THE FREE TANGENT LAW

WIKTOR EJSMONT AND FRANZ LEHNER

Abstract. Nevanlinna-Herglotz functions play a fundamental role for the study of infinitely divisible distributions in free probability [11]. In the present paper we study the role of the tangent function, which is a fundamental Herglotz-Nevanlinna function [28, 23, 52], and related functions in free probability. To be specific, we show that the function

$$\frac{\tan z}{1 - x\tan z}$$

of Carlitz and Scoville [17, (1.6)] describes the limit distribution of sums of free commutators and anticommutators and thus the free cumulants are given by the Euler zigzag numbers.

1. Introduction

Nevanlinna or Herglotz functions are functions analytic in the upper half plane having non-negative imaginary part. This class has been thoroughly studied during the last century and has proven very useful in many applications. One of the fundamental examples of Nevanlinna functions is the tangent function, see [6, 28, 23, 52]. On the other hand it was shown by by Bercovici and Voiculescu [11] that Nevanlinna functions characterize freely infinitely divisible distributions. Such distributions naturally appear in free limit theorems and in the present paper we show that the tangent function appears in a limit theorem for weighted sums of free commutators and anticommutators. More precisely, the function

$$\frac{\tan z}{1 - x\tan z}$$

arises, which was studied by Carlitz and Scoville [17, (1.6)] in connection with the combinorics of tangent numbers; in particular we recover the tangent function for $x = 0$.

In recent years a number of papers have investigated limit theorems for the free convolution of probability measures defined by Voiculescu [56, 57, 54]. The key concept of this definition is the notion of noncommutative free independence, or freeness for short. As in classical probability where the concept of independence gives rise to classical convolution, the concept of freeness leads to another operation on the measures on the real line called free convolution. Many classical results in the theory of addition of independent random variables have their counterpart in this new theory. For example the free analogue of the central limit theorem asserts that the distribution of

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}},$$

for a given family of free identically distributed random variables converges in distribution to the normal law semicircle law as n goes to infinity. More general central limit theorems were proved by Speicher [49] and by combinatorial means and provide the starting point for the present paper. We study limit theorems for sums with correlated entries, more precisely, for

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quadratic forms in free random variables. In particular, we can explicitly compute the limit distribution μ of the quadratic form

$$\frac{\sum_{k<l} a(X_kX_l + X_lX_k) + bi(X_kX_l - X_lX_k)}{n}$$

where $a^2 + b^2 = 1$ and $b \neq 0$, and its R-transform turns out to be the elementary function

$$R_\mu(z) = \frac{\tan(bz)}{b - a\tan(bz)}.$$

This is the generating function of the higher order tangent numbers of Carlitz and Scoville [17] which arose in connection with the enumeration of certain permutations. This will follow from a general limit theorem for arbitrary quadratic forms. Using these we establish a limit for of free commutators and mixed sums of commutators and anti-commutators. The respective limit laws are infinitely divisible and we call them the *free tangent law* and the *free zigzag law* according to the combinatorial interpretation of their cumulants. In addition we indicate random matrix models for these limits. The classical version of this limit theorem features the χ^2-distribution since commutators trivially vanish in classical probability.

2. Preliminaries

2.1. **Basic Notation and Terminology.** A tracial noncommutative probability space is a pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a von Neumann algebra, and $\tau : \mathcal{A} \to \mathrm{C}$ is a normal, faithful, tracial state, i.e., τ is linear and continuous in the weak* topology, $\tau(XY) = \tau(YX)$, $\tau(I) = 1$, $\tau(XX^*) \geqslant 0$ and $\tau(XX^*) = 0$ implies $X = 0$ for all $X, Y \in \mathcal{A}$. For example, the noncommutative analog of a finite probability space is the algebra of complex $N \times N$ matrices $M_N(\mathbb{C})$. The unique tracial state is the normalized trace $\tau_N(A) = \frac{1}{N}\operatorname{Tr}(A) = \frac{1}{N}\sum A_{ii}$.
The elements $X \in \mathcal{A}$ are called (noncommutative) random variables; in the present paper all random variables are assumed to be self-adjoint. Given a noncommutative random variable $X \in \mathcal{A}_{sa}$, the spectral theorem provides a unique probability measure μ_X on $\mathbb{R}$ which encodes the distribution of X in the state τ, i.e., $\tau(f(X)) = \int_{\mathbb{R}} f(\lambda)\, d\mu_X(\lambda)$ for any bounded Borel function f on $\mathbb{R}$.

2.2. **Free Independence.** A family of von Neumann subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ is called *free* if $\tau(X_1 \dots X_n) = 0$ whenever $\tau(X_j) = 0$ for all $j = 1, \dots, n$ and $X_j \in \mathcal{A}_{i(j)}$ for some indices $i(1) \neq i(2) \neq \cdots \neq i(n)$. Random variables $X_1, \dots, X_n$ are freely independent (free) if the subalgebras they generate are free. Free random variables can be constructed using the reduced free product of von Neumann algebras [55]. For more details about free convolutions and free probability theory the reader can consult the standard references [54, 44, 43].

2.3. **Free Convolution and the Cauchy-Stieltjes Transform.** It can be shown that the joint distribution of free random variables X_i is uniquely determined by the distributions of the individual random variables X_i and therefore the operation of *free convolution* is well defined: Let μ and ν be probability measures on $\mathbb{R}$, and X, Y self-adjoint free random variables with respective distributions μ and ν. The distribution of $X+Y$ is called the free additive convolution of μ and ν and is denoted by $\mu \boxplus \nu$. The analytic approach to free convolution is based on the Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-y}\, d\mu(y) \tag{2.1}$$

of a probability measure μ. The Cauchy transform is analytic on the upper half plane $\mathbb{C}^+ = \{x + iy | x, y \in \mathbb{R}, y > 0\}$ and takes values in the closed lower half plane $\mathbb{C}^- \cup \mathbb{R}$. For measures with compact support the Cauchy transform is analytic at infinity and related to the moment generating function M_X as follows:

$$M_X(z) = \sum_{n=0}^{\infty} \tau(X^n)\, z^n = \frac{1}{z}\, G_X(1/z). \tag{2.2}$$

Moreover the Cauchy transform has an inverse in some neighbourhood of infinity which has the form

$$G_\mu^{-1}(z) = \frac{1}{z} + R_\mu(z),$$

where $R_\mu(z)$ is analytic in a neighbourhood of zero and is called *R-transform*. The coefficients of its series expansion

$$R_X(z) = \sum_{n=0}^{\infty} K_{n+1}(X) z^n \tag{2.3}$$

are called *free cumulants* of the random variable X, see Section 2.11 below. for combinatorial purposes it will be convenient to consider the shift $\mathcal{C}_X(z) := zR_X(z)$, which is called the *free cumulant transform* or *free cumulant generating function*. Using this the free convolution can be computed via the identity

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z), \tag{2.4}$$

see [56].
In order to treat measures with noncompact support, it is conveniate to reformulate the identities in terms of the reciprocal Cauchy transform $F_\mu(z) = 1/G_\mu(z)$ [12]. This function has an analytic right compositional inverse F_μ^{-1} in a region

$$\Gamma_{\eta,M} = \{z \in \mathbb{C} \mid |\operatorname{Re} z| < \eta \operatorname{Im} z,\ \ \operatorname{Im} z > M\};$$

the *Voiculescu transform* is defined as the function

$$\phi_\mu(z) = F_\mu^{-1}(z) - z$$

which turns out to be $\phi_\mu(z) = R_\mu(1/z)$.

2.4. **Free infinite divisibility.** In analogy with classical probability, a probability measure μ on $\mathbb{R}$ is said to be *freely infinitely divisible* (or FID for short) if for each $n \in \{1, 2, 3, \dots\}$ there exists a probability measure μ_n such that $\mu = \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n$ (n-fold free convolution).
Free infinite divisibility of a measure μ is characterized by the property that its Voiculescu transform has a Nevanlinna-Pick representation [12]

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1 + xz}{z - x}\, d\rho(x) = \gamma + \int_{\mathbb{R}} \left(\frac{1}{z - x} + \frac{x}{1 + x^2} \right) (1 + x^2)\, d\rho(x) \tag{2.5}$$

for some $\gamma \in \mathbb{R}$ and some nonnegative finite measure ρ.
We recall a general method to compute Lévy measures from [5]. In terms of the free cumulant transform the Lévy-Khintchine representation takes the form [9]

$$\mathcal{C}_\mu(z) = cz + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz\mathbf{1}_{\{|x|<1\}}(x) \right) d\nu(x) \tag{2.6}$$

for some $c \in \mathbb{R}$, $a \geqslant 0$ and a nonnegative measure ν satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\} d\nu(x) < \infty$. The triplet (c, a, ν) is called the *free characteristic triplet*, a is called the *semicircular component* and ν is called the *free Lévy measure* of μ. The measure ρ can be calculated using the Stieltjes inversion formula

$$\int_u^v (1 + x^2)\, d\rho(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_u^v \operatorname{Im} \phi_\mu(x + i\epsilon)\, dx$$

for all points of continuity u, v of ρ. Considering the relation $R_\mu(z) = \phi(\frac{1}{z})$ and (2.6) we obtain $\frac{1+x^2}{x^2}\rho|_{\mathbb{R}\setminus\{0\}} = \nu|_{\mathbb{R}\setminus\{0\}}$ and $\rho(\{0\}) = a$. In particular, if the function $-\frac{1}{\pi}\phi_\mu(x + i\epsilon)$ converges uniformly to a continuous function $f_\mu(x)$ as $\epsilon \to 0^+$ on an interval $[u, v]$, then ρ is absolutely continuous in $[u,v]$ with density $f_\mu(x)$. Hence, ν is also absolutely continuous in $[u,v]$ with density $\frac{1+x^2}{x^2} f_\mu(x)$. Regarding atoms, their mass is given by

$$\nu(\{x\}) = \frac{1}{x^2} \lim_{\epsilon\to 0^+} i\epsilon\phi_\mu(x+i\epsilon). \tag{2.7}$$

2.5. **Wigner semicircle law.** The Wigner semicircle law has density

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx \tag{2.8}$$

on $-2 \leqslant x \leqslant 2$. Its Cauchy-Stieltjes transform is given by the formula

$$G_\mu(z) = \frac{z-\sqrt{z^2-4}}{2}, \tag{2.9}$$

where $|z|$ is big enough and where the branch of the analytic square root is determined by the condition that $\operatorname{Im}(z) > 0 \Rightarrow \operatorname{Im}(G_\mu(z)) \leqslant 0$ (see [45]).
A non-commutative random variable X distributed according to the semicircle law is called *semicircular* or *free gaussian* random variable. The reason for the latter is the fact that its free cumulants $K_r = 0$ for $r > 2$ and it appears in the free version of the central limit theorem.

2.6. **Even elements.** We call an element $X \in \mathcal{A}$ even if all its odd moments vanish, i.e., $\tau(X^{2i+1}) = 0$ for all $i \geqslant 0$. It is immediate that the vanishing of all odd moments is equivalent to the vanishing of all odd cumulants, i.e., $K_{2i+1}(X) = 0$ and thus the even cumulants contain the complete information about the distribution of an even element.

2.7. **Convergence in distribution.** In noncommutative probability we say that a sequence X_n of random variables *converges in distribution* towards X as $n \to \infty$, denoted by

$$X_n \xrightarrow{d} X$$

if we have for all $m \in \mathbb{N}$

$$\lim_{n\to\infty} \tau(X_n^m) = \tau(X^m) \text{ or equivalently } \lim_{n\to\infty} K_m(X_n) = K_m(X).$$

2.8. **Random matrices.** The semicircle law arises also as the asymptotic spectral distribution of certain random matrices.
An $N \times M$ *complex Gaussian random matrix* is a matrix $X = [x_{i,j}]_{i,j=1}^{N\times M}$ whose entries form an i.i.d. complex Gaussian family with mean zero and variance $\mathbb{E}(|x_{i,j}|^2) = \frac{1}{N}$, i.e., the real parts $\operatorname{Re} x_{ij}$ and the imaginary parts $\operatorname{Im} x_{ij}$ together form an i.i.d. family of $N(0, \frac{1}{2N})$ random variables.
An $N \times N$ *GUE* random matrix is a matrix $Y_N = [y_{ij}]_{i,j=1}^{N\times N}$ of the form $Y_N = \frac{X+X^*}{\sqrt{2}}$ where X is an $N \times N$ complex Gaussian random matrix, i.e., the family $\{y_{ii} \mid 1 \leqslant i \leqslant N\} \cup \{\operatorname{Re} y_{ij} \mid 1 \leqslant i < j \leqslant N\} \cup \{\operatorname{Im} y_{ij} \mid 1 \leqslant i < j \leqslant N\}$ is an independent family of real gaussian random variables with variance $\operatorname{Var} y_{ii} = 1/N$ and $\operatorname{Var} y_{ij} = \frac{1}{2N}$ for $i < j$. It is well known that the moments spectral distribution converge to the moments of the standard Wigner semicircle law (2.8)

$$\lim_{N\to\infty} \tau_N(Y_N^m) = \frac{1}{2\pi}\int_{-2}^{2} x^m\sqrt{4-x^2}\,dx = \begin{cases} \frac{1}{n+1}\binom{2n}{n} & \text{if } m = 2n \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

with respect to the normalized trace τ_N. In the language of section 2.7 this means that Y_N converges in distribution to a semicircular element with respect to the expectation functional τ_N.

2.9. **Convergence in eigenvalues.** Recently the concept of convergence with respect to the nonnormalized trace turned out to be useful for the study of the fine structure of random matrices [18]. We say that a sequence of $N \times N$ deterministic matrices A_N has limit distribution μ with respect to the nonnormalized trace if for every $m \in \mathbb{N}$ the moments satisfy

$$\lim_{N\to\infty} \operatorname{Tr}(A_N^m) = \int t^m d\mu(t).$$

Note that in this case μ is not necessarily a probability measure and that the limit with respect to the normalized trace τ_N is zero. Moreover the limit distribution is discrete [18, Proposition 2.10] and under certain conditions the eigenvalues converge pointwise [18, Proposition 2.8].

2.10. **Noncrossing Partitions.** We recall some facts about noncrossing partitions. For details and proofs see the lecture notes [44, Lecture 9]. Let $S \subseteq \mathbb{N}$ be a finite subset. A partition of S is a set of mutually disjoint subsets (also called *blocks*) $B_1, B_2, \dots, B_k \subseteq S$ whose union is S. Any partition π defines an equivalence relation on S, denoted by $\sim_\pi$, such that the equivalence classes are the blocks π. That is, $i \sim_\pi j$ if i and j belong to the same block of π. A partition π is called *noncrossing* if different blocks do not interlace, i.e., there is no quadruple of elements $i < j < k < l$ such that $i \sim_\pi k$ and $j \sim_\pi l$ but $i \not\sim_\pi j$.
The set of non-crossing partitions of S is denoted by $NC(S)$, in the case where $S = [n] := \{1, \dots, n\}$ we write $NC(n) := NC([n])$. $NC(n)$ is a lattice under refinement order, where we say $\pi \leqslant \rho$ if every block of π is contained in a block of ρ. The subclass of noncrossing pair partitions (i.e., noncrossing complete matchings) is denoted by $NC_2(n)$.
The maximal element of $NC(n)$ under this order is the partition consisting of only one block and it is denoted by $\hat{1}_n$. On the other hand the minimal element $\hat{0}_n$ is the unique partition where every block is a singleton. Sometimes it is convenient to visualize partitions as diagrams, for example $\hat{1}_n =$ ┌┬⋯┐ and $\hat{0}_n =$ ı ı ⋯ ı.
We will apply the product formula (2.13) below only in the case of pairwise products of random variables and in this case two specific pair partitions and their complements will play a particularly important role, namely the *standard matching* $\hat{1}_2^n =$ ⊓ ⊓⋯⊓ $\in NC(2n)$ and its shift $\nu_{0n} =$ ┌⊓ ⊓⋯⊓┐ $\in NC(2n)$.

2.11. **Free Cumulants.** Given a noncommutative probability space $(\mathcal{A}, \tau)$ the free cumulants are multilinear functionals $K_n : \mathcal{A}^n \to \mathbb{C}$ defined implicitly in terms of the mixed moments by the relation

$$\tau(X_1 X_2 \dots X_n) = \sum_{\pi \in NC(n)} K_\pi(X_1, X_2, \dots, X_n), \tag{2.10}$$

where

$$K_\pi(X_1, X_2, \dots, X_n) := \Pi_{B\in\pi} K_{|B|}(X_i : i \in B). \tag{2.11}$$

Sometimes we will abbreviate univariate cumulants as $K_n(X) = K_n(X, \dots, X)$.
Free cumulants provide a powerful technical tool to investigate free random variables. This is due to the basic property of *vanishing of mixed cumulants*. By this we mean the property that

$$K_n(X_1, X_2, \dots, X_n) = 0$$

for any family of random variables $X_1, X_2, \dots, X_n$ which can be partitioned into two mutually free nontrivial subsets. For free sequences this can be reformulated as follows. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of free random variables and $h : [r] \to \mathbb{N}$ a map. We denote by $\ker h$ the set partition which is induced by the equivalence relation

$$i \sim_{\ker h} j \iff h(i) = h(j).$$

In this notation, vanishing of mixed cumulants implies that

$$K_\pi(X_{h(1)}, X_{h(2)}, \dots, X_{h(r)}) = 0 \text{ unless } \ker h \geqslant \pi. \tag{2.12}$$

Our main technical tool is the free version, due to Krawczyk and Speicher [37] (see also [44, Theorem 11.12]), of the classical formula of James and Leonov/Shiryaev [35, 38] which expresses cumulants of products in terms of individual cumulants.

Theorem 2.1. *Let $r, n \in \mathbb{N}$ and $i_1 < i_2 < \dots < i_r = n$ be given and let*

$$\rho = \{(1, 2, \dots, i_1), (i_1 + 1, i_1 + 2, \dots, i_2), \dots, (i_{r-1} + 1, i_{r-1} + 2, \dots, i_r)\} \in NC(n)$$

be the induced interval partition. Consider now random variables $X_1, \dots, X_n \in \mathcal{A}$. Then the free cumulants of the products can be expanded as follows:

$$K_r(X_1 \dots X_{i_1}, \dots, X_{i_{r-1}+1} \dots X_n) = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \rho = \hat{1}_n}} K_\pi(X_1, \dots, X_n). \tag{2.13}$$

Our main tool is the following result from [24] which expresses cumulants of quadratic forms in even random variables in terms of the conditional expectations of the system matrix onto the diagonal matrices.

Proposition 2.2 ([24, Proposition 4.5])**.** *Let $X_1, X_2, \dots, X_n \in \mathcal{A}$ be a free family of even random variables, $\boldsymbol{X} = [X_i X_j]_{i,j=1}^n$, $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$ a scalar matrix and $Q_n = \sum a_{i,j} X_i X_j$ a quadratic form. The cumulants of Q_n are given by*

$$K_r(Q_n) = \sum_{i_1, \dots, i_r \in [n]} \operatorname{Tr}(A E_{i_1} A E_{i_2} \dots A E_{i_r}) \sum_{\substack{\pi \in NCE(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} K_\pi(X_{i_r}, X_{i_1}, X_{i_1}, X_{i_2}, \dots, X_{i_{r-1}}, X_{i_r}). \tag{2.14}$$

where by E_i we denote the projection matrix onto the i-th unit vector.

Remark 2.3. In the case of a free standard semicircular family formula (2.14) has only one contributing term and takes the particularly simple form

$$K_r(Q_n) = \operatorname{Tr}(A_n^r); \tag{2.15}$$

thus the distributions of quadratic forms in semicircular variables are easy to calculate, see [25].

Notation 2.4. For scalars $a, b, c \in \mathbb{C}$ we denote by $\left[\begin{smallmatrix} c & a \\ b & c \end{smallmatrix}\right]_n \in M_n(\mathbb{C})$ the matrix whose diagonal elements are equal to c, whose upper-triangular entries are equal to a and whose lower-triangular elements are equal to b, respectively.

2.12. **Combinatorics of tangent numbers.** The *tangent numbers*

$$T_k = (-1)^{k+1} \frac{4^k (4^k - 1) B_{2k}}{2k} \tag{2.16}$$

for $k \in \mathbb{N}$ are the Taylor coefficients of the tangent function

$$\tan z = \sum_{n=1}^{\infty} T_n \frac{z^n}{n!} = z + \frac{2}{3!} z^3 + \frac{16}{5!} z^5 + \frac{272}{7!} z^7 + \cdots ,$$

see [29, Page 287]. The tangent numbers are complemented by the *secant numbers.* Together they form the sequence of E_n of *Euler zigzag numbers* which are the Taylor coefficients of the function

$$\tan(z) + \sec(z) = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.$$

These numbers are also called *up-down numbers* [16] or *snake numbers* [8, 32] and appear in several different contexts, see for example [27, 7, 50, 51] or André's theorem [2].
Similarly, following Comtet [19, p. 260] (see also [21]) we define the *arctangent numbers* by their exponential generating function

$$\frac{(\arctan z)^k}{k!} = \sum_{n=k}^{\infty} \frac{A_n^{(k)}}{n!} z^n; \tag{2.17}$$

up to sign these are the same as the coefficients of the hyperbolic arctangent function

$$\frac{(\operatorname{atanh} z)^k}{k!} = \sum_{n=k}^{\infty} \frac{\tilde{A}_n^{(k)}}{n!} z^n \tag{2.18}$$

the latter are nonnegative and

$$A_n^{(k)} = (-i)^k i^n \tilde{A}_n^{(k)}. \tag{2.19}$$

The *higher order tangent numbers* $T_n^{(k)}$ were introduced by Carlitz and Scoville [17] as the coefficients of the Taylor series

$$\tan^{k+1} z = \sum_{n=k+1}^{\infty} T_n^{(k+1)} \frac{z^n}{n!}.$$

The generating function of the tangent polynomials $T_n(x) = \sum_{k=1}^n T_n^{(k)} x^k$ can be easily obtained from the geometric series

$$\begin{aligned} T(x,z) &= \frac{x \tan z}{1 - x \tan z} \\ &= \sum_{k=1}^{\infty} x^k \tan^k z \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{T_n^{(k)}}{n!} x^k z^n \\ &= \sum_{n=0}^{\infty} \frac{T_n(x)}{n!} z^n \end{aligned} \tag{2.20}$$

Note that our generating function slightly differs from Carlitz and Scoville's [17, Equation (1.6)], which is the expansion of the function $\frac{\tan(z)}{1 - x\tan(z)}$.
On the other hand it is well known that all derivatives of tangent and cotangent can be expressed as certain polynomials, see the side note [29, Page 287]) and the recent studies [31, 32, 15, 20]. To be specific, there is a sequence of polynomials $P_n(x)$ of degree $n+1$, $n \geqslant 0$, such that

$$\begin{aligned} \frac{d^n}{d\theta^n} \tan\theta &= P_n(\tan\theta) \\ \frac{d^n}{d\theta^n} \cot\theta &= (-1)^n P_n(\cot\theta) \end{aligned}$$

The generating function is easily derived from the Taylor series

$$\tan(\theta + z) = \sum_{n=0}^{\infty} \frac{P_n(\tan\theta)}{n!} z^n = \frac{\tan\theta + \tan z}{1 - \tan\theta \tan z}$$

to be

$$P(x,z) = \sum_{n=0}^{\infty} P_n(x) \frac{z^n}{n!} = \frac{x + \tan z}{1 - x \tan z}.$$

Comparing the generating functions we find that

$$xP(x,z) = (1+x^2)T(x,z) + x^2,$$

and from this we conclude that

$$xP_n(x) = (1+x^2)T_n(x),$$

for $n \geqslant 1$, see also [20]. Note that $P_n(x)$ is divisible by $(1+x^2)$ because of the recurrence relation

$$P_n(x) = (1+x^2)P_{n-1}'(x),\ P_0(x) = x,$$

see [29, (6.95)].

2.13. **An elementary lemma.** The moments and the spectral measures of the matrices of the underlying quadratic forms can be computed explicitly and turn out to be connected to an old problem in classical calculus. We first compute the eigenvalues of the matrix underlying the quadratic form (4.1).

Lemma 2.5. *Let $a, b \in \mathbb{R}$, $b \neq 0$ and*

$$A_n = \begin{bmatrix} 0 & a+bi & \dots & a+bi \\ a-bi & 0 & \dots & a+bi \\ \dots & \dots & \dots & \dots \\ a-ib & a-bi & \dots & 0 \end{bmatrix} \in M_n(\mathbb{C}).$$

Then the eigenvalues of the matrix A_n are given by

$$\lambda_k = b\cot\frac{\alpha + k\pi}{n} - a, \quad \text{for } 0 \leqslant k \leqslant n-1 \text{ and } \alpha = \operatorname{arccot}(a/b).$$

Proof. The characteristic polynomial $\chi_n(\lambda) = \det(\lambda I - A_n)$ satisfies the following recurrence relation. Let $w = a + bi = e^{i\alpha}$, the we have

$$\chi_n(\lambda) = \begin{vmatrix} \lambda & -w & -w & -w & \dots & -w \\ -\overline{w} & \lambda & -w & -w & \dots & -w \\ -\overline{w} & -\overline{w} & \lambda & -w & \dots & -w \\ -\overline{w} & -\overline{w} & -\overline{w} & \lambda & \dots & -w \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\overline{w} & -\overline{w} & -\overline{w} & -\overline{w} & \dots & \lambda \end{vmatrix}$$

we subtract the second row from the first row

$$= \begin{vmatrix} \lambda+\overline{w} & -\lambda-w & 0 & 0 & \dots & 0 \\ -\overline{w} & \lambda & -w & -w & \dots & -w \\ -\overline{w} & -\overline{w} & \lambda & -w & \dots & -w \\ -\overline{w} & -\overline{w} & -\overline{w} & \lambda & \dots & -w \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\overline{w} & -\overline{w} & -\overline{w} & -\overline{w} & \dots & \lambda \end{vmatrix}$$

and the second column from the first column

$$= \begin{vmatrix} 2\lambda+w+\overline{w} & -\lambda-w & 0 & 0 & \dots & 0 \\ -\lambda-\overline{w} & \lambda & -w & -w & \dots & -w \\ 0 & -\overline{w} & \lambda & -w & \dots & -w \\ 0 & -\overline{w} & -\overline{w} & \lambda & \dots & -w \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\overline{w} & -\overline{w} & -\overline{w} & \dots & \lambda \end{vmatrix}$$

$$= (2\lambda + w + \overline{w})\chi_{n-1}(\lambda) - (\lambda + w)(\lambda + \overline{w})\chi_{n-2}(\lambda)$$

and the solution of this recurrence equation (with initial values $\chi_0(\lambda) = 1$ and $\chi_1(\lambda) = \lambda$) is

$$\chi_n(\lambda) = \frac{w(\lambda + \overline{w})^n - \overline{w}(\lambda + w)^n}{w - \overline{w}}. \tag{2.21}$$

To compute the eigenvalues we may assume $|w| = 1$, i.e., $w = e^{i\alpha}$ and $\alpha = \operatorname{arccot}(a/b)$ (the general case follows by rescaling the matrix) and we substitute $z = \lambda + w$. The matrix is selfadjoint and therefore any eigenvalue λ is real, so $\overline{z} = \lambda + \overline{w}$ and we get

$$w\overline{z}^n - \overline{w}z^n = 0,$$

i.e., $\operatorname{Im}(\overline{w}z^n) = 0$. Let $z = re^{i\theta}$, then this means

$$\sin(n\theta - \alpha) = 0$$

and we conclude $\theta = \frac{\alpha+k\pi}{n}$. We return to $\lambda = z - w = re^{i\theta} - e^{i\alpha}$. This is a real number and thus the imaginary part vanishes, i.e., $r\sin\theta = \sin\alpha$, thus $r = \frac{\sin\alpha}{\sin\theta}$ and finally

$$\lambda = \sin\alpha\cot\theta - \cos\alpha$$

and in the general case where $w = a + ib$ the solutions are

$$\lambda_k = b\cot\frac{\alpha+k\pi}{n} - a, \quad 0 \leqslant k \leqslant n-1. \tag{2.22}$$

□

2.14. **Cotangent sums.** The manipulations of the eigenvalues (2.22) will lead to the following sums of cotangent powers which were explicitly evaluated in our companion paper [26, Corollary 6.4].

$$\sum_{k=1}^{n} \cot^{2m}\frac{(2k+1)\pi}{2n} = (-1)^m n + \frac{1}{(2m-1)!}\sum_{k=1}^{m} n^{2k} A_{2m}^{(2k)}\, T_{2k-1} \tag{2.23}$$

$$\begin{aligned}\sum_{k=1}^{n} \cot^{m}\frac{(4k-1)\pi}{4n} &= (-1)^{m/2} n \mathbb{1}_{m \text{ even}} + \frac{1}{2(m-1)!}\sum_{k=1}^{m} (-2n)^k A_{m}^{(k)}\, E_{k-1} \\ &= \frac{1}{2(m-1)!}\sum_{k=1}^{m} (-2n)^m A_{m}^{(m)}\, E_{m-1} + \mathcal{O}(n^{m-1}).\end{aligned} \tag{2.24}$$

3. Limit theorems and random matrix models for quadratic forms

3.1. **A general Limit Theorem.** In this section we consider limit theorems for sums of commutators and other quadratic forms of the following type.

Theorem 3.1. *Let $A_n = [a_{i,j}^{(n)}] \in M_n(\mathbb{C})$ be a sequence of selfadjoint matrices such that $\sup_{i,j,n}\lvert a_{i,j}^{(n)}\rvert < \infty$ and such that the matrix $\frac{1}{n}A_n$ has limit distribution μ with respect to the nonnormalized trace. Let X_i be free copies of a centered random variable X of variance 1, then the sequence of quadratic forms*

$$Q_n = \frac{1}{n}\sum_{i,j=1}^{n} a_{i,j}^{(n)} X_i X_j$$

converges in distribution to Y, where

$$K_r(Y) = \int t^r\, d\mu(t).$$

Remark 3.2. From [18, Proposition 2.10], we conclude that the measure μ is discrete. The limit measure in Theorem 3.1 does not depend on the specific distribution of X_i and therefore in the examples computed below we can replace the sequence X_i by a free i.i.d. sequence of standard semicircular variables, which has the advantage that formula (2.15) can be applied.

Proof. We use the product formula from Theorem 2.1:

$$\begin{aligned}K_r(Q_n) &= \frac{1}{n^r}\sum_{i_1,i_2,\dots,i_{2r}} a_{i_1,i_2}^{(n)} a_{i_3,i_4}^{(n)} \cdots a_{i_{2r-1},i_{2r}}^{(n)} K_r(X_{i_1}X_{i_2}, X_{i_3}X_{i_4}, \dots, X_{i_{2r-1}}X_{i_{2r}}) \\ &= \frac{1}{n^r}\sum_{i_1,i_2,\dots,i_{2r}} a_{i_1,i_2}^{(n)} a_{i_3,i_4}^{(n)} \cdots a_{i_{2r-1},i_{2r}}^{(n)} \sum_{\substack{\pi\in NC(2r)\\ \pi\vee\sqcap\,\sqcap\cdots\sqcap=\hat{1}_{2r}}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}, \dots, X_{i_{2r-1}}, X_{i_{2r}}) \\ &= \frac{1}{n^r}\sum_{\substack{\pi\in NC(2r)\\ \pi\vee\sqcap\,\sqcap\cdots\sqcap=\hat{1}_{2r}}} \sum_{\ker \underline{i}\geqslant\pi} a_{i_1,i_2}^{(n)} a_{i_3,i_4}^{(n)} \cdots a_{i_{2r-1},i_{2r}}^{(n)} K_\pi(X).\end{aligned}$$

By assumption X is centered and therefore only partitions without singletons contribute to this sum. Every block of such a partition π has at least size 2 and therefore $|\pi| \leqslant r$. This in turn implies that there are only $n^{|\pi|}$ allowed choices of indices $\underline{i}$ and we have the following estimate

$$\left| \frac{1}{n^r} \sum_{\ker \underline{i} \geqslant \pi} a^{(n)}_{i_1,i_2} a^{(n)}_{i_3,i_4} \cdots a^{(n)}_{i_{2r-1},i_{2r}} K_\pi(X) \right| \leqslant n^{|\pi|-r} C^r \left| K_\pi(X) \right|$$

where $C = \sum \left| a^{(n)}_{ij} \right|$. Now unless $|\pi| = r$ this converges to zero as $n \to \infty$, on the other hand, $|\pi| = r$ is only possible if π is a pair partition. The only pair partition satisfying $\pi \vee \sqcap \sqcap \cdots \sqcap = \hat{1}_{2r}$ is the partition $\pi = \nu_{0r}$ and finally we have

$$K_r(Q_n) = \frac{1}{n^r} \operatorname{Tr}(A_n^r) K_2(X)^r + \mathcal{O}(1/n) \xrightarrow[n\to\infty]{} \int t^r \, d\mu(t).$$

□

3.2. **Random matrix models.** In this subsection we construct random matrices whose limit law coincides with the limit law from Theorem 3.1. In some sense it is a simultaneous limit obtained from approximating the semicircle law on the one hand as in section 2.8 and the free central limit law on the other hand. To this end we consider compressions with random matrices. In [44, Proposition 12.18] the authors describe compound free Poisson distributions as free compressions with semicircular operators. The next proposition provides a complex version of this result, i.e., a description of compressions with circular operators. Recall that a *circular operator* is an operator C of the form $C = (X + iY)/\sqrt{2}$ where X and Y are free standard semicircular random variables.

Proposition 3.3. *Let $C_1, C_2, \ldots, C_n \in \mathcal{A}$ be a free family of circular random variables, such that $K_2(C_i, C_i^*) = 1$ which is free from $Z \in \mathcal{A}_{sa}$, $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$ be a scalar selfadjoint matrix and $T_n = \sum_{i,j}^n a_{i,j} C_i Z C_j^*$. Then the cumulants of T_n are given by*

$$K_r(T_n) = \operatorname{Tr} \otimes \tau([A \otimes Z]^r), \tag{3.1}$$

where $A \otimes Z \in M_n(\mathbb{C}) \otimes \mathcal{A}$, with functional $\operatorname{Tr} \otimes \tau$.

Proof. From the definition of T_n we see that

$$K_r(T_n) = \sum_{i_1,i_2,\ldots,i_{2r} \in [n]} \sum_{\substack{\pi \in NCE(3r) \\ \pi \vee \hat{1}_3^r = \hat{1}_{3r}}} a_{i_1,i_2} a_{i_3,i_4} \cdots a_{i_{2r-1},i_{2r}} K_\pi(C_{i_1} Z C_{i_2}^*, C_{i_3} Z C_{i_4}^*, \ldots, C_{i_{2r-1}} Z C_{i_{2r}}^*).$$

Since Z is free from the family C_i every partition with nonzero contribution can be written as $\pi = \rho \cup \sigma$ where $\rho \in NC_2(\{1, 3, 4, 6, 7, \ldots, 3r-2, 3r\})$ is a pair partition and $\sigma \in NC(\{2, 5, \ldots, 3r-1\}$ is arbitrary. Now by the argument from the proof of [44, Proposition 12.18] we conclude that the only pair partition satisfying the required condition is $\rho = \ulcorner \sqcap \sqcap \cdots \sqcap \urcorner$, while σ is arbitrary. The result is

$$\begin{aligned} &= \sum_{i_1,i_2,\ldots,i_r \in [n]} a_{i_r,i_1} a_{i_1,i_2} \cdots a_{i_{r-1},i_r} \sum_{\sigma \in NC(r)} K_\sigma(Z) \\ &= \operatorname{Tr}(A^r) \tau(Z^r) \end{aligned}$$

which is the desired formula. □

Let us now introduce some random matrix models. For notation see section 2.8.

Proposition 3.4. *Let $X_{N \times NM}$ be a complex Gaussian random matrix of size $N \times NM$ and let $D_M = [d^{(M)}_{i,j}]$ be a sequence of selfadjoint deterministic $M \times M$ matrices such that D_N has limit distribution μ with respect to the nonnormalized trace. Then for any sequence P_N of $N \times N$ (selfadjoint) deterministic matrices which converges to Z with limit distribution ν we have*

$$\lim_{M\to\infty} \lim_{N\to\infty} X_{N\times NM} [D_M \otimes P_N] X^*_{N\times NM} = Y,$$

where

$$R_Y(z) = \sum_{r=0}^{\infty} \int_{\mathbb{R}} x^{r+1} d\mu(x) \tau(Z^{r+1}) z^r$$
$$= \int\int \frac{xt}{1-xtz}\, d\mu(x)\, d\nu(t).$$

Proof. Fix M and observe that we can represent random matrix as a quadratic form in M variables by the formula

$$X_{N\times NM}[D_M \otimes P_N] X^*_{N\times NM} = \sum_{i,j=1}^{M} d_{i,j} X_{i,N} P_N X^*_{j,N},$$

where $X_{i,N}$ is a complex Gaussian random matrices (non selfadjoint) of size $N \times N$. From Voiculescu's asymptotic freeness results [57] (see also [43, Chapter 4]) we infer that

$$\sum_{i,j=1}^{M} d_{i,j} X_{i,N} P_N X^*_{j,N} \xrightarrow[N\to\infty]{} \sum_{i,j=1}^{M} d_{i,j} C_i Z C^*_j,$$

where C_i has circular distribution and C_i and Z are free. By Proposition 3.3, we have

$$K_r(\sum_{i,j=1}^{M} d_{i,j} C_i Z C^*_j) = \mathrm{Tr}(D^r_M)\tau(Z^r) \xrightarrow[M\to\infty]{} \int_{\mathbb{R}} x^r d\mu(x) \tau(Z^r),$$

which finishes the proof.

□

The following corollary provides a random matrix model for the limit law from Theorem 3.1.

Corollary 3.5. *Let $X_{N\times NM}$ be as in Proposition 3.4 and $A_M = [a^{(M)}_{i,j}]$ be a sequence of selfadjoint $M \times M$ matrices as in Theorem 3.1. Let P_N be a sequence of $N \times N$ deterministic matrices all of whose moments with respect to the normalized trace converge to* 1*, e.g., the identity matrices $P_N = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]_N$ or any projection matrix of large rank like $P_N = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]_N - \frac{1}{N}\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]_N$, then the spectral measures of*

$$\frac{1}{M} X_{N\times NM}[A_M \otimes P_N] X^*_{N\times NM}$$

converge in distribution to the limit law described in Theorem 3.1.

Next we provide a random matrix model for the limit law μ from Theorem 3.1.

Proposition 3.6. *Let X_N be standard random matrix from the GUE of size $N\times N$ and let $D_N \in M_N(\mathbb{C})$ be a sequence of selfadjoint deterministic matrices such that D_N has limit distribution μ with respect to the nonnormalized trace. Then the random matrix sequence $X_N D_N X_N$ converges to the measure μ with respect to the nonnormalized trace.*

Remark 3.7. First observe that the preceding result is a special case of [18, Theorem 5.1 (i), $k = 1$], but our proof is different. On the other hand, the spectral measures of $X_N D_N X_N$ converge to zero with respect to the normalized trace. Indeed $\lim_{N\to\infty} \mathrm{Tr}(D^m_N)/N = 0$ and a sequence of standard GUE matrices is almost surely uniformly bounded. The point here is that with respect to the nonnormalized trace $\mathrm{Tr}(\cdot)$ we obtain interesting limits.

In order to prove Proposition 3.6 we will refer to a combinatorial result from random matrix theory, which we rewrite in terms of the nonnormalized trace. To formulate this result we need the following notation.

Notation 3.8. 1. We denote by $\mathcal{P}_2(m)$ the set of pair partitions, i.e., partitions of $\{1, 2, \dots, m\}$ into blocks of size 2; this set is empty unless m is even.

2. Let $\pi \in \mathcal{P}_2(m)$ be a pair partition. To each block $\{i, j\} \in \pi$ we associate the transposition $(i\, j)$ and we identify the pair partition π with the permutation obtained as the product of these transpositions. Since they are disjoint, this permutation is well defined.
3. Let $\sigma \in \mathfrak{S}_n$ be a permutation and $\sigma = \gamma_1\gamma_2 \dots \gamma_r$ be its cycle decomposition. Then for any family of matrices $A = (A_1, A_2, \dots, A_n)$ we denote by
$$\operatorname{Tr}_\sigma(A_1, A_2, \dots, A_n) = \operatorname{Tr}_{\gamma_1}(A) \operatorname{Tr}_{\gamma_2}(A) \cdots \operatorname{Tr}_{\gamma_r}(A)$$
where for a cycle $\gamma = (i_1\, i_2\, \cdots\, i_k)$ the cyclic trace is
$$\operatorname{Tr}_\gamma(A) = \operatorname{Tr}(A_{i_1} A_{i_2} \cdots A_{i_k}).$$

Proposition 3.9. [44, Proposition 22.32] *Let X_N be a standard $N \times N$ GUE matrix as in Proposition 3.6 and D be a constant $N \times N$ matrix.*
Then we have for all $m \in \mathbb{N}$, and all $q_1, \dots, q_m \in \mathbb{N}$, that
$$\operatorname{Tr} \otimes \mathbb{E}(X_N D^{q_1} \dots X_N D^{q_m}) = \sum_{\pi \in \mathcal{P}_2(m)} \operatorname{Tr}_{\pi\gamma}(D^{q_1}, \dots, D^{q_m}) N^{-m/2},$$
where $\gamma \in \mathfrak{S}_m$ is the cyclic permutation with one cycle $\gamma = (1, 2, \dots, m)$, $\pi\gamma$ is the composition of this cycle with the permutation π associated to the pair partition according to Notation 3.8.

Proof. The m-th nonnormalized moment of $X_N D_N X_N$ is then given by
$$\operatorname{Tr} \otimes \mathbb{E}[(X_N D_N X_N)^m] = \operatorname{Tr} \otimes \mathbb{E}(\underbrace{X_N D_N X_N I_N \dots X_N D_N X_N I_N}_{m-times})$$
where I_N is the identity matrix of size $N \times N$. Put $D = D_N$ in Proposition 3.9, then $D^0 = I_N$. The advantage of this interpretation becomes apparent from the fact that in this language we can rewrite our last equation as
$$= \sum_{\pi \in \mathcal{P}_2(2m)} \operatorname{Tr}_{\pi\gamma}(D_N, I_N, \dots, D_N, I_N) N^{-m}.$$
Now let us look at the asymptotic structure of this formula. We have to determine the cycles of the permutation $\pi\gamma$ which asymptotically contribute a non-zero factor. Recall that by assumption $\lim_{N\to\infty} \operatorname{Tr}(D_N^m)$ exists for all $m \in \mathbb{N}$. In this situation the factor N^{-m} is cancelled if and only if $\pi\gamma$ contains exactly the m singleton cycles $(2), \dots, (2m)$ and each of them contributes the factor $\operatorname{Tr}(I_N) = N$. This happens if and only if $\pi = \ulcorner\sqcap\, \sqcap \cdots \sqcap\urcorner$. Indeed in order to generate the singleton cycle (2), the partition π must contain the pair $\{2, 3\}$. To generate the cycle (4), the pair $\{4, 5\}$ must occur in π and so on. It follows that asymptotically the only non-zero contribution comes from the pair partition $\pi = \ulcorner\sqcap\, \sqcap \cdots \sqcap\urcorner$ which produces the permutation $\pi\gamma = (1, 3, \dots, 2m-1)(2)(4)\dots(2m)$, and thus
$$\begin{aligned}\lim_{N\to\infty} \operatorname{Tr} \otimes \mathbb{E}((X_N D_N X_N)^m) &= \lim_{N\to\infty} \operatorname{Tr}_{(1,3,\dots,2m-1)(2)(4)\dots(2m)}(D_N, I_N, \dots, D_N, I_N) N^{-m} \\ &= \lim_{N\to\infty} \operatorname{Tr}(D_N^m) \times N^m \times N^{-m} = \int_{\mathbb{R}} x^m d\mu(x).\end{aligned}$$
□

Corollary 3.10. *Let $A_N = [a_{i,j}^{(N)}] \in M_N(\mathbb{C})$ be as in Theorem 3.1, then the spectral measures of $\frac{1}{N} X_N A_N X_N$ converge with respect to the nonnormalized trace to the measure μ.*

4. Limit Theorem of Sums of Commutators and Anticommutators

We will now illustrate the limit theorem 3.1 with some interesting computable cases and start with sums of commutators and anticommutators, the most general expression being
$$\frac{\sum_{k<l} a(X_k X_l + X_l X_k) + bi(X_k X_l - X_l X_k)}{n}. \tag{4.1}$$

4.1. **A Limit Theorem for commutators and anticommutators.** The main contribution of this paper is the following limit theorem featuring the fundamental generating function of Carlitz and Scoville [17, (1.6)].

Theorem 4.1 (Free generalized tangent law)**.** *Let $X_1, X_2, \dots, X_n \in \mathcal{A}_{sa}$ be free centered copies of a random variable with finite non-zero variance 1, then for any a, $b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b \neq 0$, the limit law*

$$Q_n = \frac{1}{n} \sum_{\substack{k,j=1 \\ k<j}}^{n} \big(a(X_k X_j + X_j X_k) + ib(X_k X_j - X_j X_k) \big) \xrightarrow{d} Y,$$

has R-transform

$$R_Y(z) = \frac{\tan(bz)}{b - a\tan(bz)}.$$

The free cumulants are given by

$$K_r(Y) = b^{r-1} \frac{T_r(a/b)}{r!} = \frac{b^r a}{r!} P_r(a/b) = (-1)^r \frac{b^r a}{r!} \cot^{(r)}(\alpha).$$

where $\alpha = \operatorname{arccot}(a/b)$ and the polynomials $P_r(x)$, $T_r(x)$ were defined in Section 2.12.

Proof. The system matrix is $\frac{1}{n} A_n = \frac{1}{n} \left[\begin{smallmatrix} 0 & a+ib \\ a-ib & 0 \end{smallmatrix}\right]_n$ from Lemma 2.5 and its characteristic polynomial is

$$\chi_n(\lambda) = \frac{w(\lambda + \frac{\bar{w}}{n})^n - \bar{w}(\lambda + \frac{w}{n})^n}{w - \bar{w}}$$

where $w = a + bi$. The cumulant generating function

$$R_{Q_n}(z) = \sum_{k=1}^{\infty} \frac{\operatorname{Tr}(A_n^k)}{n^k} z^{k-1},$$

can be obtained from the logarithmic derivative of the characteristic polynomial. Indeed if we factorize the characteristic polynomial $\chi_n(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$ then

$$\frac{\chi_n'(\lambda)}{\chi_n(\lambda)} = \sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i}$$

and

$$\frac{1}{z} \frac{\chi_n'(1/z)}{\chi_n(1/z)} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \lambda_i^k z^k = n + z R_{Q_n}(z).$$

In our case

$$\frac{\chi_n'(\lambda)}{\chi_n(\lambda)} = n \frac{w(\lambda + \frac{\bar{w}}{n})^{n-1} - \bar{w}(\lambda + \frac{w}{n})^{n-1}}{w(\lambda + \frac{\bar{w}}{n})^n - \bar{w}(\lambda + \frac{w}{n})^n}$$

and

$$\begin{aligned} R_{Q_n}(z) &= \frac{1}{z} \left(\frac{1}{z} \frac{\chi_n'(1/z)}{\chi_n(1/z)} - n \right) \\ &= \frac{n}{z} \left(\frac{w(1 + \frac{z\bar{w}}{n})^{n-1} - \bar{w}(1 + \frac{zw}{n})^{n-1}}{w(1 + \frac{z\bar{w}}{n})^n - \bar{w}(1 + \frac{zw}{n})^n} - 1 \right) \\ &= -|w|^2 \frac{(1 + \frac{z\bar{w}}{n})^{n-1} - (1 + \frac{zw}{n})^{n-1}}{w(1 + \frac{z\bar{w}}{n})^n - \bar{w}(1 + \frac{zw}{n})^n}, \end{aligned}$$

and the limit is

$$\lim_{n\to\infty} R_{Q_n}(z) = R_Y(z) = -|w|^2 \frac{e^{z\bar{w}} - e^{zw}}{we^{z\bar{w}} - \bar{w}e^{zw}},$$

and finally substituting $w = a + ib$ ($|w| = 1$), we get

$$= \frac{-\exp(z(a-ib)) + \exp(z(a+ib))}{(a+ib)\exp(z(a-ib)) - (a-ib)\exp(z(a+ib))}$$
$$= \frac{2i\sin(bz)}{-2i(a\sin(bz) - b\cos(bz))} = \frac{\tan(bz)}{b - a\tan(bz)}.$$

Thus the R-transform can be expressed in terms of the generating function of the higher order tangent numbers (2.20) as $R(z) = \frac{1}{a}T(a/b, bz)$. The rest follows from simple manipulations using the combinatorics of tangent numbers discussed in Section 2.12. □

Remark 4.2. There is another proof in terms of Newton's identities, also known as the Newton-Girard formulae, which provide a relation between two types of symmetric polynomials, namely between power sums and elementary symmetric polynomials. Observe that

$$\chi_n(\lambda) = \sum_{j=0}^{n} \lambda^j \binom{n}{j} \left(\frac{(a+ib)(a-ib)^j - (a-ib)(a+ib)^j}{2ib} \right) =: \sum_{j=0}^{n} \lambda^j c_j,$$

whose n zeros are the numbers $\lambda_k = b\cot\frac{\alpha+k\pi}{n} - a$, $k \in \{0,\dots,n-1\}$. Let $s_r^n = \lambda_1^r + \cdots + \lambda_n^r$. By Newton's formulas for roots of a polynomial, we have for $r \in \mathbb{N}$

$$s_r^n + s_{r-1}^n c_1 + \cdots + s_1^n c_{k-1} + kc_k = 0 \text{ for } k \in \{1,\dots,n+1\}.$$

Dividing both sides of above equation by n^r, and pass with n to infinity for every fixed k we get

$$\sum_{j=0}^{k-1} \tilde{s}_{r-j} \left(\frac{(a+ib)(a-ib)^j - (a-ib)(a+ib)^j}{2ibj!} \right) + k\left(\frac{(a+ib)(a-ib)^k - (a-ib)(a+ib)^k}{2ibk!} \right) = 0,$$

where $\tilde{s}_r = \lim_{n\to\infty} s_r^n/n^r$. Recall that $R_Y(z) = \sum_{r=0}^{\infty} \tilde{s}_{r+1} z^r$ and by using Cauchy product of two infinite series, we see

$$R_Y(z)\left(1 + \frac{(a+ib)}{2ib}(\exp(z(a-ib)) - 1) - \frac{(a-ib)}{2ib}(\exp(z(a+ib)) - 1)\right)$$
$$= -\frac{(a+ib)(a-ib)}{2ib}\exp(z(a-ib)) + \frac{(a-ib)(a+ib)}{2ib}\exp(z(a+ib)),$$

which after a simple computation can be written in the desired form.

Remark 4.3. From Proposition 3.6 and Theorem 4.1 for $C_N = \frac{1}{N} X_N \left[\begin{smallmatrix} 0 & a+bi \\ a-ib & 0 \end{smallmatrix}\right]_N X_N$ we obtain a random matrix approximation of the following moment generating function (with respect to the non-normalized trace)

$$\lim_{N\to\infty} M_{C_N}(z) = 1 + \frac{z\tan(bz)}{b - a\tan(bz)}.$$

4.2. **The free tangent and zigzag laws.** In this subsection we indicate yet another method to prove the limit theorem 4.1 in some special cases, namely sums of commutators and anti-commutators. These are interesting because up to rescaling the limit cumulants are equal to the tangent numbers and Euler's zigzag numbers. Secondly, these cumulants are reminiscent of certain formulae for positive integer moments random variables related to the zeta function [14, see last line of Table 1]. Thirdly, we incidentally solve a problem stated in [17]. According to Theorem 3.1, in the above proofs we can restrict our sums to pair partitions and then the sums are simply traces of powers of the matrix.

Proposition 4.4 (Free tangent law). *Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be free copies of a random variable with finite non-zero variance 1, then*

$$Q_n = \frac{1}{n} \sum_{k,j=1}^{n} i(X_k X_j - X_j X_k) \xrightarrow{d} Y,$$

where $R_Y(z) = \tan(z)$. We call the limit law μ_Y the free tangent law.

Proof. First observe that by virtue of the cancellation phenomenon see [25, Theorem 4.4], we may assume without loss of generality that X_i are even random variables and moreover by Remark 3.2 that they are semicircular. Thus the cumulants can be computed using formula (2.15) and evaluate to

$$K_r\big(i \sum_{k,j=1}^{n} (X_k X_j - X_j X_k)\big) = \operatorname{Tr}(A_n^r) \text{ where } A_n = \left[\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right]_n .$$

The eigenvalues of the matrix A_n were computed in Lemma 2.5 and they are $\lambda_k = \cot\left(\frac{\pi}{2n} + \frac{k}{n}\pi\right)$ for $k \in \{0, \ldots, n-1\}$ (including repeated eigenvalues), hence the odd cumulants vanish and the even cumulants evaluate to

$$\begin{aligned} K_{2m}\big(\sum_{k,j=1}^{n} i(X_k X_j - X_j X_k)\big) &= \sum_{k=0}^{n-1} \cot^{2m}\left(\frac{\pi}{2n} + \frac{k}{n}\pi\right) \\ &= (-1)^m n + \frac{1}{(2m-1)!} \sum_{k=1}^{m} n^{2k} A_{2m}^{(2k)} T_{2k-1} \\ &= n^{2m} \frac{T_{2m-1}}{(2m-1)!} + \mathcal{O}(n^{2m-2}) \end{aligned}$$

where we used formula (2.23), with $A_{2m}^{(2m)} = 1$. Hence

$$\lim_{n\to\infty} K_{2m}(Q_n) = \frac{T_{2m-1}}{(2m-1)!}$$

and we conclude that

$$\lim_{n\to\infty} R_{Q_n}(z) = \tan(z).$$

□

Proposition 4.5 (Free zigzag law). *Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be free copies of a centered random variable with finite non-zero variance 1, then*

$$Q_n = \frac{1}{2n} \sum_{\substack{k,l=1 \\ k<l}}^{n} \big(X_k X_l + X_l X_k + i(X_k X_l - X_l X_k)\big) \xrightarrow{d} Y,$$

where $R_Y(z) = \frac{1}{2}(\tan(z) + \sec(z) - 1)$. The density of this law is shown in Fig. 3.

Proof. The matrix $A_n = \frac{1}{2}\left[\begin{smallmatrix} 0 & 1-i \\ 1+i & 0 \end{smallmatrix}\right]_n$ corresponds to $\alpha = \operatorname{arccot}(-1) = -\frac{\pi}{4}$ and Lemma 2.5 yields

$$\lambda_k = -\frac{1}{2}\cot\left(-\frac{\pi}{4n} + \frac{k}{n}\pi\right) - \frac{1}{2}, \text{ for } k \in \{1, \ldots, n\},$$

where range of the index variable k is shifted to $\{1, \ldots, n\}$. By the binomial theorem applied for $r \geqslant 2$, we see that

$$\begin{aligned} \sum_{k=1}^{n} \left(-\cot\left(-\frac{\pi}{4n} + \frac{k}{n}\pi\right) - 1\right)^r &= (-1)^r \sum_{j=0}^{r} \binom{r}{j} \left(\sum_{k=1}^{n} \cot^{r-j}\left(-\frac{\pi}{4n} + \frac{k}{n}\pi\right)\right) \\ &= \frac{E_{r-1}}{(r-1)!} 2^{r-1} n^r + \mathcal{O}(n^{r-1}). \end{aligned}$$

by (2.24).
Finally for $r \geqslant 2$, we get

$$\begin{aligned}\tilde{K}_r &= \lim_{n\to\infty} K_r(Q_n) \\ &= \lim_{n\to\infty} \frac{1}{n^r}\sum_{k=1}^n \lambda_k^r = \lim_{n\to\infty} \frac{1}{(2n)^r}\sum_{k=1}^n \left(-\cot\left(-\frac{\pi}{4n}+\frac{k}{n}\pi\right)-1\right)^r \\ &= \frac{E_{r-1}}{2(r-1)!}.\end{aligned}$$

The first cumulant is $\tilde{K}_1 = \frac{1}{2n}\operatorname{Tr}\left(\left[\begin{smallmatrix} 0 & 1-i \\ 1+i & 0\end{smallmatrix}\right]_n\right) = 0$ and hence the desired R-transform is

$$R_Y(z) = \sum_{r=0}^\infty \tilde{K}_{r+1} z^r = \frac{1}{2}\sum_{r=1}^\infty \frac{E_r}{r!} z^r = \frac{\tan(z)+\sec(z)-1}{2}.$$

□

Remark 4.6. The above results coincide with Theorem 4.1. Indeed, if we use use the scaling appropriate for Theorem 4.1, i.e., $\frac{1}{\sqrt{2}n}$, then

$$R_{\sqrt{2}Y}(z) = \frac{\tan(\sqrt{2}z)+\sec(\sqrt{2}z)-1}{\sqrt{2}}$$

by the identity $\tan(z)+\sec(z) = \frac{1+\tan(z/2)}{1-\tan(z/2)}$, we have

$$= \frac{\tan(z/\sqrt{2})}{1/\sqrt{2}-\tan(z/\sqrt{2})/\sqrt{2}}.$$

It is interesting to compare the power series expansion of Theorem 4.1 for $a = b = \frac{1}{\sqrt{2}}$ with $\frac{\tan(\sqrt{2}z)+\sec(\sqrt{2}z)-1}{\sqrt{2}}$, because it shows the identity

$$\sum_{k=0}^{n-1} T_n^{(k+1)} = 2^{n-1}E_n.$$

This provides a new answer to a question of Carlitz and Scoville [17, equ. (2.19) on p. 418] who assert that "the numbers $\sum_{k=0}^{n-1} T_n^{(k+1)}$ are not easily evaluated"; see [21, Prop. 6] for another proof. This sequence is catalogued as $A000828$ in Sloane's database [48] and the numbers are half of the *Euler numbers of type B*, see [39].

5. Spectral radius, density, Lévy-Khinchin representation and Bercovici-Pata bijection of the tangent laws

5.1. The spectral radius of the tangent law.

Proposition 5.1. *The spectral radius of the tangent law (the limit law of Corollary 4.4) is given by*

$$\rho = \frac{1}{\varpi}(1+\sqrt{1-\varpi^2}) \simeq 2.2644374158937358461$$

where $\varpi \approx 0.7390851332$ *is the iterated cosine constant, i.e., the unique fixed point of the equation* $x = \cos x$.

Proof. Since the moments are nonnegative, Pringsheim's theorem (see [53, Sec. 7.21] or [40, Sec. 3.6]) implies that the principal singularity of the Cauchy transform lies on the positive real axis and the spectral radius can be computed as

$$\rho = \inf_{t>0} K(t)$$

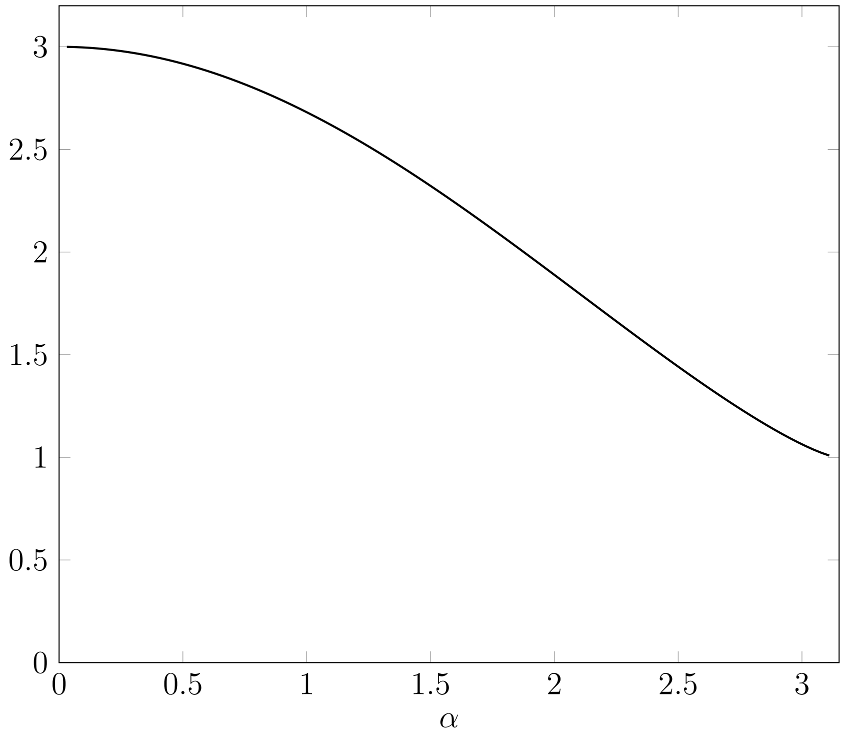

FIGURE 1. The spectral radius of the generalized free tangent laws

see [59, Ch. 9.C]. In order to compute the minimum of the function

$$K(t) = \frac{1}{t} + \tan t$$

we compute the roots of its derivative

$$K'(t) = -\frac{1}{t^2} + \frac{1}{\cos^2 t}.$$

The unique root satisfies the equation $\cos^2 t = t^2$, i.e., $t = \pm ա$ and thus

$$\rho = \frac{1}{ա} + \frac{\sin ա}{\cos ա} = \frac{1}{ա}\left(1 + \sqrt{1 - ա^2}\right)$$

□

Remark 5.2. The number ա (Armenian letter "ayb") comes up from time to time in the literature, starting at least back in the 19th century in the 4th edition of Bertrands *Traité d'algèbre* [13], continuing with numerical efforts by T.H. Miller [42] and the dedicated investigation by G.B. Arakelian [4]. This number is well known among generations of high school students who saw it appear on their electronic calculators when they started to repeatedly press the "cos" button during boring math classes, see [36, 46] for discussions.

5.2. The spectral radius of the generalized free tangent laws.

Proposition 5.3. *The spectral radius of the generalized limit law from Theorem 4.1 for $a+ib = e^{i\alpha}$, where $0 < \alpha < \pi$ is given by*

$$\rho_\alpha = \frac{1}{ա_\alpha}(\sin\alpha + \sin ա_\alpha)$$

where $ա_\alpha$ is the unique solution x of the equation

$$x = \sin(\alpha - x).$$

The dependency of the spectral radius on the parameter α is shown in Figure 1.

Proof. We proceed as in the proof of Proposition 5.1, the objective function now being

$$K(t) = \frac{1}{t} + \frac{\tan bt}{b - a\tan bt} = \frac{1}{t} + \frac{\sin bt}{\sin\alpha\cos bt - \cos\alpha\sin bt} = \frac{1}{t} + \frac{\sin bt}{\sin(\alpha - bt)}.$$

Its derivative is

$$K'(t) = -\frac{1}{t^2} + \frac{\sin^2\alpha}{\sin^2(\alpha - bt)}$$

and setting $x = bt$ the infimum is attained at the unique positive solution of the equation

$$x = \sin(\alpha - x).$$

□

5.3. **The Lévy measure of the tangent law.** The tangent function is a prominent positive definite function, see [23], and it is a fundamental example of Nevanlinna functions. Thus the free tangent law is $\boxplus$-infinitely divisible and its Lévy measure can be computed using the method from Section 2.4. To this end we consider the Voiculescu transform $\phi(z) = \tan(\frac{1}{z})$. The nontangential limit of its imaginary part is

$$\lim_{\epsilon\to 0^+} \operatorname{Im}\phi(x + i\epsilon) = \lim_{\epsilon\to 0^+} \operatorname{Im}\tan\left(\frac{1}{x + i\epsilon}\right) = 0$$

and thus the Lévy measure has no absolutely continuous part. In order to determine the atoms we compute the nontangential limits (2.7). Now

$$\lim_{\epsilon\to 0^+} i\epsilon\tan\left(\frac{1}{x + i\epsilon}\right) = 0$$

whenever x is not a pole of $\tan(1/x)$, i.e., $x \neq \frac{1}{\frac{\pi}{2}+k\pi}$, $k \in \mathbb{Z}$. On the other hand for $x = \frac{1}{\frac{\pi}{2}+k\pi}$ we get via de L'Hospital's rule

$$\lim_{\epsilon\to 0^+} i\epsilon\tan\left(\frac{1}{x + i\epsilon}\right) = \lim_{\epsilon\to 0^+} \frac{i\epsilon}{\cot\left(\frac{1}{x+i\epsilon}\right)} = \lim_{\epsilon\to 0^+} \frac{i}{\frac{-1}{\cos^2\left(\frac{1}{x+i\epsilon}\right)}\frac{-i}{(x+i\epsilon)^2}} = x^2.$$

Finally from (2.7) we infer that the Lévy measure is given by

$$\nu(\{x\}) = \begin{cases} 1 & \text{for } x = \frac{2}{n\pi} \text{ with } n \in \mathbb{Z} \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, this result can be verified as follows. From the well-known identity $\sum_{n\in\mathbb{N}\text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$ we conclude that the tangent distribution has free characteristic triplet $(0, 0, \nu)$ and we have

$$\begin{aligned}
\mathcal{C}_\mu(z) &= \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz\mathbf{1}_{\{|x|<1\}}(x)\right) d\nu(x) \\
&= \int_{\mathbb{R}} \left(\frac{(xz)^2}{1 - xz}\right) d\nu(x) \\
&= \sum_{n\in\mathbb{Z}\text{ odd}} \frac{1}{1 - \frac{2z}{n\pi}} \frac{4z^2}{n^2\pi^2} \\
&= \sum_{n\in\mathbb{N}\text{ odd}} \frac{2}{1 - \frac{4z^2}{n^2\pi^2}} \frac{4z^2}{n^2\pi^2} \\
&= \sum_{n\in\mathbb{N}\text{ odd}} \frac{8z^2}{n^2\pi^2 - 4z^2}.
\end{aligned}$$

Now Euler's well known partial fraction expansion of the contangent function [1, Ch. 25]

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}$$

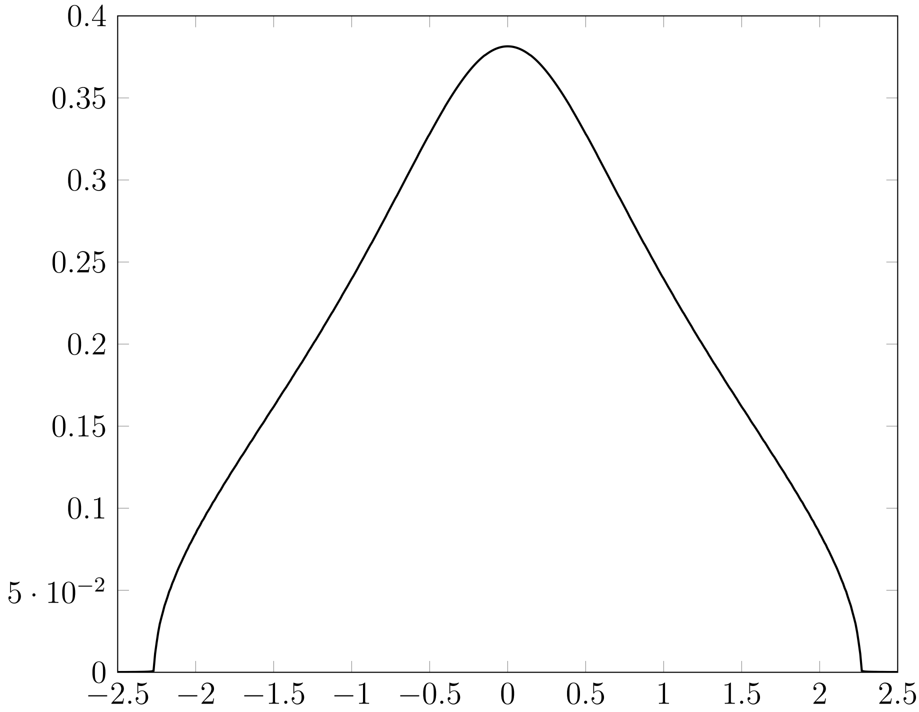

FIGURE 2. Density of the free tangent law

immediately yields a similar expansion for the tangent function

$$\tan z = \cot z - 2\cot 2z = \sum_{k=1}^{\infty} \frac{8z}{(2k-1)^2\pi^2 - 4z^2}$$

for $z \neq 0$ and thus indeed $\frac{1}{z}\,\mathcal{C}_\mu(z) = \tan(z)$.

5.4. **The Lévy measure of the generalized tangent laws.** The corresponding Lévy measure in the general case is supported on the points $x = \frac{b}{\arctan(b/a)+k\pi}$ for $k \in \mathbb{Z}$, with weight 1. This follows from the fact that $\lim_{\epsilon\to 0^+} i\epsilon \frac{\tan(\frac{b}{x+i\epsilon})}{b-a\tan(\frac{b}{x+i\epsilon})} = x^2$. We leave the formal proof to the reader.

5.5. **The density of the tangent law.** The free characteristic triplet of the free tangent law is $a = 0$ and $\nu(\mathbb{R}) = 1$ and it follows from the criterion [30, Theorem 3.4 part (2)] that the free tangent law is absolutely continuous with respect to Lebesgue measure. Moreover, Huang (see [33, Theorem 3.10] or [34]) derived a formula for the absolutely continuous part μ^{ac} by using the transform $F_\mu^{-1}(z) = z + \tan(\frac{1}{z})$. Define a continuous map on $\mathbb{R}$ by

$$\begin{aligned} v_\mu(x) :&= \inf\{y > 0 \mid \operatorname{Im}(F_\mu^{-1}(x+iy)) > 0\} \\ &= \inf\left\{ y > 0 \mid y - \frac{\sinh\frac{2y}{x^2+y^2}}{\cosh\frac{2y}{x^2+y^2} + \cos\frac{2x}{x^2+y^2}} > 0 \right\}. \end{aligned}$$

Huang proved that we can define $\psi_\mu(x) = F_\mu^{-1}(x + iv_\mu(x))$, for $x \in \mathbb{R}$, which is a homeomorphism of $\mathbb{R}$ and then we have

$$\frac{d\mu^{ac}}{dx}(\psi_\mu(x)) = \frac{v_\mu(x)}{\pi(x^2 + v_\mu^2(x))}.$$

The densities of the free tangent law and the free zigzag laws are shown in Figures 2 and 3; the densities of the generalized free tangent laws for $0 < \alpha < \pi$ are shown in Figure 4.

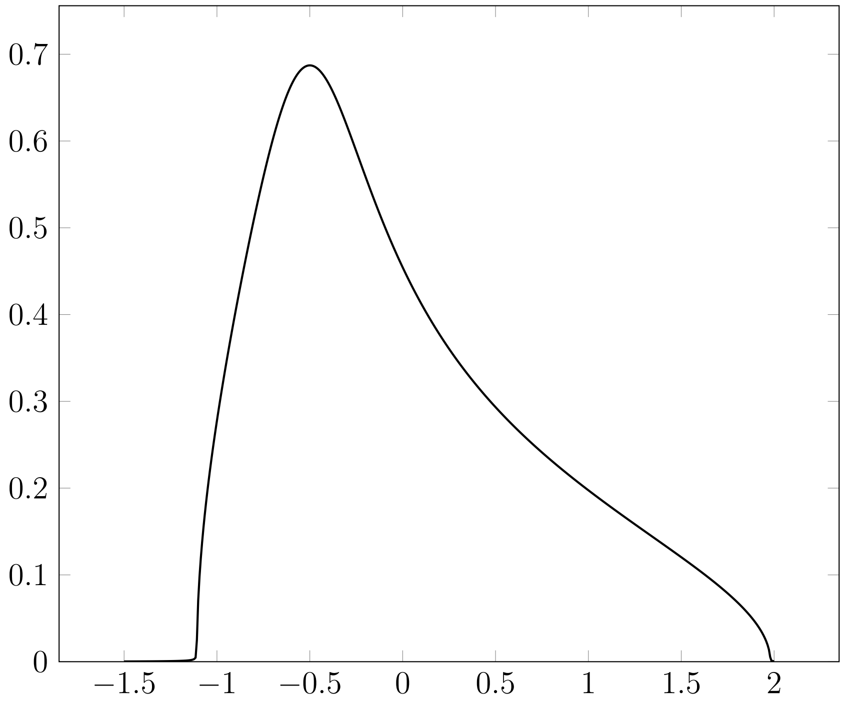

FIGURE 3. Density of the free zigzag law

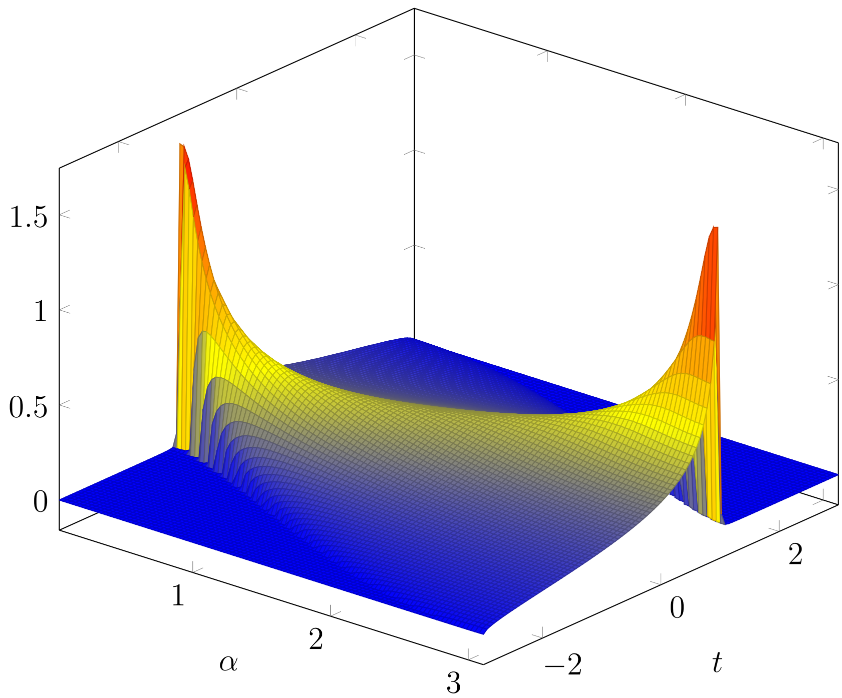

FIGURE 4. Densities of the generalized free tangent laws depending on α

5.6. **Bercovici-Pata bijection.** An important connection between free and classical infinite divisibility was established by Bercovici and Pata [10] in the form of a bijection Λ from the class of classical infinitely divisible laws to the class of free infinitely divisible laws. The easiest way to define the B-P bijection is as follows. Let μ be a probability measure in $ID(*)$ having all moments, and consider its sequence c_n of classical cumulants. Then the map Λ can be defined as the mapping that sends μ to the probability measure on $\mathbb{R}$ with free cumulants c_n.

The inverse image of the free tangent law under the Bercovici-Pata bijection has the following characteristic function

$$\log \mathbb{E}(\exp(zX)) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\frac{z^{2n}}{(2n)!}$$

and using Euler's identity $\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!}$ this is

$$= 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)!}\left(\frac{2z}{\pi}\right)^{2n} - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)!}\left(\frac{z}{\pi}\right)^{2n}$$

now using the expansion

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)!}z^{2n} = \sum_{n=1}^{\infty}\frac{\sum_{k=1}^{\infty}1/k^{2n}}{(2n)!}z^{2n} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\left(\frac{z}{k}\right)^{2n} = \sum_{k=1}^{\infty}(\cosh(z/k)-1)$$

we get further

$$\begin{aligned} &= 2\sum_{n=1}^{\infty}\left(\cosh\left(\frac{2z}{\pi n}\right)-1\right) - 2\sum_{n=1}^{\infty}\left(\cosh\left(\frac{z}{\pi n}\right)-1\right) \\ &= 2\sum_{n\in\mathbb{N}\text{ odd}}\left(\cosh\left(\frac{2z}{\pi n}\right)-1\right). \end{aligned}$$

Thus by using $\cosh(it) = \cos(t)$, we obtain the characteristic function

$$\mathbb{E}(\exp(itX)) = \exp\left[2\sum_{n\in\mathbb{N}\text{ odd}}\left(\cos\left(\frac{2t}{\pi n}\right)-1\right)\right] = \prod_{n\in\mathbb{N}\text{ odd}}\exp\left[2\left(\cos\left(\frac{2t}{\pi n}\right)-1\right)\right]$$

Note that $\exp(2\cos(t)-2)$ is the characteristic function of Skellam distribution X, i.e.,

$$P(X=k) = P(X=-k) = \frac{I_k(2)}{\exp(2)} \text{ for } k\in\mathbb{N},$$

where I_n is n-th modified Bessel function of the first kind see [47]. Hence $\exp\left[2\left(\cos\left(\frac{2t}{\pi n}\right)-1\right)\right]$ is the characteristic function of the random variable $\frac{2}{n\pi}X$ and we conclude that the classical distribution corresponding to the free tangent law under the B-P bijection is the law of the random variable $\sum_{n\in\mathbb{N}\text{ odd}}\tilde{X}_n$, where $\tilde{X}_n$ are independent random variables such that $\tilde{X}_n$ has the same distribution as $\frac{2}{n\pi}X$, $n\in\mathbb{N}$ odd.

6. Concluding Remarks

6.1. **Sums of anticommutators.** Instead of the sums of commutators one can also consider sums of the anti-commutators $\frac{1}{n}\sum_{i<j}^{n}(X_iX_j + X_jX_i)$ for sequence of standard free semicircular variables. Contrary to the case $n=1$, where the distribution of the anticommutator $XY+YX$ coincides with the distribution of the commutator $i(XY-YX)$ [44, Remark 19.8 (3)], this leads to new distributions for $n\geqslant 3$ and subsequently also in the limit as n tends to infinity. Indeed the spectrum of the corresponding matrix $A_n = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]_n$ consists of the two eigenvalues $\lambda=-1$ and $\lambda=n-1$ with respective multiplicities $n-1$ and 1. Thus in the limit the rth cumulant is equal to

$$\lim_{n\to\infty}\frac{(-1)^r(n-1)+(n-1)^r}{n^r} = \begin{cases}1 & \text{for } r\neq 1\\ 0 & \text{for } r=1.\end{cases}$$

This corresponds to the Marchenko-Pastur (or free Poisson) distribution. Observe that we can reconstruct the Marchenko-Pastur distribution from Theorem 4.1 by passing to the limit

$$\lim_{b\to 0} \frac{\tan(bz)}{b - \pm\sqrt{1-b^2}\tan(bz)} = \lim_{b\to 0} \frac{\frac{\sin(bz)}{bz}}{\frac{\cos(bz)}{z} - \pm\sqrt{1-b^2}\frac{\sin(bz)}{bz}} = \frac{z}{1-\pm z},$$

which is the R-transform of the free Poisson distribution.
Such interpolations have attracted some attention in connection with random matrices. As an application of Corollary 3.5 we present an interpolation on the unit circle $w = e^{i\alpha}$, $\alpha \in [0, 2\pi)$ between the Marchenko-Pastur law [41] $\alpha = 0$, free tangent law $\alpha = \frac{\pi}{2}$ and free zigzag law $\alpha = \frac{\pi}{4}$ in the context of random matrices

$$\lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{M} X_{N\times NM}\Big[\left[\begin{smallmatrix} 0 & w \\ \bar{w} & 0 \end{smallmatrix}\right]_M \otimes P_N \Big] X^*_{N\times NM} = Y$$

where $R_Y(z) = \frac{\tan(z \operatorname{Im} w)}{\operatorname{Im} w - \operatorname{Re} w \tan(z \operatorname{Im} w)}$. Thus we are led to measures which might be called generalized Marchenko-Pastur laws.

6.2. **The trace method for tangent numbers and the Riemann zeta function.** Propositions 4.4 and 4.5 lead to another new fact about the tangent numbers T_n, the Euler zigzag numbers E_n, the Riemann zeta function and the Bernoulli numbers for even values, namely

$$T_k = \lim_{n\to\infty} \frac{(2k-1)! \operatorname{Tr}\left(\left[\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right]_n^{2k} \right)}{n^{2k}}, \; E_k = \lim_{n\to\infty} \frac{k! \operatorname{Tr}\left(\left[\begin{smallmatrix} 0 & 1+i \\ 1-i & 0 \end{smallmatrix}\right]_n^{k+1} \right)}{2^k n^{k+1}},$$

$$\zeta(2k) = \lim_{n\to\infty} \frac{\pi^{2k} \operatorname{Tr}\left(\left[\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right]_n^{2k} \right)}{2n^{2k}(2^{2k}-1)}, B_{2k} = \lim_{n\to\infty} \frac{(2k)! \operatorname{Tr}\left(\left[\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right]_n^{2k} \right)}{(-1)^{k+1} 2^{2k}(2^{2k}-1) n^{2k}} \quad \text{for } k \in \mathbb{N}.$$

Approximation of the values of the Riemann zeta function for even integers is a popular theme, see [58, 3, 22]. It would be particularly interesting to obtain approximations for odd integers as well, but for this one would have to compute the singular values of the matrix A_n.

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(Wiktor Ejsmont) Instytut Matematyczny, Uniwersytet Wrocławski,, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: `wiktor.ejsmont@gmial.com`

(Franz Lehner) Department of Discrete Mathematics TU Graz Steyrergasse 30, 8010 Graz, Austria
E-mail address: `lehner@math.tugraz.at`