j < k < \ell \leq n$, where ℓ is chosen minimal, and k, j, and i are chosen maximal, in this order.
– θ is an occurrence of 3214. Due to the choice of i, j, k, ℓ above, a simple observation shows that $\pi_i > \pi_u > \pi_j$ for $k < u < \ell$; and $\pi_u < \pi_{u+1}$ for $k \leq u \leq \ell - 1$. Whenever the entry π_k is pushed in P_1, since P_1 avoids 123 it follows that at least one of the two entries π_i and π_j does not belong to P_1. Since $\pi_{k+1} > \pi_k$, the next step pushes π_{k+1} in P_1. (i) Assume that π_i and π_j are both in P_2. Then π_k is just below π_{k+1} in P_1 and $\pi_k < \pi_j < \pi_{k+1}$. Fact 1 implies that π is not sortable, which is a contradiction. (ii) Assume that π_j is in P_2 and π_i in P_1. Fact 1 implies that π is not sortable because $\pi_j\pi_{k+1}\pi_k$ is an occurrence of 231, which is a contradiction. (iii) Assume that π_i is in P_2 and π_j in P_1. Since $\pi_u < \pi_{u+1}$ for $k \leq u \leq \ell - 1$, the next steps of the sorting process push successively in P_1 all entries $\pi_{k+1}, \ldots, \pi_\ell$. Fact 1 implies that π is not sortable because $\pi_i\pi_\ell\pi_k$ is an occurrence of 231, which again is a contradiction. So, π cannot contain the pattern 3214.
– If θ is an occurrence of 2314 or of 4231, the proof is similar, *mutatis mutandis*, to that when θ is an occurrence of 3214.
Now let us assume that π contains the pattern $[24\bar{1}3$. This means that π contains a subsequence $\theta = \pi_1\pi_i\pi_j$, $1 < i < j$, with $\pi_1 < \pi_j < \pi_i$ and such that there is not a k in the interval (i, j) with $\pi_k < \pi_1$. We choose i and j by taking j minimal and i maximal, in this order. Due to this choice, a simple observation shows that $j = i + 1$. (i) Assume that π_i and π_j are both in P_1. In this case, P_1 contains the pattern 231 with $\pi_1\pi_j\pi_i$, and Fact 1 implies that π is not sortable, which is a contradiction. (ii) Assume that π_i is in P_2 and π_j in P_1. Since $j = i + 1 > i$, the sorting process must push π_i from P_1 to P_2 before π_j is pushed into P_1. Thus, pushing π_j in P_1 creates necessarily the pattern 123 or 132, which means that there are $u < v \leq i$, such that either $\pi_j < \pi_u < \pi_v$ or $\pi_j < \pi_v < \pi_u$, and π_u and π_v are in P_1. If π_u is greater than π_i, then the process has created a pattern 231 with $\pi_i\pi_u\pi_1$ and Fact 1 contradicts the sortability. If there is π_u such that $\pi_i > \pi_u > \pi_j$, then the process has created a pattern 231 with $\pi_j\pi_u\pi_1$ which also is a contradiction. (iii) Assume that π_i is in P_1 and π_j in P_2. At the step where π_j is pushed in P_1, the pattern 231 is created in P_1 with $\pi_j\pi_i\pi_1$ and by Fact 1 we have again a contradiction.
Conversely, we assume that π is not sortable, and we will prove that π contains the pattern 3214, 2314, 4213 or $[24\bar{1}3$. Let us consider the moment where the process is stuck. Let π_k (resp. π_ℓ) the top of the stack P_2 (resp. P_1), then we necessarily have $\pi_\ell > \pi_k$. We distinguish two cases: (i) $k < \ell$ and (ii) otherwise. In the case (i), there necessarily exists a previous step where there are i and j, $i < k < j < \ell$, such that π_i and π_k are in P_1 and π_j is the first entry of the input permutation, and such that $\theta = \pi_i\pi_k\pi_j$ is an occurrence of 321 or of 231 (otherwise we cannot push π_k in P_2). If θ is an occurrence of 321 then, using $\pi_\ell > \pi_k$, $\pi_i\pi_k\pi_j\pi_\ell$ is an occurrence of 3214 or of 4213. If θ is an occurrence of 231 then, using $\pi_\ell > \pi_k$, $\pi_i\pi_k\pi_j\pi_\ell$ is an occurrence of 2314. In the case (ii), we have $k > \ell$. In order to push π_k in P_2, we need to push firstly π_k in P_1 by keeping π_ℓ in P_1. A necessarily condition is that $\pi_1\pi_\ell\pi_k$ does not form the pattern 231 nor 321, which implies that $\pi_k > \min\{\pi_1, \pi_\ell\}$. Since $\pi_\ell > \pi_k$, this implies that $\pi_k > \pi_1$. So $\pi_1\pi_\ell\pi_k$ is an occurrence of 132. With the same reasoning, all values between π_ℓ and π_k must be greater than π_1. Thus, there is a 132-pattern $\pi_1\pi_\ell\pi_k$ that cannot be extended into $24\bar{1}3$ which implies that π does not avoid $[24\bar{1}3$. □

Corollary 1. *Permutations of length n in $Sort(123, 132)$ are enumerated by the Catalan numbers (see* A000108 *in [18]).*

Proof. Let $A_n(k)$ be the set of length n permutations starting with k in $Sort(123, 132)$, and let $A_n^1(k)$ be the subset of $A_n(k)$ consisting of permutations $\pi = \pi_1\pi_2\ldots\pi_n$ where any occurrence $\pi_1\pi_k\pi_\ell$, with $1 < k < \ell$ and $\pi_\ell = \pi_1 - 1$, $\pi_k > \pi_1$, can be extended into an occurrence $\pi_1\pi_k\pi_j\pi_\ell$, with $k < j < \ell$ and $\pi_j < \pi_\ell$. We set $A_n^2(k) = A_n(k) \backslash A_n^1(k)$.
First we prove that there is a one-to-one correspondence α between $A_n^1(k)$ and $A_n(k-1)$. Let α be the map from $A_n^1(k)$ to $A_n(k-1)$ where $\pi' = \alpha(\pi)$ is obtained from π by swapping the two entries π_1 and $\pi_1 - 1$ in π. For instance, $\alpha(45123) = 35124$. Since $\pi \in A_n^1(k)$, it is easy to see that π' avoids the pattern $[24\bar{1}3$. In addition, swapping π_1 and $\pi_1 - 1$ does not affect the avoidance of the three patterns 3214, 2314,

4213, which implies (see Theorem 1) that $\alpha(\pi) \in A_n(k-1)$. Conversely, any element in $A_n(k-1)$ is the image by α of a unique element in $A_n^1(k)$. Thus, α is a bijection.
Now, we prove that there is a one-to-one correspondence between $A_n^2(k)$ and $A_{n-1}(k)$. Let β be the map from $A_n^2(k)$ to $A_{n-1}(k)$ where $\pi' = \beta(\pi)$ is obtained from π by deleting the entry ℓ just before $k-1$, and by decreasing by one all entries of π greater than ℓ. For instance, $\beta(41532) = 4132$. It is easy to see that $\beta(\pi)$ belongs to $A_{n-1}(k)$. Conversely, we will prove that any $\pi \in A_{n-1}(k)$ is the image by β of a permutation in $A_n^2(k)$. To do this, we insert just before $\pi_i = k-1$ an entry $\ell > \pi_1 = k$ such that, after increasing by one all entries $\pi_j \geq \ell$, the obtained permutation belongs to $A_n^2(k)$. The choice of ℓ is always possible in a unique way. Indeed, it suffices to take ℓ as follows: (i) if there is no entry $\pi_u > \pi_1$ for $1 < u < i$, such that there is a $\pi_v < \pi_1$ with $u < v < i$, then we set $\ell = n$; (ii) otherwise, ℓ is the minimal entry $\pi_u > \pi_1$, $1 < u < i$, such that there is a $\pi_v < \pi_i$ with $u < v < i$. For the two cases, inserting n just before $k-1$ does not create a pattern 3214, 2314, 4213. For the case (i), if we do not choose n then we create a pattern [132 that cannot be extended to a [2413 which is banned. For the case (ii), if we do not choose π_u minimal then a pattern 2314 is created, which is banned. Therefore, ℓ is unique which implies that β is a bijection.
Finally, if we set $a_n^k = |A_n(k)|$, then due to the bijections α and β we have $a_n^k = a_n^{k-1} + a_{n-1}^k$ for $2 \leq k \leq n$. Since $A_n(1) = \{123\cdots n\}$ and $A_n(n)$ is the set of length n permutations avoiding 213 and starting with n, the initial conditions are given by $a_n^1 = 1$ and $a_n^n = c_{n-1}$ where c_n is the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. Therefore, a_n^k generates the well known Catalan's triangle (see Table 2 and [7, 10, 16]), which implies that $a_n = \sum_{k=1}^n a_n^k$ corresponds to the nth Catalan number c_n (see A000108 and A009766 in [18]). □

It is worth to mention that the classical (one) stack sorting, the 12-machine [8] and the $(123, 132)$-machine have same potency, in the sense that, for any n they sort different but equinumerous sets of length n permutations.

$k\backslash n$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7
3			2	5	9	14	20	27
4				5	14	28	48	75
5					14	42	90	165
6						42	132	297
...							...	...
$\sum$	1	2	5	14	42	132	429	1430

Table 2: Catalan's triangle obtained with $a_n^k = |A_n(k)|$ for $1 \leq n \leq 8$ and $1 \leq k \leq 6$. The five sortable permutations of length four and starting with 3 are $3124, 3241, 3412, 3142, 3421$. The image by α of the first three permutations are respectively 2134, 2341, 2413. The image by β of the last two permutations are 312 and 321.

3 The anti-unimodal machine

In this section we characterize and enumerate all sortable permutations using the $(132, 231)$-machine. In this sorting model, at each step of the process, if the stack P_1 is not empty, then it contains an anti-unimodal sequence, *i.e.* a sequence $a_1a_2\ldots a_ia_{i+1}\ldots a_j$ such that $a_1 > a_2 > \cdots > a_i < a_{i+1} < \cdots < a_j$, for some i, $1 \leq i \leq j$.

Theorem 2. *A permutation π belongs to $Sort(132, 231)$ if and only if π avoids 1324 and 2314.*

Proof. First, let us prove that a sortable permutation π contains neither the pattern 1324 nor 2314. For a contradiction, let us assume that π contains one of these patterns, and for an occurrence θ of it we pick a particular one: $\theta = \pi_i\pi_j\pi_k\pi_\ell$, $1 \le i < j < k < \ell \le n$, where ℓ is chosen minimal, k maximal and j, i minimal, in this order.
– If θ is an occurrence of 1324, then due to the choice of i, j, k, ℓ above, a simple observation shows that $\pi_u > \pi_j$ for any $u < i$. Whenever the entry π_k is pushed in P_1 and since P_1 avoids 132 and 231, at least one of the two entries π_i and π_j does not belong to P_1. (i) Assume that π_i does not belong to P_1. In this case, there is a previous step of the sorting process where π_i is the top of P_1, and the next step pushes π_i in P_2 because pushing the current entry of the input permutation creates a pattern 132 or 231. This necessarily implies that there is a $u < i$, such that $\pi_u < \pi_i$ and $\pi_u \in P_1$, which gives a contradiction with the above observation. (ii) Assume that π_j is in P_2 and π_i in P_1 (π_k has been pushed on the top of P_1). Since $\pi_\ell > \pi_j$, π_ℓ cannot be pushed in P_1 by keeping π_i in P_1 (otherwise this would create a non sortable configuration 231 with $\pi_j\pi_\ell\pi_i$, and Fact 1 gives a contradiction). Then, we need to push π_i in P_2 before pushing π_ℓ in P_1, which necessarily implies that there is a u, $k < u \le \ell$, such that $\pi_u < \pi_i$. Then, $\pi_i\pi_j\pi_u\pi_\ell$ is an occurrence of pattern 2314 which is a contradiction with the maximality of k.
– If θ is an occurrence of 2314, then the proof is similar, *mutatis mutandis*, to that of the previous case.
Conversely, let assume that π is not sortable and let us prove that π contains an occurrence of the pattern 1324 or 2314. Let us consider the step where the process is stuck. Let π_k (resp. π_ℓ) be the top of the stack P_2 (resp. P_1), then we necessarily have $\pi_\ell > \pi_k$. We distinguish two cases: (i) $k < \ell$ and (ii) otherwise.
In the case (i), there necessarily exists a previous step where there are i and j, $i < k < j < \ell$ such that π_i and π_k are in P_1 and π_j is the first entry of the input permutation, and such that $\theta = \pi_i\pi_k\pi_j$ is an occurrence of 132 or of 231 (otherwise we cannot push π_k in P_2). If θ is an occurrence of 132 then, since $\pi_\ell > \pi_k$, the pattern of $\pi_i\pi_k\pi_j\pi_\ell$ is 1324. If θ is an occurrence of 231 then, since $\pi_\ell > \pi_k$, the pattern of $\pi_i\pi_k\pi_j\pi_\ell$ is 2314.
In the case (ii), we have $k > \ell$. In order to push π_k in P_2, there is a previous step where we push firstly π_k in P_1 by keeping π_ℓ in P_1 which implies that at this step all values below π_ℓ in P_1 are greater than π_ℓ, and all values above π_ℓ and below π_k are lower than π_ℓ. Then, in order to push π_k into P_2, we need to have u and v, $\ell < u < k < v$, such that $\pi_u\pi_k\pi_v$ is an occurrence of 132 or of 231, and $\pi_u \in P_1$ and π_v is the first entry of the input. Therefore, each time we push in P_2 the top x of the stack P_1, we need to have a value y above π_ℓ and below x in P_1. This means that π_ℓ cannot be the top of the stack P_1, which is a contradiction. □

Corollary 2. *Permutations of length n in Sort$(132, 231)$ are enumerated by the large Schröder numbers (see* A006318 *in [18]).*

Proof. The enumeration of length n permutations avoiding 1324 and 2314 (or a symmetry of these patterns) can be found in [5, 21] for instance. Note that in [4], the authors provide a constructive bijection between these permutations and Schröder paths. □

4 Going further

In Section 2, we have characterized and enumerated length n sortable permutations with the $(123, 132)$-machine, which provides a new set of permutations counted by Catalan numbers. The number of such permutations starting with k generates the well known Catalan's triangle, which in turn corresponds to the number of Dyck paths of semilength n and having the first peak at height $n - k + 1$. As a future work, it would be interesting to exhibit a constructive bijection between these sets.
Also, experimental results suggest that three other (σ, τ)-machines sort permutations counted by the Catalan numbers. We leave them as open problems.

Problem 1. *A permutation π belongs to Sort$(123, 213)$ if and only if π avoids* $[24\bar{1}3$, $[4231$, $[31425$, $[421\bar{3}5$, *and any occurrence of* $[2413$ *in π is contained in a occurrence of* $[31524$ *or* $[32514$. *Moreover, permutations of length n in Sort$(123, 213)$ are enumerated by the Catalan numbers.*

Problem 2. *Permutations of length n in Sort$(132, 312)$ and in Sort$(231, 321)$ are both enumerated by the Catalan numbers.*

For the last problem, we do not succeed to predict if sortable permutations can be characterized in terms of (generalized) forbidden patterns.

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